1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

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INVERSE PROBLEMS OF PARAMETER RECOVERY IN DIFFERENTIAL EQUATION WITH MULTIPLE CHARACTERISTICS

Inverse problems - the problem of finding the causes of known or given consequences. They arise when the characteristics of an object of interest to us are not available for direct observation. These are, for example, the restoration of the characteristics of the field sources according to their given values at some points, the restoration or interpretation of the original signal from the known output signal, etc. This paper studies the solvability of finding the solution of a differential equation of inverse problems. The work is devoted to the study of the solvability in Sobolev spaces of nonlinear inverse coefficient problems for differential equations of the third order with multiple characteristics. In this paper, alongside with finding the solution of one or another differential equation, it is also required to find one or more coefficients of the equation itself for us to name them inverse coefficient problems. A distinctive feature of the problems studied in this paper is that the unknown coefficient is a numerical parameter, and not a function of certain independent variables.
Key words: Inverse problems, third-order equations, multiple characteristics, numerical parameter, solvability.

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Сиппаттамалары еселі диффененциалдық теңдеулердегі параметрді қалпына келтірудің кері есебі

Кері есептер - белгілі немесе берілген әсерлердің себептерін табу мәселесі. Олар бізді қызықтыратын объектінің сипаттамалары тікелей бақылау үшін қол жетімді болмаған кезде пайда болады. Бұл, мысалы, кейбір нүктелердегі олардың белгіленген мәндеріне сәйкес өріс көздерінің сипаттамаларын қалпына келтіру, белгілі шығыс сигналынан бастапқы сигналды қалпына келтіру немесе интерпретациялау және т.б. Берілген жұмыста біз дифференциалдық теңдеуге қойылған кері есептің шешімділігін зерттейміз. Жұмыс бірнеше сипаттамалары бар үшінші ретті дифференциалдық теңдеулер үшін сызықты

емес кері коэффициентті есептерінің Соболев кеңістігінде шешімділігін зерттеуге арналған. Бұл мақалада белгілі бір дифференциалдық теңдеудің шешімін іздеумен қатар теңдеудің бір немесе бірнеше коэффициенттерін табу да талап етіледі, сондықтан оларды кері коэффициенттік есептер деп атаймыз. Бұл жұмыста зерттелген есептердің айрықша ерекшелігі белгісіз коэффициент белгілі бір тәуелсіз айнымалылардың функциясы емес, сандық параметр болып табылады.
Кілттік сөздер: Кері есептер, үшінші ретті теңдеулер, еселі сипаттауыштар, сандық параметр, шешімділік.

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Обратные задачи восстановления параметров в дифференциальном уравнении с кратными характеристиками

Обратные задачи - задача нахождения причин известных или заданных следствий. Они возникают, когда характеристики интересующего нас объекта недоступны для непосредственного наблюдения. Это, например, восстановление характеристик источников поля в соответствии с их заданными значениями в некоторых точках, восстановление или интерпретация исходного сигнала из известного выходного сигнала и т.д. В данной работе исследуется разрешимость нахождения решения дифференциального уравнения обратных задач. Работа посвящена исследованию разрешимости в пространствах Соболева нелинейных обратных коэффициентных задач для дифференциальных уравнений третьего порядка с множественными характеристиками. В этой статье, наряду с поиском решения того или иного дифференциального уравнения, также требуется найти один или несколько коэффициентов самого уравнения, чтобы мы назвали их обратными коэффициентными задачами. Отличительной особенностью задач, изучаемых в данной работе, является то, что неизвестный коэффициент является числовым параметром, а не функцией определенных независимых переменных.
Ключевые слова: Обратные задачи, уравнения третьего порядка, кратные характеристики, числовой параметр, разрешимость.

1 Introduction

The work is devoted to the study of the solvability in Sobolev spaces of nonlinear inverse coefficient problems for differential equations of the third order with multiple characteristics. In this paper, alongside with finding the solution of one or another differential equation, it is also required to find one or more coefficients of the equation itself for us to name them inverse coefficient problems. Various aspects of the theory of inverse coefficient problems for differential equations are well covered in the world literature - see monographs [1-17]. At the same time, it should be noted that there are not many works devoted to the study of the solvability of inverse problems for differential equations with multiple characteristics - we can only name the works [17-19].

