IRSTI 27.29.19, 27.29.25, 27.39.21 DOI: https://doi.org/10.26577/JMMCS.2021.v112.i4.03

G. Dildabek[1*], M.B. Ivanova[1,2], M.A. Sadybekov[1]
[1]Institute of Mathematics and Mathematical Modeling, Kazakhstan, Almaty
[2]South Kazakhstan Medical Academy, Kazakhstan, Shymkent
[*]e-mail: dildabek@math.kz

ON ROOT FUNCTIONS OF NONLOCAL DIFFERENTIAL SECOND-ORDER OPERATOR WITH BOUNDARY CONDITIONS OF PERIODIC TYPE

In this paper we consider one class of spectral problems for a nonlocal ordinary differential operator (with involution in the main part) with nonlocal boundary conditions of periodic type. Such problems arise when solving by the method of separation of variables for a nonlocal heat equation. We investigate spectral properties of the problem for the nonlocal ordinary differential equation $\mathcal{L}y(x) \equiv -y''(x) + \varepsilon y''(-x) = \lambda y(x)$, $-1 < x < 1$. Here λ is a spectral parameter, $|\varepsilon| < 1$. Such equations are called nonlocal because they have a term $y''(-x)$ with involutional argument deviation. Boundary conditions are nonlocal $y'(-1) + ay'(1) = 0$, $y(-1) - y(1) = 0$. Earlier this problem has been investigated for the special case $a = -1$. We consider the case $a \neq -1$. A criterion for simplicity of eigenvalues of the problem is proved: the eigenvalues will be simple if and only if the number $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ is irrational. We show that if the number r is irrational, then all the eigenvalues of the problem are simple, and the system of eigenfunctions of the problem is complete and minimal but does not form an unconditional basis in $L_2(-1,1)$. For the case of rational numbers r, it is proved that a (chosen in a special way) system of eigen- and associated functions forms an unconditional basis in $L_2(-1,1)$.
Key words: Nonlocal differential operator, spectrum, eigenvalue, multiplicity of eigenvalues, eigenfunction, associated function, unconditional basis.

Г. Ділдәбек[1*], М.Б. Иванова[1,2], М.А. Садыбеков[1]
[1]Математика және математикалық моделдеу институты, Қазақстан, Алматы қ.
[2]Оңтүстік Қазақстан медициналық академиясы, Қазақстан, Шымкент қ.
[*]e-mail: dildabek@math.kz

Шекаралық шарттары периодты типті екінші ретті локалді емес дифференциалдық оператордың түбірлік функциялары туралы

Бұл жұмыста локальды емес периодты шекаралық шартты қарапайым локальды емес дифференциалды оператор (басты бөлігінде инволюция бар) үшін спектрлік есептердің бір класы қарастырылды. Мұндай есептер локальды емес жылу өткізгіштік теңдеу үшін қойылған есептерді айнымалыларды ажырату әдісімен шешкенде пайда болады. Біз $\mathcal{L}y(x) \equiv -y''(x) + \varepsilon y''(-x) = \lambda y(x)$, $-1 < x < 1$ түріндегі локалды емес қарапайым дифференциалдық теңдеуге қойылған есептердің спектрлік қасиеттерін зерттейміз. Мұндағы λ - спектрлік параметр, $|\varepsilon| < 1$. Мұндай теңдеулер локальды емес, себебі оның $y''(-x)$ түріндегі аргументтің инволютициондық ауытқуы бар мүшесі болады. $y'(-1) + ay'(1) = 0$, $y(-1) - y(1) = 0$-локальды емес шекаралық шарт болып табылады. Бұрын бұл есептің $a = -1$ кезіндегі дербес жағдайы қарастырылды. Біз бұл есептің $a \neq -1$ жағдайын қарастырамыз.

Біз бұл есептің меншікті мәндерінің қарапайымдылық критериін дәлелдедік: меншікті мәндері қарапайым болады сонда тек сонда ғана, егер $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ ирроционал болса. Егер r ирроционал болса, онда есептің меншікті мәндерінің барлығы қарапайым болатынын, бірақ $L_2(-1,1)$-де сөзсіз базис болмайтынын көрсеттік. r рационал сан кезінде, меншікті және қосылған функциялар $L_2(-1,1)$-де сөзсіз базис болатыны (арнайы таңдап алынған) дәлелденген.

Түйін сөздер: Локальды емес дифференциалдық оператор, спектр, меншікті мәндер, меншікті мәндердің еселігі, меншікті функция, сөзсіз базис.

Г. Дилдабек[1*], М.Б. Иванова[1,2], М.А. Садыбеков[1]
[1]Институт математики и математического моделирования, Казахстан, г. Алматы
[2]Южно-Казахстанская медицинская академия, Казахстан, г. Алматы
*e-mail: dildabek@math.kz

О корневых функциях нелокального дифференциального оператора второго порядка с краевыми условиями периодического типа

В настоящей работе рассматривается один класс спектральных задач для нелокального обыкновенного дифференциального оператора (с инволюцией в главной части) с нелокальными краевыми условиями. периодического типа. Такие задачи возникают при решении методом разделения переменных задач для нелокального уравнения теплопроводности. Мы исследуем спектральные свойства задачи для нелокального обыкновенного дифференциального уравнения $\mathcal{L}y(x) \equiv -y''(x) + \varepsilon y''(-x) = \lambda y(x)$, $-1 < x < 1$. Здесь λ - спектральный параметр, $|\varepsilon| < 1$. Такие уравнения называются нелокальными, так как они содержат член $y''(-x)$ с инволютиционным отклонением аргумента. Краевые условия являются нелокальными $y'(-1) + ay'(1) = 0$, $y(-1) - y(1) = 0$. Ранее эта задача была исследована для частного случая $a = -1$. Нами рассматривается случай $a \neq -1$. Нами доказан критерий простоты собственных значений задачи: собственные значения будут простыми если и только если число $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ является иррациональным. Мы показали, что если число r иррациональное, то собственные значения задачи - все простые, а система собственных функций задачи является полной и минимальной, но не образует безусловного базиса в $L_2(-1,1)$. Для случая рациональных r доказано, что (специальным образом выбранная) система собственных и присоединенных функций образует безусловный базис в $L_2(-1,1)$.

Ключевые слова: Нелокальный дифференциальный оператор, спектр, собственное значение, кратность собственных значений, собственная функция, присоединенная функция, безусловный базис.

