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INFINITE DISCRETE CHAINS AND THE MAXIMAL NUMBER OF COUNTABLE MODELS

The paper is aimed at studying the countable spectrum of small linearly ordered theories. The objectives of the research are to study the structural properties of countable linearly ordered theories, as well as to promote the solution to the well-known open problem of model theory, Vaught's conjecture, which assumes that the number of countable models of a countable complete first-order theory cannot be equal to $\aleph_1$. An important step in solving Vaught's conjecture is the search for conditions under which the theory has the maximal number of countable pairwise non-isomorphic models. By limiting ourselves to linearly ordered theories we do not get special advantages from the viewpoint of studying their countable spectrum. Therefore, in the article, a restriction on 1-types and 1-formulas of the theory is introduced. It is proved that a small countable linearly ordered theory that satisfies the restriction and has an infinite discrete chain has the maximal number of countable non-isomorphic models. To build models, the authors use the method of constructing countable models over countable sets, based on the Tarski-Vaught criterion. It is shown that it is possible to carry out the construction in such a way that the types of unnecessary elements in the resulting model are omitted, what guarantees non-isomorphism of the models and their maximal number.
Key words: small theory, linear order, countable model, number of countable models, discrete chain, omitting types.

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Шектеусіз дискретті тізбектер және саналымды модельдердің максималды саны

Бұл мақала шағын сызықтық реттелген теориялардың саналымды спектрін зерттеуге бағытталған. Зерттеудің мақсаты саналымды сызықтық реттелген теориялардың құрылымдық қасиеттерін зерттеу, сонымен қатар модельдер теориясының белгілі ашық проблемасы – бірінші ретті толық теорияның саналымды модельдерінің саны $\aleph_1$-ге тең болмайды деп болжайтын Воот гипотезасын шешуді алға жылжыту. Теорияның екеуара изоморфты емес саналымды модельдердің саны максималды болатын шарттарды іздеу Воот гипотезасын шешудегі маңызды қадам. Сызықтық реттелген теориялармен шектеле отырып, біз олардың саналымды спектрін зерттеу тұрғысынан ерекше артықшылықтарға ие болмаймыз. Сондықтан осы теорияның 1-типтері мен 1-формулаларына шектеу енгізіледі. Мақалада осы шектеуді қанағаттандыратын және шектеусіз дискретті тізбегі бар, саналымды сызықтық реттелген шағын теориясының изоморфты емес саналымды модельдердің максималды саны бар екендігі дәлелденді. Модельдерді құру үшін авторлар Тарский-Воот өлшемшартына негізделген саналымды жиындарының үстінен саналымды модельдерін құру әдісін қолданады. Құрылысты алынған модельдегі қажет емес элементтердің типтерін түсіріп жасауға болатындығы көрсетілген, бұл модельдердің изоморфизм болмауына және олардың максималды саны бар екендігіне кепілдік береді.
Түйін сөздер: шағын теория, сызықтық рет, саналымды модель, саналымды модельдердің саны, дискретті тізбе, типтерді төмендету.

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Бесконечные дискретные цепи и максимальное число счётных моделей

Данная статья направлена на изучение счётного спектра малых линейно упорядоченных теорий. Целями исследования являются изучение структурных свойств счётных линейно упорядоченных теорий, а также, продвижение решения известной открытой проблемы теории моделей – гипотезы Воота, которая предполагает, что числе счётных моделей счётной полной теории первого порядка не может равняться $\aleph_1$. Важным шагом в решении гипотезы Воота является поиск условий, при которых теория имеет максимальное число счётных попарно неизоморфных моделей. Ограничиваясь линейно упорядоченными теориями, мы не получаем особых преимуществ с точки зрения изучения их счётного спектра. Поэтому, в статье, будет введено ограничение на 1-типы и 1-формулы данной теории. В статье доказывается, что малая счётная линейно упорядоченная теория, удовлетворяющая данному ограничению и имеющая бесконечную дискретную цепь, имеет максимальное число счётных неизоморфных моделей. Для построения моделей авторы применяют метод построения счётных моделей над счётными множествами, основанный на критерии Тарского-Воота. Показывается, что можно провести построение таким образом, что типы ненужных элементов в полученной модели опускаются, что гарантирует не изоморфизм моделей и их максимальное количество.

Ключевые слова: малая теория, линейный порядок, счётная модель, число счётных моделей, дискретная цепь, опускание типов.

1 Introduction

Vaught's conjecture states that if the continuum hypothesis fails, for a countable complete theory T $I(T,\aleph_0)$ is either finite, $\aleph_0$ or $2^{\aleph}_0$. Vaught's conjecture was confirmed for various classes of theories: [1–6]. But for countable theories in general, this question is still open.

A theory T is said to be small if $|\bigcup\limits_{n<\omega} S_n(T)| = \aleph_0$. If a countable theory is not small, it has the maximal number of countable non-isomorphic models. Therefore, in the article, we restrict to studying small theories. We want to find theories that have the maximal number of countable models.

Theorem 1 *[7–9] Every countable model $\mathfrak{M}$ of a small theory T can be represented as a union of some elementary chain $(\mathfrak{M}(\bar{a}_i))_{i\in\omega}$ of prime models over the tuples $\bar{a}_i$.*

In Theorem 1 K.Zh. Kudaibergenov and S.V. Sudoplatov used a special method to inductively reconstruct a countable model of a small theory. The authors applied modified versions of this method to construct new models of small theories [10–13] as elementary submodels of an $\aleph_1$-saturated model. In the article, we consider one more application of such construction and prove that a small ordered theory that satisfies a special restriction and has a model with an arbitrarily long finite, and, therefore, with an infinite, discrete chain has the maximal countable spectrum.

