4-бөлім **Раздел 4** **Section 4**

Қолданбалы математика **Прикладная математика** **Applied Mathematics**

IRSTI 27.41.19 DOI: https://doi.org/10.26577/JMMCS.2022.v114.i2.010

D.R. Baigereyev[1*], N.B. Alimbekova[1], N.M. Oskorbin[2]
[1]S. Amanzholov East Kazakhstan University, Kazakhstan, Ust-Kamenogorsk
[2]Altai State University, Russia, Barnaul
*e-mail: dbaigereyev@gmail.com

ERROR ESTIMATES OF THE NUMERICAL METHOD FOR THE FILTRATION PROBLEM WITH CAPUTO-FABRIZIO FRACTIONAL DERIVATIVES

This paper investigates a model of fluid flow in a fractured porous medium under the assumption of a uniform distribution of fractures throughout the volume. This model is based on the use of a fractional differential analogue of Darcy's law, as well as on the assumption that the properties of rock and fluid depend on pressure and its fractional derivative. Unlike previous studies, this study uses a fractional derivative in the Caputo-Fabrizio sense with a non-singular kernel. In this paper, we propose a numerical method for solving this initial boundary value problem and theoretically investigate the order of its convergence. The formulation of a fully discrete scheme is based on application of the finite difference approximation for integer and fractional time derivatives, and the Galerkin method in the spatial variable. A second-order formula is used to approximate both integer derivative and the fractional derivative in the sense of Caputo-Fabrizio. A priori estimates are obtained for both semi-discrete and fully discrete schemes, which imply their second-order convergence in time and space variables. A number of computational experiments were carried out on the example of a model problem to validate the accuracy of the scheme. The results of the numerical tests fully confirm the outcome of the theoretical analysis.
Key words: Finite element method, fractional derivative of Caputo-Fabrizio, convergence, filtration problem, fractured porous medium.

Д.Р. Байгереев[1*], Н.Б. Алимбекова[1], Н.М. Оскорбин[2]
[1]С. Аманжолов атындағы Шығыс Қазақстан университеті, Қазақстан, Өскемен қ.,
[2]Алтай мемлекеттік университеті, Ресей, Барнаул қ.
*e-mail: dbaigereyev@gmail.com

Капуто-Фабрицио бөлшек туындылы фильтрация есебі үшін сандық әдістің қателігін бағалау

Бұл мақалада жарықшалары көлемі бойынша біртекті таралуы болжамында кеуекті ортада сұйықтықтың қозғалыс үлгісі зерттеледі. Бұл үлгі Дарси заңының бөлшек-дифференциалды баламасын қолдануға, сонымен қатар тау жынысы мен сұйықтықтың қасиеттері қысымнан және оның бөлшек туындысынан тәуелділік болжамына негізделген. Алдыңғы зерттеулерге қарағанда, бұл мақалада сингулярлық емес ядросы бар Капуто-Фабрицио мағынасындағы бөлшек туынды қолданылады. Бұл мақалада осы бастапқы шекаралық есепті шешудің сандық әдісі ұсынылған және оның жинақтылық реті теориялық тұрғыдан зерттелген. Толық дискретті сұлбаның құрылуы уақыт бойынша бүтін және бөлшек туындыларына ақырлы айырымдық жуықтауды, ал кеңістіктік айнымалысы бойынша Галеркин әдісін қолдануға негізделген. Бүтін туынды және Капуто-Фабрицио мағынасындағы бөлшек туындыны жуықтау үшін екінші ретті формула қолданылды. Априорлық бағалау жартылай дискретті және толық дискретті сұлбалар үшін алынды, олардан уақыт және кеңістіктік айнымалылары бойынша екінші ретті жинақтылық шығады. Сұлбаның дәлдігін тексеру үшін үлгі есептің мысалында бірқатар есептеу тәжірибелері жүргізілді. Сандық тәжірибелердің нәтижелері теориялық талдау нәтижелерін толық растайды.

© 2022 Al-Farabi Kazakh National University

Түйін сөздер: Ақырлы элементтер әдісі, Капуто-Фабрицио бөлшек ретті туындысы, жинақтылық, фильтрация есебі, жарықшалы кеуекті орта.

Д.Р. Байгереев[1*], Н.Б. Алимбекова[1], Н.М. Оскорбин[2]
[1]Восточно-Казахстанский университет им. С. Аманжолова, Казахстан, г. Усть-Каменогорск
[2]Алтайский государственный университет, Россия, г. Барнаул
*e-mail: dbaigereyev@gmail.com

Оценки погрешности численного метода для задачи фильтрации с дробными производными Капуто-Фабрицио

В данной статье изучается модель движения жидкости в трещиноватой пористой среде в предположении равномерного распределения трещин по объему. Данная модель основана на использовании дробно-дифференциального аналога закона Дарси и построена в предположении, что свойства породы и жидкости зависят от давления и его дробной производной. В отличие от предыдущих исследований, в настоящей статье используется дробная производная в смысле Капуто-Фабрицио с несингулярным ядром. В статье предлагается численный метод решения данной начально-краевой задачи и теоретически исследуется порядок его сходимости. Формулировка полностью дискретной схемы основана на применении конечно-разностной аппроксимации для целых и дробных производных по времени и метода Галеркина по пространственной переменной. Для аппроксимации целочисленной производной и дробной производной в смысле Капуто-Фабрицио используется формула второго порядка. Получены априорные оценки как для полудискретной, так и для полностью дискретной схем, из которых следует их сходимость со вторым порядком по временной и пространственной переменным. На примере модельной задачи проведен ряд вычислительных экспериментов для проверки точности схемы. Результаты численных тестов полностью подтверждают результаты теоретического анализа.

Ключевые слова: Метод конечных элементов, дробная производная Капуто-Фабрицио, сходимость, задача фильтрации, трещиновато-пористая среда.

1 Introduction

Fractional equations play an important role in modern science due to their extensive applications in natural and technical sciences. Interest in these equations is primarily due to their ability to describe power-law long-term memory and spatial nonlocality of complex environments and processes. Many authors confirm that models containing equations with fractional derivatives more adequately describe a particular physical process. Many studies are devoted to the study of various equations of fractional order.

This paper discusses the initial boundary value problem for the fractional differential equation

$$\frac{\partial u}{\partial t}+\bar{c}_{\phi\alpha}\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}+\bar{c}_{f\beta}\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}}-\left(F\left(\frac{\partial^{\gamma}u_x}{\partial t^{\gamma}}\right)\right)_x=\bar{f}_0,\quad t>0,\ \ x\in\Omega \tag{1}$$

in a one-dimensional domain Ω, where $\alpha,\beta\in(-1,0)$, $\gamma\in(0,1)$, $\frac{\partial^{\nu}}{\partial t^{\nu}}$ is the fractional differentiation operator in the sense of the Caputo-Fabrizio definition [1]:

$$\frac{\partial^{\nu}u}{\partial t^{\nu}}=\frac{M(\nu)}{1-\nu}\int_0^t\frac{\partial u}{\partial\theta}\exp\left(-\frac{\nu}{1-\nu}(t-\theta)\right)d\theta,\quad 0<\nu<1,\quad t>0, \tag{2}$$

where $M=M(\nu)$ is a function such that $M(0)=M(1)=0$. The important application examples of equations of the form (1) include the processes of anomalous diffusion in heterogeneous media [2–4], the flow of multiphase fluid in fractured porous formations [5,6]. In particular, in [6] an equation of the form (1) was derived to describe the pressure distribution during the flow of a single-phase fluid in a fractured porous medium, provided that the fractures are uniformly distributed over the volume. Unlike other known fluid flow models with fractional derivatives [7–9], the peculiarity of the model under consideration is that the model retains the structure of classical integer order filtration equations when the fractional differentiation order is replaced by an integer order.

