IRSTI 27.03.66 DOI: https://doi.org/10.26577/JMMCS2025126205

Aigerim Dauletiyarova
SDU University, Kaskelen, Kazakhstan
e-mail: d_aigera95@mail.ru

ON O-MINIMALITY FOR EXPANSIONS OF A DENSE MEET-TREE

This paper aims to define the notion of o-minimality for partially ordered sets. Originally, the notion of o-minimality was introduced for linearly ordered sets in the following way: A linearly ordered structure is said to be o-minimal if any definable subset is a finite union of intervals and points. For partially ordered sets, this definition does not work. One of the main reasons for this is that the complement of an interval need not be a finite union of intervals, as happens in linearly ordered sets. Here we suggest a notion of a generalized interval which makes possible defining o-minimality for such a partial case of partially ordered sets as a dense meet-tree in a classical way: an expansion of a dense meet-tree is said to be o-minimal if any definable subset is a finite union of generalized interval and points. We think that this approach allows us to transfer the machinery for investigating o-minimality for linearly ordered structures to partially ordered structures.
Key words: Ehrenfeucht's theory, small theory, linearly ordered set, partially ordered set, o-minimality.

Айгерім Дәулетиярова
СДУ университеті, Қаскелең, Қазақстан
e-mail: d_aigera95@mail.ru

Тығыз кездесу ағашының кеңейтілген құрылымдарындағы о-минималдылығы туралы

Бұл жұмыстың мақсаты жартылай реттелген жиындар үшін о-минималдылық түсінігін анықтау болып табылады. О-минималдылық түсінігі бастапқыда сызықты реттелген жиындар үшін келесідей енгізілген болатын: егер сызықты реттелген құрылымның әрбір формульді ішкі жиыны интервалдар мен нүктелердің ақырлы бірігуі болса, онда осы сызықты реттелген құрылым о-минималды деп аталады. Бұл анықтама жартылай реттелген жиындар үшін орындалмайды. Мұның басты себептерінің бірі интервалдың толықтауышы сызықты реттелген жиындардағы сияқты интервалдардың ақырлы бірігуі ретінде әрдайым бола бермейді. Мұнда біз классикалық жолмен жартылай реттелген жиындардың мысалы ретінде тығыз кездесу ағашы үшін о-минималдылығын анықтауға мүмкіндік беретін жалпыланған интервал түсінігін ұсынамыз: егер әрбір формульді ішкі жиын жалпыланған интервал мен нүктелердің ақырлы бірігуі болса, онда тығыз кездесу ағашының кеңеюі о-минималды деп аталады. Бұл тәсіл сызықты реттелген құрылымдар үшін о-минималдылықты зерттеу аппаратын жартылай реттелген құрылымдарға ауыстыруға мүмкіндік береді деп есептейміз.
Түйін сөздер: Эренфойхт теориясы, шағын теория, сызықтық реттелген жиын, жартылай реттелген жиын, о-минималдылық.

Айгерим Даулетиярова
Университет СДУ, Каскелен, Казахстан
e-mail: d_aigera95@mail.ru

Об о-минимальности для обогащений плотного дерева встреч

Целью данной статьи является определение понятия о-минимальности для частично упорядоченных множеств. Первоначально понятие о-минимальности было введено для линейно упорядоченных множеств следующим образом: линейно упорядоченная структура называется о-минимальной, если любое формульное подмножество является конечным объединением интервалов и точек. Для частично упорядоченных множеств это определение не работает. Одной из главных причин этого является то, что дополнение интервала не обязательно должно быть конечным объединением интервалов, как это происходит в линейно упорядоченных множествах.

Здесь мы предлагаем понятие обобщенного интервала, которое позволяет определить o-минимальность для такого частичного случая частично упорядоченных множеств как плотное дерево встреч классическим способом: обогащение плотного дерева встреч называется o-минимальным, если любое формульное подмножество является конечным объединением обобщенного интервала и точек. Мы считаем, что этот подход позволяет нам перенести аппарат исследования o-минимальности для линейно упорядоченных структур на частично упорядоченные структуры.
Ключевые слова: эренфойхтова теория, малая теория, линейно упорядоченное множество, частично упорядоченное множество, o-минимальность.