1 Introduction

It is well known that many spectral problems for ordinary differential operators arise in using the method of separation of variables (Fourier method) to solve initial-boundary value problems for evolution equations. Due to the fact that spectral properties of self-adjoint problems are well studied and the system of eigenvectors of self-adjoint operators forms an orthonormal basis, researchers use self-adjoint boundary conditions to model various processes of natural science. And in order to use more complex differential equations and/or more complex boundary conditions in modeling, it is necessary to develop the spectral theory of nonlocal operators. Such operators, as a rule, are non-self-adjoint. Therefore, nothing is known about their spectral properties and additional research is required in each particular case.

In this paper we consider one class of spectral problems for a nonlocal ordinary differential operator (with involution in the main part) with nonlocal boundary conditions of periodic type. Such problems arise when solving by the method of separation of variables for a nonlocal heat equation.

For example, one can consider a problem of modeling thermal diffusion process which is close to one described in the article of Cabada and Tojo [1], where the example is given that describes a specific physical situation. We consider a closed metal wire (length 2) that is wrapped around a thin sheet of insulation material. Assume that the position $x = 0$ is the most low in the wire, and the wire goes around the insulation up to the left to the point $x = -1$ and to the right to the point $x = 1$. Since the wire is closed, then the points $x = -1$ and $x = 1$ physically coincide. It is assumed that the insulating layer is slightly permeable. Hence, the value of the temperature $u(x, t)$ (at point x of the wire at time t) on one side of the insulation affects the diffusion process on the other side of the insulation, at point $(-x, t)$. For this reason, the standard heat equation is modified by adding an additional term $\varepsilon u_{xx}(-x, t)$ to the "classical" term $u_{xx}(x, t)$. Thus, this process is described by the nonlocal heat equation

$$u_t(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x,t), \tag{1}$$

in $\Omega = \{(x,t) : -1 < x < 1, 0 < t < T\}$. Here $f(x,t)$ is a function of influence of an external source; $|\varepsilon| < 1$ is a coefficient depending on the permeability of the insulating layer $t = 0$ is an initial moment of time; $t = T$ is a final moment of time.

The initial temperature distribution in the wire is considered known:

$$u(x,0) = \tau(x), \quad -1 \le x \le 1. \tag{2}$$

Since the wire is closed, it is natural to assume that the temperature at the ends of wire is the same:

$$u(-1,t) = u(1,t), \quad 0 \le t \le T. \tag{3}$$

If we consider the case when an additional external thermal effect occurs at the junction of the ends of the wire then boundary conditions of periodic type but non-self-adjoint, arise. Consider the process, where the temperature flux at one end at each time t is proportional to the rate of change of the average temperature over the entire wire. After non-singular transformations such a boundary condition can be reduced to the form

$$u_x(-1,t) + a u_x(1,t) = 0, \quad 0 \le t \le T. \tag{4}$$

Here a is a certain coefficient characterizing the proportionality of the temperature flux at one end and the rate of change of the average temperature over the entire wire.

Such a mathematical model can serve as a direct justification of the need to consider the nonlocal differential equations and the nonlocal boundary conditions for them. Our paper is devoted to the investigation of spectral properties of the problem arising when solving the formulated problem (1)–(4) by the method of separation of variables.

2 Materials and methods

3 Spectral problem

The use of the Fourier method for solving problem (1)–(4) leads to a spectral problem for the operator $\mathcal{L}$ given by the differential expression

$$\mathcal{L}y\,(x) \equiv -y''\,(x) + \varepsilon y''\,(-x) = \lambda y\,(x)\,, \; -1 < x < 1, \tag{5}$$

and the boundary conditions of periodic type

$$\begin{cases} U_1(y) \equiv y'(-1) + ay'(1) = 0, \\ U_2(y) \equiv y(-1) - y(1) = 0, \end{cases} \tag{6}$$

where λ is a spectral parameter.

Spectral problems for Eq. (5) were first considered, apparently, in [2], [3]. Cases of the Dirichlet and Neumann boundary$a = -1$ were considered. Cases of the boundary conditions when the system of root vectors forms a Riesz basis in L_2 were singled out. Here we consider a case $a \neq -1$. This case has not yet been investigated before.

Close spectral problems were considered in the works [4]- [9]. In [4] for Eq. (5) a problem with the nonlocal conditions

$$y(-1) = 0, \quad y'(-1) = y'(1)$$

was studied. It was proved that if $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ is irrational, then the system of eigenfunctions is complete and minimal in $L_2(-1,1)$ but is not an unconditional basis. For rational r, a method for choosing associated functions for which the system of root functions of the problem is the unconditional basis in $L_2(-1,1)$ was indicated. A similar result was proved in [5] for the case of the space $L_p(-1,1)$.

A problem for Eq. (5) with the nonlocal boundary conditions

$$y(-1) = \beta y(1), \quad y'(-1) = y'(1)$$

was investigated in [8] for the case of the space $L_2(-1,1)$ and in [9] for the space $L_p(-1,1)$. In these papers it was also shown that the multiplicity of eigenvalues depends on the rationality or irrationality of the number r.

Since for Eq. (5) the spectral theory of boundary value problems is not yet fully formed, then each separate case of boundary conditions must be considered separately. The spectral problems with the nonlocal conditions (6) have not been previously considered. In this connection, we note the works [10]– [18] in which close problems related with spectral properties of nonlocal problems were considered.

4 General solution of equation (5)

To construct a general solution of equation (5), consider the Cauchy problem with data at the interior point

$$\mathcal{L}y(x) \equiv -y''(x) + \varepsilon y''(-x) - \lambda y(x) = f(x), \ -1 < x < 1, \tag{7}$$

$$y(0) = A, \ y'(0) = B, \tag{8}$$

with arbitrary constants A and B. Here $f(x) \in C[-1,1]$.

By direct calculation it is easy to show that this problem (7) to (8) is equivalent to the integral equation

$$y(x) + \lambda \int_{-x}^{x} k(x,t) y(t) dt = A + Bx - \int_{-x}^{x} k(x,t) f(t) dt, \tag{9}$$

with the integral operator

$$\int_{-x}^{x} k(x,t)\varphi(t)dt \equiv \frac{1}{1-\alpha^2}\left\{\alpha\int_{-x}^{0}(x+t)\varphi(t)dt + \int_{0}^{x}(x-t)\varphi(t)dt.\right\}$$

Let us show that the integral equation (9) has a unique solution. For this, we introduce a new function

$$Y(x) = y(x)e^{-\mu|x|},$$

where $\mu > 0$ is a positive parameter which we will choose below.

