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SOME BOUNDARY VALUE PROBLEMS FOR THE INHOMOGENEOUS CAUCHY-RIEMANN EQUATION IN INTERSECTION OF TWO CIRCLES WITH SAME RADIUS

This article examines boundary value problems for the Cauchy–Riemann equations in the intersection region of two circles of equal radius. Using the parquet reflection method and the Cauchy–Pompeiu integral formula, a Schwarz representation formula is derived. The parquet reflection method is applied to solving a number of boundary value problems for specific domains whose boundaries consist of circular arcs. This contributes to the method for solving the Schwarz boundary value problem for the nonhomogeneous Cauchy-Riemann equation and also plays an important role in solving other boundary value problems. If the boundary of a domain has the specified properties, a new domain can be created in the plane by reflecting the original domain about its boundary, and by reflecting this new domain about its boundary, another domain is found. By repeating this process, we obtain fragments of the plane, the union of which provides a covering for the complex plane. The covering can be achieved with a single reflection, several successive reflections, or infinitely repeated reflections. Furthermore, the Schwarz problem for the inhomogeneous Cauchy–Riemann equation is studied; explicit solution formulas are obtained, and solvability conditions for the boundary value problem under consideration are established.
Key words: inhomogeneous Cauchy-Rimann equation, Cauchy-Pompeiu formula, Schwarz boundary value problem, integral representation of the solution, parquet reflection method.

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Бірдей радиусты екі шеңбердің қиылысуындағы біртекті емес Коши-Риман теңдеуі үшін кейбір шеттік есептер

Бұл мақалада біз тең радиусты екі шеңбердің қиылысуындағы Коши-Риман теңдеулері үшін шекаралық мән есептерін зерттейміз. Паркеттік шағылысу техникасын және Коши-Помпей интегралдық формуласынын қолдана отырып, Шварц өрнектеу формуласы алынды. Паркеттік шағылысу әдісі шекаралары шеңбер доғаларынан тұратын белгілі бір облыстар үшін бірқатар шеттік есептерді шешуде қолданылады. Бұл біртекті емес Коши-Риман теңдеуі үшін Шварцтың шеттік есебін шешу әдісіне өз үлесін қосады, сондай-ақ басқа да шеттік есептерді шешуде маңызды рөл атқарады. Егер облыс шекарасы аталған қасиеттерге ие болса, бастапқы облысты оның шекарасына қатысты шағылдыру арқылы жазықтықтың жаңа облысы құрылуы мүмкін, ал осы жаңа облысты оның шекарасына қатысты шағылдыру арқылы басқа облыс табылады. Осы процесті қайталай отырып, біз бірігуі комплексті жазықтықты толық жабуды қамтамасыз ететін жазықтық фрагменттеріне қол жеткіземіз. Жабу бір шағылумен, бірнеше жүйелі шағылумен немесе шексіз қайталанатын шағылулармен жүзеге асырылуы мүмкін. Сонымен қатар, біз біртекті емес Коши-Риман теңдеуі үшін Шварц шеттік мән есебін қарастырамыз және шешім мен шешілу шарттары үшін айқын өрнектерді аламыз.
Түйін сөздер: біртекті емес Коши-Риман теңдеуі, Коши-Помпей интегралдық формулалары, Шварц шекаралық есебі, шешімдердің интегралдық өрнегі, паркеттік шағылыстыру әдісі.

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Некоторые краевые задачи для неоднородного уравнения Коши-Римана на пересечении двух окружностей с одинакового радиуса

В этой работе исследуются краевые задачи для уравнений Коши–Римана в области пересечения двух окружностей равного радиуса. С применением метода паркетного отражения и интегральной формулы Коши–Помпею выведена формула представления Шварца. Метод паркетного отражения применяется для решения ряда краевых задач для конкретных областей, границы которых состоят из дуг окружностей. Это вносит свой вклад в метод решения краевой задачи Шварца для неоднородного уравнения Коши-Римана, а также играет важную роль при решении других краевых задач. Если граница области обладает указанными свойствами, может быть создана новая область плоскости путем отражения исходной области относительно ее границы, и путем отражения этой новой области относительно ее границы будет найдена другая область. Повторяя этот процесс, мы достигаем фрагментов плоскости, объединение которых обеспечивает покрытие для комплексной плоскости. Покрытие может быть выполнено с одним отражением, несколькими последовательными отражениями или бесконечно повторяющимися отражениями. Наряду с этим изучается задача Шварца для неоднородного уравнения Коши–Римана; получены явные формулы решения и установлены условия разрешимости рассматриваемой краевой задачи.
Ключевые слова: неоднородное уравнение Коши-Римана, формулы представления Коши-Помпейю, краевая задача Шварца, интегральное представление решений, метод паркетного отражения.