2 Main Part

We use Gothic letters $\mathfrak{A}$, $\mathfrak{M}$, $\mathfrak{N}$, ... to denote structures, and we use capital letters (A, M, N, ...) for universes of those structures.

For subsets $A, B \subseteq M$ of a structure $\mathfrak{M}$ of a linearly ordered theory T we use the following notations:

$$A^+ := \{\gamma \in M \mid \text{for all } a \in A,\ \mathfrak{M} \models a < \gamma\};$$
$$A^- := \{\gamma \in M \mid \text{for all } a \in A,\ \mathfrak{M} \models \gamma < a\}.$$

We write $A < B$ if for all $a \in A$, $b \in B$ $\mathfrak{M} \models a < b$. If A and B are C-definable ($C \subseteq M$), then A^+, A^- and $A < B$ are also C-definable.

For a 2-formula $\varphi(x, y)$ denote $\varphi(x, y)^- := \forall z\big(\varphi(z, y) \to z > x\big)$, and $\varphi(x, y)^+ := \forall z\big(\varphi(z, y) \to z < x\big)$.

Definition 1 *The set A is convex in a set $B \supseteq A$ if for all $a, b \in A$ and all $c \in B$, $a < c < b$ implies $c \in A$. The set A is convex if it is convex in M.*

Definition 2 *A formula $\varphi(x, \bar{y}, \bar{a})$ is said to be a convex formula, if for every $\bar{b} \in M$ $\varphi(M, \bar{b}, \bar{a})$ is convex in every model of $\mathfrak{M} \models T$ containing $\bar{b}$ and $\bar{a}$.*

Definition 3 *[14] 1) A convex closure of a formula $\varphi(x, \bar{a})$ is the formula:*

$$\varphi^c(x, \bar{a}) := \exists y_1 \exists y_2 \big(\varphi(y_1, \bar{a}) \wedge \varphi(y_2, \bar{a}) \wedge (y_1 \leq x \leq y_2)\big).$$

2) A convex closure of a type $p(x) \in S_1(A)$ is the following type:

$$p^c(x) := \{\varphi^c(x, \bar{a}) \mid \varphi(x, \bar{a}) \in p\}.$$

Similarly, $tp^c(\alpha/A) := \{\varphi^c(x, \bar{a}) \mid \varphi(x, \bar{a}) \in tp(\alpha/A)\}$.

In weakly o-minimal theories, and, therefore, in o-minimal theories, $p^c(\mathfrak{M}) = p(\mathfrak{M})$ for every $p \in S_1(A)$.

Definition 4 *[15, 16] Let $\mathfrak{M}$ be a linearly ordered structure, $A \subseteq M$, $\mathfrak{M}$ be $|A|^+$-saturated, and $p \in S_1(A)$ be a non-algebraic type.*

1) An A-definable formula $\varphi(x, y)$ is p–preserving if there are α, γ_1, $\gamma_2 \in p(\mathfrak{M})$ such that $p(\mathfrak{M}) \cap \big(\varphi(M, \alpha) \setminus \{\alpha\}\big) \neq \emptyset$ and $\gamma_1 < \varphi(M, \alpha) < \gamma_2$.

3) A p-preserving formula $\varphi(x, y)$ convex to the right (left) on the type p if there is $\alpha \in p(\mathfrak{M})$ for which $p(\mathfrak{M}) \cap \varphi(M, \alpha)$ is convex, α is the left (right) endpoint of the set $\varphi(M, \alpha)$, and $\alpha \in \varphi(M, \alpha)$.

We introduce special notations for the following formulas:

$$S(x, y) := x < y \wedge \forall t\big(x \leq t \leq y \to t = x \vee t = y\big);$$

$$S_0(x, y) := (x = y);$$

and for all $n \geq 1$,

$$S_n(x,y) := \exists z_1, \exists z_2, ..., \exists z_{n+1}\big(x = z_1 < z_2 < ... < z_{n+1} = y \wedge \bigwedge_{1\leq i\leq n} S(z_i, z_{i+1})\big);$$

$$S_{-n}(x,y) := \exists z_1, \exists z_2, ..., \exists z_{n+1}\big(x = z_1 < z_2 < ... < z_{n+1} = y \wedge \bigwedge_{1\leq i\leq n} S(z_{i+1}, z_i)\big).$$

Definition 5 *Let $p_1, p_2 \in S_1(A)$. We say that p_2 is attached to p_1, if there exists $n \in \mathbb{Z}$ such that for some $\delta_1 \in p_1(\mathfrak{M})$ the element $\delta_2 \in S_n(\delta_1, M)$ realizes p_2.*

The relation of attachment of types is an equivalence relation on the set $S_1(A)$.

Restriction 1 *We restrict to theories T such that for every $M \models T$ and every $A \subseteq M$, where $\mathfrak{M}$ is $|A|^+$-saturated, there is no infinite family $P \subseteq S_1(A)$ of pairwise attached non-isolated types such that for every A-definable formula φ such that φ and $\neg\varphi$ belong to at least one type of P each,*

1) each φ and $\neg\varphi$ is in an infinite number of types from P;

2) for every δ realizing some type from P, for every $n \in \mathbb{Z}$ there exist $m_n^1, m_n^2 > n$ and $m_n^3, m_n^4 < n$ such that $S_{m_n^1}(\delta, M) \cap \varphi(M) \neq \emptyset$, $S_{m_n^2}(\delta, M) \cap \neg\varphi(M) \neq \emptyset$, $S_{m_n^3}(\delta, M) \cap \varphi(M) \neq \emptyset$, and $S_{m_n^4}(\delta, M) \cap \neg\varphi(M) \neq \emptyset$.

All weakly o-minimal theories satisfy the Restriction 1.