Then for $Y(x)$ we obtain the integral equation

$$Y(x) + \lambda\int_{-x}^{x} k_1(x,t)Y(t)dt = \psi(x), \tag{10}$$

where it is indicated

$$k_1(x,t) = k(x,t)e^{-\mu[|x|-|t|]},$$

$$\psi(x) = (A+Bx)\,e^{-\mu|x|} - \int_{-x}^{x} k_1(x,t)f_1(t)dt,$$

$$f_1(x) = f(x)e^{-\mu|x|}.$$

By I_λ denote the integral operator in the left-hand side of (10). Estimating its norm in $L_2(-1,1)$, we have

$$\|I_\lambda\| \le \frac{|\lambda|}{1-\alpha^2}\frac{\sqrt{2\mu-1+e^{-2\mu}}}{\mu}.$$

Hence it is easy to see that for any λ we always can choose a positive number $\mu > 0$ such that the operator norm will be less than one: $\|I_\lambda\| \le \delta < 1$. Therefore, with this choice of μ, equation (10) has the unique solution $Y(x) \in L_2(-1,1)$.

That is why equation (9) has the unique solution $y(x) \in L_2(-1,1)$. Further, it is easy to justify by the classical formula that $y(x) \in C^2[-1,1]$ for $f(x) \in C[-1,1]$. Thus, it is proved

Lemma 1. *For any values of the parameter λ, of the constants A and B and for any function $f(x) \in C[-1,1]$ the Cauchy problem (7) to (8) has the unique solution $y(x) \in C^2[-1,1]$.*

As follows from this lemma, the general solution of equation (5) is two-parameter. As fundamental solutions we choose two functions $c(x,\lambda) \in C^2[-1,1]$ and $s(x,\lambda) \in C^2[-1,1]$ which are solutions of equation (5) and satisfy the Cauchy conditions:

$$c(0,\lambda) = s'(0,\lambda) = 1,\ c'(0,\lambda) = s(0,\lambda) = 0.$$

The existence of such solutions is ensured by Lemma 1.

By direct calculation it is easy to obtain these solutions explicitly:

$$c(x,\lambda) = \cos(\mu_1 x),\ s(x,\lambda) = \frac{1}{\mu_2}\sin(\mu_2 x),\quad \mu_1 = \sqrt{\frac{\lambda}{1-\varepsilon}},\quad \mu_2 = \sqrt{\frac{\lambda}{1+\varepsilon}}.$$

It is also easy to verify that the chosen solutions have the following symmetry properties:

$$c(-x,\lambda) = c(x,\lambda), \quad s(-x,\lambda) = -s(x,\lambda), \quad -1 \le x \le 1. \tag{11}$$

Thus, the general solution of equation (5) has the form:

$$y\,(x,\lambda) = C_1 c(x,\lambda) + C_2 s(x,\lambda) \tag{12}$$

with arbitrary constants C_1 and C_2.

5 Eigenvalues of problem (5) to (6)

First of all, it is easy to see that $\lambda = 0$ is an eigenvalue of problem (5) to (6). The corresponding eigenfunction has the form:

$$y_0(x) = 1.$$

Consider a case $\lambda \ne 0$. Satisfying the general solution (12) of equation (5) to the boundary conditions (6), we get the linear system

$$\begin{cases} C_1 U_1\,(c(x,\lambda)) + C_2 U_1\,(s(x,\lambda)) = 0, \\ C_1 U_2\,(c(x,\lambda)) + C_2 U_2\,(s(x,\lambda)) = 0. \end{cases} \tag{13}$$

Its determinant will be the characteristic determinant of the spectral problem (5) to (6):

$$\triangle(\lambda) \equiv \begin{vmatrix} U_1\,(c(x,\lambda)) & U_1\,(s(x,\lambda)) \\ U_2\,(c(x,\lambda)) & U_2\,(s(x,\lambda)) \end{vmatrix} = 0.$$

Therefore, taking into account the symmetry conditions (11), we calculate

$$\triangle(\lambda) \equiv 2(1-a)c'(1,\lambda)s(1,\lambda) = 0. \tag{14}$$

First of all from (14) we get that for $a = 1$ each number λ is the eigenvalue of problem (5) to (6), regardless of the value of ε. In this case system (13) has the form

$$\begin{cases} C_2 s'(1,\lambda) = 0, \\ C_2 s(1,\lambda) = 0. \end{cases}$$

Since $|s'(1,\lambda)| + |s(1,\lambda)| > 0$, then it follows that $C_2 = 0$. Thus, it is proved

Lemma 2. *For $a = 1$ each number λ is the eigenvalue of problem (5) to (6). Corresponding eigenfunctions have the form*

$$y\,(x,\lambda) = \cos\left(\mu_1 x\right), \ \mu_1 = \sqrt{\frac{\lambda}{1-\varepsilon}}. \tag{15}$$

Now consider the case when $a \ne 1$. Then from (14) we obtain $c'(1,\lambda)s(1,\lambda) = 0$. Therefore, taking into account the explicit form of fundamental solutions, we have

$$\sin\left(\mu_1\right)\sin\left(\mu_2\right) = 0.$$

Thus, problem (5) to (6) has two series of the eigenvalues

$$\lambda_k^{(1)} = (1-\varepsilon)(k\pi)^2, \quad k = 0, 1, 2, ...,$$
$$\lambda_n^{(2)} = (1+\varepsilon)(n\pi)^2, \quad n = 1, 2, ... \tag{16}$$

Lemma 3. *Problem* (5) *to* (6) *has multiple eigenvalues if and only if the number* $r = \sqrt{(1-\varepsilon)\,/\,(1+\varepsilon)}$ *is rational.*

Proof. Indeed, suppose that any two eigenvalues from different series coincide:

$$\lambda_k^{(1)} = \lambda_n^{(2)}.$$

This is equivalent to the equality

$$(1-\varepsilon)(k\pi)^2 = (1+\varepsilon)(n\pi)^2.$$

That is, the coincidence of eigenvalues is possible if and only if for some $k_0, n_0 \in N$ $r = n_0/k_0$ holds. That is, only if the value r is rational.

6 Spectral problem for irrational numbers r

Let r be an irrational number. Then, by virtue of Lemma 3, all eigenvalues of problem (5) to (6) are simple and are given by the formulas (16). By direct calculation from (13) we get that

$$y_k^{(1)}(x) = \cos(k\pi x),$$
$$y_n^{(2)}(x) = (1+a)r\cos(n\pi)\cos\left(\tfrac{n\pi x}{r}\right) + (a-1)\sin\left(\tfrac{n\pi}{r}\right)\sin(n\pi x), \tag{17}$$

where $k = 0, 1, 2, ...$ and $n = 1, 2, ...$, correspond to these eigenvalues.

Lemma 4. *The system of functions* (17) *is complete and minimal in* $L_2(-1,1)$.