1 Introduction

Complex analysis is a rapidly developing field of modern mathematics. The study of boundary value problems is particularly significant, as they are of interest not only from a theoretical perspective but also have wide practical applications in various fields. The explicit solutions are important not only for applications in engineering, mathematical physics, fluid dynamics etc., but also influence the general theory for arbitrary domains.

In recent years, boundary value problems have attracted sustained interest from the scientific community. Significant results have been obtained in the theory of boundary value problems for complex partial differential equations, which has contributed to the further development of research in the fields of generalized analytic functions, Riemann–Hilbert problems, mathematical physics, and other related disciplines. Among these works, research devoted to the Riemann problem in the space of p-summable functions with infinite index is noteworthy, a Riemann-Hilbert BVP in a bounded sector, smooth solution to the second initial-boundary value problem for a parabolic system in a domain with nonsmooth lateral boundary, BVPs (Schwarz, Dirichlet and Neumann) for the Cauchy-Riemann equation in the dics [1], harmonic Green's functions are studied for a plane domain whose boundary is formed by two tangent circles [2], harmonic BVPs in half disc and half ring [3], operator estimates for planar domains with irregularly curved boundary the Dirichlet and Neumann conditions [4], Schwarz BVPs for polyanalytic equation in a sector ring [5], BVP for stationary stokes equations with impermeability boundary condition [6], BVP of holomorphic vector functions in 1D QCs [7], a third boundary value problem for the fractional diffusion equation, posed in a half-strip domain, is considered [8], the Riemann–Hilbert problem for first-order elliptic systems on a plane with constant coefficients of the highest derivatives is studied [9], Dirichlet problem for an elliptic equation with three singular coefficients [10], Schwarz BVP for the equation in Cauchy-Riemann a rectangle, the Riemann–Hilbert problem for first-order elliptic systems on a plane with constant coefficients of the highest derivatives is studied. In works [12, 13] Green's functions for the Dirichlet problem are obtained for triharmonic and polyharmonic equations in a multidimensional unit sphere; their correct restrictions associated with polyharmonic operators are also established.

In [14], well-posed problems for the Moisil-Teodorescu system are presented for the case of a spherical layer and inside a torus. In [15], two theorems on a priori estimates for solutions to nonlinear equations in a finite-dimensional space are obtained.

In [16, 17], a uniqueness criterion is obtained for the solution of time-nonlocal problems for the differential-operator equation $l(\cdot) - A$ with a wave operator A with a shift. In [18], classes of unique solvability are established for a time-nonlocal problem for multidimensional quasi-hyperbolic equations.

The parquet reflection method is applied in the study and solution of a number of boundary value problems in special domains whose boundaries consist of circular arcs. An approach to solving the Schwarz problem for the homogeneous Cauchy–Riemann equation is developed and also, play important roles in dealing with Dirichlet and Neumann BVPs, in other words, When solving boundary value problems for homogeneous Cauchy–Riemann equations, the parquet reflection method, which underlies the construction of the Schwarz kernel, is used. If the boundary of a domain satisfies the appropriate conditions, then reflecting the original domain with respect to its boundary yields a new planar domain. Further repetition of this procedure leads to the construction of a sequence of domains, the union of which yields a covering of the complex plane. Such a covering can arise as a result of a single reflection, several successive reflections, or an infinitely continuing process of reflections. [2]. In many areas of the theory of partial differential equations and mathematical physics, obtaining explicit integral forms for solutions to boundary value problems plays an important role. The Schwarz problem is one of the key problems in complex analysis and serves as a typical example of a boundary value problem for elliptic equations and systems. Its essence lies in determining a function satisfying a given differential equation within a domain, given known values on the boundary. Similarly, using reflected points, a Dirichlet representation formula is derived for the domain under consideration, and a solution to the Dirichlet problem for the Cauchy-Riemann equation is constructed. See [1].

