1-бөлім | Раздел 1 | Section 1

Математика | Математика | Mathematics

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE WELL-POSEDNESS OF A BOUNDARY VALUE PROBLEM FOR A LINEAR LOADED HYPERBOLIC EQUATION

Problems for loaded hyperbolic equations have acquired particular relevance in connection with the study of the stability of vibrations of the wings of an aircraft loaded with masses, and in the calculation of the natural vibrations of antennas loaded with lumped capacities and self-inductions. Loaded differential equations have a number of features that must be taken into account when setting problems for these equations and creating methods for their solution. One of the features of loaded differential equations is that such equations can be undecidable without additional conditions. The main idea of the research work is to expand the class of solvable boundary value problems and develop methods that provide a numerical-analytical solution. The paper considers a boundary value problem for a linear hyperbolic equation with a mixed derivative, where the load points are set in terms of the spatial variable. By introducing unknown functions, the problem is reduced to an equivalent boundary value problem for a linear loaded hyperbolic equation of the first order. With the help of the well posed of the equivalent boundary value problem, the well posed of the original problem is established. The paper presents the necessary and sufficient conditions for the well-posedness of a periodic boundary value problem for a linear loaded hyperbolic equation with two independent variables.

Key words: well-posedness solvability, necessary and sufficient conditions, loaded hyperbolic equation, linear hyperbolic equation, semi-periodic boundary value problem.

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Сызықтық жүктелген гиперболалық теңдеу үшін периодты шеттік есептің корректілі шешілімділігінің қажетті және жеткілікті шарттары

Жүктелген гиперболалық теңдеулер үшін есептер массалармен жүктелген ұшақ қанаттарының тербелістерінің тұрақтылығын зерттеуге және шоғырланған сыйымдылықтар мен өзін-өзі индукциялармен жүктелген антенналардың өзіндік тербелістерін есептеуге байланысты ерекше өзекті болып табылды. Жүктелген дифференциалдық теңдеулердің бірқатар ерекшеліктері бар, оларды осы теңдеулер үшін есептер шығару және оларды шешу әдістерін құру кезінде ескеру қажет. Жүктелген дифференциалдық теңдеулердің бір ерекшелігі мұндай теңдеулер қосымша шарттарсыз шешілмеуі мүмкін.

Бұл зерттеудің негізгі мақсаты шешілімді шекаралық есептер класын кеңейту және аналитикалық шешім беретін әдістерді құрастыру болып табылады. Жұмыста жүктеме нүктелері кеңістіктік айнымалыға қойылған аралас туындылы сызықтық гиперболалық теңдеу үшін шекттік есеп қарастырылады. Белгісіз функцияларды енгізу арқылы есеп бірінші ретті сызықтық жүктелген гиперболалық теңдеу үшін эквивалентті шеттік есепке келтіріледі. Эквивалентті шеттік есептің дұрыс шешілімділігінің көмегімен бастапқы есептің дұрыс шешілімділігі алынады. Жұмыста аралас туындысы бар сызықтық жүктелген гиперболалық теңдеу үшін жартылай периодты шекаралық есептің дұрыс шешілімділігінің қажетті және жеткілікті шарттары алынған.

Key words: корректілі шешілімділік, қажетті және жеткілікті шарттар, жүктелген гиперболалық теңдеу, сызықтық гиперболалық теңдеу, жартылай периодты шеттік есеп.

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Необходимые и достаточные условия корректной разрешимости краевой задачи для линейного нагруженного гиперболического уравнения