Proof Consider an arbitrary function $f(x)$ orthogonal to system (17). Since it is orthogonal to all functions $y_k^{(1)}(x)$, $k = 0, 1, 2, ...$, then we have

$$0 = \int_{-1}^{1} f(x)\cos(k\pi x)dx = \int_0^1 \Big\{ f(x) + f(-x) \Big\} \cos(k\pi x)dx.$$

But the system $\{\cos(k\pi x), k = 0, 1, 2, ...\}$ forms a basis in $L_2(0,1)$. Therefore, $f(x)+f(-x) = 0$ holds almost everywhere on the interval $(-1,1)$. That is, this function is odd.

Therefore, from the orthogonality of $f(x)$ to all functions $y_n^{(2)}(x)$, $n = 1, 2, ...$ we obtain

$$0 = \int_{-1}^{1} f(x) y_n^{(2)}(x)dx = (a-1)\sin\left(\frac{n\pi}{r}\right) \int_{-1}^{1} f(x)\sin(n\pi x)dx.$$

Since $a \neq 1$ and the number r is irrational, then from this we have

$$0 = \int_{-1}^{1} f(x)\sin(n\pi x)dx = \int_0^1 \Big\{ f(x) - f(-x) \Big\} \sin(n\pi x)dx.$$

But the system $\{\sin(n\pi x), n = 1, 2, ...\}$ forms a basis in $L_2(0,1)$. Therefore, $f(x) - f(-x) = 0$ holds almost everywhere on the interval $(-1,1)$. That is, this function is even.

Thus, the function $f(x)$ turns out to be simultaneously even and odd almost everywhere on the interval $(-1,1)$. Consequently, $f(x) = 0$ holds almost everywhere on the interval $(-1,1)$. This proves the completeness of the system of functions (17) in $L_2(-1,1)$.

Since the system under consideration (17) is a system of eigenfunctions of an linear operator, then it has a biorthogonal system consisting of eigenfunctions of an adjoint operator. We will not dwell here on a specific form of this system and the adjoint operator. But from the existence of the biorthogonal system follows the minimality of the system of functions (17) in $L_2(-1,1)$. Lemma is proved.

Now let us show that despite the fact that the system of functions (17) is complete and minimal in $L_2(-1,1)$, it does not form an unconditional basis. For this, we use the necessary condition for the basis property from [19].

Lemma 5. ([19], Th. 3.135, s. 219) *Let $\{u_j\}$ be a closed and minimal system in a Hilbert space H. If the system $\{u_j\}$ is an unconditional basis in H, then the strict inequality holds*

$$\limsup_{j\to\infty} \left| \left\langle \frac{u_j}{\|u_j\|}, \frac{u_{j+1}}{\|u_{j+1}\|} \right\rangle \right| < 1, \tag{18}$$

where $\langle \cdot, \cdot \rangle$ is the inner products in H.

By virtue of this lemma, for the unconditional basis property in $L_2(-1,1)$ of the system of functions (17), it is necessary to satisfy the strict inequality

$$\limsup_{j\to\infty} \left| \left\langle \frac{y_{k_j}^{(1)}}{\left\|y_{k_j}^{(1)}\right\|}, \frac{y_{n_j}^{(2)}}{\left\|y_{n_j}^{(2)}\right\|} \right\rangle \right| < 1 \tag{19}$$

for all possible infinitely increasing subsequences k_j and n_j.

Calculating the norms of the eigenfunctions, we obtain

$$\left\|y_k^{(1)}\right\| = 1,$$

$$\left\|y_n^{(2)}\right\|^2 = (1+a)^2 r^2 \left\{1 + \frac{r}{2n\pi} \sin\left(\frac{2n\pi}{r}\right)\right\} + (1-a)^2 \sin^2\left(\frac{n\pi}{r}\right).$$

Therefore,

$$\left\|y_n^{(2)}\right\|^2 = (1+a)^2 r^2 + (1-a)^2 \sin^2\left(\frac{n\pi}{r}\right) + O\left(\frac{1}{n}\right) \tag{20}$$

for $n \to \infty$.

Calculate the inner products in $L_2(-1,1)$:

$$\left|\left\langle y_k^{(1)}, y_n^{(2)} \right\rangle\right| = |1+a| r \left| \int_{-1}^{1} \cos(k\pi x) \cos\left(\frac{n\pi x}{r}\right) dx \right|$$

$$= |1+a| r \left| \frac{\sin\left(k - \frac{n}{r}\right)\pi}{\left(k - \frac{n}{r}\right)\pi} + O\left(\frac{1}{k+n}\right) \right|$$

for $k, n \to \infty$.

According to the Dirichlet's approximation theorem (see, example, [20], Th. 1A, p. 34), for any irrational number α there exists an infinite set of irreducible fractions $\frac{p}{q}$ (where p and q are integers) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Choosing here $\alpha = \frac{1}{r}$, we get that there exist infinite subsequences of the natural numbers k_j and n_j such that

$$\left|\frac{1}{r} - \frac{k_j}{n_j}\right| < \frac{1}{n_j^2}.$$

For these subsequences we will have

$$\left|k_j - \frac{n_j}{r}\right| < \frac{1}{n_j}.$$

Therefore, there exists the limit

$$\lim_{j\to\infty} \frac{\sin\left(k_j - \frac{n_j}{r}\right)\pi}{\left(k_j - \frac{n_j}{r}\right)\pi} = 1.$$

From this we have that the limit exists

$$\lim_{j\to\infty} \left|\left\langle y_{k_j}^{(1)}, y_{n_j}^{(2)}\right\rangle\right| = |1+a|r.$$

From (20) it is easily seen that the limit exists

$$\lim_{j\to\infty} \left\|y_{n_j}^{(2)}\right\| = |1+a|r.$$

Finally, substituting everything obtained in (21), we get

$$\lim_{j\to\infty} \left|\left\langle \frac{y_{k_j}^{(1)}}{\left\|y_{k_j}^{(1)}\right\|}, \frac{y_{n_j}^{(2)}}{\left\|y_{n_j}^{(2)}\right\|}\right\rangle\right| = 1 \tag{21}$$

for our chosen (according to the Dirichlet's approximation theorem) infinitely increasing subsequences k_j and n_j. That is, the necessary condition of the unconditional basis property (3.3) is not satisfied. Thus, the following lemma is proved.

