IRSTI 27.03.45 DOI: https://doi.org/10.26577/JMMCS.2022.v113.i1.03

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# ON UNIVERSAL NUMBERINGS OF GENERALIZED COMPUTABLE FAMILIES

The paper investigates the existence of universal generalized computable numberings of different families of sets and total functions. It was known that for every set $A$ such that $\emptyset' \leq_T A$, a finite family $\mathcal{S}$ of $A$-c.e. sets has an $A$-computable universal numbering if and only if $\mathcal{S}$ contains the least set under inclusion. This criterion is not true for infinite families. For any set $A$ there is an infinite $A$-computable family of sets $\mathcal{S}$ without the least element under inclusion that has an $A$-computable universal numbering; moreover, the family $\mathcal{S}$ consist pairwise not intersect sets. If $A$ is a hyperimmune set, then an $A$-computable family $F$ of total functions which contains at least two elements has no $A$-computable universal numbering. And if $\deg_T(A)$ is hyperimmune-free, then every $A$-computable finite family of total functions has an $A$-computable universal numbering. In this paper for a hyperimmune-free oracle $A$ we show that any infinite effectively discrete family of sets has an $A$-computable universal numbering. It is also proved that if family $\mathcal{S}$ contains all co-finite sets and does not contain at least one co-c.e. set, then this family has no $\Sigma_2^{-1}$-computable universal numbering.
**Key words:** Rogers semilattice, Ershov hierarchy, computable numbering, universal numbering, hyperimmune set.

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**Жалпыланған есептелімді үйірлердің универсал нөмірлеулері**

Бұл жұмыста жиындардан және барлық жерде анықталған функциялардан құралған әр түрлі үйірлердің универсал жалпыланған есептелімді нөмірлеулері болуы туралы сұрақтар зерттеледі. Бұрыннан белгілі кез келген $A$ жиыны үшін, бұл жерде $\emptyset' \leq_T A$, $A$-e.c. $\mathcal{S}$ ақырлы үйірінде $A$-есептелімді универсал нөмірлеу болады, сонда тек сонда ғана $\mathcal{S}$ үйірі ішкі жиын қатынасы бойынша ең кіші элементті қамтыса. Бұл критерий шексіз үйірлер үшін орындалмайды. Кез келген $A$ жиыны үшін $A$-есептелімді универсал нөмірлеуі бар ішкі жиын қатынасы бойынша ең кіші элементті қамтымайтын жиындардан құралған шексіз $A$-есептелімді $\mathcal{S}$ үйірі табылады, сонымен қатар $\mathcal{S}$ үйірінде барлық элементтер жұптық қиылыспайды. Егер $A$– гипериммунды жиын болса, онда кем дегенде екі элементі бар барлық жерде анықталған функциялардан құралған $A$-есептелімді $\mathcal{F}$ үйірінде $A$-есептелімді универсал нөмірлеуі болмайды. Ал егер $\deg_T(A)$ гипериммунды-бос болса, онда әрбір барлық жерде анықталған функциялардан құралған үйірде $A$-есептелімді универсал нөмірлеуі болады. Бұл жұмыста біз $A$-гипериммунды-бос оракулымен есептелетін жиындардан құралған кез келген шексіз эффективті дискретті үйірлерде $A$-есептелімді универсал нөмірелуі болатынын көрсеттік. Сондай-ақ, егер $\mathcal{S}$ үйірі барлық толықтауы ақырлы жиындарды қамтыса және кем дегенде бір толықтауы рекурсив саналымды жиынды қамтымаса, онда бұл үйірдің универсал $\Sigma_2^{-1}$-есептелімді нөмірлеуі болмайтыны дәлелденді.

**Түйін сөздер:** Роджерс жарты торы, Ершов иерархиясы, есептелімді нөмірлеу, универсал нөмірлеу, гипериммунды жиын.

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**Об универсальных нумерациях обобщенно вычислимых семейств**