1 Introduction

This article aims to apply the notion of o-minimality to partially ordered structures. We start with the dense meet tree [2, 4] as a sufficiently tame partial order to examine our approach, where a dense meet-tree means a lower semilinear order $<$ in which each pair of elements a, b has a greatest common lower bound, their meet $a \sqcap b$ without the least and greatest elements such that:

(a) for each pair of incomparable elements, their join does not exist;

(b) for each pair of distinct comparable elements, there is an element between them;

(c) for each element a there exist infinitely many pairwise incomparable elements greater than a, whose infimum is equal to a.

The first paper on o-minimality for partially ordered sets was by Carlo Toffalori [6], who gave the following definition. A partially ordered structure is *o-minimal* if each its definable over some set X subset is a finite Boolean combination of sets defined by formulae $x \leq a$ or $x \geq b$, where these a and b are in the algebraic closure of X. As we know, all other notions of o-minimality and weak o-minimality of partially ordered structures are based on this definition, for instance, [3]. We suggest another approach, which was first discovered by S. Sudoplatov and V. Verbovskiy in [5] for weak o-minimality of partially ordered structures.

The standard notion of o-minimality for totally ordered structures is not convenient for partially ordered structures because of the following reasons. In a totally ordered set the complement of an interval is just a union of at most two intervals, while in partially ordered sets this is no more true. That is why Toffalori suggested using a Boolean combination of intervals in place of a finite union of intervals. Here we suggest another approach: we do not change "a finite union", but we change the notion of an interval, and we introduce the notion of a generalized interval. So, our definition of an o-minimal partially ordered set is the following.

Definition 1 *A partially ordered structure is said to be o-minimal if each of its definable subsets is a finite union of generalized intervals and points.*

In the rest of the paper, we discuss the notion of a generalized interval.

The aim of this paper is to find a way of extending the notion of o-minimality to partially ordered structure, because the notion of o-minimality and its generalizations, as o-stability [1, 8, 9] already proved its own fruitfulness. Perhaps, it will be complicated to extend the notion of stability in a direct way to partially ordered structures, but we can also use a more general notion of relative stability [7], where the scheme of creating different classes of theories was suggested.

2 Definable subsets of DMT

This section introduces the concept of generalized intervals, which extends the classical representation of intervals. This concept allows us to work with our structure, namely the dense meet-tree.

To begin with, we give a definition below.

Definition 2 (V. V. Verbovskiy) *Let $(M, <, \sqcap)$ be a model of the DMT theory. A subset of M is said to be a generalized interval if it is either an interval or is equal to*

$$\bigcup_{a \in A} (a, +\infty)$$

for some definable with parameters in the signature $\{<, \sqcap\}$ subset A of M on one of the following forms:

1. $A = (c, +\infty) \setminus ([a_1]_c \cup \cdots \cup a_{nc})$, *for positive integer n and some elements $a_1, \ldots, a_n \in (c, +\infty)$;*
2. $A = (c, a)$*;*
3. $A = \{(b, c) : b \in (-\infty, a),\ c \in (b, +\infty)\}$

For our reasoning, we need the following definition.

Definition 3 An element b is said to be a *partial infimum* of a set A if there exists a partition of A into sets A^+ and A^0 such that $A^+ \neq \emptyset$, $b = \inf A^+$ and $b \parallel a_0$ whenever $a_0 \in A^0$.

Observe that the elementary theory of a dense meet tree admits quantifier elimination, so in the above definition we can use just subsets which are definable by a quantifier-free formula with parameters.

Let $\mathcal{M} = (M; <, \sqcap)$ be a dense meet-tree. For every $c \in M$ it is possible to define an equivalence relation above c, that is, on the set $(c, +\infty)$:

$$a \sim_c b \Leftrightarrow a \sqcap b > c.$$

We call an equivalence class of this equivalence relation an *open cone above c*.