Despite the fact that there are many analytical methods [10, 11] for solving problems for fractional differential equations, such equations are difficult to solve using these methods in many cases. Therefore the development of numerical methods based on the features of fractional derivatives and fractional equations is relevant. There are many numerical methods for solving fractional differential equations arising in fluid mechanics, and these methods differ mainly in the approach in which integer and fractional derivatives are discretized. These methods include the finite difference methods [12–14], compact difference scheme [15–17], finite element methods [18–20], finite volume schemes [21], mixed finite element schemes [22] and others. However, it is rather difficult to obtain a high-order approximation in time due to the peculiarities of the fractional derivatives.

In [23, 24], the authors considered the issues of the numerical solution of fractional differential equations, to which the filtration equations are reduced, using the methods of the theory of difference schemes, and they carried out a rigorous theoretical study of the convergence order of the proposed schemes. In the previous work [25], two finite element schemes of the convergence order $O\left(\tau^{2-\nu}\right)$, $\nu = \max\{\alpha, \beta, \gamma\}$, $\alpha, \beta, \gamma \in (0, 1)$ were constructed for solving the initial boundary value problem for an equation of the form (1) with a fractional Caputo derivative. In this paper, we continue this endeavor, but unlike [25], we use a fractional derivative in the sense of Caputo-Fabrizio and assume that its use provides a more realistic description of the fluid flow process and helps to better capture the dynamic behavior of real phenomena as discussed in works [26,27]. In addition, the use of the Caputo-Fabrizio derivative eliminates the difficulty of a degenerate singular kernel, which makes it difficult to apply approximate methods of its discretization. When constructing numerical methods for solving fractional-order equations, approximation formulas are used. With regard to the derivative in the sense of Caputo-Fabrizio, for example, the $L1$ formula of order $O\left(\tau^2\right)$, the $L1-2$ formula of order $O\left(\tau^3\right)$ [28] are known, where τ is a time step.

The purpose of this paper is to construct and study a finite element method for solving an initial boundary value problem for the equation with a fractional derivative in the sense of Caputo-Fabrizio, describing the pressure distribution during fluid flow through a fractured porous medium with a uniform distribution of fractures over the volume [6]. The paper defines a semi-discrete formulation of the problem with respect to time, obtained using the approximation of the fractional order derivative, and a fully discrete formulation of the problem. Theoretical a priori estimates are obtained for the convergence order of both semi-discrete and fully discrete schemes. Finally, the results of numerical tests are presented to verify the results of theoretical analysis.

2 Materials and methods

2.1 Formulation of the problem

In $Q_T = \overline{\Omega} \times [0, T]$, where $\Omega = (0, 1)$, the following initial boundary value problem is considered:

$$\frac{\partial u}{\partial t} + \bar{c}_{\phi\alpha} \frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}} + \bar{c}_{f\beta} \frac{\partial^{\beta+1} u}{\partial t^{\beta+1}} - \left(F \left(\frac{\partial^{\gamma} u_x}{\partial t^{\gamma}} \right) \right)_x = \bar{f}_0, \quad 0 < t < T, \ \ x \in \Omega, \tag{3}$$

$$u(0, t) = u(1, t) = 0, \quad 0 < t < T, \tag{4}$$

$$u(x, 0) = u_0(x), \quad x \in \overline{\Omega}, \tag{5}$$

where $\alpha, \beta \in (-1, 0), \gamma \in (0, 1)$, $\bar{c}_{\phi\alpha}$, $\bar{c}_{f\beta}$, $\bar{f}_0$ are some positive constants, and the fractional differentiation operator $\frac{\partial^{\nu}}{\partial t^{\nu}}$, $0 < \nu < 1$ is defined in (2). Let us assume that:

(A1) F is a differentiable function defined on Ω such that

$$F(u) = \mu u + \varphi_0(x, t), \tag{6}$$

where φ_0 is a given function, and μ is a positive constant.

(A2) Suppose that the problem (3)-(5) has a unique solution that has sufficient number of derivatives required for conducting the theoretical analysis.

Definition 1 *A weak solution to the problem (3)-(5) is the function $u \in H^1(0, T; H_0^1(\Omega))$, $u(x, 0) = u_0(x)$, satisfying the identity*

$$\left(\frac{\partial u}{\partial t}, v \right) + \bar{c}_{\phi\alpha} \left(\frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}}, v \right) + \bar{c}_{f\beta} \left(\frac{\partial^{\beta+1} u}{\partial t^{\beta+1}}, v \right) + \left(F \left(\frac{\partial^{\gamma} u_x}{\partial t^{\gamma}} \right), v_x \right) = \left(\bar{f}_0, v \right) \tag{7}$$

for any $v \in H_0^1(\Omega)$, where $\alpha, \beta \in (-1, 0), \gamma \in (0, 1)$.

2.2 Discretization of the problem

First, let us discretize the problem (3)-(5) with respect to the temporal variable. To this end, we divide the time interval $[0, T]$ by points $t_n = n\tau$, $n = 0, 1, ..., N_t$, $N_t \tau = T$ and let u^n denote the semi-discrete approximation of u with respect to the temporal variable. We use the following approximation formula for the Caputo-Fabrizio fractional derivative.

Lemma 1 *The Caputo-Fabrizio fractional derivative $\dfrac{\partial^{\nu} u}{\partial t^{\nu}}$ of order ν, $0 < \nu < 1$ at $t = t_n$ is approximated by [28]*

$$\left. \frac{\partial^{\nu} u}{\partial t^{\nu}} \right|_{t=t_n} = \Delta^{\nu} u^n + r_n^{\nu}, \tag{8}$$

$$\Delta^{\nu} u^n = \sum_{s=1}^{n} d_{n,s}^{\nu} \left(u^s - u^{s-1} \right)$$

where

$$d_{n,s}^{\nu} = \frac{M(\nu)}{\tau\nu}\left(\exp\left(-\sigma_\nu\left(t_n - t_s\right)\right) - \exp\left(-\sigma_\nu\left(t_n - t_{s-1}\right)\right)\right), \quad \sigma_\nu = \frac{\nu}{1-\nu},$$

and the following relation holds for the approximation error r_n^{ν}:

$$|r_n^{\nu}| \le \frac{(1-\nu)M(\nu)}{2\nu^2}\max_{0\le t\le t_n}\left|\frac{\partial^2 u}{\partial t^2}\right|\tau^2.$$

It is easy to show that the coefficients $d_{n,s}^{\nu}$ satisfy the following properties:
a) $d_{n,s}^{\nu}$ are strictly positive for all $1 \le s \le n$;
b) The sequence $\left\{d_{n,s}^{\nu}\right\}_{s=1}^{n}$ is increasing;
c) $d_{n,n}^{\nu} = O(1)$.
Approximate the first-order derivative at $t = t_n$ in the following form:

$$\frac{\partial u}{\partial t}(t_n) = \begin{cases} \frac{1}{2\tau}\left(3u^n - 4u^{n-1} + u^{n-2}\right) + O\left(\tau^2\right), & n \ge 2 \\ \frac{1}{\tau}\left(u^1 - u^0\right) + O(\tau), & n = 1. \end{cases}$$

Let us define a semi-discrete formulation of the problem (3)-(5):

Problem 1 *Let $u^i \in H_0^1(\Omega)$, $i = 0, 1, ..., n-1$ be known, $u^0 = u_0(x)$. Find $u^n \in H_0^1(\Omega)$ satisfying the identity:*

a) when $n = 1$:

$$\frac{1}{\tau}\left(u^1 - u^0, v\right) + \bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}u^1, v\right) + \bar{c}_{f\beta}\left(\Delta^{\beta+1}u^1, v\right) + \left(F\left(\Delta^{\gamma}u_x^1\right), v_x\right) = \left(\bar{f}_0, v\right), \tag{9}$$

b) when $n \ge 2$:

$$\frac{1}{2\tau}\left(3u^n - 4u^{n-1} + u^{n-2}, v\right) + \bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}u^n, v\right) + \bar{c}_{f\beta}\left(\Delta^{\beta+1}u^n, v\right) + \\ + \left(F\left(\Delta^{\gamma}u_x^n\right), v_x\right) = \left(\bar{f}_0, v\right), \tag{10}$$

for all $v \in H_0^1(\Omega)$, where $\alpha, \beta \in (-1, 0)$, $\gamma \in (0, 1)$.