В работе исследуется вопросы существования универсальных обобщенно-вычислимых нумерации различных семейтв множеств и всюду определенных функций. Было известно, что для любого множество $A$ такого, что $\emptyset' \leq_T A$, конечное семейство $A$ -в.п. множеств $\mathcal{S}$ имеет $A$ -вычислимую универсальную нумерацию тогда и только тогда, когда $\mathcal{S}$ содержит наименьшее по включению множество. Данный критерий не верно для бесконечных семейств. Для любого множества $A$ существует бесконечное $A$-вычислимое семейство множеств $\mathcal{S}$ без наименьшего по включению элемента имеющий $A$-вычислимую универсальную нумерацию, более того в семействе $\mathcal{S}$ все элементы попарно не пересекаются. Если $A$ — гипериммунное множество, тогда $A$-вычислимое семейство $F$ всюду определенных функций содержащий не менее двух элементов не имеет универсальной $A$-вычислимой нумерации. А если $\deg_T(A)$ гипериммунно-свободно, тогда каждое $A$-вычислимое конечное семейство всюду определенных функций имеет $A$-вычислимую универсальную нумерацию. В данной работе показывается, что любое бесконечное эффективно дискретное семейство множеств с гипериммунно-свободным оракулом $A$ имеет $A$-вычислимую универсальную нумерацию. Также доказывается, что если семейство $\mathcal{S}$ содержит все ко-конечные множества и не содержит хотябы один ко-в.п. множество, тогда данное семейство не имеет универсальную $\Sigma_2^{-1}$-вычислимую нумерацию.

**Ключевые слова:** полурешётка Роджреса, иерархия Ершова, вычислимая нумерация, универсальная нумерация, гипериммунное множество.

## 1 Introduction

In this paper, we consider some issues, mainly related to universal numberings. The interest in the study of such numberings is due to the fact that the universal computable numbering of any family contains information about all its computable numberings. Until now, various results have been obtained on generalized computable numberings in the arithmetic hierarchy and in the Ershov hierarchy. Most of the results in numbering theory are related to the study of properties of Rogers semilattices. Recall some definitions of the theory of numberings (see, for example, [1], [2] for details). Any surjective mapping $\alpha$ of the set $\omega$ of natural numbers onto a nonempty set $S$ is called a *numbering* of $S$. Let $\alpha$ and $\beta$ be numberings of $S$. We say that a numbering $\alpha$ is *reducible* to a numbering $\beta$ (written $\alpha \leq \beta$ ) if there exists a computable function $f$ such that $\alpha(n) = \beta(f(n))$ for any $n \in \omega$. We say that the numberings $\alpha$ and $\beta$ are *equivalent* (written $\alpha \equiv \beta$) if $\alpha \leq \beta$ and $\beta \leq \alpha$.

Let $A$ be a set of natural numbers, and let $\mathcal{S}$ be a family of $A$-computably enumerable (in Ershov hierarchy $\Sigma^0_{n+1}$-c.e.) set. We say that a numbering $\alpha$ of $S$ is $A$-*computable* (in Ershov hierarchy $\Sigma^0_{n+1}$-*c.*) if its universal set $\{< x, n > | x \in \alpha(n)\}$ is $A$-c.e. (in Ershov hierarchy $\Sigma^0_{n+1}$-c.e.). Let $Com(\mathcal{S})$ be the set of all $A$-computable (in Ershov hierarchy $\Sigma^0_{n+1}$-c.) numberings of the family $\mathcal{S}$. The numbering reducibility relation is a preorder on $Com^A(\mathcal{S})$, (in Ershov hierarchy $Com^0_{n+1}(\mathcal{S})$-c.) which in the usual way induces some quotient structure $\mathcal{R}(\mathcal{S})$ (in Ershov hierarchy $\mathcal{R}^0_{n+1}(\mathcal{S})$), which is an upper semilattice and is called the *Rogers semilattice* of $A$-computable (in Ershov hierarchy $\Sigma^0_{n+1}$-c.) numberings of the family $\mathcal{S}$.

An $A$-computable numbering of a family $\mathcal{S}$ is *universal* if any $A$-computable numbering of $S$ is reducible to that numbering. Denote by $A^{<x>}$ the set $\{y :< x, y >\in A\}$. An $A$-c.e. set with Godel number $e$ is denoted by $W^A_e$.

A computable family $\mathcal{S}$ is called *effectively discrete* if there exists such a strongly computable sequence of finite sets such that [3]:

(1) for any $A \in \mathcal{S}$ there exists $T_i \subseteq A_i$;
(2) $T_i \subseteq T_j \Rightarrow T_i = T_j$;
(3) $T_i \subseteq A_j \in A$ and $T_i \subseteq A_p \in A$ then $A_j = A_p$.