Lemma 6. *Let r be irrational. Then the system of eigenfunctions (3.2) of the spectral problem (1.8), (1.9) is complete and minimal but does not form an infinite basis in $L_2(-1,1)$.*

7 Spectral problem for rational numbers r

Now consider the case when r is a rational number. Then there exist natural numbers n_0 and k_0 such that $r = \frac{n_0}{k_0}$. In this case, as follows from (16), problem (5) to (6) has an infinite number of double eigenvalues

$$\lambda_{k_0 j}^{(1)} = \lambda_{n_0 j}^{(2)}, \quad j \in \mathbb{N}. \tag{22}$$

As mentioned above, the spectral problems for Eq. (5) with periodic boundary conditions ($a = -1$) were considered in [2], [3]. For periodic problems it was shown that root subspaces, consisting of two eigenfunctions, correspond to the double eigenvalues. Here we consider the case $a \neq -1$.

For $a \neq -1$, one eigenfunction and one associated function correspond to the double eigenvalues (22).

By direct calculation it is easily shown that for the cases when $\frac{n}{k} \neq \frac{n_0}{k_0}$, problem (5) to (6) has the eigenfunctions

$$\begin{aligned} y_k^{(1)}(x) &= \cos(k\pi x), \\ y_n^{(2)}(x) &= (1+a) r \cos(n\pi) \cos\left(\tfrac{n\pi x}{r}\right) + (a-1)\sin\left(\tfrac{n\pi}{r}\right)\sin(n\pi x), \end{aligned} \tag{23}$$

where $k = 0, 1, 2, ...$ and $n = 1, 2, ...$, except the cases when $k = k_0 j$, $n = n_0 j$ for some j.

And for those cases when $\frac{n}{k} = \frac{n_0}{k_0}$ (that is, when $k = k_0 j$, $n = n_0 j$ for some j), problem (5) to (6) has the eigenfunctions $y_{k_0 j}^{(1)}(x)$ and the corresponding associated functions $y_{n_0 j,1}(x)$:

$$\begin{aligned} y_{k_0 j}^{(1)}(x) &= \cos\left(k_0 j \pi x\right), \\ y_{n_0 j,1}(x) &= -\tfrac{1}{2k_0 j\pi(1-\varepsilon)}\left\{x\sin(k_0 j\pi x) + \tfrac{1-a}{1+a}\tfrac{1}{r}(-1)^{(n_0+k_0)j}\sin\left(n_0 j\pi x\right)\right\}. \end{aligned} \tag{24}$$

Here we mean by the associated functions (according to M.V. Keldysh) solutions of the inhomogeneous equation

$$\mathcal{L} y_{k,1}(x) \equiv -y_{k,1}''(x) + \varepsilon y_{k,1}''(-x) = \lambda_k^{(1)} y_{k,1}(x) + y_k^{(1)}(x), \; -1 < x < 1, \tag{25}$$

satisfying the boundary conditions (6).

It is well known that the associated functions are not defined uniquely. Functions of the form

$$\widetilde{y}_{k_0 j,1}(x) = y_{k_0 j,1}(x) + C_j y_{k_0 j}^{(1)}(x)$$

for any constants C_j are also associated functions of problem (5) to (6) corresponding to the eigenvalues $\lambda_{k_0 j}^{(1)}$ and the eigenfunctions $y_{k_0 j}^{(1)}(x)$. "Problem of choosing associated functions" is also well known. This problem is related to the fact that with one choice of the constants C_j the system can form a basis, and with other choice of these constants the system does not form an unconditional basis. To avoid this problem, we fix such a choice of associated functions by formula (24).

Lemma 7. *The system of eigen- and associated functions* (23) *to* (24) *of problem* (5) *to* (6) *is complete and minimal in* $L_2(-1,1)$.

The proof is similar to the proof of Lemma 4. Consider an arbitrary function $f(x)$ orthogonal to the system of functions (23) to (24). Since it is orthogonal to all functions $y_k^{(1)}(x)$, $k = 0,1,2,...$, then, as in the proof of Lemma 4, we have that $f(x)+f(-x) = 0$ (that is, this function is even) holds almost everywhere on the interval $(-1,1)$.

Further, from the orthogonality of $f(x)$ to all functions $y_n^{(2)}(x)$ from (23) we get that

$$\int_{-1}^{1} f(x)\sin(n\pi x)dx = 0 \tag{26}$$

for all $n = 1,2,...$, except the cases when $n = n_0 j$ for some j.

It follows from the oddness of $f(x)$ that it is orthogonal to the functions $x\sin(k_0 j\pi x)$. Therefore, from the orthogonality of $f(x)$ to all functions $y_{k_0 j,1}(x)$ from (24) we get that (26) holds and for the cases when $n = n_0 j$ for some j.

Since the system $\{\sin(n\pi x), n = 1,2,...\}$ forms the basis in $L_2(0,1)$, then $f(x)-f(-x) = 0$ (that is, this function is even) holds almost everywhere on the interval $(-1,1)$.

Thus, the function $f(x)$ turns out to be simultaneously even and odd almost everywhere on the interval $(-1,1)$. Consequently, $f(x) = 0$ holds almost everywhere on the interval $(-1,1)$. This proves the completeness of the system of functions (23) to (24) in $L_2(-1,1)$.

Since the system under consideration (23) to (24) is the system of eigen- and associated functions of a linear operator, then it has a biorthogonal system consisting of eigen- and associated functions of an adjoint operator. We will not dwell here on a specific form of this system and the adjoint operator. But from the existence of the biorthogonal system follows the minimality of the system of functions (23) to (24) in $L_2(-1,1)$. Lemma is proved.

Now let us prove that system (23) to (24) forms the unconditional basis in $L_2(-1,1)$. For this we need a biorthogonal system. It is a system of eigen- and associated functions of the adjoint problem:

$$\mathcal{L}^* v\left(x\right) \equiv -v''\left(x\right) + \varepsilon v''\left(-x\right) = \lambda v\left(x\right), \ -1 < x < 1, \tag{27}$$

$$\left\{ \begin{array}{l} V_1\left(v\right) \equiv v'\left(-1\right) - v'\left(1\right) = 0, \\ V_2\left(v\right) \equiv (a-\varepsilon) v\left(-1\right) + (1-a\varepsilon) v\left(1\right) = 0. \end{array} \right. \tag{28}$$

Since the eigenvalues (16) of problem (5) to (6) are real, then they are also and the eigenvalues of the adjoint problem (27) to (28). The system of eigen- and associated functions of this problem can be constructed explicitly.

The eigenfunction

$$v_0(x) = \frac{1}{2} - \frac{(1+a)}{2(1-a)} r^2 x \tag{29}$$

corresponds to a zero eigenvalue.