Lemma 1 *An equivalence class for the equivalence relation $\sim_c$ is expressible as written below:*

$$[a]_c = \bigcup_{d \in (c,a)} (d, +\infty).$$

Proof. Assume that $b \in [a]_c$. By definition, $a \sqcap b > c$. Since the order is dense, there exists $d_0 \in (c, a \sqcap b)$. Then $b \geq a \sqcap b > d_0$ and $b \in (d_0, +\infty)$. Hence, $[a]_c \subseteq \bigcup_{d \in (c,a)} (d, +\infty)$.

Let $b \in \bigcup_{d \in (c,a)} (d, +\infty)$. Then $b \in (d, +\infty)$ for some element $d \in (c, a)$. It means that $d < b$ and $d < a$, so $c < d \leq a \sqcap b$, that is, $a \sim_c b$. We proved the inverse inclusion and, thus, the equality of two sets. □

Note that for each class $[a]_c$ there exists its infimum.

The difference between a total ordering and a partial one is the existence of incomparable elements. So, we express the set of incomparable to a elements as a union of intervals.

Lemma 2 *For any element a the following holds:*

$$a \parallel x \Leftrightarrow x \in \bigcup_{b\in(-\infty,a)} \bigcup_{c\in(b,+\infty)\setminus[a]_b} (c,+\infty)$$

Proof. First, we show the necessary condition. Let $b = x \sqcap a$. It means that $x \notin [a]_b$, namely x is outside the class a over b. Then there is an element c between b and x such that $b < c < x$. Note that $c \notin [a]_b$. So, $x \in (c,+\infty)$ and this interval is one of the intervals of the union.

Now we show the sufficient condition. Let $x \in (c,+\infty)$ for some $c \in (b,+\infty) \setminus [a]_b$ where $b < a$. Assume the contrary, that x and a are comparable. If $a \leq x$, then a and c are comparable, because both are less than x. Then $a \sqcap c = \min(a,c) > b$. It means that $c \in [a]_b$, for a contradiction. Let $x < a$. Since $c < x$ we obtain that $c < a$ and $a \sqcap c = c > b$, for a contradiction. □

Note that the set of all incomparable elements to some element does not have infimum.

Thus, using the notations about the class of equivalence relations and incomparable elements, we can express the complements of the equivalence class above c.

Below we use the following notation. Let $\psi(x)$ be a formula. Then

$$(\psi(x))^{\mathcal{M}} = \{a \in M : \mathcal{M} \models \psi(a)\}.$$

Lemma 3 *For any elements a and c with $c < a$ the following holds:*

$$\overline{[a]}_c = (-\infty, c] \cup (x \parallel c)^{\mathcal{M}} \cup \bigcup_{d\in(c,+\infty)\setminus[a]_c} (d,+\infty).$$

In particular,

$$\overline{[a]}_c \cap (c,+\infty) = \bigcup_{d\in(c,+\infty)\setminus[a]_c} (d,+\infty).$$

In particular, the set $\overline{[a]}_c$ does not have infimum and $c = \overline{[a]}_c \cap (c,+\infty)$.

Proof. Let $b \notin \overline{[a]}_c$. Then we have the following possibilities:

$$\neg(b \sqcap a > c) \Leftrightarrow (b \sqcap a = c) \vee (b \sqcap a < c) \vee (b \sqcap a \parallel c)$$

We consider each disjunct separately:

1. If $b \sqcap a = c$ then $b \geq c$ and $b \notin [a]_c$. Now we write the set of all such b's as follows:

$$[c,+\infty) \setminus [a]_c = \{c\} \cup \bigcup_{d\in(c,+\infty)\setminus[a]_c} (d,+\infty)$$

Let $b \in (c,+\infty) \setminus [a]_c$. Since the order is dense, there exists $d \in (c,b)$. Note that $d \in [b]_c \neq [a]_c$, then $d \notin [a]_c$ and $b \in (d,+\infty)$.