A function $f$ dominates a function $g$ if $f(x) \geq g(x)$ for all $x$. A degree $a$ is *hyperimmune* if there is a function $f \leq_T a$ which is not dominated by any recursive function: otherwise $a$ is *hyperimmune-free* [4].

The Ershov hierarchy consists of finite and infinite levels. The finite levels of the hierarchy consists of the $n$-c.e. or $\Sigma^{-1}_n$ sets for $n \in \omega$.

A set $A$ is $\Sigma^{-1}_n$ set, if either $n = 0$ and $A = \emptyset$, or $n > 0$ and there are c.e. sets $R_0 \supseteq R_1 \supseteq .... \supseteq R_{n-1}$ such that [5]

$$A = \bigcup_{i=0}^{[\frac{n-1}{2}]} (R_{2i} \setminus R_{2i+1}) \quad \text{(Here if } n \text{ is odd number then } R_n \neq \emptyset)$$

The class of $\Sigma^{-1}_1$ sets coincide with the class c.e. sets, $\Sigma^{-1}_2$ sets can be written as $R_1 \setminus R_2$, where $R_1 \supseteq R_2$ c.e. sets, therefore they are also called $d$-c.e. sets.

The $n$-c.e. sets are exactly those sets constitute the level $\Sigma^{-1}_n$ of the Ershov hierarchy.

## 2 Literature review

In [6] the theory of numbering was studied for the first time and the concepts of computable numbering were proposed for constructive languages describing a numbered family of objects that were called generalized computable. Also, in fact, in [7] [also [8–10]] the work of Badaev and Goncharov was started to study the numberings $A$-computable, where $A$ is a given oracle. Hyperimmune-free degrees have been studied extensively beginning with the work of Miller and Martin [11], and Jockusch and Soare 12. Despite the abundance of work dedicated to the generalized computable numberings, there are a number of open questions about Rogers semilattice of generalized computable families.

## 3 Material and methods

### 3.1 The formulation of the problem

Let us pass now to recalling the available results about the existence of $A$-computable universal numberings. The following results about criteria for finite and infinite families of $A$-c.e. sets with $\emptyset' \leq_T A$ were obtained in [7].

**Theorem 1.** *[7] For every set $A$ such that $\emptyset' \leq_T A$, a finite family $\mathcal{S}$ of $A$-c.e. sets has an $A$-computable universal numbering if and only if $\mathcal{S}$ contains the least set under inclusion.*

*Proof.* We will give a brief of the proof [13].

Let $\mathcal{S} = \{A_1, A_2, ..., A_n\}$ be an $A$-computable family. Choose a family of finite sets $F_1, ..., F_n$ with the following property: for all $1 \leq i,\ j \leq n$, we have

$$F_i \subseteq F_j \Leftrightarrow A_i \subseteq A_j \Leftrightarrow F_i \subseteq A_j$$

Suppose that $A_1$ is a least element of $\mathcal{S}$ and $F_1 = \emptyset$. Let $\mathfrak{C}$ be the set of all chains, i.e. strictly increasing sequences $F_{i_1} \subset ... \subset F_{i_k}$. There is a maximal chain $\mathcal{C} = \{i_0 < i_1 < ... < i_k\}$ and denote by $A_{\mathcal{C}}$ the set of the family corresponding to $F_{\mathcal{C}}$ (i.e. $A_{\mathcal{C}} = A_i$ if and only if $F_{\mathcal{C}} = F_i$). It is easy (see [14] for details) to build a $A$-computable numbering $\alpha$ such that, for every $e$,

(1) We found maximal $i_k$ with $F_{i_k} \subseteq W_e^A$. Then, clearly, $\alpha(e) = A_{i_k}$;

(2) We found number $i_k$ such that $F_{i_k} \subseteq W_e^A$. If $W_e^A \in (\mathcal{S})$ then $W_e^A = \alpha(e)$.

Assume now $\rho(e) = W_e^A$. Thus, for any $A$-computable numbering $\beta$ of the family $\mathcal{S}$, there exists a computable function $f(x)$ such that $\beta(x) = \rho(f(x))$ for all $x$. By constructing $A$-computable numbering $\alpha$ implies that $\beta(x) = \rho(f(x)) = \alpha(f(x))$, and therefore, $\alpha$ is universal $A$-computable numbering of the family $\mathcal{S}$.