Theorem 2 *Let $\mathfrak{M}$ be a sufficiently saturated model of a small linearly ordered theory T that satisfies Restriction 1. If in $\mathfrak{M}$ there exists an infinite discrete chain, then $I(T, \aleph_0) = 2^{\aleph_0}$.*

Proof of Theorem 2. For all $n \geq 1$ denote

$$P_n(x,y) := \exists z_1 \exists z_2 ... \exists z_n \Big(x = z_1 < z_2 < ... < z_n = y\Big) \wedge \forall z \Big(x < z < y \rightarrow \exists z_1 \exists z_2 \big(S(z_1, z) \wedge S(z, z_2)\big)\Big).$$

Consider the consistent set $p_0 := \{P_n(x,y)\}_{n\geq 1}$, and let $(\alpha_0, \beta_0) \in q_0(M)$. For $n < \omega$ denote $Q_n(z, \alpha_0, \beta_0) := \exists x_1 \exists x_2 ... \exists x_n \exists y_1 \exists y_2 ... \exists y_n (\alpha_0 = x_1 < x_2 < ... < x_n < z < y_n < ... < y_2 < y_1 = \beta_0)$. Then $q(z) := \{Q_n(z, \alpha_0, \beta_0)\}_{n<\omega}$ is locally consistent.

The given theory T is small, therefore the theory $T \cup tp(\alpha_0\beta_0)$ is small as well. Since in a countable model there is only a countable number of places to choose a realization of a 2-type, if we prove that $T \cup tp(\alpha_0\beta_0)$ has $2^{\aleph_0}$ countable models, then T also has. So, for convenience, we add the elements α_0 and β_0 to our language and replace T with the theory $T \cup tp(\alpha_0\beta_0)$.

There are two cases.

1) Suppose that there exist $\delta_1, \delta_2 \in q(\mathfrak{M})$ with $tp(\delta_1) = tp(\delta_2) =: r$, and there exists $c \geq 1$ such that $\mathfrak{M} \models S_c(\delta_1, \delta_2)$. Then for every $m \in \mathbb{Z}$, $S_{mc}(\delta_1, M) \subseteq r(\mathfrak{M})$.

For $m \in \mathbb{N}$ denote $S^m(x,y) := \forall z (S_{mc}(y,z) \rightarrow y \leq x \leq z)$.

Let $p_1(x,y) := p_0(x,y) \cup r(x) \cup r(y) \cup \Big\{\varphi_R(y,x) \mid \varphi_R$ is a convex to the right on the type r formula such that for all $m \in \mathbb{N}$ and all $\alpha \in r(\mathfrak{M})$, $\big((S^m(\alpha, M) \cap r(\mathfrak{M})) \subseteq \varphi_R(M, \alpha)\big)\Big\} \cup$

$\Big\{\varphi_L(x,y) \mid \varphi_L(x,y)$ is a convex to the left on r formula such that for all $n \in \mathbb{N}$ and all $\beta \in r(\mathfrak{M})$, $\big((S^{-m}(\beta, M) \cap r(\mathfrak{M})) \subseteq \varphi_L(M,\beta)\big)\Big\}$. Consistence of $p_1(x,y)$ can be verified directly. If T is weakly o-minimal, then p_1 is a complete 2-type.

For $\gamma, \gamma_1, \gamma_2 \in q(\mathfrak{M})$ and $n \in \mathbb{Z}$ we define neighborhoods of elements and intervals between neighborhoods:

$$V_{q(\mathfrak{M})}(\gamma) := \{\gamma' \in q(\mathfrak{M}) \mid S_m(\gamma,\gamma') \text{ for some } m \in \mathbb{Z}\};$$
$$V^n_{q(\mathfrak{M})}(\gamma) := \{\gamma' \in q(\mathfrak{M}) \mid S_{mn}(\gamma,\gamma') \text{ for some } m \in \mathbb{Z}\};$$
$$(V_q(\gamma_1), V_p(\gamma_2))_{q(\mathfrak{M})} := \{\gamma' \in q(\mathfrak{M}) \mid V_{q(\mathfrak{M})}(\gamma_1) < \gamma' < V_{q(\mathfrak{M})}(\gamma_2)\}.$$

Then $V_{r(\mathfrak{M})}(\delta_1) = V^n_{q(\mathfrak{M})}(\delta_1) \subseteq r(\mathfrak{M})$. When no ambiguity appears, we omit $\mathfrak{M}$ from the indexes.

Lemma 1 *There exists a complete type $p(x,y) \supseteq p_1(x,y)$ such that for all $(\alpha,\beta) \in p(\mathfrak{M})$ and all $\delta_1, \delta_2 \in (V_r(\alpha), V_r(\beta))_{r(\mathfrak{M})}$, $tp^c(\delta_1/\alpha\beta) = tp^c(\delta_2/\alpha\beta)$.*

Proof of Lemma 1. Towards a contradiction suppose that the lemma is not true. Let $p(x,y)$ be a type extending p_1, and let (α,β) be a tuple realizing p. Then there exist a convex $\{\alpha,\beta\}$-formula $\psi(x,y,\alpha,\beta)$ and $\delta_1,\delta_2 \in (V_r(\alpha), V_r(\beta))_{r(\mathfrak{M})}$ such that $\delta_1 \in \psi(M,\alpha,\beta) < \delta_2$. We can choose this formula so that the set $\psi(M,\alpha,\beta)$ coincides with the set $(\psi(M,\alpha,\beta)^+)^-$ and has δ_1 as its left endpoint. Notice that there are δ_1' and δ_2' such that $\delta_1' \in \psi(M,\alpha,\beta) < \delta_2'$ and $\delta_2' \in S(\delta_1', M)$.