Задачи для нагруженных гиперболических уравнений приобрели особую актуальность в связи с изучением устойчивости вибраций крыльев самолета, нагруженного массами, и при расчете собственных колебаний антенн, нагруженных сосредоточенными емкостями и самоиндукциями. Нагруженные дифференциальные уравнения имеют ряд особенностей, которые должны быть учтены при постановке задач для этих уравнений и создании методов их решений. Одним из особенностей нагруженных дифференциальных уравнений является то, что такие уравнения могут быть неразрешимыми без дополнительных условий. Основной целью данного исследования заключается в том, чтобы расширить класс разрешимых краевых задач и разработать методы которые дают аналитический вид решения задачи. В работе рассматривается краевая задача для линейного гиперболического уравнения со смешанной производной, где точки нагрузки ставятся по пространственной переменной. Задача путем введения неизвестных функции сводиться к эквивалентной краевой задаче для линейного нагруженного гиперболического уравнения первого порядка. С помощью корректной разрешимости эквивалентной краевой задачи устанавливается корректная разрешимость исходной задачи. В работе получены необходимые и достаточные условия корректной разрешимости полупериодической краевой задачи для линейного нагруженного гиперболического уравнения со смешанной производной.
Ключевые слова: корректная разрешимость, необходимые и достаточные условия, нагруженное гиперболическое уравнение, линейное гиперболическое уравнение, полупериодическая краевая задача.

1 Introduction

1.1 Problem statement

In the domain $\bar{\Omega} = [0, T] \times [0, \omega]$, we consider the semi-periodic boundary value problem for the linear loaded hyperbolic equation

$$\frac{\partial^2 u}{\partial t \partial x} = A(x,t)\frac{\partial u}{\partial x} + B(x,t)\frac{\partial u}{\partial t} + C(x,t)u + f(x,t) + A_0(x,t)\frac{\partial u(x_0,t)}{\partial x}, \qquad (1)$$

$$u(0,t) = \psi(t), \quad t \in [0,T], \tag{2}$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega], \tag{3}$$

where $A(x,t)$, $B(x,t)$, $C(x,t)$, and $f(x,t)$ are continuous on $\bar{\Omega}$, $\psi(t)$ is continuously differentiable on $[0,T]$ and satisfies the condition $\psi(0) = \psi(T)$, and x_0 is the load point. Let $C(\overline{\Omega})$ be the space of continuous functions $u : \overline{\Omega} \to R$ on $\overline{\Omega}$ with the norm $\|u\|_C = \max\limits_{\overline{\Omega}} |u(x,t)|$. By $C^{1,1}_{x,t}(\overline{\Omega})$ we denote the space of continuous and continuously differentiable functions $u(x,t)$ on $\bar{\Omega}$ with the norm $\|u\|_0 = \max(\|u\|_C, \|u_x\|_C, \|u_t\|_C)$. $C^1\big([0,T]\big)$ denotes the space of continuous and differentiable function $\psi(t)$ on $[0,T]$ with the norm $\|\psi\|_1 = \max\big(\max\limits_{t\in[0,T]} |\psi(t)|, \max\limits_{t\in[0,T]} |\dot{\psi}(t)|\big)$.

A function $u(x,t) \in C(\overline{\Omega})$, that has partial derivatives

$$\frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t},$$

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} \in C(\overline{\Omega}),$$

is called a solution of problem (1)-(3), if it satisfies equation (1) for all $(x,t) \in \overline{\Omega}$, take the value $\psi(t)$, $t \in [0,T]$ on the characteristic $x = 0$ and has equal values on the characteristics $t = 0$, $t = T$ for all $x \in [0,\omega]$.

Loaded hyperbolic partial differential equations with non-local boundary conditions arise in many fields of science and technology. The general definition of the loaded equation was given by Nakhushev. Loaded differential equations have a number of features that should be taken into account when setting problems for these equations and creating methods for their solution. One of the features of loaded differential equations is that such equations can be unsolvable unless additional conditions are imposed. There are some examples of linear loaded ordinary differential equations and loaded hyperbolic equations with mixed derivatives that have no solution.Boundary value problems for loaded differential equations have been studied by many authors [1-9].

Definition 1. *The boundary value problem (1)-(3) is called well-posed if for any $f(x,t) \in C(\overline{\Omega})$, continuous and continuously-differentiable on $[0,T]$ functions $\psi(t)$, it has a unique solution $u(x,t)$ and the inequality*

$$\|u\|_0 \le K \max\big\{\|\psi\|_1, \|f\|_C\big\},$$

is valid, where K is a constant, independent of $f(x,t)$ and $\psi(t)$.