Suppose that the family $\mathcal{S}$ has no least element. Let

$$\mathcal{S}_0 \leftrightharpoons \mathcal{S} \cup \{\emptyset\}$$

and by the above argument let $\alpha_0$ be universal in $Com^A(\mathcal{S}_0)$. Let us define a $A$-computable function $f : \omega \to \{x | \alpha_0(x) \neq \emptyset\}$.

Then $\alpha = \alpha_0 \circ f \in Com^A(\mathcal{S})$. Let now $\beta \in Com^A(\mathcal{S})$, and define

$$\beta_0 \leftrightharpoons \begin{cases} \emptyset & \text{if } x = 0, \\ \beta(x-1) & \text{otherwise} \end{cases}$$

and let $h$ be a computable function such that $\beta_0 = \alpha_0 \circ h$. Hence, for every $x$

$$\beta(x) = \beta_0(x+1) = \alpha_0(h(x+1)).$$

But $h(x+1) \in range(f)$. Let

$$k(x) \leftrightharpoons \mu y(f(y) = h(x+1)).$$

It follows that $\beta(x) = \alpha_0(f(k(x)))$, i.e. $\beta = \alpha \circ k$. Since $k$ is $A$-computable, it follows that $\beta \leq \alpha$, hence $\alpha$ is $A$-universal in $Com^A(\mathcal{S})$. □

Badaev and Goncharov also showed that for $\emptyset' \leq_T A$ the presence of the least set under inclusion is neither necessary nor sufficient for an infinite family of $A$-c.e., sets to have an $A$-computable universal numbering [7].

**Corollary 1.** *[7] For every set $A$, there is an $A$-computable family that contains the least set under inclusion but has no $A$-computable universal numbering.*

The following theorem gives a negative answer to the question of Podzorov in [15]: Is it true that if the Rogers semilattice of a family of arithmetical sets has the greatest element then the family itself has the least set under inclusion?

**Theorem 2.** *[7] For every set $A$, there is an infinite $A$-computable family $\mathcal{S}$ of sets with pairwise disjoint elements such that $\mathcal{S}$ has an $A$-computable universal numbering.*

Also, in [7] the following questions were posed: Does the statement of Theorem 1 for finite families of sets remain valid if $\emptyset <_T A <_T \emptyset'$ or $A$ is Turing incomparable with $\emptyset'$?

In [8–10], gives answers with hyperimmune and hyperimmune-free oracles. If $\emptyset <_T A <_T \emptyset'$ or $\emptyset' \leq_T A$, then $\deg_T(A)$ is hyperimmune.

**Theorem 3.** *[8] Let $A$ be a hyperimmune set. If $A$-computable family $F$ of total functions contains at least two elements, then $F$ has no universal $A$-computable numbering.*

**Theorem 4.** *[8] If $\deg_T(A)$ is hyperimmune-free. Then every $A$-computable finite family of total functions has an $A$-computable universal numbering.*

**Theorem 5.** *[9] Let $\deg_T(A)$ is hyperimmune-free. Then any finite family of $A$-c.e. sets has a universal $A$-computable numbering.*

**Corollary 2.** *[9] For a set $A$ the following conditions are equivalent.*

*(1)* $\deg_T(A)$ *is hyperimmune;*
*(2) there exists a finite family of $A$-c.e. sets which does not have universal $A$-computable numberings;*
*(3) every finite family of $A$-c.e. sets without a least element under inclusion has no universal $A$-computable numberings.*

After these results, the unsolved question was whether there exists an infinite family of total functions with a hyperimmune-free oracle that has an $A$-computable universal numbering. In [10] gives a positive answer to this question and it is proved.

**Theorem 6.** *[10] There exists an infinite $A$-computable family $\mathcal{F}$ of total functions, where Turing degree of the set $A$ is hyperimmune-free, such that $\mathcal{F}$ has an $A$-computable universal numbering.*

In [16], some generalization of this result was obtained for an infinite $A$-computable effectively discrete family of total functions.

**Theorem 7.** *[16] Let $\mathcal{S}$ be an infinite $A$-computable effectively discrete family of total functions, where $A$—hyperimmune-free, then family $\mathcal{S}$ has an $A$-computable universal numbering.*

## 4 Main results

In this section, we have proved the results for a family of sets.