By direct calculation it is easily shown that for those cases when $\frac{n}{k} \neq \frac{n_0}{k_0}$, problem (27) to (28) has the eigenfunctions

$$\begin{aligned} &v_k^{(1)}(x) = \cos(k\pi x) - \tfrac{1+a}{1-a} r^2 \tfrac{(-1)^k}{\sin(rk\pi)} \sin(rk\pi x), \\ &v_n^{(2)}(x) = -\tfrac{1}{1-a} \tfrac{1}{\sin\left(\frac{n\pi}{r}\right)} \sin(n\pi x), \end{aligned} \tag{30}$$

corresponding to the eigenvalues $\lambda_k^{(1)}$ and $\lambda_n^{(2)}$, where $k = 0, 1, 2, ...$ and $n = 1, 2, ...$, except the cases when $k = k_0 j$, $n = n_0 j$ for some j.

And for the cases when $\frac{n}{k} = \frac{n_0}{k_0}$ (that is, when $k = k_0 j$, $n = n_0 j$ for some j), problem (27) to (28) has the eigenfunctions $v_{n_0 j}^{(2)}(x)$ and the associated functions corresponding to them $v_{k_0 j,1}(x)$:

$$\begin{aligned} &v_{n_0 j}^{(2)}(x) = -k_0 j\pi(1-\varepsilon)\tfrac{1+a}{1-a} r(-1)^{(n_0+k_0)j} \sin\left(n_0 j\pi x\right), \\ &v_{k_0 j,1}(x) = -\tfrac{1+a}{1-a} r^2 (-1)^{(n_0+k_0)j} x\cos(n_0 j\pi x) + \cos\left(k_0 j\pi x\right). \end{aligned} \tag{31}$$

When constructing this system of eigen- and associated functions of the adjoint problem, we have normalized the eigenfunctions so that the biorthogonality conditions

$$\left\langle y_k^{(1)}, v_k^{(1)} \right\rangle = 1, \quad \left\langle y_n^{(2)}, v_n^{(2)} \right\rangle = 1,$$

hold for all $k = 0, 1, 2, ...$ and $n = 1, 2, ...$, except the cases when $k = k_0 j$, $n = n_0 j$ for some j.

And for the cases when $\frac{n}{k} = \frac{n_0}{k_0}$ (that is, when $k = k_0 j$, $n = n_0 j$ for some j), we have required the fulfilment of the biorthogonality conditions

$$\left\langle y_{k_0 j}^{(1)}, v_{k_0 j,1} \right\rangle = 1, \quad \left\langle y_{n_0 j,1}, v_{n_0 j}^{(2)} \right\rangle = 1.$$

Here by $\langle \cdot, \cdot \rangle$ we denote the inner product in $L_2(-1, 1)$.

For what follows, we need to estimate the norms of the constructed eigen- and associated functions. By direct calculation we find

$$\left\| y_k^{(1)} \right\| = 1; \; \left\| y_n^{(2)} \right\|^2 = (1+a)^2 r^2 \left\{ 1 + \frac{r}{2n\pi} \sin\left(\frac{2n\pi}{r} \right) \right\} + (1-a)^2 \sin^2\left(\frac{n\pi}{r} \right);$$

$$\left\| v_k^{(1)} \right\|^2 = 1 + \left(\frac{1+a}{1-a} \right)^2 \frac{r^2}{\sin^2\left(rk\pi\right)}; \; \left\| v_n^{(2)} \right\|^2 = \frac{1}{(1-a)^2} \frac{1}{\sin^2\left(\frac{n\pi}{r}\right)};$$

$$\left\| y_{n_0 j}^{(1)} \right\| = 1; \; \left\| y_{n_0 j,1} \right\|^2 = \frac{1}{\left(2k_0 j\pi(1-\varepsilon)\right)^2} \left\{ \frac{1}{3} - \frac{1}{2\left(k_0 j\right)^2} + \left(\frac{1-a}{1+a} \right)^2 \frac{1}{r^2} \right\};$$

$$\left\| v_{n_0 j}^{(2)} \right\|^2 = \left(2k_0 j\pi(1-\varepsilon)\right)^2 \left(\frac{1+a}{1-a} \right)^2 r^2;$$

$$\left\| v_{k_0 j,1} \right\|^2 = 1 + \left(\frac{1+a}{1-a} \right)^2 r^4 \left\{ \frac{1}{3} + \frac{1}{2\left(k_0 j\right)^2} \right\}.$$

Analyzing these explicit formulas, we see that only the asymptotic behavior of multipliers $\sin\left(\frac{n\pi}{r}\right)$ and $\sin(rk\pi)$ is not obvious. Let us show that these multipliers are strictly separated from zero.

Lemma 8. *If r is a rational number: $r = \frac{n_0}{k_0}$, then for all values of the indices n and k, when $n \neq n_0 j$ and $k \neq k_0 j$, the inequalities hold*

$$\left| \sin\left(\frac{n\pi}{r} \right) \right| \geq \left| \sin\left(\frac{\pi}{n_0} \right) \right|, \quad \left| \sin\left(rk\pi\right) \right| \geq \left| \sin\left(\frac{\pi}{k_0} \right) \right|. \tag{32}$$

The proof will be carried out by the method used in [7], [8], [9]. Since $n \neq n_0 j$, then the representation $n = n_0 j + i$ holds for some $j, i \in \mathbb{N}$, $1 \leq i \leq n_0 - 1$. Therefore, $\frac{n}{r} = k_0 j + \frac{k_0 i}{n_0}$. Since $\frac{n}{k} \neq \frac{n_0}{k_0}$, then this number $\frac{n}{r} = k_0 j + \frac{k_0 i}{n_0}$ is not an integer. Consequently, we have:

$$\left|\sin\left(\frac{n\pi}{r}\right)\right| = \left|\sin\left(\pi\left(\frac{n}{r} - k_0 j\right)\right)\right| = \left|\sin\left(\frac{\pi}{n_0} k_0 i\right)\right| \geq \left|\sin\left(\frac{\pi}{n_0}\right)\right|.$$

The second inequality from (32) is proved similarly. Since $k \neq k_0 j$, then the representation $k = k_0 j + i$ holds for some $j, i \in \mathbb{N}$, $1 \leq i \leq k_0 - 1$. Therefore, $rk = n_0 j + \frac{n_0 i}{k_0}$. Since $\frac{n}{k} \neq \frac{n_0}{k_0}$, then this number $rk = n_0 j + \frac{n_0 i}{k_0}$ is not an integer. Hence we have:

$$|\sin(rk\pi)| = |\sin(\pi(rk - n_0 j))| = \left|\sin\left(\frac{\pi}{k_0} n_0 i\right)\right| \geq \left|\sin\left(\frac{\pi}{k_0}\right)\right|.$$

Lemma 9. *If r is a rational number: $r = \frac{n_0}{k_0}$, then each of the systems* (23) *to* (24) *and* (29) *to* (31)*, after the normalization in $L_2(-1,1)$, satisfies a Bessel type inequality and hence forms an unconditional basis in $L_2(-1,1)$.*

Note that the system $\{\varphi_j\}$ has the Bessel property in a Hilbert space H, if there exists a constant $B > 0$ such that the Bessel type inequality

$$\sum_j |\langle f, \varphi_j \rangle|^2 \leq B \|f\|^2$$

holds for all elements $f \in H$.