1.2 Equivalent problem

Let us introduce new unknown functions $v(x,t) = \dfrac{\partial u(x,t)}{\partial x}$ and $w(x,t) = \dfrac{\partial u(x,t)}{\partial t}$.
The problem (1)-(3) is reduced it to the equivalent problem:

$$\frac{\partial v}{\partial t} = A(x,t)v + B(x,t)w + C(x,t)u + f(x,t) + A_0(x,t)v(x_0,t), \tag{4}$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega], \tag{5}$$

$$u(x,t) = \psi(t) + \int_0^x v(\xi,t)d\xi,\ w(x,t) = \dot{\psi}(t) + \int_0^x v_t(\xi,t)d\xi \tag{6}$$

The triple of continuous functions $\{u(x,t), w(x,t), v(x,t)\}$ on $\bar{\Omega}$ is called solution of problem (4)-(6), if $v(x,t) \in C(\bar{\Omega})$ has the continuous derivative with respect to t on $\bar{\Omega}$ and satisfies the family of periodic boundary value problems (4), (5), where the functions $u(x,t)$ and $w(x,t)$ are related to $v(x,t)$ and $\dfrac{\partial v(x,t)}{\partial t}$ by the functional relations (6).

The problems (1)-(3) and (4)-(6) are equivalent in the sense that if $u(x,t)$ is solution of problem (1)-(3), then the triple of functions $\{u(x,t), w(x,t), v(x,t)\}$ will be solution of problem (4)-(6). Vice versa, if a triple of functions $\{u^*(x,t), w^*(x,t), v^*(x,t)\}$ is a solution of problem (4)-(6), then $u^*(x,t)$ will be a solution of problem (1)-(3).

We consider the family of periodic boundary value problems for loaded ordinary differential equations

$$\frac{dv}{dt} = A(x,t)v + A_0(x,t)v(x_0,t) + F(x,t),\ (x,t) \in \overline{\Omega} \tag{7}$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega], \tag{8}$$

where the functions $A(x,t), A_0(x,t)$ and $F(x,t)$ are continuous on $\overline{\Omega}$.

A function $v(x,t) \in C(\overline{\Omega})$ having the continuous derivative with respect to t is called a solution to the boundary value problem (7), (8) if it satisfies the system of equations (7) for all $(x,t) \in \overline{\Omega}$ and the boundary condition (8) for $x \in [0,\omega]$.

Definition 2. *The boundary value problem (7),(8) is called well posed if for any function $F(x,t)$ it has a unique solution $v(x,t)$ and the estimate*

$$\max_{t\in[0,T]} |v(x,t)| \le K \max_{t\in[0,T]} |F(x,t)|,$$

is valid, where K is a const, independent of $F(x,t)$.

Similarity Theorem 3 [10, C.23], we can show that the semi-periodic boundary value problem (1)-(3) is well-posed if and only if the periodic boundary value problem (7), (8) is wellposed.

2 Materials and methods

2.1 The well-posedness of the equivalent problem

The following statement establishes the necessary and sufficient conditions for the well-posedness of problem (7), (8).

Theorem 1. *The problem (7), (8) is well posed if and only if, for some $\delta_1 > 0, \delta_2 > 0$ the following inequalities holds:*

1) $\left| \int_0^T A(x,\tau)d\tau \right| \ge \delta_1$ *for all* $x \in [0,\omega]$,

2) $\left| \int_0^T \left[A(x_0,\tau) + A_0(x_0,\tau)\right] d\tau \right| \ge \delta_2, x_0 \in [0,\omega].$