**Theorem 8.** *Let $\mathcal{S}$ be an infinite $A$-computable effectively discrete family of sets, where $A$—hyperimmune-free, then family $\mathcal{S}$ has an $A$-computable universal numbering.*

*Proof.* The following proof is based on the ideas of [9, 10]. Let $\mathcal{S}$ be an infinite $A$-computable effective discrete family of sets, say $\mathcal{S} = \{A_i\}_{i\in\omega}$. Then it is possible to find a strongly computable sequence of finite sets $T_i | i \in \omega$ such that

(1) $T_i \subseteq A_i$;
(2) $T_i \subseteq T_j \Rightarrow T_i = T_j$;
(3) $T_i \subseteq A_j \in A$ and $T_i \subseteq A_p \in A$ then $A_j = A_p$.

We define an $A$-computable numbering $\beta$ as follows: for every $e$,

$$\beta(< e, x, s >) = \begin{cases} A_k & \text{if } (\exists k < s)[T_k \subseteq (W^A_{e,s})^{<x>}], \\ A_0 & \text{otherwise} \end{cases}$$

It is clear that $\beta$ is a $A$-computable numbering of the family $\mathcal{S}$. Now let $\alpha$ be an arbitrary $A$-computable numbering of $\mathcal{S}$. We need to show that $\alpha \leq \beta$. Fix an index $e$ for which $\alpha(x) = (W^A_{e,s})^{<x>}$ with any $x$.

Let $g$ be $A$-computable function, and define

$$g(x) = \mu_s[(\exists k < s)[T_k \subseteq (W^A_{e,s})^{<x>}].$$

Since $A$ is hyperimmune-free sets, it is follows that there exists a computable function $f$ such that $g(x) \leq f(x)$ for all $x$. It means that for all $e$ and $x$ satisfy the following

$$\beta < e, x, f(x) >= A_k = (W^A_{e,f(x)})^{<x>} = \alpha(x)$$

Hence, $\beta$ is $A$-computable universal numbering $Com^A(\mathcal{S})$ .

□

**Theorem 9.** *Let $\mathcal{A}$ be a family of all co-finite sets. If $\mathcal{S}$ be a $\Sigma^{-1}_2$-computable family such that $\mathcal{A} \subseteq \mathcal{S}$ and there is co-c.e. set $B$ that $B \notin \mathcal{S}$, then $\mathcal{R}^{-1}_2(\mathcal{S})$ has no universal numbering.*

*Proof.* Let $\nu \in \mathcal{R}^{-1}_2(\mathcal{S})$ be arbitrary numbering. We will construct a numbering $\mu \in Com^{-1}_2(\mathcal{S})$ which not reduced to $\nu$. Let $\mu(2x) = \nu(x)$ and $\mu(2x+1)$ defined as following. In construction of $\mu(2x+1)$ we additional constructed function $r_s$.

Stage 0. Put $\mu_0(2x+1) = \omega$ and $r_0 = 1$.

Stage s+1. We will consecutively implement the following stages:

(1) If $\varphi_{x,s}(2x+1) \uparrow$ then go to the next stage
(2) If $\varphi_{x,s}(2x+1) \downarrow= y$, and $\mu_s(2x+1) \restriction r_s = \nu_s(y) \restriction r_s$ then $r_{s+1} = r_s + 1$ and $\mu_{s+1}(2x+1) \restriction r_s = B_s \restriction r_s$
(3) If $\varphi_{x,s}(2x+1) \downarrow= y$ and $\mu_s(2x+1) \restriction r_s \neq \nu_s(y) \restriction r_s$ then $\mu_{s+1}(2x+1) = \mu_s(2x+1)$ and $r_{s+1} = r_s$.

The description of the construction is over. Let

$$\nu(2x+1) = \bigcap_s \mu_s(2x+1).$$

**Lemma 1.** *If for $x$ case (2) in construction is hold infinitely often then $\nu(y) = B$.*

*Proof.* Let for $x$ case (2) is hold infinitely often and $\nu(y) \neq B$, where $y = \varphi_x(2x+1)$. Consider two case

(1) Let $\exists z$, $z \in \nu(y)$ and $z \notin B$.

Every stage when case (2) is hold the function $r_s$ will be increasing. Then we can find stage $s^*$ such that $z \in \nu_{s^*}(y)$ and $r_{s^*} \geq z$. Since $z \notin B$ then $z \notin B_s$ for every $s \in \omega$. So for every $s^{'} > s^*$, we have $z \notin \mu_{s^{'}}(2x+1) \upharpoonright r_{s^{'}}$ and $z \in \nu_{s^{'}}(y)$. So the condition of case (2) is not hold for any $s^{'} > s^*$. It is contradicted that case (2) is hold infinitely often.