A distinctive feature of the problems studied in this paper is that the unknown coefficient is a numerical parameter, and not a function of certain independent variables. Similar problems were studied earlier, but only for classical parabolic, hyperbolic and elliptic equations - see papers [20-29]. The inverse problems of determination for differential equations with multiple characteristics, together with the solution of numerical parameters, which are the coefficients of the equation itself, have not been previously studied.

It should be noted that differential equations with constant coefficients are often obtained by mathematical modeling of processes taking place in a homogeneous medium - see papers [30, 31]. If in this case the coefficients characterizing certain properties of the environment are unknown quantities, then we will automatically obtain inverse problems with unknown parameters. All constructions and arguments in this work will be carried out using Lebesgue spaces L_p and Sobolev spaces W_p^l. The necessary information about functions from these spaces can be found in monographs [32-34].

2 Problem statement

Let Ω be an interval on the Ox, Q be a rectangle $\{(x,t): \; x \in \Omega, \; t \in (0,T)\}$ $(0 < T < +\infty)$. Further, let $N(x)$ and $f(x,t)$ be given functions defined for $x \in \overline{\Omega}$, $t \in [0,T]$, A and β be given real numbers.

Inverse Problem I: it to find the function $u(x,t)$ and a positive number α connected in the rectangle Q by the equation

$$\alpha u_t - u_{xxx} + \beta u = f(x,t), \tag{1}$$

when the following conditions for the function $u(x,t)$ are met.

$$u(x,0) = 0, \quad x \in \Omega; \tag{2}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0, \quad t \in (0,T); \tag{3}$$

$$\int\limits_{\Omega} N(x)u(x,T)\,dx = A. \tag{4}$$

Inverse Problem II: it to find the function $u(x,t)$ and a positive number α connected in the rectangle Q by the equation

$$u_t - \alpha u_{xxx} + \beta u = f(x,t), \tag{5}$$

when the conditions (2)–(4) for the function $u(x,t)$ are met .

In inverse problems I and II, conditions (2) and (3) are the conditions of the general initial-boundary value problem for a third-order differential equation with multiple characteristics, while condition (4) is the initial integral overdetermination condition, the presence of which is explained by the presence of an additional unknown value of the coefficient (parameter).

Differential equations (1) and (5) have a simple model form. Possible generalizations of these equations and possible enhancements and generalizations of the obtained results will be described at the end of the work.

3 Solvability of Inverse Problem I.

The study of the solvability of the inverse problem will be carried out by a method based on the transition from the original problem to a new problem for a nonlinear integro–differential equation – see works [26–29].

Let R_1 be a given positive number, $\varphi_1(v)$ be the function

$$\varphi_1(v) = \int\limits_Q N(x)[v_{xxx}(x,t) - \beta v(x,t)]\,dx\,dt.$$

Let us consider the following problem: to find a function $u(x,t)$ which is a solution to the equation in the rectangle Q

$$\frac{R_1 + \varphi_1(u)}{A} u_t - u_{xxx} + \beta u = f(x,t) \tag{6}$$

and conditions (2) and (3) must be satisfied for that function.

In the boundary value problem (6), (2), (3), equation (6) is an integro–differential equation, called in some sources as "loaded" [35, 36].

Let μ_0 be a number from the interval $(0,1)$. Let us consider the following:

$$f_1(x,t) = (T-t)f(x,t),$$

$$K_1 = \frac{T^{1/2}}{\beta}\|N\|_{L_2(\Omega)}\|f_{xxx}\|_{L_2(Q)}, \;\; K_2 = \frac{2\beta A T^{1/2}}{(1-\mu_0)}\|N\|_{L_2(\Omega)}\|f_1\|_{L_2(Q)},$$