Proof. Sufficiency. We consider the periodic boundary value problems (7), (8). For a fixed $x \in [0,\omega]$, we solve the differential equation (7). Its general solution is written as

$$v(x,t) = \exp\Big(\int_0^t A(x,\tau)d\tau\Big)\times$$

$$\times\Big[C(x)+\int_0^t F(x,\tau)\exp\Big(-\int_0^\tau A(x,\tau_1)d\tau_1\Big)d\tau+$$

$$+\int_0^t A_0(x,\tau)v(x_0,\tau)\exp\Big(-\int_0^\tau A(x,\tau_1)d\tau_1\Big)d\tau\Big],\ t\in[0,T] \tag{9}$$

where $C(x)$ is a function continuous on $[0,\omega]$. Given the condition 2) of the theorem and the boundary condition (8), we find $C(x)$. Substituting it into (9), we find the solution of problem (7), (8) in the following form

$$v(x,t)=\frac{\exp\Big(\int_0^t A(x,\tau)d\tau\Big)}{1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)}\int_0^T\Big[F(x,\tau)+A_0(x,\tau)v(x_0,\tau)\Big]\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau+$$

$$+\int_0^t\Big[F(x,\tau)+A_0(x,\tau)v(x_0,\tau)\Big]\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau,\ t\in[0,T] \tag{10}$$

Setting $x=x_0$, in problem (7), (8), we obtain the problem

$$\frac{dv(x_0,t)}{dt}=\big[A(x_0,t)+A_0(x_0,t)\big]v(x_0,t)+F(x_0,t),\ \ (x_0,t)\in\overline{\Omega}, \tag{11}$$

$$v(x_0,0)=v(x_0,T),\quad x_0\in[0,\omega]. \tag{12}$$

We then find the solution of problem (11), (12):

$$v(x_0,t)=\exp\Big(\int_0^t\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)\Big[C(x_0)+$$

$$+\int_0^t F(x_0,\tau)\exp\Big(-\int_0^\tau\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau,t\in[0,T]. \tag{13}$$

Substituting (13) into condition (12), we get

$$C(x_0)=C(x_0)\exp\Big(\int_0^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)+$$

$$+\int_0^T F(x_0,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau$$

When the conditions (2) $\Big|\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big|\geq\delta_2$ of the theorem 1 is fulfilled, we get that $\exp\Big(\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)\neq 1$. Then the function $C(x_0)$ is defined as follows:

$$C(x_0)=\frac{1}{1-\exp\Big(\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)}\times$$

$$\times\int_0^T F(x_0,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau.$$

Thus, the solution of the boundary value problem (11), (12) has the representation

$$v(x_0,t)=\frac{\exp\Big(\int_0^t\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)}{1-\exp\Big(\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)}\times$$

$$\times\int_0^T F(x_0,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big])d\tau_1\Big)d\tau+$$

$$+\int_0^t F(x_0,\tau)\exp\Big(\int_\tau^t\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau. \tag{14}$$

Substituting (14) into the right-hand of (10), we get

$$v(x,t)=\frac{\exp\Big(\int_0^t A(x,\tau)d\tau\Big)}{1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)}\int_0^T F(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau+$$

$$+\int_0^t F(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau+$$

$$+\frac{\exp\Big(\int_0^t A(x,\tau)d\tau\Big)}{1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)}\int_0^T A_0(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)\times$$

$$\times\Bigg\{\frac{\exp\Big(\int_0^\tau\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}{1-\exp\Big(\int_0^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}\times$$

$$\times\int_0^T F(x,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big])d\tau_1\Big)d\tau+$$

$$\times\int_0^\tau F(x_0,\tau_1)\exp\Big(\int_{\tau_1}^\tau\big[A(x_0,\tau_2)+A_0(x_0,\tau_2)\big]d\tau_2\Big)d\tau_1 d\tau\Bigg\}+$$

$$+\int_0^t A_0(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)\Bigg\{\frac{\exp\Big(\int_0^\tau\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}{1-\exp\Big(\int_0^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}\times$$

$$\times\int_0^T F(x,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big])d\tau_1\Big)d\tau+$$

$$\times\int_0^\tau F(x_0,\tau_1)\exp\Big(\int_{\tau_1}^\tau\big[A(x_0,\tau_2)+A_0(x_0,\tau_2)\big]d\tau_2\Big)d\tau_1 d\tau\Bigg\},\ t\in[0,T] \tag{15}$$