Conversely, let

$$b \in \{c\} \cup \bigcup_{d\in(c,+\infty)\setminus[a]_c} (d,+\infty)$$

If $b = c$, then $b \in [c, +\infty] \setminus [a]_c$. Assume that $b \neq c$. We choose $d \in (c, +\infty) \setminus [a]_c$ so that $b \in (d, +\infty)$. Then $b > d > c$ implies $b > c$ and $b \in [d]_c \neq [a]_c$. That is why $b \notin [a]_c$.

2. Since $b \sqcap a < c$ and $a > c$, this implies $b \sqcap a = b \sqcap c$ and $b \sqcap c < c$, so $b < c$ or $b \parallel c$. The set of all b's such that $b < c$ is an interval $(-\infty, c)$. The set of all b's such that $b \parallel c$ is a generalized interval. So, the set of all b's such that $b \sqcap a < c$ is a union of an interval and a generalized interval.

3. Since $a > c$ and $a \geq b \sqcap a$, then c and $b \sqcap a$ are comparable, so $b \sqcap a \parallel c$ is inconsistent. $\square$

Now, we describe definable subsets of DMT. It is well-known that the theory of DMT admits quantifier elimination [4], so any formula in one free variable x is a Boolean combination of formulae of the following kinds:

(1) $t(x,\overline{u}) = t(x,\overline{v})$	(5) $t(x,\overline{u}) \neq t(x,\overline{v})$
(2) $t(x,\overline{u}) < t(x,\overline{v})$	(6) $t(x,\overline{u}) \not< t(x,\overline{v})$
(3) $t(x,\overline{u}) > t(x,\overline{v})$	(7) $t(x,\overline{u}) \not> t(x,\overline{v})$
(4) $t(x,\overline{u}) \parallel t(x,\overline{v})$	(8) $t(x,\overline{u}) \not\parallel t(x,\overline{v})$

The formulae with negation can be transformed by the following equivalences:

$$\begin{aligned}
t(x,\overline{u}) \neq t(x,\overline{v}) \quad &\Leftrightarrow \quad [t(x,\overline{u}) < t(x,\overline{v})] \vee [t(x,\overline{u}) > t(x,\overline{v})] \vee [t(x,\overline{u}) \parallel t(x,\overline{v})];\\
t(x,\overline{u}) \not< t(x,\overline{v}) \quad &\Leftrightarrow \quad [t(x,\overline{u}) > t(x,\overline{v})] \vee [t(x,\overline{u}) = t(x,\overline{v})] \vee [t(x,\overline{u}) \parallel t(x,\overline{v})];\\
t(x,\overline{u}) \not> t(x,\overline{v}) \quad &\Leftrightarrow \quad [t(x,\overline{u}) < t(x,\overline{v})] \vee [t(x,\overline{u}) = t(x,\overline{v})] \vee [t(x,\overline{u}) \parallel t(x,\overline{v})];\\
t(x,\overline{u}) \not\parallel t(x,\overline{v}) \quad &\Leftrightarrow \quad [t(x,\overline{u}) < t(x,\overline{v})] \vee [t(x,\overline{u}) > t(x,\overline{v})] \vee [t(x,\overline{u}) = t(x,\overline{v})].
\end{aligned}$$

So, we can assume that any formula is a disjunction of conjunctions of formulae of the kinds (1)–(4).