The principle of parqueting and reflection is suitable for constructing harmonic Green's functions for certain planar domains with boundaries consisting of subarcs of circles and straight lines. (Continued) Reflection on the boundary parts should ensure parqueting of the complex plane. In the paper [19], the situation with an apple from Almaty, modified by a half-disk, is considered. In this work, harmonic Green and Neumann functions are constructed for these regions.

2 The complex Gauss theorems

Complex analysis makes extensive use of the complex differentiation operators ∂_z and $\partial_{\bar{z}}$ which are expressed in terms of real partial derivative operators ∂_x and ∂_y as

$$2\partial_{\bar z} = \partial_x + i\partial_y, \quad 2\partial_z = \partial_x - i\partial_y. \tag{1}$$

Formally, these relations are derived by considering the variables

$$\bar z = x - iy, \quad z = x + iy, \quad x, y \in \mathbb{R},$$

as independent, followed by the application of the rule for differentiating a complex function. Let a complex-valued function $w = u + iv$ be defined by two real-valued functions u and v depending on real variables x and y. For brevity, we will denote such a function by $w(z)$, although in general it depends not only on $\bar z$. If, in some open region of the complex plane $\mathbb{C}$, the function w is independent of $\bar z$, then it is analytic. In this case, the function w satisfies the first-order Cauchy–Riemann system of equations

$$u_y = -v_x, \quad u_x = v_y. \tag{2}$$

This condition is equivalent to the relation

$$\partial_{\bar z} w = 0, \tag{2$'$}$$

which follows directly from the equality

$$2\partial_{\bar z} w = (u + iv)(\partial_x + i\partial_y) = \partial_x u - \partial_y v + i(\partial_x v + \partial_y u). \tag{3}$$

Furthermore, in this case, the equality holds

$$2\partial_z w = (u + iv)(\partial_x - i\partial_y) = \partial_x u + \partial_y v + i(\partial_x v - \partial_y u) = 2\partial_x w = -2i\partial_y w = 2w'. \tag{4}$$

Using the complex derivatives introduced above, the classical real Gauss divergence theorem for functions of two real variables continuously differentiable in some regular domain can be naturally rewritten in complex form. Here, a regular domain is understood to be a bounded domain D with a smooth boundary ∂D, and the functions under consideration are assumed to be continuous in the closure $\overline{D} = D \cup \partial D$.

Theorem 1 *Let the function $w \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$ be defined in a regular domain D of the complex plane $\mathbb{C}$*

$$\int_D w_z(z)\, dx\, dy = -\frac{1}{2i} \int_{\partial D} w(z)\, d\bar z \tag{5}$$

and

$$\int_D w_{\bar z}(z)\, dx\, dy = \frac{1}{2i} \int_{\partial D} w(z)\, dz. \tag{5$'$}$$

Remark 1 Cauchy's theorem for analytic functions is a special case of formula (5′), namely:

$$\int_\gamma w(z)\, dz = 0.$$

Indeed, if γ is a simple closed smooth curve, and D denotes the interior region bounded by it, then the indicated integral is equal to zero, since relation (2′) holds in this case.

3 Cauchy-Pompeiu representation formulas

Similar to the derivation of the Cauchy formula from the Cauchy theorem using relations (5) and (5′), formulas for the Cauchy–Pompeiu representation can be established. Let $D \subset \mathbb{C}$ — be a regular domain and the function $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Denoting $\zeta = \xi + i\eta$ for $z \in D$ we have

$$w(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{w(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D \frac{w_{\bar\zeta}(\zeta)\,d\xi\,d\eta}{\zeta - z}. \tag{6}$$

and

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D}\frac{w(\zeta)\,d\bar\zeta}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D \frac{w_{\zeta}(\zeta)\,d\xi\,d\eta}{\overline{\zeta - z}}. \tag{6′}$$

hold. Let M — be a domain in the complex plane. It formed by the intersection of two circles: $C_1 = \{z \in \mathbb{C} : |z-1| = \sqrt{2}\}$ and $C_2 = \{z \in \mathbb{C} : |z+1| = \sqrt{2}\}$. The boundary ∂M of the domain M consist of two cicular arcs.