To formulate a fully discrete scheme, we define a discrete space $V_h \subset H_0^1$:

$$V_h = \left\{v_h \in H_0^1(\Omega) \cap C^0\left(\overline{\Omega}\right) \;\middle|\; v_h\Big|_e \in P_1(e), \ \forall e \in \mathcal{K}_h\right\},$$

where $\mathcal{K}_h$ is a quasi-uniform domain triangulation in Ω.
Define the projection operator $Q_h : H_0^1(\Omega) \to V_h$, satisfying

$$\left(\left(Q_h u - u\right)_x, u_{h,x}\right) = 0 \quad \forall u \in H_0^1(\Omega), \ u_h \in V_h.$$

The projection operator has the following properties:

$$\|u - Q_h u\|_0 + h\|u - Q_h u\|_1 \le Ch^2\|u\|_2 \quad \forall u \in H_0^1(\Omega) \cap H^2(\Omega), \tag{11}$$

where $\|\cdot\|_q$ denotes the norm in $H^q(\Omega)$.
Let us define the fully discrete scheme for the problem (3)-(5) as follows.

Problem 2 *Let $u_h^i \in V_h$, $i = 0, 1, ..., n-1$ be given, $u_h^0 = Q_h u_0$. Find $u_h^n \in V_h$ satisfying the following identities:*

a) when $n = 1$:

$$\frac{1}{\tau}\left(u_h^1 - u_h^0, v_h\right) + \bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1} u_h^1, v_h\right) + \bar{c}_{f\beta}\left(\Delta^{\beta+1} u_h^1, v_h\right) + \left(F\left(\Delta^{\gamma} u_{h,x}^1\right), v_{h,x}\right) = \left(\bar{f}_0, v_h\right), \quad (12)$$

b) when $n \geq 2$:

$$\frac{1}{2\tau}\left(3u_h^n - 4u_h^{n-1} + u_h^{n-2}, v_h\right) + \bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1} u_h^n, v_h\right) + \bar{c}_{f\beta}\left(\Delta^{\beta+1} u_h^n, v_h\right) +$$

$$+ \left(F\left(\Delta^{\gamma} u_{h,x}^n\right), v_{h,x}\right) = \left(\bar{f}_0, v_h\right) \quad (13)$$

for any $v_h \in V_h$, where $\alpha, \beta \in (-1, 0)$, $\gamma \in (0, 1)$.

2.3 Study of convergence of the discrete schemes

Lemma 2 *Let $\{u^i\}_{i=0}^{N_t}$, $u^i \in L^2(\Omega)$ be the sequence of functions. For any $u^n \in L^2(\Omega)$, $n \geq 1$,*

$$\left(\Delta^{\nu} u^n, u^n\right) \geq \Phi_n - \Phi_{n-1} - \frac{1}{2} d_{n,1}^{\nu} \left\|u^0\right\|_0^2, \quad (14)$$

where $\Phi_n = \frac{1}{2}\sum_{i=1}^{n} d_{n,i}^{\nu}\left\|u^i\right\|_0^2$, $n \geq 1$, $\Phi_0 = 0$.

Proof. First, let us show that

$$\left(\Delta^{\nu} u^n, u^n\right) \geq \frac{1}{2}\Delta^{\nu}\left\|u^n\right\|_0^2. \quad (15)$$

Consider the difference $A = \left(\Delta^{\nu} u^n, u^n\right) - \frac{1}{2}\Delta^{\nu}\left\|u^n\right\|_0^2$. Using the definition of the discrete analogue of the Caputo-Fabrizio fractional derivative (8), we obtain the chain of equalities

$$A = \sum_{s=1}^{n} d_{n,s}^{\nu}\left(u^s - u^{s-1}, u^n\right) - \sum_{s=1}^{n} d_{n,s}^{\nu}\left(u^s - u^{s-1}, \frac{u^s + u^{s-1}}{2}\right) =$$

$$= \sum_{s=1}^{n} d_{n,s}^{\nu}\left(u^s - u^{s-1}, \frac{1}{2}\left(u^s - u^{s-1}\right) + \sum_{k=s+1}^{n}\left(u^k - u^{k-1}\right)\right) =$$

$$= \frac{1}{2}\sum_{s=1}^{n} d_{n,s}^{\nu}\left(\left(u^s - u^{s-1}\right)^2, 1\right) + \sum_{k=2}^{n} d_{n,s}^{\nu}\left(u^k - u^{k-1}, \sum_{s=1}^{k-1}\left(u^s - u^{s-1}\right)\right). \quad (16)$$

Further, it is easy to show that

$$u^k - u^{k-1} = \frac{\zeta^k - \zeta^{k-1}}{d_{n,k}^{\nu}}, \quad k = 1, 2, ..., n,$$

where $\sum_{s=1}^{k} d_{n,s}^{\nu}\left(u_i^s - u_i^{s-1}\right) = \zeta^k$. Then from (16) we get

$$A = \frac{1}{2d_{n,1}^{\nu}}\left\|\zeta^1\right\|_0^2 + \sum_{k=2}^{n}\frac{1}{2d_{n,k}^{\nu}}\left(\left\|\zeta^k\right\|_0^2 - \left\|\zeta^{k-1}\right\|_0^2\right) =$$

$$= \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{1}{d_{n,k}^{\nu}} - \frac{1}{d_{n,k+1}^{\nu}}\right)\left\|\zeta^k\right\|_0^2 + \frac{1}{2d_{n,n}^{\nu}}\left\|\zeta^n\right\|_0^2 \geq 0,$$

whence the validity of the inequality (15) follows.

Let us now prove the inequality (14). Transform the right-hand side of (15) using the definition of a discrete analog of the derivative:

$$\frac{1}{2}\Delta^{\nu}\left\|u^n\right\|_0^2 = \frac{1}{2}\sum_{s=1}^{n} d_{n,s}^{\nu}\left\|u^s\right\|_0^2 - \frac{1}{2}\sum_{s=1}^{n} d_{n-1,s-1}^{\nu}\left\|u^{s-1}\right\|_0^2 =$$

$$= \Phi_n - \Phi_{n-1} - \frac{1}{2}d_{n,1}^{\nu}\left\|u^0\right\|_0^2.$$

The lemma is proved.

Let us turn to the study of the question of the convergence of the method. Below we sometimes use the notation $u(t) = u(\cdot, t)$.

Theorem 1 *Under the assumptions (A1)-(A2) the solution u^n of Problem 1 converges to the solution of the problem (3)-(5) and the following inequality holds:*

$$\left\|u(t_n) - u^n\right\|_0 + \tau\sqrt{2c_0T^{-1}}\left\|u(t_n) - u^n\right\|_1 \leq C\tau^2,$$

where $c_0 = \min\left\{\bar{c}_{\phi\alpha} d_{n,1}^{\alpha+1}, \bar{c}_{f\beta} d_{n,1}^{\beta+1}, \mu d_{n,1}^{\gamma}\right\}$.