The operation $\sqcap$ is idempotent, commutative, and associative. That is why any term $t(x,\bar{u})$ is equal to $x \sqcap \tilde{t}(\bar{u})$ for some term $\tilde{t}$. So, we obtain the following types of atomic formulae:

(1.1) $x = v$

(1.2) $x \sqcap u = x$

(1.3) $x \sqcap u = v$

(1.4) $x \sqcap u = x \sqcap v$

(2.1) $x < v$

(2.2) $x \sqcap u < x$

(2.3) $x \sqcap u < v$

(2.4) $x \sqcap u < x \sqcap v$

(3.1) $x > v$

(3.2) $x \sqcap u > x$

(3.3) $x \sqcap u > v$

(3.4) $x \sqcap u > x \sqcap v$

(4.1) $x \parallel v$

(4.2) $x \sqcap u \parallel x$

(4.3) $x \sqcap u \parallel v$

(4.4) $x \sqcap u \parallel x \sqcap v$

We consider each case separately and show that each formula can be described as a point, an interval, or a generalized interval.

Cases (2.2), (3.2), (4.2), and (4.4) are false. For Case (1.1) we have a point, Cases (2.1) and (3.1) give the intervals. Also, Case (1.2) is equivalent to $x \leq u$, and Case (4.1) is the generalized interval.

Now we look at the remaining cases in more detail.

Case (1.3). We obtain the following

$$x \sqcap u = v \Leftrightarrow [u = v \wedge x \geq v] \vee [u > v \wedge x = v] \vee [u > v \wedge x > v \wedge \neg(x \sim_v u)]$$

Here, the first disjunct $u = v \wedge x \geq v$ defines an interval and the second disjunct defines a single point. From Lemma 3 it follows that the third disjunct define a generalized interval.

Case (2.3). We see that x belongs either to the interval or to the generalized interval

$$\begin{aligned} x \sqcap u < v \quad \Leftrightarrow \quad & [u < v] \vee [u \geq v \wedge (x < v \vee x \parallel v)] \vee \\ & \vee [u \parallel v \wedge (x \leq u \sqcap v \vee x \parallel u \sqcap v \vee (x > u \sqcap v \wedge \neg(x \sim_{u \sqcap v} u)))] \end{aligned}$$

Obviously, each disjunct defines an interval or a generalized interval.

Case (3.3). Since $u \leq v$ is impossible in this case, we can see that

$$x \sqcap u > v \quad \Leftrightarrow \quad [u > v \wedge x \sim_v u]$$

Case (4.3). This case is possible only under the condition $u \parallel v$:

$$x \sqcap u \parallel v \quad \Leftrightarrow \quad [u \parallel v \wedge x \sim_{u \sqcap v} u]$$

Case (1.4). Here we have two possibilities: $x \sqcap u < u \sqcap v$ and $x \sqcap u = u \sqcap v$. The first case is similar to *Case* (2.3). So, we consider $x \sqcap u = x \sqcap v = u \sqcap v$.

This case is written as follows

$$x \sqcap u = x \sqcap v = v \sqcap u \Leftrightarrow x = u \sqcap v \vee \\ \vee (x > u \sqcap v \wedge \neg(x \sim_{u \sqcap v} u) \wedge \neg(x \sim_{u \sqcap v} v))$$

Case (2.4). It is certain that $u \geq v$ is false, then we get the following

$$x \sqcap u < x \sqcap v \Leftrightarrow [u < v \wedge x \sim_u v] \vee [u \parallel v \wedge x \sim_{u \sqcap v} v]$$

Case (3.4). Similar to *Case* (2.4).

Theorem 1 *Notion of o-minimality for an expansion of the DMT theory is correct, that is, any Boolean combination of sets which are either a point or a generalized interval can be expressed as a finite union of points and generalized intervals.*

For any definable set there exist at most finitely many partials infima.

Proof. We prove both statements of this theorem by the simultaneous induction in the complexity of the construction of a Boolean combination.

Note that any set defined by an atomic formula has at most one infimum because it is either a point, or an interval, or a generalized interval. Because of quantifier elimination, it is sufficient to consider just an arbitrary Boolean combination of atomic formulas.

1. It is obvious that the union of generalized intervals and points is a finite union of generalized intervals and points. Obviously, any finite union of sets that have at most finitely many partial infima also has at most finitely many infima.