Domain M (consist of two cicular arcs)

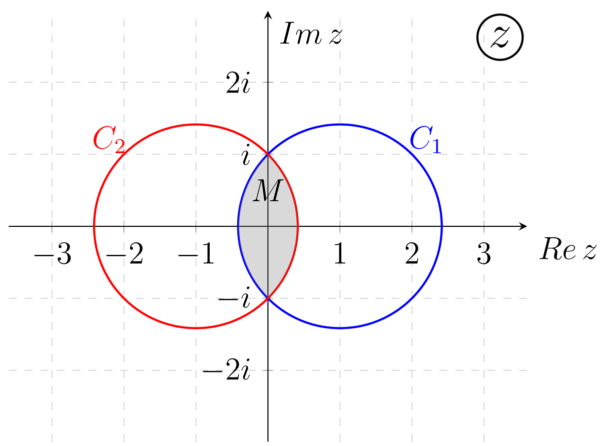

Figure 1: domain M

Recall that the Pompeiu operator is defined by the relation:

$$Tf(z) = -\frac{1}{\pi}\int_M \frac{f(\zeta)}{\zeta - z}d\xi d\eta,$$

where $\zeta = \xi + i\eta$. The extended complex plane is the complex plane to which the point at infinity is attached

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

The Cauchy-Pompeiu formula is the fundamental tool for complex BVPs which just has to be properly modified. Now we state the Cauchy-Pompeiu theorem.

Theorem 2 *Any $\omega \in C^1(M,\mathbb{C}) \cap C(\overline{M},\mathbb{C})$ for a regular complex domain $M \subset \mathbb{C}$ can be represented as*

$$\frac{1}{2\pi i}\int_{\partial M}\omega(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_M \omega_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - z} = \begin{cases}\omega(z), & z \in M,\\ 0, & z \notin \overline{M},\end{cases}$$

and

$$-\frac{1}{2\pi i}\int_{\partial M}\omega(\zeta)\frac{d\bar\zeta}{\overline{\zeta - z}} - \frac{1}{\pi}\int_M \omega_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}} = \begin{cases}\omega(z), & z \in M,\\ 0, & z \notin \overline{M},\end{cases}$$

where $\zeta = \xi + i\eta$.

The Poisson kernel for the unit disc D has the property

$$\lim_{z\to\zeta,|z|<1}\frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)\left[\frac{\zeta}{\zeta - z} + \frac{\bar\zeta}{\bar\zeta - \bar z} - 1\right]\frac{d\zeta}{\zeta} = \gamma(\zeta),$$

for $\gamma \in C(\partial D,\mathbb{R})$. See [1].

In this paper, explicit representations of solutions are obtained and solvability conditions are established for some basic boundary value problems in domain M. To achieve this goal, a Schwarz representation formula is first derived from a modified Cauchy–Pompeiu formula. This modification is implemented by applying the Cauchy–Pompeiu formula to points constructed using the parquet reflection method. Next, using the Schwarz representation formula, a solution to the Schwarz boundary value problem for the homogeneous Cauchy–Riemann equation is found.

4 The representation formula for M

In this section, using the parquet reflection method, we determine the points in the complex plane that belong to the region under consideration and are necessary for constructing the Schwarz representation formula.

The reflection of any point $z \in M$ with respect to the boundary ∂C_1 is defined as follows

$$|z-1| = \sqrt{2} \Rightarrow (\bar z - 1)(z-1) = 2 \Rightarrow z = \frac{\bar z + 1}{\bar z - 1}.$$

Similarly, the reflections of these two points with respect to ∂C_2 are expressed by the formulas

$$-\frac{1}{z}, \frac{-\bar z + 1}{\bar z + 1}.$$

Thus, successive reflections lead to a parquet covering of the entire complex plane. On this basis, the Schwarz representation formula is derived by combining the Cauchy-Pompeiu integral representation formula applied to the above points.

Theorem 3 *Any function $\omega \in C^1(M;\mathbb{C}) \cap C(\overline{M};\mathbb{C})$ can be represented as*

$$\begin{aligned}\omega(z) &= \frac{1}{2\pi i}\int_{\partial M\cap\partial C_1}\operatorname{Re}\omega(\zeta)\left[\frac{2(\zeta-1)}{\zeta-z}-1+\frac{2z(\zeta-1)}{\zeta z+1}-1\right]d\zeta\\ &+\frac{1}{2\pi i}\int_{\partial M\cap\partial C_2}\operatorname{Re}\omega(\zeta)\left[\frac{2(\zeta+1)}{\zeta-z}-1+\frac{2z(\zeta+1)}{\zeta z+1}-1\right]d\zeta\\ &+\frac{2}{\pi}\int_{\partial M\cap\partial C_1}\operatorname{Im}\omega(\zeta)\frac{d\zeta}{\zeta-1}+\frac{2}{\pi}\int_{\partial M\cap\partial C_2}\operatorname{Im}\omega(\zeta)\frac{d\zeta}{\zeta+1}\\ &-\frac{1}{\pi}\int_M\left\{\omega_{\bar\zeta}(\zeta)\left[\frac{1}{\zeta-z}+\frac{z}{\zeta z+1}\right]\right.\\ &\left.+\overline{\omega_{\bar\zeta}(\zeta)}\left[\frac{\bar z-1}{\zeta(\bar z-1)-\bar z-1}-\frac{\bar z+1}{\zeta(\bar z+1)+\bar z-1}\right]\right\}d\xi\,d\eta,\end{aligned}\tag{7}$$