Proof. Denote $w^n = u(t_n) - u^n$. Consider the difference of identity (7) at $t = t_n$ and identities (9), (10) and choose $v = w^n$:

a) when $n = 1$:

$$\left(\frac{\partial u}{\partial t}(t_1) - \frac{u^1 - u^0}{\tau}, w^n\right) + \bar{c}_{\phi\alpha}\left(\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}(t_1) - \Delta^{\alpha+1}u^1, w^n\right) +$$

$$+\bar{c}_{f\beta}\left(\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}}(t_1) - \Delta^{\beta+1}u^1, w^n\right) + \left(F\left(\frac{\partial^{\gamma}u_x}{\partial t^{\gamma}}(t_1)\right) - F\left(\Delta^{\gamma}u_x^1\right), w_x^n\right) = 0; \qquad (17)$$

b) when $n \geq 2$:

$$\left(\frac{\partial u}{\partial t}(t_n) - \frac{3u^n - 4u^{n-1} + u^{n-2}}{2\tau}, w^n\right) + \bar{c}_{\phi\alpha}\left(\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}(t_n) - \Delta^{\alpha+1}u^n, w^n\right) +$$

$$+\bar{c}_{f\beta}\left(\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}}(t_n) - \Delta^{\beta+1}u^n, w^n\right) + \left(F\left(\frac{\partial^{\gamma}u_x}{\partial t^{\gamma}}(t_n)\right) - F\left(\Delta^{\gamma}u_x^n\right), w_x^n\right) = 0. \qquad (18)$$

Let us estimate the terms in (17) and (18):

$$\left(\frac{\partial u}{\partial t}(t_1)-\frac{u^1-u^0}{\tau},w^n\right)\geq\frac{1}{2\tau}\left\|w^1\right\|_0^2-\frac{1}{2\tau}\left\|w^0\right\|_0^2+\frac{\tau}{2}\left(\frac{\partial^2 u}{\partial t^2}(\zeta_1),w^1\right),$$

$$\left(\frac{\partial u}{\partial t}(t_n)-\frac{3u^n-4u^{n-1}+u^{n-2}}{2\tau},w^n\right)\geq$$

$$\geq\frac{1}{4\tau}\left(\left\|w^n\right\|_0^2+\left\|2w^n-w^{n-1}\right\|_0^2+\left\|w^n-2w^{n-1}+w^{n-2}\right\|_0^2\right)-$$

$$-\frac{1}{4\tau}\left(\left\|w^{n-1}\right\|_0^2+\left\|2w^{n-1}-w^{n-2}\right\|_0^2\right)+\frac{\tau^2}{3}\left(\frac{\partial^3 u}{\partial t^3}(\zeta_n),w^n\right),$$

$$\left(\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}(t_n)-\Delta^{\alpha+1}u^n,w^n\right)\geq\left(r_n^{\alpha+1},w^n\right)+\left(\Phi_n^{\alpha+1}-\Phi_{n-1}^{\alpha+1}\right)-\frac{1}{2}d_{n,1}^{\alpha+1}\left\|w^0\right\|_0^2,$$

$$\left(\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}}(t_n)-\Delta^{\beta+1}u^n,w^n\right)\geq\left(r_n^{\beta+1},w^n\right)+\left(\Phi_n^{\beta+1}-\Phi_{n-1}^{\beta+1}\right)-\frac{1}{2}d_{n,1}^{\beta+1}\left\|w^0\right\|_0^2,$$

$$\left(F\left(\frac{\partial^\gamma u_x}{\partial t^\gamma}(t_n)\right)-F\left(\Delta^\gamma u_x^n\right),w_x^n\right)\geq\mu\left(r_n^\gamma,w_x^n\right)+\mu\left(\Phi_n^\gamma-\Phi_{n-1}^\gamma\right)-\frac{1}{2}\mu d_{n,1}^\gamma\left\|w_x^0\right\|_0^2,$$

where

$$\Phi_n^{\alpha+1}=\frac{1}{2}\sum_{s=1}^{n}d_{n,s}^{\alpha+1}\left\|w^s\right\|_0^2,\qquad r_n^{\alpha+1}=\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}(t_n)-\Delta^{\alpha+1}u(t_n),$$

$$\Phi_n^{\beta+1}=\frac{1}{2}\sum_{s=1}^{n}d_{n,s}^{\beta+1}\left\|w^s\right\|_0^2,\qquad r_n^{\beta+1}=\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}}(t_n)-\Delta^{\beta+1}u(t_n),$$

$$\Phi_n^{\gamma+1}=\frac{1}{2}\sum_{s=1}^{n}d_{n,s}^{\gamma}\left\|w_x^s\right\|_0^2,\qquad r_n^{\gamma+1}=\frac{\partial^{\gamma}u_x}{\partial t^{\gamma}}(t_n)-\Delta^{\gamma}u_x(t_n),$$

and $\Phi_0^r=0$. Taking into account the obtained estimates in (17) and (18), we arrive at the following inequalities:

$$\left\|w^1\right\|_0^2+2\tau\Phi_1\leq\left\|w^0\right\|_0^2+2\tau\Phi_0+2\tau^2\left|\left(\frac{\partial^2 u}{\partial t^2}(\zeta_1),w^1\right)\right|+$$

$$+2\tau\bar{c}_{\phi\alpha}\left|\left(r_1^{\alpha+1},w^1\right)\right|+2\tau\bar{c}_{f\beta}\left|\left(r_1^{\beta+1},w^1\right)\right|+2\tau\mu\left|\left(r_1^\gamma,w_x^1\right)\right|, \tag{19}$$

$$\left\|w^n\right\|_0^2+\left\|2w^n-w^{n-1}\right\|_0^2+4\tau\Phi_n+\left\|w^n-2w^{n-1}+w^{n-2}\right\|_0^2\leq$$

$$\leq\left\|w^{n-1}\right\|_0^2+\left\|2w^{n-1}-w^{n-2}\right\|_0^2+4\tau\Phi_{n-1}+\frac{4\tau^3}{3}\left|\left(\frac{\partial^3 u}{\partial t^3}(\zeta_n),w^n\right)\right|+4\tau\bar{c}_{\phi\alpha}\left|\left(r_n^{\alpha+1},w^n\right)\right|+$$

$$+4\tau\bar{c}_{f\beta}\left|\left(r_n^{\beta+1},w^n\right)\right|+4\tau\mu\left|\left(r_n^\gamma,w_x^n\right)\right|, \tag{20}$$

where the notation

$$\Phi_n=\bar{c}_{\phi\alpha}\Phi_n^{\alpha+1}+\bar{c}_{f\beta}\Phi_n^{\beta+1}+\mu\Phi_n^{\gamma+1}$$

is used. By estimating the last four terms on the right-hand side of (20), applying the Cauchy inequality, we obtain

$$\|w^n\|_0^2 + \left\|2w^n - w^{n-1}\right\|_0^2 + 4\tau\Phi_n \le \left\|w^{n-1}\right\|_0^2 + \left\|2w^{n-1} - w^{n-2}\right\|_0^2 + 4\tau\Phi_{n-1} +$$

$$+\frac{4\tau^3}{3}\left\|\frac{\partial^3 u}{\partial t^3}(\zeta_n)\right\|_0 \|w^n\|_0 + 4\tau\bar{c}_{\phi\alpha}\left\|r_n^{\alpha+1}\right\|_0 \|w^n\|_0 +$$

$$+4\tau\bar{c}_{f\beta}\left\|r_n^{\beta+1}\right\|_0 \|w^n\|_0 + 4\tau\mu\left\|r_n^{\gamma}\right\|_0 \|w_x^n\|_0 . \qquad (21)$$

