Бөлім 1 Раздел 1 Section 1

Математика Математика Mathematics

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IN-PLANE MOTION OF INCOMPRESSIBLE FLUID CONSIDERING AN INTERNAL SMALL BOUNDARY

We consider plane-radial motion, where the filtration velocity vectors are directed along the radii toward the borehole axis. As the borehole is approached, the pressure and filtration rate gradients sharply increase [1]. In this case, the radii change in geometric progression, while the pressure on the isobars changes in arithmetic progression [15]. At the same time, the mathematical model of such filtration flow movement can be described using Laplace's equation. For example, A.N. Chekalin [5] by way of a mathematical experiment based on solving the simplest problems, introduced a correction coefficient for the Laplace difference equation and studies the movement of filtration flow in the Cartesian coordinate system. A.A. Andreev [2], [4] experiments with the application of the Dirac delta function. Based on the theory of distributions functions, this paper proves the modeling of plane-radial motion in a Cartesian coordinate grid based on the Dirac delta function. An analysis of the results of the numerical and analytical solution of the problem is performed.
Key words: borehole, distributions, reservoir thickness filtration, numerical solution, finite difference method, analytical solution.

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Ішкі кіші элементті ескерегендегі сығылмайтын сұйықтың жазықтықтағы қозғалысы

Біз жазық-радиалды ағынды қарастырамыз, мұнда сүзу жылдамдығының векторлары радиустар бойымен ұңғыма осіне бағытталған. Ұңғыма оқпанына жақындаған сайын қысым мен сүзу жылдамдығының градиенттері күрт артады [1]. Бұл жағдайда радиустар геометриялық прогрессия түрінде өзгереді, ал изобаралардағы қысым арифметикалық прогрессия түрінде өзгереді [15]. Мұндай сүзу ағыны қозғалысының математикалық моделін Лаплас теңдеуін пайдаланып сипаттауға болады. Мысалы, А.Н. Чекалин [5] қарапайым есептерді шешуге негізделген математикалық экспериментті пайдаланып, Лаплас айырым теңдеуіне түзету коэффициентін енгізді және декарттық координаттар жүйесіндегі сүзу ағынының қозғалысын зерттеді. А.А. Андреев [2], [4] Дирак дельта функциясымен тәжірибе жасады. Бұл мақалада таралу функциялары теориясының негізінде Дирак дельта функциясын қолдана отырып, декарттық координаттар жүйесіндегі жазық-радиалды қозғалысты модельдеу дәлелденеді. Есептің сандық және аналитикалық шешімінің нәтижелеріне талдау жүргізіледі.
Түйін сөздер: ұңғыма, үлестірімдер, түзілу қалыңдығы бойынша сүзу, сандық шешім, ақырлы-айырымдық әдіс, аналитикалық шешім.

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Движение несжимаемой жидкости в плоскости с учетом внутреннего малого элемента

Мы рассматриваем плоско-радиальное движение, где векторы скорости фильтрации направлены вдоль радиусов к оси скважины. По мере приближения к скважине градиенты давления и скорости фильтрации резко возрастают [1]. В этом случае радиусы изменяются в геометрической прогрессии, а давление на изобарах в арифметической прогрессии [15]. При этом математическую модель такого движения фильтрационного потока можно описать с помощью уравнения Лапласа. Например, А.Н. Чекалин [5] посредством математического эксперимента, основанного на решении простейших задач, ввел поправочный коэффициент для разностного уравнения Лапласа и изучил движение фильтрационного потока в декартовой системе координат. А.А. Андреев [2], [4] экспериментирует с применением дельта-функции Дирака. На основе теории функций распределения в данной работе доказывается моделирование плоско-радиального движения в декартовой системе координат на основе дельта-функции Дирака. Проводится анализ результатов численного и аналитического решения задачи.
Ключевые слова: скважина, распределения, фильтрация по толщине пласта, численное решение, метод конечных разностей, аналитическое решение.

1 Introduction

Consider the filtration of an incompressible fluid in the Ω plane with boundary $\partial\Omega$. Let ω be a circle of radius $r_c \ll \operatorname{diam}\Omega$ with boundary $\partial\omega$. Let us define the problem of finding the harmonic function $p(x,y)$ in a two-connected region $\Omega_\varepsilon = \Omega/\partial\omega$.