Proof By virtue of the above estimates of the eigen- and associated functions, to justify the Bessel property, it suffices to prove the Bessel property of the following three type of systems ($j \in \mathbb{N}$):

$$\cos(j\pi x), \quad \sin(j\pi x); \tag{33}$$

$$\cos\left(\frac{k_0}{n_0} j\pi x\right), \quad \sin\left(\frac{k_0}{n_0} j\pi x\right); \tag{34}$$

$$x\cos(j\pi x), \quad x\sin(j\pi x). \tag{35}$$

System (33) is orthonormal in $L_2(-1,1)$ and hence satisfies the Bessel type inequality with constant $B = 1$. The Bessel property of system (35) follows from the Bessel property of system (33), because the multiplier x is bounded. Finally, system (34) is a Bessel system by virtue of the following assertion proved in [7], [8], [9].

Lemma 10. ([7], [8], [9]) *Let $\{\gamma_j\}$ be a sequence of complex numbers such that*

$$\sup_j |Im(\gamma_j)| < \infty, \quad \sup_{t \geq 1} \sum_{j: \left|Re(\gamma_j) - t\right| \leq 1} 1 < \infty. \tag{36}$$

Then each of the systems $\{\sin(\gamma_j x)\}$ and $\{\cos(\gamma_j x)\}$ is a Bessel system in $L_2(-1,1)$.

System (34) satisfies condition (36) because

$$\mathrm{Im}(\gamma_j) = 0, \qquad \sum_{j: \left|\mathrm{Re}(\gamma_j) - t\right| \leq 1} 1 \leq 2m_0 + 1.$$

The unconditional basis property of the systems (23) to (24) and (29) to (31) follows from the well-known Bari theorem [21]. The proof of Lemma 9 is complete.

8 Formulation of main result

Combining all the results, we formulate them together in the form of one theorem.

Theorem *Let $a \neq -1$. Then the spectral problem* (5) *to* (6) *has the following properties.*
$\star$ *For $a = 1$ each number λ will be an eigenvalue of problem* (5) *to* (6)*. Corresponding eigenfunctions are of the form* (15)*.*
$\star$ *Problem* (5) *to* (6) *has double eigenvalues if and only if the number $r = \sqrt{(1-\varepsilon)\,/\,(1+\varepsilon)}$ is rational.*
$\star$ *If r is an irrational number, then all eigenvalues of problem* (5) *to* (6) *are simple, and its system of eigenfunctions* (17) *is complete and minimal but does not form an unconditional basis in $L_2(-1,1)$.*
$\star$ *If r is a rational number, then there exists an infinite countable subsequence of eigenvalues of problem* (5) *to* (6) *which are double. The rest of the eigenvalues of problem* (5) *to* (6) *(there are also infinite countable number of them) are simple. One eigenfunction and one associated function correspond to each double eigenvalue. The system of eigen- and associated functions* (23) *to* (24) *of problem* (5) *to* (6) *is complete and minimal in $L_2(-1,1)$. The associated functions of problem* (5) *to* (6) *can be chosen in such a special way that this special system of eigen- and associated functions forms an unconditional basis in $L_2(-1,1)$.*

9 Conclusions

Thus, in this paper, we consider one class of spectral problems for a nonlocal ordinary differential operator (with involution in the main part) with nonlocal boundary conditions of periodic type. The main result of the work is to study the questions of the unconditional basis property of the system of root vectors of the given differential operator. We have proved the criterion for the simplicity of the eigenvalues of the problem. In addition, it have been proved that the system of root vectors forms an unconditional basis only in the case of multiple eigenvalues. Therefore, (in the case of multiple eigenvalues) this system of root vectors can be further used to solve problems of nonlocal heat conduction with nonlocal boundary conditions of periodic type.

10 Acknowledgement

The work was supported by grant funding of scientific and technical programs and projects of the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. AP08957619).

References

[1] Cabada A., Tojo F., Adrian F. Differential equations with involutions // Workshop on Differential Equations. - 2015.

[2] Kopzhassarova A. A., Lukashov A. L., Sarsenbi A. M. Spectral Properties of non-self-adjoint perturbations for a spectral problem with involution // Abstract and Applied Analysis. - 2012. - V. 2012/ - Art. ID 576843/ P. 1-6. DOI: 10.1155/2012/576843.

[3] Kopzhassarova A.A., Lukashov A.L., Sarsenbi A.M. Spectral Properties of non-self-adjoint perturbations for a spectral problem with involution // Abstract and Applied Analysis. - 2012. - V. 2012. - Art. ID 590781. DOI: 10.1155/2012/590781.

[4] Kritskov L. V., Sarsenbi A. M. Spectral properties of a nonlocal problem for the differential equation with involution // Differential Equations. - 2015. - V. 51, №8. - P. 984-990.

[5] Kritskov L. V., Sarsenbi A. M. Basicity in L_p of root functions for differential equations with involution // Electronic Journal of Differential Equations. - 2015. - V. 2015, №278. - P. 1-9.

[6] Baskakov A. G, Krishtal I. A., Romanova E. Y. Spectral analysis of a differential operator with an involution // Journal of Evolution Equations. - 2017. - V. 17, №2. - P. 669-684.

[7] Kritskov L. - V., Sarsenbi A. M. Riesz basis property of system of root functions of second-order differential operator with involution // Differential Equations. - 2017. - V. 53, №1. - P. 33-46.

[8] Kritskov L. - V., Sadybekov M. A., Sarsenbi A. M. Nonlocal spectral problem for a second-order differential equation with an involution // Bulletin of the Karaganda University-Mathematics. - 2018. - V. 91, №3. - P. 53-60.

[9] Kritskov L. - V., Sadybekov M. A., Sarsenbi A. M. Properties in L_p of root functions for a nonlocal problem with involution // Turkish Journal of Mathematics. - 2019. - V. 43, №1. - P. 393-401.