Sum the inequality (21) for n from 2 to n to get

$$\|w^n\|_0^2 + 4\tau\Phi_n \le 5\left\|w^1\right\|_0^2 + 4\tau\Phi_1 +$$

$$+\frac{C}{\varepsilon_1}\left(\tau^{5/2}\left\|\frac{\partial^3 u}{\partial t^3}(\zeta_n)\right\|_0 + \tau\left\|r_n^{\alpha+1}\right\|_0 + \tau\left\|r_n^{\beta+1}\right\|_0\right)^2 + \varepsilon_1\tau\|w^n\|_0^2 + \frac{C}{\varepsilon_2}\tau\left\|r_n^{\gamma}\right\|_0^2 + \varepsilon_2\tau\|w_x^n\|_0^2 +$$

$$+\frac{C}{\varepsilon_3}\sum_{i=2}^{n-1}\left(\tau^2\left\|\frac{\partial^3 u}{\partial t^3}(\zeta_i)\right\|_0 + \bar{c}_{\phi\alpha}\left\|r_i^{\alpha+1}\right\|_0 + \bar{c}_{f\beta}\left\|r_i^{\beta+1}\right\|_0\right)^2 +$$

$$+\varepsilon_3\tau^2\sum_{i=2}^{n-1}\left\|w^i\right\|_0^2 + \frac{C}{\varepsilon_4}\sum_{i=2}^{n}\left\|r_i^{\gamma}\right\|_0^2 + \varepsilon_4\tau^2\sum_{i=2}^{n}\left\|w_x^i\right\|_0^2$$

or

$$\|w^n\|_0^2 + 4\tau\Phi_n \le 5\left\|w^1\right\|_0^2 + 4\tau\Phi_1 +$$

$$+2\tau\left(\left(\bar{c}_{\phi\alpha}d_{n,1}^{\alpha+1} + \bar{c}_{f\beta}d_{n,1}^{\beta+1}\right)\|w^n\|_0^2 + \mu d_{n,1}^{\gamma}\|w_x^n\|_0^2\right) +$$

$$+\tau^2\sum_{i=2}^{n-1}\left(\left(\bar{c}_{\phi\alpha}d_{i,1}^{\alpha+1} + \bar{c}_{f\beta}d_{i,1}^{\beta+1}\right)\left\|w^i\right\|_0^2 + \mu d_{i,1}^{\gamma}\left\|w_x^i\right\|_0^2\right) + C\tau^4.$$

Considering that $\left(\bar{c}_{\phi\alpha}d_{n,1}^{\alpha+1} + \bar{c}_{f\beta}d_{n,1}^{\beta+1}\right)\|w^n\|_0^2 + \mu d_{n,1}^{\gamma}\|w_x^n\|_0^2 \le \Phi_n$, it follows that

$$\|w^n\|_0^2 + 2\tau\Phi_n \le 5\left\|w^1\right\|_0^2 + 4\tau\Phi_1 + \tau^2\sum_{i=2}^{n-1}\Phi_i + C\tau^4.$$

Applying the discrete Gronwall's lemma, we obtain

$$\|w^n\|_0^2 + 2\tau\Phi_n \le C\left(\left\|w^1\right\|_0^2 + \tau\Phi_1 + \tau^4\right). \qquad (22)$$

Let us now evaluate terms in (19):

$$\left\|w^1\right\|_0^2 + 2\tau\Phi_1 \le \left\|w^0\right\|_0^2 + 2\tau\Phi_0 + 2\tau^2\left\|\frac{\partial^2 u}{\partial t^2}(\zeta_1)\right\|_0 \left\|w^1\right\|_0 + 2\tau\bar{c}_{\phi\alpha}\left\|r_1^{\alpha+1}\right\|_0 \left\|w^1\right\|_0 +$$

$$+ 2\tau\bar{c}_{f\beta}\left\|r_1^{\beta+1}\right\|_0 \left\|w^1\right\|_0 + 2\mu\left\|r_1^{\gamma}\right\|_0 \cdot \tau\left\|w_x^1\right\|,$$

or

$$\left\|w^1\right\|_0^2 + 4\tau\Phi_1 \le \frac{\tau^2}{2}\mu d_{1,1}^{\gamma}\left\|w_x^1\right\|_0^2 + C\tau^4.$$

Noticing that $\frac{1}{2}\mu d_{1,1}^{\gamma}\tau\left\|w_x^1\right\|_0^2 \le \Phi_1$, we get

$$\left\|w^1\right\|_0^2 + 3\tau\Phi_1 \le C\tau^4. \tag{23}$$

By substituting (23) into (22), and applying elementary transformations, we arrive at the assertion of the theorem.

Theorem 2 *Under the assumptions (A1)-(A2) there exists $\tau_0 > 0$ such that for all $\tau \le \tau_0$ the solution u_h^n of Problem 2 converges to the solution of Problem (3)-(5) and the following inequality holds:*

$$\|u(t_n) - u_h^n\|_0 + 2\tau\sqrt{c_0 T^{-1}}\,\|u(t_n) - u_h^n\|_1 \le C\left(\tau^2 + h^2\right),$$

where $c_0 = \min\left\{\bar{c}_{\phi\alpha} d_{n,1}^{\alpha+1}, \bar{c}_{f\beta} d_{n,1}^{\beta+1}, \mu d_{n,1}^{\gamma}\right\}$.

Proof. Let $u^n - u_h^n = (u^n - Q_h u^n) + (Q_h u^n - u_h^n) = \vartheta^n + \eta^n$.
Consider the difference of (10) and (13) and choose $v_h = \eta^n$:

$$\|\eta^n\|_0^2 + \left\|2\eta^n - \eta^{n-1}\right\|_0^2 - \left\|\eta^{n-1}\right\|_0^2 - \left\|2\eta^{n-1} - \eta^{n-2}\right\|_0^2 + \left\|\eta^n - 2\eta^{n-1} + \eta^{n-2}\right\|_0^2 +$$

$$+4\tau\bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}\left(\vartheta^n + \eta^n\right), \eta^n\right) + 4\tau\bar{c}_{f\beta}\left(\Delta^{\beta+1}\left(\vartheta^n + \eta^n\right), \eta^n\right) +$$

$$+4\tau\left(F\left(\Delta^{\gamma} u_x^n\right), \eta_x^n\right) - 4\tau\left(F\left(\Delta^{\gamma} u_{h,x}^n\right), \eta_x^n\right) + 4\tau\left(\frac{3\vartheta^n - 4\vartheta^{n-1} + \vartheta^{n-2}}{2\tau}, \eta^n\right) = 0. \tag{24}$$

Consider the term

$$4\tau\bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}\left(\vartheta^n + \eta^n\right), \eta^n\right) = 4\tau\bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}\eta^n, \eta^n\right) + 4\tau\bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}\vartheta^n, \eta^n\right) = K_1 + K_2.$$