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\sigma\frac{\partial p}{\partial y}\right)=0, \quad (x,y)\in\Omega_\varepsilon, \tag{1}$$

A constant pressure is maintained at the outer boundary of $\partial\Omega$.

$$p(x,y)=p_0, \quad (x,y)\in\partial\Omega, \tag{2}$$

At the small inner boundary, the pressure is set to $p_c < p_0$.

$$p(x_c,y_c)=p_{c,}, \quad (x_c,y_c)\in\partial\omega. \tag{3}$$

2 Analytical solution of the problem (1)–(3).

Substituting variables $x = r\cos\theta$, $y = r\sin\theta$ and considering $r = \sqrt{(x-x_c)^2+(y-y_c)^2}$, equation (1) to (3) write in the polar coordinate system ($\sigma = \text{const}$)

$$\frac{d^2p}{dr^2}+\frac{1}{r}\frac{dp}{dr}=0, \quad r_c<r<R, \tag{4}$$

$$p(r_c)=p_c, \quad p(R)=p_0. \tag{5}$$

The solution of the obtained problem (4), (5) will be

$$p(r)=p_c+\frac{p_0-p_c}{\ln\left(\dfrac{R}{r_c}\right)}\ln\left(\frac{r}{r_c}\right). \tag{6}$$

The obtained solution (6) allows us to analyze the numerical calculation of the solution of the problem (1)–(3). Here the volumetric flow rate of the borehole will be determined by Dupuis' formula

$$q = \frac{2\pi\sigma}{\ln\dfrac{R_k}{r_c}}(p_0 - p_c),$$

where $\sigma = \dfrac{kH}{\mu}$ is coefficient of hydraulic conductivity.

Let's construct a numerical solution of the problem (1)–(3) based on the distributions function using the Dirac delta-function [6], [7].

In the case of filtration of homogeneous liquid, the liquid flow rate is obtained by

$$q = \sigma \int_{\ell} \frac{\partial p}{\partial n} d\ell,$$

where ℓ-length of the perimeter of the borehole, $\sigma = \dfrac{kH}{\mu}$ is coefficient of hydraulic conductivity of isotropic reservoir.

Assuming that the borehole is circular of radius r_c, we have

$$q = \sigma r_c \int_0^{2\pi} \frac{dp}{dr} d\varphi,$$

Taking the mean value of $\dfrac{dp}{dr}$ out from under the integral sign, we obtain

$$q = 2\pi\sigma r_c \frac{dp}{dr}, \tag{7}$$

The problem is reduced to the calculation of fluid flow rate through the side surface of a circular cylinder. According to (7) for a production borehole we have

$$q = 2\pi\sigma r \frac{dp}{dr},$$

Separating and integrating the variables in the last expression we obtain:

$$p(r) = \frac{q}{2\pi\sigma}\ln r + C_1. \tag{8}$$

Let's construct the difference method for solving equation (1) on a square grid $x = ih,\ y = jh,\ i,j = \pm1, \pm2, \ldots;$ Then equation (1) on the difference cell $D_{i,j} = \left\{x_{i-\frac{1}{2}} \le x \le x_{i+\frac{1}{2}},\ y_{j-\frac{1}{2}} \le y \le y_{j+\frac{1}{2}}\right\}$ in the case of a five-point template with error $O(h^2)$ will be of the form, [8]

$$\sigma_{i+\frac{1}{2}}(p_{i+1} - p_i) + \sigma_{i-\frac{1}{2}}(p_{i-1} - p_i) + \sigma_{j+\frac{1}{2}}(p_{j+1} - p_j) + \sigma_{j-\frac{1}{2}}(p_{j-1} - p_j) = 0.$$

In cell $D(i_c, j_c)$, here there is a borehole, its outer boundary is formed by line segments $x = x_c \pm \frac{1}{2}(h + r_c)$, $y = y_c \pm \frac{1}{2}(h + r_c)$ Figure 1. Let us draw a circle with radius $h_c = x = y$. Here h_c is shown as a large arrow in Figure 1.

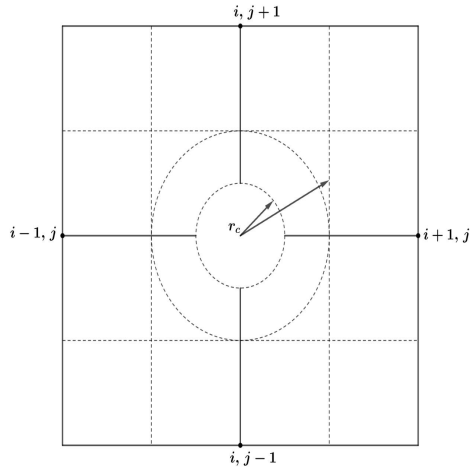

Figure 1. The small arrow corresponds to the radius of the borehole, and the large arrow shows the radius corresponding to that drawn on the square difference grid.