Uniqueness. Assume the opposite, let $v^*(x,t)$ and $\overline{v}(x,t)$ be two solutions of the periodic boundary value problem (7), (8). Then their difference $\Delta v(x,t)=v^*(x,t)-\overline{v}(x,t)$ satisfies the periodic boundary value problem for ordinary differential equation

$$\frac{d\Delta v}{dt}=A(x,t)\Delta v+A_0(x_0,t)\Delta v(x_0,t), \tag{16}$$

$$\Delta v(x,0) = \Delta v(x,T). \tag{17}$$

The general solution of equation (16) is of the form

$$\Delta v(x,t) = C(x)\cdot \exp\Big(\int_0^t A(x,\tau)d\tau\Big) + \int_0^T A(x_0,t)\exp\Big(\int_t^\tau A(x,\tau_1)d\tau_1\Big)\times$$

$$\times \exp\Big(\int_0^t [A(x_0,\tau)+A_0(x_0,\tau)]d\tau\Big)dt.$$

From the boundary condition (17), we get

$$C(x)\Bigg[1-\exp\Big(\int_0^T A(x,t)dt\Big) + C(x_0)\Big(\int_0^T A(x_0,t)\exp\Big(\int_t^\tau A(x,\tau_1)d\tau_1\Big)\Big)\times$$

$$\times \exp\Big(\int_0^\tau [A(x_0,\tau)+A_0(x_0,\tau)]d\tau\Big)\Bigg] = 0.$$

For all $x\in[0,\omega]$, we have

$$\Big|\int_0^T A(x,t)dt\Big| \ge \delta_1 > 0 \;\text{ and }\; \Big|\int_0^T \big[A(x_0,t)+A_0(x_0,t)\big]dt\Big| \ge \delta_2 > 0.$$

Then

$$\Big|1-\exp\Big(\int_0^T A(x,t)dt\Big)\Big| \ne 0 \quad\text{and}\quad \Big|\int_0^T \big[A(x_0,t)+A_0(x_0,t)\big]dt\Big| \ne 0.$$

This implies that the function $C(x)$ is equal to zero for all $x\in[0,\omega]$. Then $\Delta v(x,t)\equiv 0$, i.e. problem (16),(17) has only the trivial solution. Therefore, $v^*(x,t)=\overline{v}(x,t)$ for all $x\in[0,\omega]$. Let us show that the inequality

$$\frac{1}{\Big|1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)\Big|} \le \frac{e^{\delta_1}}{e^{\delta_1}-1} \tag{18}$$

is correct.
Let us consider two cases: (a) $\int_0^T A(x,\tau)d\tau \le -\delta_1$, and (b) $\int_0^T A(x,\tau)d\tau \ge \delta_1 > 0$.

In the case (a), $\exp\Big(\int_0^T A(x,\tau)d\tau\Big) \le e^{-\delta_1}$, $1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big) \ge 1-e^{-\delta_1}$,

and $\Big[1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)\Big]^{-1} \le \dfrac{1}{1-e^{-\delta_1}} = \dfrac{e^{\delta_1}}{e^{\delta_1}-1}.$

In the case (b) $\exp\Big(\int_0^T A(x,\tau)d\tau\Big) \ge e^{\delta_1}$, $\exp\Big(\int_0^T A(x,\tau)d\tau\Big)-1 \ge e^{\delta_1}-1$,

and $\Big[\exp\Big(\int_0^T A(x,\tau)d\tau\Big)-1\Big]^{-1} \le \dfrac{1}{e^{\delta_1}-1}.$

From these inequalities, we obtain (18).