Using Lemma 2, we get:

$$K_1 \ge 4\tau\bar{c}_{\phi\alpha}\left(\Phi_n^{\alpha+1} - \Phi_{n-1}^{\alpha+1}\right) - 2\tau\bar{c}_{\phi\alpha} d_{n,1}^{\alpha+1}\left\|\eta^0\right\|_0^2,$$

$$\begin{aligned}
K_2 &\le 4\tau\bar{c}_{\phi\alpha}\left\|\Delta^{\alpha+1}\vartheta^n\right\|_0 \|\eta^n\|_0 \le \\
&\le 4\tau\bar{c}_{\phi\alpha}^2\left\|\sum_{s=1}^{n} d_{n,s}^{\alpha+1}\left(\vartheta^s - \vartheta^{s-1}\right)\right\|_0^2 + 2\tau\|\eta^n\|_0^2 = \\
&= 4\tau\bar{c}_{\phi\alpha}^2\int_{\Omega}\left(\sum_{s=1}^{n} d_{n,s}^{\alpha+1}\int_{t_{s-1}}^{t_s}\vartheta_t d\theta\right)^2 dx + 2\tau\|\eta^n\|_0^2 \le \\
&\le 4T\tau\left(\bar{c}_{\phi\alpha} d_{n,n}^{\alpha+1}\right)^2\int_0^T \|\vartheta_t\|_0^2 d\theta + 2\tau\|\eta^n\|_0^2,
\end{aligned}$$

where $\Phi_n^{\alpha+1}=\dfrac{1}{2}\sum\limits_{s=1}^{n}d_{n,s}^{\alpha+1}\left\|\eta^s\right\|_0^2$. Similarly,

$$4\tau\bar{c}_{f\beta}\left(\Delta^{\beta+1}\left(\vartheta^n+\eta^n\right),\eta^n\right)\geq 4\tau\bar{c}_{f\beta}\left(\Phi_n^{\beta+1}-\Phi_{n-1}^{\beta+1}\right)-2\tau\bar{c}_{f\beta}d_{n,1}^{\beta+1}\left\|\eta^0\right\|_0^2-$$

$$-4T\tau^2\left(\bar{c}_{f\beta}d_{n,n}^{\beta+1}\right)^2\int_0^T\left\|\vartheta_t\right\|_0^2d\theta-2\tau\left\|\eta^n\right\|_0^2,$$

where $\Phi_n^{\beta+1}=\dfrac{1}{2}\sum\limits_{s=1}^{n}d_{n,s}^{\beta+1}\left\|\eta^s\right\|_0^2$. Estimate the remaining terms as follows:

$$\begin{aligned}4\tau\left(F\left(\Delta^\gamma u_x^n\right),\eta_x^n\right)-4\tau\left(F\left(\Delta^\gamma u_{h,x}^n\right),\eta_x^n\right)&=4\tau\mu\left(\Delta^\gamma\left(\vartheta_x^n+\eta_x^n\right),\eta_x^n\right)=\\&=4\tau\mu\left(\Phi_n^\gamma-\Phi_{n-1}^\gamma\right)-2\tau\mu d_{n,1}^\gamma\left\|\eta_x^0\right\|_0^2,\end{aligned}$$

$$\begin{aligned}K_6&\equiv 4\tau\left(\frac{3\vartheta^n-4\vartheta^{n-1}+\vartheta^{n-2}}{2\tau},\eta^n\right)\leq\\&=2\tau\left\|\frac{\int_{t_{n-1}}^{t_n}\vartheta_td\theta-\int_{t_{n-2}}^{t_{n-1}}\vartheta_td\theta}{2\tau}\right\|_0^2+2\tau\left\|\eta^n\right\|_0^2=\\&=\frac{1}{2}\left(\int_{t_{n-1}}^{t_n}\int_\Omega\vartheta_t^2dxd\theta+\int_{t_{n-2}}^{t_{n-1}}\int_\Omega\vartheta_t^2dxd\theta\right)+2\tau\left\|\eta^n\right\|_0^2=\\&=\frac{1}{2}\int_{t_{n-2}}^{t_n}\left\|\vartheta_t\right\|_0^2d\theta+2\tau\left\|\eta^n\right\|_0^2,\end{aligned}$$

where $\Phi_n^\gamma=\dfrac{1}{2}\sum\limits_{s=1}^{n}d_{n,s}^\gamma\left\|\eta_x^s\right\|_0^2$. Then it follows from (24) that

$$\left\|\eta^n\right\|_0^2+\left\|2\eta^n-\eta^{n-1}\right\|_0^2+\left\|\eta^n-2\eta^{n-1}+\eta^{n-2}\right\|_0^2+4\tau\Phi_n\leq$$

$$\leq\left\|\eta^{n-1}\right\|_0^2+\left\|2\eta^{n-1}-\eta^{n-2}\right\|_0^2+4\tau\Phi_{n-1}+4T\tau\left(\bar{c}_{\phi\alpha}d_{n,n}^{\alpha+1}\right)^2\int_0^T\left\|\vartheta_t\right\|_0^2d\theta+$$

$$+4T\tau^2\left(\bar{c}_{f\beta}d_{n,n}^{\beta+1}\right)^2\int_0^T\left\|\vartheta_t\right\|_0^2d\theta+6\tau\left\|\eta^n\right\|_0^2+\frac{1}{2}\int_{t_{n-2}}^{t_n}\left\|\vartheta_t\right\|_0^2d\theta,\tag{25}$$

where

$$\Phi_n=\bar{c}_{\phi\alpha}\Phi_n^{\alpha+1}+\bar{c}_{f\beta}\Phi_n^{\beta+1}+\mu\Phi_n^\gamma.$$

Sum the inequality (25) for n from 2 to n to obtain

$$\left\|\eta^n\right\|_0^2+4\tau\Phi_n\leq 5\left\|\eta^1\right\|_0^2+4\tau\Phi_1+6\tau\left\|\eta^n\right\|_0^2+6\tau\sum_{i=2}^{n-1}\left\|\eta^i\right\|_0^2+Ch^4,$$

whence, for sufficiently small τ, we obtain

$$\|\eta^n\|_0^2 + 4\tau\Phi_n \le 5\left\|\eta^1\right\|_0^2 + 4\tau\Phi_1 + 6\tau\sum_{i=2}^{n-1}\left\|\eta^i\right\|_0^2 + Ch^4.$$

By applying the discrete Gronwall's lemma, we get

$$\|\eta^n\|_0^2 + 4\tau\Phi_n \le C\left(\left\|\eta^1\right\|_0^2 + \tau\Phi_1 + h^4\right). \tag{26}$$

Considering the difference of (9) and (12), choosing $v_h = \eta^1$ and using a similar technique for estimating the terms in the resulting identity, we arrive at

$$\left\|\eta^1\right\|_0^2 + \tau\Phi_1 \le C\tau^2\left\|\vartheta^1\right\|_0^2 + \frac{1}{2}\left\|\eta^1\right\|_0^2 + \frac{3\tau^2}{2}\left\|\frac{\vartheta^1-\vartheta^0}{\tau}\right\|_0^2,$$

therefore,

$$\frac{1}{2}\left\|\eta^1\right\|_0^2 + \tau\Phi_1 \le C\left(\tau^4 + h^4\right). \tag{27}$$

Combining (26) and (27), we obtain

$$\|\eta^n\|_0^2 + 4c_0\tau\sum_{s=1}^{n}\|\eta^s\|_1^2 \le C\left(\tau^4 + h^4\right),$$

whence the assertion of the theorem follows.

3 Results

To check the accuracy of the scheme, computational experiments were carried out using a model problem as an example.