Similarly, calculating the fluid flow rate through the side surface of a circular cylinder with radius h_c, we obtain in the form (7)

$$q = 2\pi\sigma h_c \frac{dp}{dh_c}$$

Integrating, we have

$$p(h_c) = \frac{q}{2\pi\sigma} \ln h_c + C_2. \tag{9}$$

Definition 1 *A function $p(r)$ on a set $\omega \subset \Omega_\varepsilon$ is called a mapping $p(r) : \omega \to \Omega_\varepsilon$.*

Definition 2 *Let $c \subset \Omega_\varepsilon$ be such that for any $\delta > 0$ there exists an $r \in \omega$, such that $0 < |r - c| < \delta$. A function $p(h_c)$ is said to tend to $p(r_c)$ when r, tends to c, and writes $\lim\limits_{r\to c} p(h_c) = p(r_c)$, if*

$$\forall \varepsilon > 0 \ \exists \ \delta > 0 \ \forall \ 0 < |r - c| < \delta : \ |p(h_c) - p(r_c)| < \varepsilon.$$

The existence of the right $\lim\limits_{r\to c+0} p(h_c) = p(r_c)$ means that

$$\forall \varepsilon > 0 \ \exists \ \delta > 0 \ \forall \ 0 < r - c < \delta : \ |p(h_c) - p(r_c)| < \varepsilon.$$

The equality of $\lim\limits_{r\to +\infty} p(h_c) = p(r_c)$ means that

$$\forall \varepsilon > 0 \ \exists \ \delta > 0 \ \forall \ r > \delta : \ |p(h_c) - p(r_c)| < \varepsilon.$$

Similarly, we can formulate the definition of the left limit $\lim\limits_{r\to c-0} p(h_c) = p(r_c)$ and the $\lim\limits_{r\to -\infty} p(h_c) = p(r_c)$, [9].

If the locally integrable functions $p_\nu(h_c)$ $(\nu = 1, 2, \dots)$ in each bounded region converge uniformly to the locally integrable function $p(r)$, then the corresponding functionals $(p_\nu(h_c) -$

$p(r)$), converge to the sought-for functional, [10]. Indeed, let $\varphi(r)$ be some finite function, i.e., outside some region it is zero. In other words, for every finite function $\varphi(r)$ there exists a limit of a numerical sequence, [11]. Thus, a sequence of distributions functions $p_1, p_2, \ldots, p_\nu, \ldots$, by definition converges to a distributions function p, if for any basic function $\varphi(r)$

$$\lim_{r\to\infty}(p_\nu, \varphi) = (p, \varphi). \tag{10}$$

If the flow rate of incompressible fluid from a single point source is $q_\nu = q(r_c) = q(h_c)$, then from (8) and (9)we have the formula

$$p_\nu(h_c) - p(r) = \frac{q_\nu}{2\pi\sigma}\ln\frac{h_c}{r}.$$

Whence we have

$$q_\nu = \frac{2\pi\sigma}{\ln\dfrac{h_c}{r_c}}(p_\nu(h_c) - p(r)). \tag{11}$$

Now, if we compute the weak limit of the sequence of functions $q_\nu(r)$, $\nu \to 0$, for every fixed continuous function φ, there will be a functional which is the limit of the numerical sequence $\int q_\nu(r)\varphi(r)dr$ at $\nu \to 0$, [12]. Thus, for any continuous function $\varphi(r)$ the limit relation is true

$$\lim_{\nu\to 0}\int q_\nu(r)\varphi(r)dr \to (\delta, \varphi), \tag{12}$$

where the symbol (δ, φ) denotes the number $\varphi(0)$ is value of the functional δ on the function φ.

If there is a mass q, concentrated at the point (x_c, y_c) then the corresponding density should be assumed to be $q\delta(x-x_c, y-y_c)$. Thus, the density created by a material point cannot be described within the classical notion of function. Then a linear continuous functional in the form of a Dirac δ-function is attracted to describe it.

Let's write the difference approximation of the problem (1)–(3) at constant coefficient $\sigma = \text{const}$ taking into account (14) in the following form

$$p_{\bar{x}x} + p_{\bar{y}y} = \frac{2\pi}{\ln\dfrac{h_c}{r_c}}(p - p_c)\delta(x - x_c, y - y_c), \quad (x, y) \in \Omega_\varepsilon, \tag{13}$$

A constant pressure is maintained at the outer boundary of $\partial\Omega$:

$$p(x, y) = p_0, (x, y) \in \partial\Omega, \tag{14}$$

where p is a dummy function and p_c is a given function related to the selection of incompressible fluid.