[10] Kirane M., Sadybekov M. A., Sarsenbi A. A. On an inverse problem of reconstructing a subdiffusion process from nonlocal data // Mathematical Methods in the Applied Sciences. - 2019. - V. 42, №6. - P. 2043-2052.

[11] Vladykina - V. E., Shkalikov A. A. Spectral Properties of Ordinary Differential Operators with Involution // Doklady Mathematics. - 2019. - V. 99, №1. - P. 5-10.

[12] Ashyralyev A., Sarsenbi A. M. Well-posedness of a parabolic equation with nonlocal boundary condition // Boundary Value Problems. - 2015. - V. 2015, №1.

[13] Ashyralyev A., Sarsenbi A. M. Well-Posedness of a Parabolic Equation with Involution // Numerical Functional Analysis and Optimization. - 2017. - V. 38, №1. - P. 1-10.

[14] Orazov I., Sadybekov M. A. One nonlocal problem of determination of the temperature and density of heat sources // Russian Mathematics. - 2012. - V. 56, №2. - P. 60-64.

[15] Orazov I., Sadybekov M. A. On a class of problems of determining the temperature and density of heat sources given initial and final temperature // Siberian Mathematical Journal. - 2012. - V. 53, №1. - P. 146-151.

[16] Kurdyumov - V. P.,, Khromov A. P. The Riesz bases consisting of eigen and associated functions for a functional differential operator with variable structure // Russian Mathematics. - 2010. - V. 54, №2. - P. 39-52.

[17] Sarsenbi A. M., Tengaeva A. On the basis properties of root functions of two generalized eigenvalue problems // Differential Equations. - 2012. - V. 48, №2. - P. 306-308.

[18] Sadybekov M. A., Sarsenbi A. A. Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution // Differential Equations. - 2012. - V. 48, №8. - P. 1112-1118.

[19] Ruzhansky M., Sadybekov M. A., Suragan D. Spectral Geometry of Partial Differential Operators // New York: Taylor & Francis Group. - 2020.

[20] Schmidt W. M. Diophantine Approximations and Diophantine Equations // Mathematics. - 1991.

[21] Bari N. K. Biorthogonal Systems and Bases in Hilbert Space // Uchenye Zapiski Moskovskogo Gosudarstvennogo Universiteta. - 1951. - V. 148, №4 - P. 69-107. [in Russian]

Список литературы

[1] Cabada A., Tojo F., Adrian F. Differential equations with involutions // Workshop on Differential Equations. - 2015.

[2] Kopzhassarova A. A., Lukashov A. L., Sarsenbi A. M. Spectral Properties of non-self-adjoint perturbations for a spectral problem with involution // Abstract and Applied Analysis. - 2012. - V. 2012/ - Art. ID 576843/ P. 1-6. DOI: 10.1155/2012/576843.

[3] Kopzhassarova A.A., Lukashov A.L., Sarsenbi A.M. Spectral Properties of non-self-adjoint perturbations for a spectral problem with involution // Abstract and Applied Analysis. - 2012. - V. 2012. - Art. ID 590781. DOI: 10.1155/2012/590781.

[4] Kritskov L. - V., Sarsenbi A. M. Spectral properties of a nonlocal problem for the differential equation with involution // Differential Equations. - 2015. - V. 51, №8. - P. 984-990.

[5] Kritskov L. - V., Sarsenbi A. M. Basicity in L_p of root functions for differential equations with involution // Electronic Journal of Differential Equations. - 2015. - V. 2015, №278. - P. 1-9.

[6] Baskakov A. G, Krishtal I. A., Romanova E. Y. Spectral analysis of a differential operator with an involution // Journal of Evolution Equations. - 2017. - V. 17, №2. - P. 669-684.

[7] Kritskov L. - V., Sarsenbi A. M. Riesz basis property of system of root functions of second-order differential operator with involution // Differential Equations. - 2017. - V. 53, №1. - P. 33-46.

[8] Kritskov L. - V., Sadybekov M. A., Sarsenbi A. M. Nonlocal spectral problem for a second-order differential equation with an involution // Bulletin of the Karaganda University-Mathematics. - 2018. - V. 91, №3. - P. 53-60.

[9] Kritskov L. - V., Sadybekov M. A., Sarsenbi A. M. Properties in L_p of root functions for a nonlocal problem with involution // Turkish Journal of Mathematics. - 2019. - V. 43, №1. - P. 393-401.

[10] Kirane M., Sadybekov M. A., Sarsenbi A. A. On an inverse problem of reconstructing a subdiffusion process from nonlocal data // Mathematical Methods in the Applied Sciences. - 2019. - V. 42, №6. - P. 2043-2052.

[11] Vladykina - V. E., Shkalikov A. A. Spectral Properties of Ordinary Differential Operators with Involution // Doklady Mathematics. - 2019. - V. 99, №1. - P. 5-10.

[12] Ashyralyev A., Sarsenbi A. M. Well-posedness of a parabolic equation with nonlocal boundary condition // Boundary Value Problems. - 2015. - V. 2015, №1.

[13] Ashyralyev A., Sarsenbi A. M. Well-Posedness of a Parabolic Equation with Involution // Numerical Functional Analysis and Optimization. - 2017. - V. 38, №1. - P. 1-10.

[14] Orazov I., Sadybekov M. A. One nonlocal problem of determination of the temperature and density of heat sources // Russian Mathematics. - 2012. - V. 56, №2. - P. 60-64.

[15] Orazov I., Sadybekov M. A. On a class of problems of determining the temperature and density of heat sources given initial and final temperature // Siberian Mathematical Journal. - 2012. - V. 53, №1. - P. 146-151.

[16] Kurdyumov - V. P.„ Khromov A. P. The Riesz bases consisting of eigen and associated functions for a functional differential operator with variable structure // Russian Mathematics. - 2010. - V. 54, №2. - P. 39-52.

[17] Sarsenbi A. M., Tengaeva A. On the basis properties of root functions of two generalized eigenvalue problems // Differential Equations. - 2012. - V. 48, №2. - P. 306-308.

[18] Sadybekov M. A., Sarsenbi A. A. Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution // Differential Equations. - 2012. - V. 48, №8. - P. 1112-1118.

[19] Ruzhansky M., Sadybekov M. A., Suragan D. Spectral Geometry of Partial Differential Operators // New York: Taylor & Francis Group. - 2020.

[20] Schmidt W. M. Diophantine Approximations and Diophantine Equations // Mathematics. - 1991.

[21] Бари Н. К. Биортогональные системы и базисы в гильбертовом пространстве // Учение записки Московского государственного университета. - 1951. - Т. 148, №4 - С. 69-107.