For $P(x,y) \in p$, $R(x) \in r$, $k,l,m < \omega$ such that $l+m<k$ denote

$$\Lambda_{P,R,k,l,m}(x,y) := \Big(x < y \wedge \neg\, S^k(x,y) \wedge P(x,y) \wedge$$
$$\exists z_1 \exists z_2 \big(\psi(z_1,x,y) \wedge \neg\psi(z_2,x,y) \wedge S^1(z_1,z_2)\big)\Big) \to \exists z_1\exists z_2\Big(x < z_1 < z_2 < y \wedge$$
$$R(z_1) \wedge R(z_2) \wedge S(z_1,z_2) \wedge \neg S^l(x,z_1) \wedge \neg S^m(z_2,y) \wedge \psi(z_1,x,y) \wedge \neg\psi(z_2,x,y)\Big).$$

Let $\langle P_i\rangle_{i<\omega}$ and $\langle R_j\rangle_{j<\omega}$ be two infinite sequences of formulas from the types p and r respectively such that

$$\mathfrak{M} \models \forall x \forall y \big(P_{i+1}(x,y) \to P_i(x,y)\big),\ \mathfrak{M} \models \forall x\big(R_{j+1}(x) \to R_j(x)\big),$$
$$p(\mathfrak{M}) = \bigcap_{i<\omega} P_i(M), \text{ and } r(\mathfrak{M}) = \bigcap_{j<\omega} R_j(M).$$

By compactness, we can see the following:

There are increasing sequences $\langle i(n)\rangle$, $\langle j(n)\rangle_{n<\omega}$, $\langle k(n)\rangle_{n<\omega}$, $\langle l(n)\rangle_{n<\omega}$, $\langle m(n)\rangle_{n<\omega}$ such that for every $m<\omega$ $\mathfrak{M} \models \forall x\forall y \Lambda_{P_{i(n)},R_{j(n)},k(n),l(n),m(n)}(x,y)$.

Then the formula $\psi(x,\alpha,\beta)$ should divide some neighborhood in r: there exists $\gamma \in r(\mathfrak{M})$ such that $V_r(\alpha) < V_r(\gamma) < V_r(\beta)$ and $r(\mathfrak{M}) \cap \psi(M,\alpha,\beta) \cap V_r(\gamma) \neq \emptyset$.

Let $G(x,\alpha,\beta) := \psi(x,\alpha,\beta) \wedge \exists y \big(S^1(x,y) \wedge \neg\psi(y,\alpha,\beta)\big)$. If necessary, we can narrow down G by adding conjunction with a formula of $r(x)$. This guarantees that the future formulas ψ_τ will act the same way as $\psi(x,\alpha,\beta)$. Every first order property that holds for realizations of some type should hold for all realizations of some of its formula. The formula $G(x,\alpha,\beta)$ defines a single element $\psi(M,\alpha,\beta)\cap r(\mathfrak{M})$ whose S^1-successor is in $\neg\psi(M,\alpha,\beta)$. Without loss of generality, let this element be γ. Then the following two cases cases are possible:

Case A. $tp(\gamma/\alpha)=tp(\beta/\alpha)$ and $tp(\gamma/\beta)=tp(\alpha/\beta)$;

Case B. $tp(\gamma/\alpha)\neq tp(\beta/\alpha)$ or (and) $tp(\gamma/\beta)\neq tp(\alpha/\beta)$.

In Case A, for every $m<\omega$ there exist $n<\omega$, $m_1<\omega$, $m_1>m$, such that

$$\mathfrak{M}\models\forall x\forall y\Big(\big(P_{m_1}(x,y)\wedge\neg S^n(x,y)\big)\rightarrow$$
$$\exists z_1\exists z_2\big(P_m(x,z_1)\wedge P_m(z_1,y)\wedge S^1(z_1,z_2)\wedge\psi(z_1,x,y)\wedge\neg\psi(z_2,x,y)\big)\Big).$$

Then, the definable set of $\alpha<x<\beta$ is divided into three disjoint sets definable by formulas $\Psi(x,\alpha,\beta)$, $\varphi_0(x,\alpha,\beta):=\alpha<x<\Psi(M,\alpha,\beta)$ and $\varphi_1(x,\alpha,\beta):=\Psi(M,\alpha,\beta)<x<\beta$ with

$$\varphi_0(M,\alpha,\beta)<\Psi(M,\alpha,\beta)<\varphi_1(M,\alpha,\beta).$$

Define

$$\psi_0(x,\alpha,\beta):=\exists z(\psi(x,\alpha,z)\wedge\Psi(z,\alpha,\beta));$$
$$\psi_1(x,\alpha,\beta):=\exists z(\psi(x,z,\beta)\wedge\Psi(z,\alpha,\beta)).$$

The formulas ψ_0 and ψ_1 are defined correctly since, in Case A, $tp(\alpha\delta)=tp(\delta\beta)=p$. In Case B γ still realizes some complete extension of the type p_0. By supposing that Lemma 1 is not true, there exists a formula $\psi^0(x,\alpha,\gamma)$ and (or) a formula $\psi^1(x,\gamma,\beta)$ that has the same properties as $\psi(x,\alpha,\beta)$. Then, in the definitions of ψ_0 and ψ_1 replace $\psi(x,y)$ with $\psi^0(x,y)$ and $\psi^1(x,y)$ respectively. Then ψ_0 divides the interval $(V_q(\alpha),V_q(\gamma))_p$, and ψ_1 divides $(V_q(\gamma),V_q(\beta))_q$.