$$K_0 = \frac{1}{2\mu_0}\left(K_1 + \sqrt{K_1^2 + 4\mu_0 K_2}\right).$$

Theorem 1 *Let the following conditions be satisfied*

$$N(x) \in L_2(\Omega); \;\; \beta > 0, \quad A > 0; \;\; \frac{\partial^k f(x,t)}{\partial x^k} \in L_2(Q), \quad k = \overline{0,3},$$

$$f(0,t) = f_x(0,t) = f_{xx}(1,t) = 0 \quad npu \quad t \in (0,T);$$

$$\exists\, \mu_0 \in (0,1): \;\; K_0 \le R_1.$$

Then the boundary value problem (6), (2), (3) has the following $u(x,t)$ solution, $u(x,t) \in L_2(0,T;W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $|\varphi_1(u)| \le \mu_0 R_1$.

Proof. For the number $\mu = \mu_0 R_1$ let us define a function $G_\mu(\xi)$:

$$G_\mu(\xi) = \begin{cases} \xi, & if \;\; |\xi| \le \mu, \\ \mu, & if \;\; \xi > \mu, \\ -\mu, & if \;\; \xi < -\mu. \end{cases}$$

Let us consider the following problem: to find a function $u(x,t)$ that is a solution to the equation in rectangle Q and such that conditions (2) and (3) are satisfied for it.

$$\frac{R_1 + G_\mu[\varphi_1(u)]}{A} u_t - u_{xxx} + \beta u = f(x,t) \tag{6_μ}$$

Using the regularization method, a priori estimates and the fixed point method we will show that this problem has a regular solution (that is, a solution that has all its derivatives generalized according to S.L. Sobolev).

Let ε be a positive number. Let us consider the following problem: *to find a function $u(x,t)$ which is a solution to the equation*

$$\frac{R_1 + G_\mu(\varphi_1(u))}{A} u_t - u_{xxx} - \varepsilon u_{xxxxxxx} + \beta u = f(x,t) \tag{7}$$

in the rectangle Q and such that conditions (2) and (3), as well as the condition (8) are satisfied

$$u_{xxx}(0,t) = u_{xxxx}(1,t) = u_{xxxxx}(1,t) = 0, \quad t \in (0,T). \tag{8}$$

Let V be the set of functions $v(x,t)$ such that $v(x,t) \in L_2(0,T;W_2^6(\Omega))$, $v_t(x,t) \in L_2(0,T;L_2(\Omega))$, and the function $v(x,t)$ satisfies conditions (2), (3), and (8). Let us give this set the following norm

$$\|v\|_V = \left(\int\limits_Q (v^2 + v_t^2 + v_{xxxxxx}^2)\, dx\, dt \right)^{1/2}.$$

Obviously, the set V with this norm will be a Hilbert space.

For a function $v(x,t)$ from the space V, we will consider the following problem: to find a function $u(x,t)$ which is a solution to the equation

$$\frac{R_1 + G_\mu(\varphi_1(v))}{A} u_t - u_{xxx} - \varepsilon u_{xxxxxxx} + \beta u = f(x,t) \tag{9}$$

and such that conditions (2), (3), and (8) are satisfied for it. In this problem, differential equation (9) is a linear parabolic equation of the sixth order, while boundary conditions (3) and (8) are self-adjoint. Consequently, this problem is solvable in the space V (this fact can be proved directly using the classical Galerkin method with the choice of a special basis).

The solvability in the space V of the boundary value problem (9), (2), (3), (8) means that this problem generates an operator Φ acting from the space V and associating the function $v(x,t)$ from V with the solution $u(x,t)$ of the boundary value problem (9), (2), (3), (8). Let us show that this operator has fixed points in the space V. First, let us note that for the solutions of the boundary value problem (9), (2), (3), (8) there is a priori estimate

$$\|u\|_V \leq K\|f\|_{L_2(Q)} \tag{10}$$

(natural for parabolic equations), where the constant K is determined only by the numbers β, T and ε. It follows from this estimate that the operator Φ takes a closed ball of radius $R^* = K\|f\|_{L_2(Q)}$ of the space V into itself.

Let us show now that the operator Φ is continuous on the space V.