2. We consider the intersection of two finite unions of generalized intervals and points. Since

$$\bigcup_i A_i \cap \bigcup_j B_j = \bigcup_{i,j} (A_i \cap B_j)$$

it is sufficient to consider the intersection of generalized intervals and points.

The intersection of generalized intervals with a point either is empty or a point. The intersection of generalized intervals with an interval of the form (a, b), where $a \in M \cup \{-\infty\}, b \in M$ is a subset of a linearly ordered set (a, b), and in linearly ordered sets the intersection of intervals either is empty or an interval.

We consider an intersection of the form: $(a, +\infty) \cap (b, +\infty)$. If a and b are comparable, then this is $(\max(a, b), +\infty)$, otherwise it is empty. We can see that in these two cases the intersection of two sets has at most one infimum.

We consider the intersection of generalized intervals:

$$\left(\bigcup_{a \in A} (a, +\infty) \right) \cap \left(\bigcup_{b \in B} (b, +\infty) \right) = \bigcup_{a \in A} \bigcup_{b \in B} ((a, +\infty) \cap (b, +\infty)) = \bigcup_{c \in C} (c, +\infty)$$

where $C = \{c \in A \cup B \mid c = \max(a, b)$ for some $(a, b) \in A \times B$ such that a and b are comparable$\}$. Therefore, the intersection of two generalized intervals is itself a generalized interval.

Note that if both A and B have at most finitely many partial infima, then C also has at most finitely many partial infima, then $\bigcup_{c \in C} (c, +\infty)$ has at most finitely many infima.

We consider the intersection of a generalized interval with an interval of the form $(b, +\infty)$, but the last interval can be written as $\bigcup_{b\in\{b\}}(b, +\infty)$, thus we get into the previous case.

3. Now we consider the complements of the finite union of generalized intervals and points. $\overline{\bigcup_i A_i} = \bigcap_i \overline{A}_i$, therefore, it is sufficient to consider only the complement of a point, an interval, and a generalized interval.

We should also consider the complement of intervals since we know that the complement of intervals is a finite union of intervals, which is what the below states.

$$\begin{array}{rcl}\overline{(-\infty, a)} &=& [a, +\infty) \cup (x \parallel a)^{\mathcal{M}}\\ \overline{(a, +\infty)} &=& (-\infty, a] \cup (x \parallel a)^{\mathcal{M}}\\ \overline{(a, b)} &=& (-\infty, a] \cup [b, +\infty) \cup (x \parallel a)^{\mathcal{M}} \cup [(a, +\infty) \cup (b \parallel x)^{\mathcal{M}}]\end{array}$$

Note that the complements of a point a is $x < a \cup x > a \cup x \parallel a$.

As one can see, any of the above sets has at most one infimum.

So, we consider the complement of a generalized interval:

$$\overline{\bigcup_{a\in A}(a, +\infty)} = \bigcap_{a\in A} \overline{(a, +\infty)} = \bigcap_{a\in A}((-\infty, a] \cup (x \parallel a)^{\mathcal{M}}) =$$
$$= \bigcap_{a\in A}(-\infty, a] \cup \bigcap_{a\in A}(x \parallel a)^{\mathcal{M}})$$

Note that $(-\infty, a] \cap (-\infty, b] = (-\infty, a \sqcap b]$ for any a and b. Since A is definable in $(M, \leq, \sqcap)$ and $Th(\mathcal{M})$ admits quantifier elimination, there exists $c = \inf\{a \sqcap b : a, b \in A\}$ or $\bigcap_{a\in A}(-\infty, a] = \emptyset$. So, $\bigcap_{a\in A}(-\infty, a]$ being non-empty is equal to $(-\infty, c)$, provided that $c = \min\{a \sqcap b : a, b \in A\}$, and to $(-\infty, c)$ otherwise.