Let

$$\Psi_0(x,\alpha,\beta):=\psi_0(x,\alpha,\beta)\wedge\exists y(\neg\psi_0(y,\alpha,\beta)\wedge S^1(x,y));$$
$$\Psi_1(x,\alpha,\beta):=\psi_1(x,\alpha,\beta)\wedge\exists y(\neg\psi_1(y,\alpha,\beta)\wedge S^1(x,y)).$$

The formulas Ψ_0 and Ψ_1 define singletons γ_0 and γ_1 in $\psi_0(M,\alpha,\beta)\cap r(\mathfrak{M})$ and $\psi_1(M,\alpha,\beta)\cap r(\mathfrak{M})$ whose S^1-successors are in $\neg\psi_0(M,\alpha,\beta)\cap r(\mathfrak{M})$ and $\neg\psi_1(M,\alpha,\beta)\cap r(\mathfrak{M})$ respectively. Each of those singletons can satisfy an analogue of either the Case A or the Case B. Suppose that they satisfy the Case A: $tp(\alpha,\gamma_0)=tp(\gamma_0,\gamma)=tp(\gamma,\gamma_1)=tp(\gamma_1,\beta)=p$. In Case B, replace ψ with suitable formulas the same way as in the definitions of ψ_0 and ψ_1.

We have $V_r(\alpha)<V_r(\gamma_0)<V_r(\gamma)<V_r(\gamma_1)<V_r(\beta)$.

Denote $\varphi_{00}(x,\alpha,\beta):=\alpha<x<G_0(M,\alpha,\beta)$, $\varphi_{01}(x,\alpha,\beta):=\Psi_0(M,\alpha,\beta)<x<\Psi(M,\alpha,\beta)$, $\varphi_{10}(x,\alpha,\beta):=\Psi(M,\alpha,\beta)<x<\Psi_1(M,\alpha,\beta)$, and $\varphi_{11}(x,\alpha,\beta):=\Psi_1(N,\alpha,\beta)<x<\beta$.

We continue defining such formulas by induction:

$$\psi_{\tau 0}(x):=\exists z_1\exists z_2\big(\Psi_\mu(z_1)\wedge\Psi_\tau(z_2)\wedge\psi(x,z_1,z_2)\big);$$
$$\psi_{\tau 1}(x):=\exists z_1\exists z_2\big(\Psi_\tau(z_1)\wedge\Psi_\delta(z_2)\wedge\psi(x,z_1,z_2)\big).$$

If τ consists only of zeros, denote

$$\psi_{\tau 0}(x) := \exists z_2\big(\Psi_\tau(z_2) \wedge \psi(x,\alpha,z_2)\big);$$

If τ consists only of ones, denote

$$\psi_{\tau 1}(x) := \exists z_1\big(\Psi_\tau(z_1) \wedge \psi(x,z_1,\beta)\big),$$

Above, μ (δ) is obtained by removing the one or more digit from τ such that Ψ_μ is the left (right) "closest" to Ψ_τ. In Case B, replace ψ in the definitions with a suitable formula as before.

Define

$\Psi_{\tau 0}(x) := \exists z(\psi_{\tau 0}(x) \wedge S(x,z) \wedge \neg\psi_{\tau 0}(z))$; $\Psi_{\tau 1}(x) := \exists z(\psi_{\tau 1}(x) \wedge S(x,z) \wedge \neg\psi_{\tau 1}(z))$; $\varphi_{\tau 0}(x) := \Psi_\mu(M) < x < \Psi_\tau(M)$ and $\varphi_{\tau 1}(x) := \Psi_\tau(M) < x < \Psi_\delta(M)$. Then we obtain $\varphi_{\tau 0}(M) < \Psi_\tau(M) < \varphi_{\tau 1}(M)$; $\varphi_{\tau 0}(M) \cup \varphi_{\tau 1}(x) \subset \varphi_\tau(M)$.

This way, for arbitrary $\nu \in 2^\omega$ we constructed the following set of 1-$\alpha\beta$-formulas: $r_\nu := \{\varphi_{\nu(n)}(x) \mid n < \omega\}$. It is obviously consistent. But this is impossible in a small theory. A contradiction. □ Lemma 1

A 1-type $r \in S_1(A)$ is said to be irrational if $r^c(\mathfrak{M})^+$ and $r^c(\mathfrak{M})^-$ are both non-definable sets. By Lemma 1 there is a type $q(x,y)$ such that for every $\alpha,\beta \in p(\mathfrak{M})$ if $tp(\alpha,\beta) = q(x,y)$, for every $\gamma \in (V_{p,\varphi}(\alpha), V_{p,\varphi}(\beta))_p$, $tp^c(\gamma/\alpha\beta)(\mathfrak{M}) = (V_{p,\varphi}(\alpha)^+ < x < V_{p,\varphi}(\beta)^-)(\mathfrak{M})$ is an irrational type. From this follows that the type $tp(\gamma/\alpha\beta)$ is non-principal.

It follows from Lemma 1, that for every formula $\psi(z,x,y)$ such that $\mathfrak{M} \models \forall x\forall y\big(\psi(M,x,y) < y \wedge \forall z(\psi(z,x,y) \leftrightarrow \psi(z,x,y)^{+-})\big)$, there are the following two possibilities:

1) There exists $k_0 < \omega$ for which $\forall z\big(\neg S^{k_0}(x,z) \to \neg\psi(z,x,y)\big) \in p(x,y)$, or, equivalently, for $T_1[\psi](x,y) := \forall x_1\Big(S^1(x_1,x) \to \exists z\big(\psi(z,x,y) \wedge \neg\psi(z,x_1,y)\big)\Big) \wedge \forall y_1\Big(S^1(y,y_1) \to \forall z\big(\neg\psi(z,x,y) \leftrightarrow \neg\psi(z,x,y_1)\big)\Big)$, $T_1[\psi](x,y) \in p$.