(2) Let $\exists z$, $z \notin \nu(y)$ and $z \in B$.

Suppose that there is a step $s^*$ such that $z \notin \nu_{s^*}(y)$, $r_{s^*} \geq z$ and $z \notin B$. Let's choose the smallest step $s^{'}$ and $\forall s^{'} > s^*$ and $z \notin \mu_{s^{'}}(2x+1) \upharpoonright r_{s^{'}}$. We come to a contradiction with properties of the case (2) in construction, that $\mu_{s^{'}}(2x+1) \upharpoonright r_{s^{'}} \neq \nu_{s^{'}}(y)$ such that $z \notin B_{s^{'}} \upharpoonright r_{s^{'}}$.

□

**Lemma 2.** *If* $\varphi_x(2x+1)\downarrow$ *then* $\lim_s(r_s)$ *exists and* $\mu(2x+1) \neq \nu(\varphi_x(2x+1))$.

*Proof.* Since $B \notin \mathcal{S}$ then from Lemma 1 there is stage $s^{'}$ that for all $s > s^{'}$ which hold only condition of case (3). It means that the function $r_s$ has a limit and let it will be $r$. Then $\mu(2x+1) \upharpoonright r \neq \nu(y) \upharpoonright r$. Consequently, $\mu(2x+1) \neq \nu(\varphi_x(2x+1))$. □

**Lemma 3.** $\mu$ *is* $\Sigma_2^{-1}$*-computable numbering of family* $\mathcal{S}$.

*Proof.* $\theta_\mu = \{(x,y)\colon y \in \mu(x)\}$. Let $A_1$, $A_2$ computable enumerable sets such that $\theta_\nu = A_1 \setminus A_2$. Then

$$B_1 = \{(2x,y)\colon (x,y) \in A_1\} \cup \{(2x+1,y)\colon x,y \in \omega\},$$

$$B_2 = \{(2x,y)\colon (x,y) \in A_2\} \cup \{(2x+1,y)\colon \exists s[y \notin \mu_s(2x+1)]\}.$$

It is not difficult to see $\theta_\mu = B_1 \setminus B_2$ and $B_1$, $B_2$ are computably enumerable sets. Consequently, $\theta_\mu \in \Sigma_2^{-1}$.

Now show that $\mu(x) \in \mathcal{S}$ for all $x$. If $x = 2k$ then $\mu(2k) = \nu(k)$. So $\nu(k) \in \mathcal{S}$ then $\mu(x) \in \mathcal{S}$ . If $x = 2k+1$ then $\mu(2k+1) = \omega$ if $\varphi_k(2k+1)\uparrow$ and it is gives co-finite. If $\varphi_k(2k+1)\downarrow$ by Lemma 2 there exists $\lim_s(r_s)$ and by constraction $\mu(2k+1)$ contain $[\lim_s(r_s), \infty)$. Consequently, $\mu(2k+1)$ is co-finite. Since family $\mathcal{A} \subseteq \mathcal{S}$ contain all co-finite sets, it means that $\mu(x) \in \mathcal{S}$. □

If $\mu \leq \nu$ then there exists computable function $f$ such that $\mu(x) = \nu(f(x))$ for all $x$. Let $f = \varphi_e$ for some $e$. Since $f$ is total, $\varphi_e(2e+1)\downarrow$. From Lemma 2 $\mu(2e+1) \neq \nu(\varphi_e(2e+1))$. It is contradiction. □

**Corollary 3.** *If* $\mathcal{S}$ *is the family of all c.e. sets, then* $\mathcal{R}_2^{-1}(\mathcal{S})$ *has no universal numbering.*

## 5 Conclusion

In conclusion, we proved that if an infinite $A$-computable effectively family of sets, where $A$ is a hyperimmune set, then the family has an $A$-computable universal numbering. It was also proved that in $\mathcal{R}_2^{-1}(\mathcal{S})$ has no universal numbering if $\mathcal{A}$ be a family of all co-finite sets and if $\mathcal{S}$ be a $\Sigma_2^{-1}$-computable family such that $\mathcal{A} \subseteq \mathcal{S}$ and there is co-c.e. set $B$ that $B \notin \mathcal{S}$.

## 6 Acknowledgement

The work was supported by grant funding of scientific and technical programs and projects of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08856493 Positive graphs and computable reducibility on them, as a mathematical model of databases, 2020-2022).

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