Let $\{v_n(x,t)\}_{n=1}^{\infty}$ be a sequence of functions from the space V converging to a function $v_0(x,t)$. If we put $u_n = \Phi(v_n)$, $u_0 = \Phi(v_0)$, $\overline{v}_n = v_n - v_0$, $\overline{u}_n = u_n - u_0$, then we will have the following equality

$$\frac{R_1 + G_\mu(\varphi_1(v_0))}{A}\overline{u}_{nt} - \overline{u}_{nxxx} - \varepsilon\overline{u}_{nxxxxxxx} + \beta\overline{u}_n = \frac{1}{A}[G_\mu(\varphi_1(v_0)) - G_\mu(\varphi_1(v_n))]. \qquad (11)$$

Since the function $G_\mu(\xi)$ is Lipschitz, and the inequality $|G_\mu(\xi)| \le |\xi|$, then we will have the following estimate

$$|G_\mu(\varphi_1(v_0)) - G_\mu(\varphi_1(v_n))| \le |\varphi_1(\overline{v}_n)|. \qquad (12)$$

Now repeating for equality (11) the proof of estimate (10), taking into account inequality (12) and taking into account that $\varphi_1(\overline{v}_n) \to 0$ as $n \to \infty$ (due to the convergence of the sequence $v_n(x,t)$ in the space V to the function $v_0(x,t)$)), we see that the convergence takes place $\overline{u}_n(x,t) \to 0$ as $n \to \infty$. And this means that the operator Φ is continuous everywhere in the space V.

Let us now show that the operator Φ is compact.

Let $\{v_n(x,t)\}_{n=1}^{\infty}$ be an arbitrary bounded sequence of functions from the space V. Since the embedding $W_2^1(Q) \subset L_2(Q)$ is compact [32–34], it follows from the sequence $\{v_n(x,t)\}_{n=1}^{\infty}$ that a subsequence $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$, strongly converging in the space $L_2(Q)$ to some function $v_0(x,t)$ belonging to the space V. Let us note that the boundedness in the space V of the sequence $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$ and the strong convergence of the sequence $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$ in the space $L_2(Q)$ implies that the sequence $\{v_{n_k xxx}(x,t)\}_{k=1}^{\infty}$ is fundamental in the space $L_2(Q)$. Indeed, we will have the following equality

$$\int\limits_Q (v_{n_k xxx} - v_{n_l xxx})^2\,dx\,dt = -\int\limits_Q (v_{n_k} - v_{n_l})(v_{n_k xxxxxx} - v_{n_l xxxxxx})\,dx\,dt.$$

It follows from this equality that the sequence $\{v_{n_k xxx}(x,t)\}_{k=1}^{\infty}$

If we put $u_{n_k} = \Phi(v_{n_k})$, $v_{kl}(x,t) = v_{n_k}(x,t) - v_{n_l}(x,t)$, $u_{kl}(x,t) = u_{n_k}(x,t) - u_{n_l}(x,t)$, then we will have the following equality

$$\frac{R_1 + G_\mu(\varphi_1(u_{n_k}))}{A}u_{klt} - u_{klxxx} - \varepsilon u_{klxxxxxxx} + \beta u_{kl} = \frac{1}{A}\varphi_1(v_{kl})u_{n_l t}. \qquad (11)$$

Repeating for this equality the proof of estimate (10) and taking into account the fundamental nature of the sequences $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$ and $\{v_{n_k xxx}(x,t)\}_{k=1}^{\infty}$ in the space $L_2(Q)$, we see that the sequence $\{u_{n_k}(x,t)\}_{k=1}^{\infty}$ is fundamental in the space V.

Thus, from any sequence $\{v_n(x,t)\}_{n=1}^{\infty}$ bounded in the space V, one can extract a subsequence $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$ such that the sequence $\{\Phi(v_{n_k})\}_{k=1}^{\infty}$ is fundamental in the space V. And this means that the operator Φ is compact in the space V

Everything proved above means that for the operator Φ on the ball of radius R^* of the space V, all conditions of Schauder's theorem are satisfied. Consequently, the operator Φ has at least one fixed point in the space V. Obviously, this fixed point will represent the solution of the boundary value problem (7), (2), (3), (8).