Now we consider $\bigcap_{a\in A}(x \parallel a)^{\mathcal{M}})$. Observe that

$$(x \parallel a)^{\mathcal{M}} \cap (x \parallel b)^{\mathcal{M}} = (x \parallel a)^{\mathcal{M}}$$

for any pair $a < b$. So, if A contains the least element, say, c, we obtain $\bigcap_{a\in A}(x \parallel a)^{\mathcal{M}} = (x \parallel c)^{\mathcal{M}}$. If A is not bounded below, we obtain $\bigcap_{a\in A}(x \parallel a)^{\mathcal{M}} = \emptyset$. So, we assume that A is bounded below. By induction hypothesis A has at most finitely many partial infima, say $c_1, \ldots, c_n$. Let $A_1, \ldots, A_n$ be a partition of A such that $c_i = \inf A_i$ for each i. As we have noticed, if $c_i \in A_i$, then $\bigcap_{a\in A_i}(x \parallel a)^{\mathcal{M}} = (x \parallel c_i)^{\mathcal{M}}$.

Assume that $c_i \notin A_i$. Then obviously, $\bigcap_{a\in A_i}(x \parallel a)^{\mathcal{M}} \supseteq (x \parallel c_i)^{\mathcal{M}}$. Indeed, if an element is comparable with some a, then it is comparable to c. Now we consider an element d that is comparable to c. If $d \leq c$, then by transitivity $d < a$ for each $a \in A_i$. So, we consider only those d, that $d > c$. We denote $D = \{d > c : d \notin A_i\}$.

By the quantifier elimination result it holds that either D is contained in finitely many $\sim_c$-classes or D contains cofinitely many $\sim_c$-classes. Also, at most finitely many $\sim_c$-classes intersect D but not subsets of D. Let D_1 consist of those $\sim_c$-classes, that are subsets of D and $D_2 = D \setminus D_1$. Then $\bigcap_{a\in A_i}(x \parallel a)^{\mathcal{M}} \supseteq (x \parallel c_i)^{\mathcal{M}} \cup D_1$.

Let $d' \in D_2$ and $d = \inf(-\infty, d') \cap D_2$. If $d > c_i$ then we obtain a similar situation as before, we consider $\sim_d$-classes. So, we assume that $d = c_i$. We obtain $(c_i, d'] \subseteq D_2$, this means that $(c_i, d'] \cap A_i = \emptyset$. By the definition of D_2, we have $[d']_{c_i} \not\subseteq D_2$.

In order to obtain A_i from $(c_i, +\infty)$ we remove finitely many subsets definable by a conjunction of atomic formulas. We can remove

1. an equivalence class $[u]_v$ for some $v \geq c_i$ and u,
2. an interval of the form $(v, +\infty)$ for some $v > c_i$,
3. an interval of the form (v, u) for some $u > v \geq c_i$,
4. the set of all elements that are not comparable with v for some $v > c_i$.

We have already described the way to deal with Case (1).

Assume we have removed an interval of the form $(v, +\infty)$ for some $v > c_i$. Since the order is dense, there exists $u \in (c_i, v)$. Then $(x \parallel t)^{\mathcal{M}} \supseteq (x \parallel u)^{\mathcal{M}}$ whenever $t > u$. Then

$$\bigcap_{a \in A_i} (x \parallel a)^{\mathcal{M}} = \bigcap_{a \in A_i \cup (v,\infty)} (x \parallel a)^{\mathcal{M}}$$

Assume that we have removed an interval of the form (v, u) for some $u > v \geq c_i$. If $v > c$ we proceed as above just replacing (v, ∞) with (v, u). So, we assume that $v = c_i$, that is, we have removed an interval of the form (c_i, u). Let b be the supreme of all $t > u$ such that $(c_i, t) \subseteq D_2$ and $(w, +\infty) \cap A_i \neq \emptyset$ for each $w \in (c_i, t)$. So, we have removed an interval of the form (c_i, b) or $(c_i, b]$ and this set is a maximal connected set that contains u, is a subset of D_2 and any element of D_2 that is comparable with some element of (c_i, b) (or $(c_i, b]$) then this element is comparable with all elements of (c_i, b) (or $(c_i, b]$). In this case we obtain

$$\bigcap_{a \in A_i} (x \parallel a)^{\mathcal{M}} \supseteq (x \parallel c_i)^{\mathcal{M}} \cup \big([b]_c \setminus [b, +\infty)\big)$$

or $\bigcap_{a \in A_i} (x \parallel a)^{\mathcal{M}} \supseteq (x \parallel c_i)^{\mathcal{M}} \cup \big([b]_c \setminus (b, +\infty)\big)$ depending which kind of an interval do we have: (c_i, b) or $(c_i, b]$.