Example 1 *In $Q_1 = \overline{\Omega}\times[0,1]$, where $\Omega = (0,1)$, consider the problem*

$$\frac{\partial u}{\partial t} + \frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}} + \frac{\partial^{\beta+1}u}{\partial t^{\beta+1}} - \frac{\partial^{\gamma}u_{xx}}{\partial t^{\gamma}} = \bar{f}_0, \quad 0<t<1, \quad x\in\Omega, \tag{28}$$

$$\bar{f}_0(x,t) = -\frac{2}{\gamma}\cdot\left(\exp\left(t\left(\gamma-1\right)/\left(\gamma-2\right)\right) - \exp\left(t/2\right)\right) +$$

$$+x\left(x-1\right)\left[\frac{\exp\left(\alpha t/\left(\alpha-1\right)\right)-\exp\left(t/2\right)}{\alpha+1} + \frac{\exp\left(\beta t/\left(\beta-1\right)\right)-\exp\left(t/2\right)}{\beta+1} - \frac{\exp\left(t/2\right)}{2}\right]$$

$$u\left(x,0\right) = x\left(1-x\right), \quad x\in\overline{\Omega}, \tag{29}$$

$$u\left(0,t\right) = u\left(1,t\right) = 0, \quad 0<t<1, \tag{30}$$

where $\alpha,\beta\in(-1,0)$, $\gamma\in(0,1)$.

The exact solution to the problem is $u(x,t) = x(1-x)\exp(t/2)$.

The first series of computational experiments was carried out to compare the convergence order of the scheme with respect to the time step with a fixed value of the spatial step, $h = 1/20000$. For this, the time step value was gradually halved from 1/10 to 1/640, and the convergence order was evaluated as $(\ln(R_{2\tau}/R_\tau))/\ln 2$, where R_τ is the L^2-error of the approximate solution calculated with the use of the time step τ. Tables 1-3 outline the results of the analysis for different values of the fractional derivative orders, $\alpha = \beta \in \{-0.9, -0.5, -0.1\}$ and $\gamma \in \{0.1, 0.5, 0.9\}$. It can be clearly seen from the presented values that the convergence order does not depend on the fractional derivative orders for all considered cases, and its value approaches 2. This behavior agrees well with the theoretically predicted order with respect to the time step obtained in Theorem 2.

Similarly, the second series of computational experiments was conducted in order to compare the convergence order with respect to the spatial step with a fixed temporal step, $\tau = 1/20000$. The corresponding L^2-errors and convergence orders are presented in Tables 4-6. As it follows from numerical experiments, the actual convergence order for all considered cases is close to 2. Hence, the results obtained fully confirm the theoretically predicted order obtained in Theorem 2.

Table 1: Error analysis with respect to the temporal step, $\gamma = 0.1$

τ	$\alpha = \beta = -0.9$		$\alpha = \beta = -0.5$		$\alpha = \beta = -0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	1.5578×10^{-4}	-	1.9224×10^{-4}	-	3.2447×10^{-3}	-
1/20	3.7507×10^{-5}	2.05	4.4459×10^{-5}	2.11	7.6783×10^{-4}	2.08
1/40	9.1597×10^{-6}	2.03	1.0491×10^{-5}	2.08	1.8135×10^{-4}	2.08
1/80	2.2580×10^{-6}	2.02	2.5154×10^{-6}	2.06	4.2878×10^{-5}	2.08
1/160	5.5977×10^{-7}	2.01	6.1017×10^{-7}	2.04	1.0196×10^{-5}	2.07
1/320	1.3920×10^{-7}	2.01	1.4951×10^{-7}	2.03	2.4359×10^{-6}	2.07
1/640	3.4467×10^{-8}	2.01	3.7320×10^{-8}	2.00	5.9093×10^{-7}	2.04

Table 2: Error analysis with respect to the temporal step, $\gamma = 0.5$

τ	$\alpha = \beta = -0.9$		$\alpha = \beta = -0.5$		$\alpha = \beta = -0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	3.1783×10^{-4}	-	3.4405×10^{-4}	-	3.2313×10^{-3}	-
1/20	6.7777×10^{-5}	2.23	7.2527×10^{-5}	2.25	7.6007×10^{-4}	2.09
1/40	1.4763×10^{-5}	2.20	1.5600×10^{-5}	2.22	1.7861×10^{-4}	2.09
1/80	3.2883×10^{-6}	2.17	3.4378×10^{-6}	2.18	4.1985×10^{-5}	2.09
1/160	7.4652×10^{-7}	2.14	7.7628×10^{-7}	2.15	9.8740×10^{-6}	2.09
1/320	1.7842×10^{-7}	2.06	1.7728×10^{-7}	2.13	2.3550×10^{-6}	2.07
1/640	4.2906×10^{-8}	2.06	4.1230×10^{-8}	2.10	5.7194×10^{-7}	2.04

Table 3: Error analysis with respect to the temporal step, $\gamma = 0.9$

τ	$\alpha=\beta=-0.9$		$\alpha=\beta=-0.5$		$\alpha=\beta=-0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	1.0618×10^{-3}	-	1.0576×10^{-3}	-	1.0644×10^{-3}	-
1/20	2.5121×10^{-4}	2.08	2.4874×10^{-4}	2.09	2.5491×10^{-4}	2.06
1/40	5.9316×10^{-5}	2.08	5.8445×10^{-5}	2.09	6.1094×10^{-5}	2.06
1/80	1.4018×10^{-5}	2.08	1.3736×10^{-5}	2.09	1.4812×10^{-5}	2.04
1/160	3.3405×10^{-6}	2.07	3.2921×10^{-6}	2.06	3.6240×10^{-6}	2.03
1/320	7.9946×10^{-7}	2.06	7.8824×10^{-7}	2.06	8.9512×10^{-7}	2.02
1/640	1.9589×10^{-7}	2.03	1.9291×10^{-7}	2.03	2.2394×10^{-7}	2.00

Table 4: Error analysis with respect to the spatial step, $\gamma = 0.1$

h	$\alpha=\beta=-0.9$		$\alpha=\beta=-0.5$		$\alpha=\beta=-0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	1.0955×10^{-6}	-	1.2410×10^{-6}	-	1.3720×10^{-6}	-
1/20	2.7388×10^{-7}	2.00	3.1026×10^{-7}	2.00	3.4538×10^{-7}	1.99
1/40	6.8469×10^{-8}	2.00	7.7565×10^{-8}	2.00	8.6946×10^{-8}	1.99
1/80	1.7117×10^{-8}	2.00	1.9257×10^{-8}	2.01	2.1737×10^{-8}	2.00
1/160	4.2498×10^{-9}	2.01	4.7810×10^{-9}	2.01	5.4341×10^{-9}	2.00
1/320	1.0551×10^{-9}	2.01	1.1788×10^{-9}	2.02	1.3491×10^{-9}	2.01
1/640	2.6014×10^{-10}	2.02	2.9064×10^{-10}	2.02	3.3496×10^{-10}	2.01

Table 5: Error analysis with respect to the spatial step, $\gamma = 0.5$

h	$\alpha=\beta=-0.9$		$\alpha=\beta=-0.5$		$\alpha=\beta=-0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	9.3002×10^{-7}	-	1.0559×10^{-6}	-	1.1687×10^{-6}	-
1/20	2.3251×10^{-7}	2.00	2.6581×10^{-7}	1.99	2.9421×10^{-7}	1.99
1/40	$5,8126 \times 10^{-8}$	2.00	6.6453×10^{-8}	2.00	7.4065×10^{-8}	1.99
1/80	1.4532×10^{-8}	2.00	1.6499×10^{-8}	2.01	1.8516×10^{-8}	2.00
1/160	3.6078×10^{-9}	2.01	4.0961×10^{-9}	2.01	4.6291×10^{-9}	2.00
1/320	8.9572×10^{-10}	2.01	1.0099×10^{-9}	2.02	1.1493×10^{-9}	2.01
1/640	2.2085×10^{-10}	2.02	2.4901×10^{-10}	2.02	2.8533×10^{-10}	2.01