Let us show that under the conditions of the theorem for the solutions $u(x,t)$ of the boundary value problem (7), (2), (3), (8), there are a priori estimates uniform in the parameter E, which will allow us to establish the existence of solutions to the boundary value problem (6), (2), (3). We multiply equation (7) by the function $-u_{xxxxxx}(x,t)$ and integrate over the rectangle Q. Applying the formula for integration by parts both on the left and on the right in the obtained equality, we obtain the first estimate uniform in ε

$$\left(\int\limits_Q u_{xxx}^2\,dx\,dt\right)^{1/2} \le \frac{1}{\beta}\left(\int\limits_Q f_{xxx}^2\,dx\,dt\right)^{1/2}. \tag{13}$$

At the next step, we multiply equation (7) by the function $(T-t)u(x,t)$ and integrate over the rectangle Q. Taking into account the inequality $R_1 + G_\mu(\varphi_1(u)) \ge (1-\mu_0)R_1$ and applying Helder's inequality , we obtain that for the solutions $u(x,t)$ of the boundary value problem (7), (2), (3), (8), the second estimate uniform in ε is satisfied

$$\left(\int\limits_Q u^2\,dx\,dt\right)^{1/2} \le \frac{2A}{(1-\mu_0)R_1}\left(\int\limits_Q f_1^2\,dx\,dt\right)^{1/2}. \tag{14}$$

We should note that at the first step, one more estimate uniform in ε is derived

$$\varepsilon\int\limits_Q u_{xxxxxx}^2\,dx\,dt \le \frac{1}{\beta}\int\limits_Q f_{xxx}^2\,dx\,dt. \tag{15}$$

Finally, there is also the obvious last estimate in ε,

$$\int\limits_Q u_t^2\,dx\,dt \le C_0\int\limits_Q f^2\,dx\,dt, \tag{16}$$

where the constant C_0 is determined by the numbers β, μ_0, R_1 и T. Estimates (13) - (16), as well as the reflexivity property of the Hilbert space, allow, after choosing a sequence $\{\varepsilon_m\}_{m=1}^\infty$ of positive numbers that monotonically tends to zero, to go over to the family of solutions to boundary value problems (7), (2), (3), (8) with $\{\varepsilon_m\}_{m=1}^\infty$ to a weakly converging subsequence and then, in the limit, obtain a solution $u(x,t)$ of the boundary value problem (6_μ), (2), (3), and the solution for which the estimates (13) , (14) and (16). For this solution, due to the inequality

$$|\varphi_1(u)| \le T^{1/2}\|N\|_{L_2(\Omega)}\left(\int\limits_Q u_{xxx}^2\,dx\,dt\right)^{1/2} + \beta T^{1/2}\|N\|_{L_2(\Omega)}\left(\int\limits_Q u^2\,dx\,dt\right)^{1/2},$$

of estimates (13) and (14), as well as the condition $K_0 \le R_1$, we will obtain the following estimate

$$|\varphi_1(u)| \le \mu_0 R_1.$$

Therefore, for the found solution $u(x,t)$ of the boundary value problem (6_μ), (2), (3), the equation $G_\mu(\varphi_1(u)) = \varphi_1(u)$ will be satisfied. This means that the found function $u(x,t)$ will be the desired solution of the boundary value problem (6), (2), (3). The theorem is proved.

Theorem 2 *Let the following conditions be satisfied*

$$N(x) \in L_2(\Omega); \ \ \beta > 0, \quad A > 0; \ \frac{\partial^k f(x,t)}{\partial x^k} \in L_2(Q), \quad k = \overline{0,3},$$

$$f(0,t) = f_x(0,t) = f_{xx}(1,t) = 0 \quad npu \quad t \in (0,T);$$

$$\exists\, \mu_0 \in (0,1): \ K_0 \le \int\limits_Q N(x) f(x,t)\, dx\, dt.$$

Then inverse problem I has a solution $\{u(x,t), \alpha\}$ *such that* $u(x,t) \in L_2(0,T; W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $\alpha > 0$.