2) There exists $l_0 < \omega$ for which $\forall z\Big(\big(x < z \wedge \neg S^{l_0}(z,y)\big) \to \psi(z,x,y)\Big) \in p(x,y)$, or, equivalently, for $T_2[\psi](x,y) := \forall y_1\Big(\varphi(y_1,y) \to \exists z\big(\psi(z,x,y) \wedge \neg\psi(z,x,y_1)\big)\Big) \wedge \forall x_1\Big(S^1(x_1,x) \to \forall z\big(\psi(z,x,y) \leftrightarrow \psi(z,x_1,y)\big)\Big)$, $T_2[\psi](x,y) \in p$.

The formulas $\forall x\forall y(T_1[\psi](x,y) \to \neg T_2[\psi](x,y))$ and $\forall x\forall y\big(T_2[\psi](x,y) \to \neg T_1[\psi](x,y)\big)$ are in the theory T. Therefore, $\big(T_1[\psi](x,y) \vee T_2[\psi](x,y)\big) \in p$.

Then $p_1(x,y) := p_0(x,y) \cup \Big\{T_1[\psi](x,y) \vee T_2[\psi](x,y) \mid \mathfrak{M} \models \forall x\forall y\Big(\psi(M,x,y) < y \wedge \forall z\big(\psi(z,x,y) \leftrightarrow \psi(z,x,y)^{+-}\big)\Big)\Big\}$ should be consistent, and every its extension to a complete type should satisfy the property from Lemma 1.

We can generalize Lemma 1 the following way:

Lemma 2 *For every n ($n < \omega$), there exists an n-type $p^n(x_1,\ldots,x_n)$ such that for every increasing sequence $\alpha_1,\alpha_2,\ldots,\alpha_n$ of realizations of p in $\mathfrak{M}$ such that $tp(\alpha_1,\alpha_2,\ldots,\alpha_n) = p$, for every i ($1 \le i < n$), and every $\delta_1,\delta_2 \in (V_{q,\varphi}(\alpha_i), V_{q,\varphi}(\alpha_{i+1}))_q$,*

$$tp^c(\delta_1/\bar\alpha) = tp^c(\delta_2/\bar\alpha), \text{ and}$$
$$tp^c(\delta_1/\bar\alpha)(\mathfrak{M}) = \{m \in q(\mathfrak{M}) \mid V_q(\alpha_i)^+ < m < V_q(\alpha_{i+1})^-\}.$$

Here $\bar\alpha := (\alpha_1,\alpha_2,\ldots,\alpha_n)$.

Let $tp(\bar{\delta}) = p(\bar{x})$ satisfy Lemma 2. Then for every $i(1 \leq i < n)$, and every formula $\psi_i(z, \bar{\delta})$ with $\mathfrak{M} \models \delta_i < \psi_i(M, \bar{\delta}) < \delta_{i+1}$ and

$$\mathfrak{M} \models \forall x \forall y \Big(\exists z \psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \to \forall z \big(\psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \leftrightarrow \psi_i(z, x, y, \bar{\delta}_n^{i,i+1})^{+-} \big) \Big)$$

one of the following two cases is possible:

1) There is $k_0 < \omega$ for which $\forall z \big(\neg S^{k_0}(x, z) \to \neg \psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \big) \in p(x, y, \bar{\delta}_n^{i,i+1})$, or, equivalently, for

$$T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) := \forall x_1 \Big(S^1(x_1, x) \to \exists z \big(\psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \wedge \neg \psi_i(z, x_1, y, \bar{\delta}_n^{i,i+1}) \big) \Big) \wedge$$
$$\forall y_1 \Big(\varphi(y_1, y) \to \forall z \big(\neg \psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \leftrightarrow \neg \psi_i(z, x, y_1, \bar{\delta}_n^{i,i+1}) \big) \Big),$$

$T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \in p(x, y, \bar{\delta}_n^{i,i+1})$.

2) There exists $l_0 < \omega$ for which $\forall z \Big(\big(x < z \wedge \neg S^{l_0}(z, y) \big) \to \psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \Big) \in p(x, y, \bar{\delta}_n^{i,i+1})$, or, equivalently, for

$$T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) := \forall y_1 \Big(S^1(y, y_1) \to \exists z \big(\psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \wedge \neg \psi_i(z, x, y_1, , \bar{\delta}_n^{i,i+1}) \big) \Big) \wedge$$
$$\forall x_1 \Big(S^1(x_1, x) \to \forall z \big(\psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \leftrightarrow \psi(z, x_1, y) \big) \Big),$$

$T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \in p(x, y, \bar{\delta}_n^{i,i+1})$.

Because

$$\mathfrak{M} \models \forall x \forall y \big(T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \to \neg T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \big) \wedge \forall x \forall y \big(T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \to$$
$$\neg T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \big),$$

$\big(T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \vee T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \big) \in p(x, y, \bar{\delta}_n^{i,i+1})$.

Therefore, the following set is consistent:

$$p_1^n(\bar{x}) := p_0^n(\bar{x}) \cup \Big\{ T_1[\psi_i](\bar{x}) \vee T_2[\psi_i](\bar{x}) \mid \mathfrak{M} \models \forall x \forall y \Big(\psi_i(x, x, y, \bar{x}_n^{i,i+1}) \wedge \psi_i(M, \bar{x}) <$$
$$y \wedge \forall z \big(\psi(z, x, y, \bar{x}_n^{i,i+1}) \leftrightarrow \psi(z, x, y, \bar{x}_n^{i,i+1})^{+-} \big) \Big) \Big\}.$$

Every complete extension of p_1^n should satisfy the property in Lemma 2.

Denote $\Delta_n := \{ \langle i_1, i_2, \ldots, i_n \rangle \mid i_1 < i_2 < \cdots < i_n < \omega \}$. For every $\mu \in S_n$ let $\bar{x}_\mu := x_{\mu(1)} x_{\mu(2)} \ldots x_{\mu(n)}$. Lemma 2 implies consistence of the set $\bigcup\limits_{n<\omega,\ \mu \in S_n} q_1^n(\bar{x}_\mu) := \Gamma$.