Now we consider the last case: we have removed from $(c, +\infty)$ the set of all elements that are not comparable with v for some $v > c_i$. Note that

$$(c, +\infty) \setminus (x \parallel v)^{\mathcal{M}} = (c, v) \cup \{v\} \cup (v, +\infty)$$

Also we observe that

$$\bigcap_{a \in (c,v) \cup [v,+\infty)} (x \parallel a)^{\mathcal{M}} = \bigcap_{a \in (c,v)} (x \parallel a)^{\mathcal{M}} = (x \parallel c_i)^{\mathcal{M}} \cup \big((c, +\infty) \setminus [v]_c\big)$$

Since we can make only finitely many removals from $(c, +\infty)$, we end with finitely many steps describing $\bigcap_{a \in A_i} (x \parallel a)^{\mathcal{M}}$.

Also we can see that this operation cannot give a definable set with infinitely many partial infima. □

So, the next is clear.

Theorem 2 $(M, <, \sqcap)$ *is o-minimal, that is, any of its definable subsets is a finite union of generalized intervals and points.*

Acknowledgment

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23484665).

References

[1] B.S. Baizhanov, V.V. Verbovskii, O-stable theories. *Algebra and Logic*, **50(3)** (2011), 211–225.

[2] A.B. Dauletiyarova, V.V. Verbovskiy, On the number of countable models of constant and unary predicates expansions of the dense meet-tree theory. *Siberian Electronic Mathematical Reports*, **21(2)** (2024), 755–770.

[3] L. Newelski, R. Wencel, Definable sets in Boolean-ordered o-minimal structures. I. *The Journal of Symbolic Logic*, **66(4)** (2001), 1821–1836.

[4] M.G. Peretyat'kin, On complete theories with a finite number of denumerable models. *Algebra and Logic*, **12(5)** (1973), 310–326.

[5] S.V. Sudoplatov, V.V. Verbovskiy, On the concepts of convexity and weak o-minimality for partially ordered structures. *Herald of the Kazakh-British Technical University*, **21(1)** (2024), 75–84. *(in Russian)*

[6] C. Toffalori, Lattice ordered o-minimal structures. *Notre Dame Journal of Formal Logic*, **39(4)** (1998), 447–463.

[7] V.V. Verbovskiy, On definability of types and relative stability. *Mathematical Logic Quarterly*, **65** (2019), 332–346.

[8] V.V. Verbovskiy, A.B. Dauletiyarova, Piecewise monotonicity for unary functions in o-stable groups. *Algebra and Logic*, **60** (2021), 15–25.

[9] V.V. Verbovskiy, A.D. Yershigeshova, On non-essentiality of an o-stable expansion of $(\mathbb{Z}, <, +)$. *Lobachevskii Journal of Mathematics*, **44** (2023), 5485–5492.

Авторлар туралы мәлімет:

Дәулериярова Айгерім – СДУ университетінің докторанты, аға оқытушысы (Қаскелең, Қазақстан, электрондық пошта: d_aigera95@mail.ru).

Информация об авторах:

Даулериярова Айгерим – докторант, старший преподаватель университета СДУ (Каскелен, Казахстан, электронная почта: d_aigera95@mail.ru).

Information about authors:

Dauletiyarova Aigerim – PhD student, senior lecturer of SDU University (Kaskelen, Kazakhstan, e-mail: d_aigera95@mail.ru).

Received: February 13, 2025
Accepted: June 9, 2025