Here the volume flow rate of the borehole is determined by the formula

$$q = \frac{2\pi\sigma}{\ln\frac{h_c}{r_c}}(p - p_c). \tag{15}$$

Then we write expression (13) taking into account (15) in the form [3]:

$$p_{\bar{x}x} + p_{\bar{y}y} = q\delta(x - x_c, y - y_c), \tag{16}$$

Problem (1)–(3) was solved with the following data. The downhole pressure $p_c = 100\,kgf/cm^2$, and the supply circuit pressure $p_0 = 130\,kgf/cm^2$, the distance to the supply circuit $R_k = 1000\,m$. Formation permeability coefficient $\kappa = 0.8\,D$, dynamic fluid viscosity coefficient $\mu = 4\,mPa \cdot s$, formation thickness $H = 10\,m$, borehole radius $r_c = 12.4\,cm$. Table 1shows the solution of the problem using formula (6). The approximate solution of the problem (13), (14) is constructed on a square grid with a step $h = 100\,m$. The results of the solution are shown in Table 2.

Table 1. Results of the solution to the problem (4), (5).

100.0	122.3	124.6	126.0	126.9	127.7	128.3	128.8	129.3	129.6	130.0
122.3	123.5	125.0	126.2	127.0	127.8	128.3	128.8	129.3	129.7	130.0
124.6	125.0	125.8	126.6	127.3	127.9	128.5	128.9	129.4	129.7	130.0
126.0	126.2	126.6	127.1	127.7	128.2	128.7	129.1	129.5	129.8	130.0
126.9	127.0	127.3	127.7	128.1	128.5	128.9	129.3	129.6	129.9	130.0
127.7	127.8	127.9	128.2	128.5	128.8	129.2	129.5	129.8	130.0	130.0
128.3	128.3	128.5	128.7	128.9	129.2	129.5	129.7	130.0	130.0	130.0
128.8	128.8	128.9	129.1	129.3	129.5	129.7	130.0	130.0	130.0	130.0
129.3	129.3	129.4	129.5	129.6	129.8	130.0	130.0	130.0	130.0	130.0
129.6	129.7	129.7	129.8	129.9	130.0	130.0	130.0	130.0	130.0	130.0
130.0	130.0	130.0	130.0	130.0	130.0	130.0	130.0	130.0	130.0	130.0

Numerical values of the analytical solution by formula (3).

Table 2. Results of the solution to the problem (13), (14).

100.0	122.3	124.6	125.9	126.9	127.6	128.2	128.7	129.1	129.6	130.0
122.3	123.7	125.1	126.1	127.0	127.6	128.2	128.7	129.2	129.6	130.0
124.6	125.1	125.8	126.5	127.2	127.8	128.3	128.8	129.2	129.6	130.0
125.9	126.1	126.5	127.1	127.6	128.0	128.5	128.9	129.3	129.6	130.0
126.9	127.0	127.2	127.6	127.9	128.3	128.7	129.0	129.4	129.7	130.0
127.6	127.6	127.8	128.0	128.3	128.6	128.9	129.2	129.5	129.7	130.0
128.2	128.2	128.3	128.5	128.7	128.9	129.1	129.3	129.6	129.8	130.0
128.7	128.7	128.8	128.9	129.0	129.2	129.3	129.5	129.7	129.8	130.0
129.1	129.2	129.2	129.3	129.4	129.5	129.6	129.7	129.8	129.9	130.0
129.6	129.6	129.6	129.6	129.7	129.7	129.8	129.8	129.9	129.9	130.0
130.0	130.0	130.0	130.0	130.0	130.0	130.0	130.0	130.0	130.0	130.0

Finite-difference solution obtained by Peaceman-Reckford method, [13], [14], [16]. Here the number of successive approximations was $S = 821$.

Conclusion

Based on the analysis results, calculations were performed to determine the volumetric flow rate using the Dupuis formula (3), [14] and a numerical solution using the Dirac delta function (15). Based on the results of the analytical solution, we obtained a flow rate of $q = 0.4132E+02\,t/day$. Based on the results of the analytical solution, we obtained a flow rate of $q = 0.4132E+02\,t/day$. If we take the difference grid step $h_c = h$ as an approximation, we obtain an increased flow rate value $q_\nu = 0.4427E + 02\, t/day$, [13]. A numerical experiment makes it possible to determine that the best approximation is obtained with an incomplete difference grid step $h_c = 0.3 \cdot h$.

The results of the numerical experiment show that the sequence $\{q_\nu(r)\}$ of approximation results converges to the obtained analytical solution of the problem.In this case, the sequence $q_\nu(r)$ of continuous functions converges to the generalized function $\varphi(r)$ if and only if it is a fundamental sequence of the generalized function $\varphi(r)$.

Authors contribution

A.K.: methodology; writing—original draft preparation; supervision; project administration. K.I.: writing—review and editing; supervision. All authors have read and agreed to the published version of the manuscript.

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