Proof. In the boundary value problem (6), (2), (3), we will take the number $\int\limits_Q N(x) f(x,t)\, dx\, dt$ as the number R_1. According to Theorem 1, this problem has a regular solution $u(x,t)$. Let us define the numberα:

$$\alpha = \frac{R_1 + \varphi_1(u)}{A}. \tag{17}$$

It is obvious that the number will be positive and that the number α and the function $u(x,t)$ are related in the rectangle Q by equation (1). Let us show that the function $u(x,t)$ will satisfy the overdetermination condition (4).

A consequence of equation (1) is the equality

$$\alpha u(x,t) - \int\limits_0^t [u_{xxx}(x,\tau) - \beta u(x,\tau)]\, d\tau = \int\limits_0^t f(x,\tau)\, d\tau.$$

Setting $t = T$ in this equality, then multiplying it by the function $N(x)$ and integrating over Ω, we obtain the ratio

$$\alpha \int\limits_\Omega N(x) f(x,T)\, dx = R_1 + \varphi_1(u).$$

On the other hand, representation (17) gives the equality

$$\alpha A = R_1 + \varphi_1(u).$$

From the two obtained equations and from the positiveness of the number α and we will see that for the solution $u(x,t)$ of the boundary value problem (6), (2), (3) with the above number R_1, the overdetermination condition (4) is satisfied. Consequently, the function $u(x,t)$ and the number α give the desired solution to Inverse Problem I. The theorem is proved.

4 Solvability of Inverse Problem II

We will use a method based on the transition from the studied inverse problem to some direct problem for a nonlinear loaded differential equation again.

Let R_1 be a given positive number, $N_1(x)$ is a function for which the following equalities are satisfied.

$$N_1'''(x) = N(x), \quad N_1(0) = N_1'(0) = N_1''(1) = 0,$$

$\varphi_2(v)$ where

$$\varphi_2(v) = \int\limits_{\Omega} N_1(x)v(x,T)\,dx + \beta \int\limits_{Q} N_1(x)v(x,t)\,dx\,dt.$$

Let us consider the following problem: to find a function $u(x,t)$ which is a solution to the equation

$$u_t - \frac{R_1 - \varphi_2(v)}{A} u_{xxx} + \beta u = f(x,t) \tag{18}$$

in rectangle Q and such that conditions (2) and (3) are satisfied. This problem is an auxiliary direct problem for a loaded differential equation.

Let μ_0 be a number from $(0,1)$. We will then have the following

$$f_2(x,t) = (T-t)f_{tt}(x,t),$$

$$M_1 = \sqrt{\frac{2}{\beta}}\|N_1\|_{L_2(\Omega)}\|f_t\|_{L_2(Q)},$$

$$M_2 = \frac{\beta A T^{1/2}\|N_1\|_{L_2(\Omega)}}{(1-\mu_0)}(\|f_t\|_{L_2(Q)} + 2\|f_2\|_{L_2(Q)}),$$

$$M_0 = \frac{1}{2\mu_0}\left(M_1 + \sqrt{M_1^2 + 4\mu_0 M_2}\right).$$

Theorem 3 *Let the following conditions be satisfied:*

$$N(x) \in L_2(\Omega); \ \ \beta > 0, \quad A > 0;$$

$$\frac{\partial^{k+l} f(x,t)}{\partial x^k \partial t^l} \in L_2(Q), \quad k = \overline{0,3}, \quad l = \overline{0,2}, \quad \frac{\partial^{k+l} f(0,t)}{\partial x^k \partial t^l} = 0, \quad k = \overline{0,1},$$

$$l = \overline{0,2}, \quad t \in (0,T), \quad \frac{\partial^{2+l} f(1,t)}{\partial x^2 \partial t^l} = 0, \quad l = \overline{0,2}, \quad t \in (0,T),$$

$$\frac{\partial^l f(x,0)}{\partial t^l} = 0, \quad l = \overline{0,1}, \quad x \in \Omega;$$

$$\exists\, \mu_0 \in (0,1): \ M_0 \le R_1.$$

Then the boundary value problem (18), (2), (3) has a solution $u(x,t)$ such that $u(x,t) \in L_2(0,T; W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $|\varphi_1(u)| \le \mu_0 R_1$.