Let $\mathfrak{N}$ be an $\aleph_1$-saturated elementary extension of $\mathfrak{M}$. There exists a countable ordered set $D \subset N$ that satisfies Γ and is ordered by the type of ω. Consider the Ehrefeucht-Mostovski type $EM(\omega/D) = \{ \varphi(x_1, ..., x_n) \mid$ for every $\mu \in S_n, n < \omega, \mathfrak{N} \models \varphi(\bar{a}_\mu) \}$. Then, $\Gamma(x_1, x_2, \ldots, x_n, \ldots) \subseteq EM(\omega/D)$. The Standard Lemma [17] implies that for every infinite linear ordering there is an indiscernible sequence $\langle d_j \rangle_{j \in J}$. Since $p_1^n(x_\mu) \subseteq EM(\omega/D)$, every finite sequence of J of length n forms a tuple $\bar{d}_\mu$ that satisfies the property from Lemma 2. This allows us to conduct the construction of $2^{\aleph_0}$ countable models.

Let $\mu := \langle \mu_1, \mu_2, ..., \mu_i, ... \rangle_{i<\omega}$, $\mu_i \in \{0, 1\}$, be an infinite sequence of zeros and ones. And let $K_\mu := \{\kappa_1, \kappa_2\} \cup \{\kappa_{2i-1,j} \mid i \in \mathbb{N}, j \in \mathbb{Q}\} \cup \{\kappa_{2i,1}, \kappa_{2i,2} \mid i \in \mathbb{N}, \mu_i = 0\} \cup \{\kappa_{2i,1}, \kappa_{2i,2}, \kappa_{2i,3} \mid i \in \mathbb{N}, \mu_i = 1\}$ be an indiscernible subset of $r(\mathfrak{N})$ that exists by the previous statements, and such that the sets $V_{r(\mathfrak{N})}(\kappa_{i,j})$ are disjoint and ordered lexicographically by the indices i, j, and $\kappa_1 < \kappa_{i,j} < \kappa_2$ for all i and j. Fix some enumeration $K_\mu = \{\kappa_1, \kappa_2, ..., \kappa_i, ...\}$. For $n < \omega$ denote $\bar{\kappa}_n := (\kappa_1, \kappa_2, ..., \kappa_n)$. We use an analogical notation $\bar{c}_n := (c_1, c_2, ..., c_n)$ later as well.

Now we construct a countable model $\mathfrak{A}_\mu \prec \mathfrak{N}$ such that $K_\mu \subseteq A_\mu$. We want the models obtained for different sequences μ to be non-isomorphic.

Step 1. Let $\Phi_1 := \{\varphi_{1,i}(x) \mid i < \omega\}$ be the set of all non-equivalent 1-formulas over $\emptyset$ such that $\mathfrak{N} \models \exists x \varphi_{1,i}(x)$ for all $i < \omega$. First we find a witness for $\varphi_{1,1}(x)$. Since the theory T is small, $\varphi_{1,1}$ has a principal over $\emptyset$ subformula $\varphi_{1,1,0}$. The formula $\varphi_{1,1,0}$, in turn, has a principal subformula $\varphi_{1,1,1}$ over κ_1. The formula $\varphi_{1,1,1}$ has a principal subformula over κ_2, and so on. We obtain a principal over parameters nested sequence of formulas $\varphi_{1,1,i}$, that has to be realized in the $\aleph_1$-saturated model $\mathfrak{N}$. Denote this realization by c_1. Denote $C_1 := \{c_1\}$, and $K_1 := \{\kappa_1\}$.

Next, we continue the same procedure. To satisfy the Tarski-Vaught criterion, on each step we form a new set of parameters, and realize one formula over each of the existing sets of parameters.

At the end of step n we have defined the finite nested sets $C_1 \subseteq C_2 \subseteq ... \subseteq C_n$, and, for every i, $2 \leq i \leq n$, the family Φ_i of all $C_{i-1} \cup K_{i-1}$-definable 1-formulas that have witnesses in $\mathfrak{N}$.

Step n+1. Firstly, realize one new formula from Φ_1, then from $\Phi_2, ..., \Phi_{n-1}$. For $1 \leq m \leq n$ let i_m be the smallest index such that $\varphi_{m,i_m} \in \Phi_m$ was not considered before. Construct a nested sequence of principal over parameters formulas: $\varphi_{m,i_m}(N, \bar{\kappa}_m, \bar{c}^{m_1}) \supseteq \varphi_{m,i_{m,0}}(N, \bar{\kappa}_i, \bar{c}^{m_1}) \supseteq \varphi_{m,i_{m,1}}(N, \bar{\kappa}_{m+1}, \bar{c}^{m_2}) \supseteq ... \supseteq \varphi_{m,i_{m,j}}(N, \bar{\kappa}_{m+j}, \bar{c}^{m_2}) \supseteq ...$, where $\bar{c}^{m_1}$ is the tuple enumerating the set C_m ($\bar{c}_{\frac{(m+1)m}{2}}$ to be exact), and $\bar{c}^{m_2} = \bar{c}_{\frac{(n+1)n}{2}+m-1}$ is the tuple enumerating all the c's obtained so far. Choose a realization $c_{\frac{(n+1)n}{2}+m} \in N$ of this sequence. Then $c_{\frac{(n+1)n}{2}+m}$ is principal over K_{m-1} and the c_j's for $j < \frac{(n+1)n}{2} + m$.

Let Φ_{n+1} be the family of all $K_n \cup C_n$-definable 1-formulas that are satisfiable in $\mathfrak{N}$. Choose $c_{\frac{(n+1)n}{2}+n+1}$ as before, as a realization of a chain of nested principal subformulas. Denote $C_{n+1} := \{c_1, c_2, ..., c_{\frac{(n+1)n}{2}+n+1}\}$.