Let us show the inequality is correct

$$\frac{1}{\left|1-\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\right|}\le\frac{e^{\delta_2}}{e^{\delta_2}-1}. \tag{19}$$

In view of (18) and (19), we get the following estimate for $v(x,t)$:

$$\|v(x,\cdot)\|_1\le\frac{e^{\alpha T}\cdot e^{\delta_1}}{e^{\delta_1}-1}\cdot\frac{e^{\alpha T}-1}{\alpha}\|F(x,\cdot)\|_1+\frac{e^{\alpha T}-1}{\alpha}\|F(x,\cdot)\|_1+$$

$$+\frac{e^{\delta_1}}{e^{\delta_1}-1}\cdot\frac{\alpha_0\cdot(e^{\alpha T}-1)}{\alpha}33\left[\frac{e^{\delta_2}}{e^{\delta_2}-1}\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\|F(x,\cdot)\|_1+\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\|F(x,\cdot)\|_1\right]+$$

$$+\frac{\alpha_0(e^{\alpha T}-1)}{\alpha}\cdot\left[\frac{e^{\delta_2}}{e^{\delta_2}-1}\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\|F(x,\cdot)\|_1+\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\|F(x,\cdot)\|_1\right]\le$$

$$\le\left\{1+\frac{e^{\alpha T}\cdot e^{\delta_1}}{e^{\delta_1}-1}+\alpha_0\cdot\Big(1+\frac{e^{\delta_1}}{e^{\delta_1}-1}\Big)\cdot\Big(1+\frac{e^{\delta_2}}{e^{\delta_2}-1}\Big)\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\right\}\times$$

$$\times\frac{e^{\alpha T}-1}{\alpha}\cdot\|F(x,\cdot)\|_1=K_1(\alpha,\alpha_0,\delta_1,\delta_2,T)\|F(x,\cdot)\|_1, \tag{20}$$

where $\alpha=\max\limits_{(x,t)\in\overline{\Omega}}|A(x,t)|$, $\alpha_0=\max\limits_{(x,t)\in\overline{\Omega}}|A_0(x,t)|$, and

$$K_1(\alpha,\alpha_0,\delta_1,\delta_2,T)=$$

$$=\left\{1+\frac{e^{\alpha T}\cdot e^{\delta_1}}{e^{\delta_1}-1}+\alpha_0\cdot\Big(1+\frac{e^{\delta_1}}{e^{\delta_1}-1}\Big)\cdot\Big(1+\frac{e^{\delta_2}}{e^{\delta_2}-1}\Big)\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\right\}\cdot\frac{e^{\alpha T}-1}{\alpha}.$$

Thus, by definition, problem (7), (8) is well posed.

Necessity. Let problem (7), (8) be well posed and let K be a constant satisfying inequality (20).

Since the family of periodic boundary value problems (7), (8) is well posed, we take $F(x,t)=1$ and consider the boundary value problem for the ordinary differential equation

$$\frac{dv}{dt}=A(x,t)v+A_0(x,t)v(x_0,t)+1, \tag{21}$$

$$v(x,0)=v(x,T) \tag{22}$$

Let us assume that there is $\widetilde{x}\in[0,\omega]$ for which $\int_0^T A(\widetilde{x},t)dt=0$ and $\int_0^T[A(\widetilde{x}_0,t)+A_0(\widetilde{x}_0,t)]dt=0$. The well-posedness of problem (7), (8) implies the existence of a unique $v_1(x,t)$ problem (21), (22). The function $v_1(x,t)$ satisfies the differential equation (21) for all $x,x_0\in[0,\omega]$, then for $x=\widetilde{x},x_0=\widetilde{x}_0$ we have

$$v_1(\widetilde{x},t)=\exp\Big(\int_0^t A(\widetilde{x},\tau)d\tau\Big)\Big[v_1(\widetilde{x},0)+\int_0^t\exp\Big(-\int_0^\tau A(\widetilde{x},\tau_1)d\tau_1\Big)d\tau\Big]+$$

$$+\exp\Big(\int_0^t A(\widetilde{x},\tau)d\tau\Big)\int_0^t A(\widetilde{x}_0,\tau)\exp\Big(\int_0^\tau A(\widetilde{x},\tau_1)d\tau_1\Big)$$

$$\times\exp\int_0^t\big[A(\widetilde{x}_0,\tau_1)+A_0(\widetilde{x}_0,\tau_1)\big]d\tau_1\times$$

$$\times\Big[\widetilde{v}_1(\widetilde{x}_0,0)+\int_0^t\exp\Big(-\int_0^\tau\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau\Big]d\tau. \quad (23)$$