Proof. Let μ be the number $\mu_0 R_1$, $\widetilde{\varphi}_2(v)$ be the functional.

$$\widetilde{\varphi}_2(v) = \int\limits_{Q} N_1(x)v(x,t)\,dx\,dt + \beta \int\limits_{Q} \left(\int\limits_0^t N_1(x)w(x,\tau)\,d\tau \right) dx\,dt.$$

Let us consider the problem: to find a function $w(x,t)$ that is a solution of the equation

$$w_t - \frac{R_1 - G_\mu(\widetilde{\varphi}_2(w))}{A} w_{xxx} + \beta w = f_{tt}(x,t) \tag{19}$$

in the rectangle Q and such that conditions (2) and (3) are satisfied for it. Repeating the proof of the solvability of problem (6_μ), (2), (3), it is easy to show that under the conditions of the theorem this problem has a solution $w(x,t)$ such that $w(x,t) \in L_2(0,T;W_2^3(\Omega))$, $w_t(x,t) \in L_2(Q)$. We define the function $v(x,t)$ as:

$$v(x,t) = \int_0^t w(x,\tau)\,d\tau.$$

Since $f_t(x,0) = 0$, then for the function $v(x,t)$ in the rectangle Q the equation

$$v_t - \frac{R_1 - G_\mu(\varphi_2(v))}{A} v_{xxx} + \beta v = f_t(x,t), \tag{20}$$

will be satisfied and conditions (2) and (3) will also be satisfied. Let us show that the required a priori estimates hold for the functions $w(x,t)$ and $v(x,t)$. Let us multiply equation (19) by the function $(T-t)w(x,t)$ and integrate over the rectangle Q. After simple transformations, we obtain the estimate

$$\left(\int_Q w^2\,dx\,dt\right)^{1/2} \le 2\left(\int_Q f_2^2\,dx\,dt\right)^{1/2}. \tag{21}$$

At the next step, we multiply equation (20) by the function $v(x,t)$ and integrate over the rectangle Q. The consequence of the obtained equality will be the second estimate

$$\int_\Omega v^2(x,T)\,dx \le \frac{2}{\beta}\int_Q f_t^2\,dx\,dt. \tag{22}$$

Next, we multiply equation (20) by the function $-v_{xxx}(x,t)$ and integrate over the rectangle Q. Using the inequality $R_1 - G_\mu(\varphi_1(v)) \ge (1-\mu_0)R_1$, applying Helder's inequality and taking into account the estimate (21), we obtain inequality

$$\left(\int_Q v_{xxx}^2\,dx\,dt\right)^{1/2} \le \frac{A}{(1-\mu_0)R_1}\left(\|f_t\|_{L_2(Q)} + 2\|f_2\|_{L_2(Q)}\right). \tag{23}$$

Inequalities (22) and (23) make it possible to estimate $|\varphi_2(v)|$:

$$|\varphi_2(v)| \le \|N_1\|_{L_2(\Omega)}\left(\int_\Omega v^2\,dx\right)^{1/2} + \beta T^{1/2}\|N_1\|_{L_2(\Omega)}\left(\int_Q v^2\,dx\,dt\right)^{1/2} \le$$

$$\leq \sqrt{\frac{2}{\beta}}\|N_1\|_{L_2(\Omega)}\|f_t\|_{L_2(Q)} + \beta T^{1/2}\|N_1\|_{L_2(\Omega)}\left(\int\limits_Q v_{xxx}^2\,dx\,dt\right)^{1/2} \leq$$

$$\leq M_1 + \frac{M_2}{R_1}. \tag{24}$$

The resulting estimate and the inequality $M_0 \leq R_1$ from the conditions of the theorem mean that the inequality $|\varphi_2(v)| \leq \mu_0 R_1$ is satisfied, and then the equality $G_\mu(\varphi_2(v)) = \varphi_2(v)$ is satisfied. The last equality means that the solution $v(x,t)$ to equation (20) is a solution to the equation