Let $A_\mu := K \cup \bigcup_{i<\omega} C_i$. By Tarski-Vaught criterion A_μ is a universe of an elementary substructure of $\mathfrak{N}$.

It can be easily verified by induction that for every $i < \omega$ the type $tp(c_i/K_n)$ is principal starting from a sufficient $n < \omega$. For instance, for every $n \geq i-1$. Then $r(\mathfrak{A}_\mu) \setminus \bigcup_{\kappa \in K_\mu} V_{p(\mathfrak{M})}(\kappa) = \emptyset$, since otherwise, if some element of $\mathfrak{A}_\mu$ was in this set, it would have a principal type over some tuple from K_μ, but, by Lemma 2, this is impossible.

Since the theory T is small, the theory $T \cup tp(\alpha_0\beta_0, \kappa_1, \kappa_2)$ is small as well. The number of different infinite sequences μ of zeros and ones equals to $2^{\aleph_0}$, thereofre $I(T \cup tp(\alpha_0\beta_0, \kappa_1, \kappa_2), \aleph_0) = 2^{\aleph_0}$, and $I(T, \aleph_0) = 2^{\aleph_0}$.

2) Suppose that for all $\delta_1, \delta_2 \in q(\mathfrak{M})$ with $tp(\delta_1) = tp(\delta_2)$, and all $n \in \mathbb{Z}$, $\mathfrak{M} \models \neg S_n(\delta_1, \delta_2)$. Then for every complete type $q_1 \supseteq q$ over $\{\alpha, \beta\}$ and every $\delta \in q_1(\mathfrak{M})$, δ is the only realization of q_1 in its neighborhood $V(\delta) := \bigcup_{n \in \mathbb{N}} S_n(\delta, M)$.

2.1) Suppose that there exists $q_1 \in S_1(T)$ such that $q \subseteq q_1$, and q_1 has exactly n attached types for some $n \in \mathbb{N}$. This case contradicts with 2): for every $\delta \in q_1(\mathfrak{M})$ its neighborhood

$V(\delta)$ is infinite, but every its element should have a type attached to q_1 and realized by a singleton.

2.2) Let Π be the set of all types from $S_1(T)$ that extend q and have an infinite number of attached types. We consider the case when Π is infinite. Let $\Pi = P \cup R$, where P contains all the principal types from Π, and R – all non-principal types from Π. Since T is small, P and R can be either countable, finite or empty. Fix enumerations $P = \{p_1, p_2, ..., p_i, ...\}_{i<\omega}$, $R = \{r_1, r_2, ..., r_i, ...\}_{i<\omega}$.

2.2.1) Suppose that P is not empty. Consider, for instance, $p_1 \in P$. Since all the types of Π have unique realizations inside of a single neighbourhood, each r_n will be different from the other types of R by a formula $\psi_n(x) := \exists x_1(\varphi_1(x_1) \wedge S_n(x, x_1))$, where φ_1 is the isolating formula of p_1, and n is the distance between realizations of p_1 and r_n.

2.2.2) Suppose that $P = \emptyset$.

2.2.2.1) Suppose there exists a formula φ that belongs to only a finite number of types from Π. Then, analogically with 2.2.1), for every $n < \omega$ we can find a formula that distinguishes r_n from all other types form Π: $\psi_n(x) := \exists x_1 \exists x_2 ... \exists x_m(\bigwedge\limits_{1 \leq i \leq m-1} S_{n_i}(x_i, x_{i+1}) \wedge \bigwedge\limits_{1 \leq i \leq n} \varphi(x_i) \wedge S_n(x, x_1))$, where m is the number of types φ belongs to, and $n, n_1, ..., n_{m-1}$ are sufficient integers.

In cases 2.2.1) and 2.2.2.1), the set $\Gamma := \{\neg\varphi_i \mid i < \omega,\ \varphi_i$ is an isolating formula of $p_i\} \cup \{\neg\psi_i \in r_i \mid i < \omega\}$ is a locally consistent set of negations of representatives of all types of Π, and every its completion to a 1-type is not in Π. Then, by 2.1), the proof is done.

2.2.2.2) Let 2.2.2.1) be not true. Then every $\emptyset$-definable formula φ will belong either to no types from Π, or to all of them, or both φ and $\neg\varphi$ belong to an infinite number of types from Π. This contradicts with Restriction 1.

□

Corollary 1 *Let $\mathfrak{M}$ be a model of a small linearly ordered theory T that satisfies Restriction 1. Let for every $n < \omega$ there is $m_n \geq n$ such that in $\mathfrak{M}$ there exists a discrete chain of length m_n. Then $I(T, \aleph_0) = 2^{\aleph_0}$.*

Proof of Corollary 1. By compactness, there exists an infinite discrete chain in some elementary extension $\mathfrak{M}'$ of $\mathfrak{M}$. Then, by Theorem 2, $I(T, \aleph_0) = 2^{\aleph_0}$.

□

Corollary 2 *Let T be a small linearly ordered theory that satisfies Restriction 1 and such that $I(T, \aleph_0) < 2^{\aleph_0}$.*

1) There exists $n_T \in \mathbb{N}$ such that in every model of T length of every discrete chain is less than n_T.

2) If T has no finite models, every model of T is densely ordered up to finite discrete chains.

3 Conclusions

In the article, in order to avoid a fictitious linear order, a special restriction on theories was given. It was proved that if a small linearly ordered theory satisfies this restriction and has an infinite discrete chain, then it has $2^{\aleph_0}$ countable non-isomorphic models. Two corollaries of this theorem were given. Description of all cases of maximality of the countable spectrum ultimately leads to consideration of all possible countable spectra of complete theories with a definable linear order.

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