From (23), setting $t=T$, we get

$$v(\widetilde{x},T)=v(\widetilde{x},0)+\int_0^T\exp\Big(-\int_0^\tau A(\widetilde{x},\tau_1)d\tau_1\Big)d\tau+\int_0^T A(\widetilde{x}_0,\tau)\exp\Big(\int_0^\tau A(\widetilde{x},\tau_1)d\tau_1\Big)\times$$

$$\times\Big[\widetilde{v}_1(\widetilde{x}_0,0)+\int_0^T\exp\Big(-\int_0^\tau\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau\Big]d\tau.$$

It follows from our assumption, that $v(\widetilde{x},0)\neq v(\widetilde{x},T)$. The boundary condition (8) does not hold, hence we get that problem (7), (8) has no solution. Thus, we have a contradiction. Assuming that there is $\widetilde{x}\in[0,\omega]$ such that $\int_0^x A(\widetilde{x},t)dt=0$ and $\int_0^T[A(\widetilde{x}_0,t)+A_0(\widetilde{x}_0,t)]dt=0$ are not valid. It follows that if problem (7), (8) is well posed, then we have $\big|\int_0^T A(x,\tau)dt\big|\neq 0$ and $\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\neq 0$ for all $x,x_0\in[0,\omega]$. Since the functions $A(x,t)$ and $A_0(x,t)$ are continuous on $\overline{\Omega}$, then $\widetilde{A}(x)=\int_0^T A(x,t)dt$ and $\widetilde{A}(x_0)=\int_0^T[A(x_0,t)+A_0(x_0,t)]dt$ are also continuous functions on $[0,\omega]$. Hence $\big|\int_0^T A(x,t)dt\big|\neq 0$ and $\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\neq 0$ for all $x\in[0,\omega]$. From the well-known theorem [11, p. 175 - 176], it follows that there are $\delta_1>0,\delta_2>0$ such that the inequalities $\big|\int_0^T A(x,t)dt\big|\geq\delta_1$ and $\big|\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\big|\geq\delta_2$ hold for all $x,x_0\in[0,\omega]$. Theorem is proved.

2.2 Well-posedness of the main problem

Theorem 2. *Problem (1)-(3) is well posed if and only if for some $\delta>0$ following inequality hold:*
1. $\big|\int_0^T A(x,\tau)d\tau\big|\geq\delta_1$ *for all* $x\in[0,\omega]$.
2. $\Big|\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big|\geq\delta_2, x_0\in[0,\omega]$.

Proof. Necessity. Let problem (1)-(3) be well posed. By Theorem 3 from [10, p.23], we obtain that problem (7), (8) is well posed. Then Theorem 1 implies the existence of $\delta_1>0,\delta_2>0$ such that the inequalities $\Big|\int_0^T A(x,\tau)d\tau\Big|\geq\delta_1$ and $\Big|\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\Big|\geq\delta_2$ hold for all $x\in[0,\omega]$.

Sufficiency. Let there exist $\delta_1>0$ and $\delta_2>0$ such that $\Big|\int_0^T A(x,\tau)d\tau\Big|\geq\delta_1$ and $\Big|\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\Big|\geq\delta_2$ for all $x\in[0,\omega]$. Then, by Theorem 1, we obtain the well-posedness of problem (7), (8). The equivalence of boundary value problems (7), (8) and (1)-(3), and Theorem 3 from [10, p.23] imply the well-posedness of the boundary value problem (1)-(3). Theorem 2 is proved.

3 Conclusions

In this research, we considered a boundary value problem for a linear loaded hyperbolic equation with a mixed derivative, where the load points are set in terms of the spatial variable. An explicit form of the solution of an equivalent boundary value problem for a linear loaded equation is constructed. With its help, necessary and sufficient conditions of well-posedness solvability are obtained for a linear loaded hyperbolic equation.

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