$$v_t - \frac{R_1 - \varphi_2(v)}{A} v_{xxx} + \beta v = f_t(x,t).$$

Let us define the function $u(x,t)$ in the following form:

$$u(x,t) = \int\limits_0^t v(x,\tau)\,d\tau.$$

Since $f(x,0) = 0$, the function $u(x,t)$ will be a solution to equation (18). The function $u(x,t)$ belongs to the required class, conditions (2) and (3) are fulfilled for it, as well as the inequality $|\varphi_2(u_t)| \leq \mu_0 R_1$ is satisfied. The theorem is proved. We will set the following

$$R_1 = \int\limits_\Omega N_1(x) f(x,T)\,dx.$$

Theorem 4 *Let R_1 R1 be positive, and all conditions of Theorem 3 are satisfied for it and for a given function $f(x,t)$ and for β number. Then inverse problem II has a solution $\{u(x,t),\alpha\}$ such that $u(x,t) \in L_2(0,T;W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $\alpha > 0$.*

Proof. For the indicated number R_1, let us consider the boundary value problem (18), (2), (3). According to Theorem 3, this problem has a solution $u(x,t)$ such that $u(x,t) \in L_2(0,T;W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $|\varphi_2(u_t)| \leq \mu_0 R_1$. We will define the number α:

$$\alpha = \frac{R_1 - \varphi_2(u_t)}{A}. \tag{25}$$

It is obvious that this number and the function $u(x,t)$ will be related in the rectangle Q by equation (5), and that the number α will be positive. Further, the function $u(x,t)$ will satisfy the overdetermination condition (4) - this is proved quite similarly as the proof of the fulfillment of the overdetermination condition in Theorem 2. Therefore, the solution $u(x,t)$ to the boundary value problem (18), (2), (3) and the number defined by formula (25) give the desired solution to Inverse Problem II.

The theorem is proved.

5 Remarks and additions

1. As mentioned above, the paper considers model equations with multiple characteristics. It is easy to carry out all the reasoning and obtain theorems on the solvability of inverse problems of I and II types for more general equations. For example, in equations (1) and (5) the coefficient β can be a function of the variables x and t, there can be lower terms (derivatives with respect to the variable x of the first and second orders), the coefficient of the third derivative in equation (1) and the coefficient of the derivative with respect to the time variable in equation (5) can be functions of independent variables. The number of calculations and conditions in similar more general problems increases significantly, but the essence of the results on solvability remains the same.

2. Along with inverse problems I and II, it is not difficult, practically repeating all reasoning and calculations, to study the solvability of inverse problems with the setting for $x = 1$ the value of the solution (and not the second derivative of the solution). All statements of Theorems 1–4 on the existence of solutions remain valid, only the condition $f_{xx}(1,t) = 0$ will need to be replaced by the condition $f(1,t) = 0$ (we only specify that in the regularized problem for equation (7) additional boundary conditions will have the form $u_{xxx}(0,t) = u_{xxx}(1,t) = u_{xxxx}(1,t) = 0$ при $t \in (0,T)$).

3. It is easy to establish that the conditions $K_0 \leq R_1$ of Theorems 1 and 2, μ_0 of Theorems 3 and 4 are satisfied for a fixed value of ?0 and for a given function $f(x,t)$ if the number β is large, but the number A is small.

4. Theorems 1 and 3 on the solvability of initial-boundary value problems for "loaded" differential equations with multiple characteristics (as well as similar theorems with the replacement of the condition $u_{xx}(1,t) = 0$ by the condition $u(1,t) = 0$) have, in our opinion, an independent value.

6 Conclusion

In this paper we investigate the inverse problem for differential equations with multiple characteristics. The paper shows the results of the study of solvability in Sobolev spaces. Inverse problems arise when the characteristics of the object of interest are not available for direct observation. A distinctive feature of the problems studied in this paper is that the unknown coefficient is a numerical parameter rather than a function of some independent variables. Similar problems have been studied before, but only for classical parabolic, hyperbolic, and elliptic equations. It should be noted that inverse problems of determination for differential equations with multiple characteristics, together with the solution of numerical parameters that are coefficients of the equation itself, have not been studied before.

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