Table 6: Error analysis with respect to the spatial step, $\gamma = 0.9$

h	$\alpha = \beta = -0.9$		$\alpha = \beta = -0.5$		$\alpha = \beta = -0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	7.7391×10^{-7}	-	8.8972×10^{-7}	-	1.0233×10^{-6}	-
1/20	1.9482×10^{-7}	1.99	2.2243×10^{-7}	2.00	2.5761×10^{-7}	1.99
1/40	4.8706×10^{-8}	2.00	5.5607×10^{-8}	2.00	6.4849×10^{-8}	1.99
1/80	1.2176×10^{-8}	2.00	1.3806×10^{-8}	2.01	1.6212×10^{-8}	2.00
1/160	3.0231×10^{-9}	2.01	3.4276×10^{-9}	2.01	4.0531×10^{-9}	2.00
1/320	7.5055×10^{-10}	2.01	8.4510×10^{-10}	2.02	1.0063×10^{-9}	2.01
1/640	1.8505×10^{-10}	2.02	2.0837×10^{-10}	2.02	2.4983×10^{-10}	2.01

4 Conclusion

Thus, the constructed numerical method allows obtaining an approximate solution to the problem of fluid flow in a fractured porous medium with the second order in both time and spatial variable. The results of computational experiments carried out for various orders of fractional derivatives and grid configurations fully confirm the results of theoretical analysis. The methods used and the conclusions drawn, described in the work, can be used to solve other classes of fractional differential equations.

5 Acknowledgement

The work was supported by grant funding of scientic and technical programs and projects of the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. AP08053189, 2020-2022).

References

[1] Caputo M. and Fabrizio M., "A new definition of fractional derivative without singular kernel", *Progress in Fractional Differentiation and Applications,* 2 (2015): 73–85.

[2] Chechkin A. V., Gorenflo R. and Sokolov I. M., "Fractional diffusion in inhomogeneous media", *Journal of Physics A: Mathematical And General,* 38 (2005): 679–984.

[3] Esen A., Ucar Y., Yagmurlu M. and Tasbozan O., "Solving fractional diffusion and fractional diffusionwave equations by Petrov Galerkin finite element method", *Turkish World Mathematical Society Journal of Applied and Engineering Mathematics,* 4, no. 2 (2014): 155–168.

[4] Zhu A., Wang Y. and Xu Q., "A weak Galerkin finite element approximation of two-dimensional subdiffusion equation with time fractional derivative", *AIMS Mathematics,* 5, no. 5 (2020): 4297–4310.

[5] Abiola O. D., Enamul H. M., Kaseem M. and Sidqi A. A., "A modified memory based mathematical model describing fluid flow in porous media", *Computers and Mathematics with Applications,* 73, no. 6 (2017): 1385–1402.

[6] Gazizov R. K. and Lukashchuk S. Yu., "Drobno-differentsialnyi podkhod k modelirovaniyu protsessov filtratsii v slozhnykh neodnorodnykh poristykh sredakh [Fractional differential approach to modeling filtration processes in complex inhomogeneous porous media]", *Vestnik UGATU,* 21, no. 4 (2017): 104–112 (in Russian).

[7] Caffarelli L. and Vazquez J. L., "Nonlinear porous medium flow with fractional potential pressure", *Archive for Rational Mechanics and Analysis,* 202, no. 2 (2011): 537–565.

[8] Gazizov R. K., Kasatkin A. A. and Lukashchuk S. Yu., "Symmetries and exact solutions of fractional filtration equations", *AIP Conference Proceedings,* 1907, no. 020010 (2017): 1–9.

[9] Meilanov R. R., Akhmedov E. N. and Beybalaev V. D., "To the theory of nonlocal nonisothermal filtration in porous medium", *IOP Conf. Series: Journal of Physics: Conf. Series,* 946, no. 012076 (2018): 1–8.

[10] El Amin M. F., Radwan A. G. and Sun S., "Analytical solution for fractional derivative gasflow equation in porous media", *Results in Physics,* 7 (2017): 2432–2438.

[11] Ray S. S., "Exact solutions for time fractional diffusion wave equations by decomposition method", *Phys. Scr.,* 75 (2007): 53–61.

[12] Zhang Y. N., Sun Z. Z. and Liao H. L., "Finite difference methods for the time fractional diffusion equation on nonuniform meshes", *Journal of Computational Physics,* 265 (2014): 195–210.

[13] Qiao H. L., Liu Z. G. and Cheng A. J., "Two unconditionally stable difference schemes for time distributed order differential equation based on Caputo–Fabrizio fractional derivative", *Advances in Difference Equations,* 2020 (2020): 1–17.

[14] Alikhanov A. A., "A new difference scheme for the time fractional diffusion equation", *Journal of Computational Physics,* 208 (2015): 424–438.

[15] Du R., Cao W. R. and Sun Z. Z., "A compact difference scheme for the fractional diffusion wave equation", *Applied Mathematical Modelling,* 34 (2010): 2998–3007.

[16] Xu T., Lu S., Chen W. and Chen H., "Finite difference scheme for multiterm variable order fractional diffusion equation", *Advances in Difference Equations,* 1 (2018): 1-13.

[17] Huang J., Tang Y., Wang W. and Yang J. "A compact difference scheme for time fractional diffusion equation with Neumann boundary conditions", *Communications in Computer and Information Science,* 323 (2012): 273–284.

[18] Zhang C., Liu H. and Zhou Z. J., "A priori error analysis for timestepping discontinuous Galerkin finite element approximation of time fractional optimal control problem", *Journal of Scientific Computing,* 80, no. 2 (2019): 993–1018.

[19] Liu J. and Zhou Z., "Finite element approximation of time fractional optimal control problem with integral state constraint", *AIMS Mathematics,* 6, no. 1 (2020): 979–997.

[20] Liu K., Feckan M., O'Regan D. and Wang J. R., "Hyers–Ulam stability and existence of solutions for differential equations with Caputo–Fabrizio fractional derivative", *Mathematics,* 333, no. 7 (2019): 1-14.

[21] Coronel Escamilla A., Gomez Aguilar J.F., Torres L. and Escobar Jimenez R.F., "A numerical solution for a variable order reaction diffusion model by usingfractional derivatives with nonlocal and nonsingular kernel", *Physica A: Statistical Mechanics and its Applications,* 491 (2017): 406–424.

[22] Liu Y., Du Y. and Li H., "A twogrid mixed finite element method for a nonlinear fourthorder reaction diffusion problem with time fractional derivative", *Computers and Mathematics with Application,* 70 (2015): 2474–2492.

[23] Alimbekova N. B., Baigereyev D. R. and Madiyarov M. N., "Study of a numerical method for solving a boundary value problem for a differential equation with a fractional time derivative", *Izvestiya of Altai State University,* 114, no. 4 (2020): 64–69.

[24] Alimbekova N. B., Berdyshev A. S. and Baigereyev D. R., "Study of initial boundary value problem for two dimensional differential equation with fractional time derivative in the sense of Caputo", *Third International Conference on Material Science, Smart Structures and Applications: (ICMSS 2020)* (2021): 1–6.

[25] Baigereyev D., Alimbekova N., Berdyshev A. and Madiyarov M., "Convergence analysis of a numerical method for a fractional model of fluid flow in fractured porous media", *Mathematics,* 9, no. 2179 (2021): 1-24.

[26] Caputo M., "Models of flux in porous media with memory", *Water Resources Research,* 36, no. 3 (2000): 693– 705.

[27] Agarwal R., Yadav M. P., Baleanu D. and Purohit S. D., "Existence and uniqueness of miscible flow equation through porous media with a non singular fractional derivative", *AIMS Mathematics,* 5, no. 2 (2020): 1062–1073.

[28] Akman T., Yildiz B. and Baleanu D., "New discretization of Caputo-Fabrizio derivative", *Computational and Applied Mathematics,* 37, no. 3 (2017): 3307-3333.