where $\zeta=\xi+i\eta$.

Proof 1 *The points constructed using the parquet reflection method are written as follows* $z\in M$ *and* $\frac{\bar z+1}{\bar z+1}, -\frac{1}{z}, \frac{-\bar z+1}{\bar z+1}\notin\overline{M}$. *So by the Cauchy-Pompeiu formula*

$$w(z)=\frac{1}{2\pi i}\int_{\partial M}\omega(\zeta)\frac{d\zeta}{\zeta-z}-\frac{1}{\pi}\int_M\omega_{\bar\zeta}(\zeta)\frac{d\xi d\eta}{\zeta-z}.\tag{8}$$

$$0=\frac{1}{2\pi i}\int_{\partial M}\omega(\zeta)\frac{(\bar z-1)d\zeta}{\zeta(\bar z-1)-\bar z-1}-\frac{1}{\pi}\int_M\omega_{\bar\zeta}(\zeta)\frac{(\bar z-1)d\xi d\eta}{\zeta(\bar z-1)-\bar z-1}.\tag{9}$$

$$0=\frac{1}{2\pi i}\int_{\partial M}\omega(\zeta)\frac{zd\zeta}{\zeta z+1}-\frac{1}{\pi}\int_M\omega_{\bar\zeta}(\zeta)\frac{zd\xi d\eta}{\zeta z+1}.\tag{10}$$

$$0=\frac{1}{2\pi i}\int_{\partial M}\omega(\zeta)\frac{(\bar z+1)d\zeta}{\zeta(\bar z+1)+\bar z-1}-\frac{1}{\pi}\int_M\omega_{\bar\zeta}(\zeta)\frac{(\bar z+1)d\xi d\eta}{\zeta(\bar z+1)+\bar z-1}.\tag{11}$$

By adding (8), (10) *and the complex conjugates of* (9), (11), *the formula* (7) *is obtained.*

Theorem 3 makes it possible to construct a solution to the corresponding boundary value problem for the homogeneous Cauchy–Riemann equation in the domain M.

5 Schwarz BVP for M

We introduce the Schwarz integral on M as follows

$$\begin{aligned}S_M\gamma(z) &:= \frac{1}{2\pi i}\int_{\partial M\cap\partial C_1}\gamma(\zeta)\left[\frac{\zeta-1}{\zeta-z}+\frac{\bar\zeta-1}{\bar\zeta-\bar z}+1+\frac{z(\zeta-1)}{\zeta z+1}+\frac{\bar z(\bar\zeta-1)}{\bar\zeta\bar z+1}+1\right]\frac{d\zeta}{\zeta-1}\\ &+\frac{1}{2\pi i}\int_{\partial M\cap\partial C_2}\gamma(\zeta)\left[\frac{\zeta+1}{\zeta-z}+\frac{\bar\zeta+1}{\bar\zeta-\bar z}+1+\frac{z(\zeta+1)}{\zeta z+1}+\frac{\bar z(\bar\zeta+1)}{\bar\zeta\bar z+1}+1\right]\frac{d\zeta}{\zeta+1},\end{aligned}\tag{12}$$

where $\zeta = \xi + i\eta$.

Based on the formulas of representation (7), the solvability condition and the solution of the Schwarz-type boundary value problem for the homogeneous Cauchy–Riemann equation in the domain M are derived. To study the boundary behavior, we formulate the following theorem.

Theorem 4 *Let $\gamma \in C(\partial M, \mathbb{C})$, then for $t \in \partial M$*

$$\lim_{z\to t} S_M\gamma(z) = \gamma(\zeta).$$

Proof 2 *We consider the boundary behavior, for $z \in \partial M \cap \partial C_1$:*

$$\frac{\bar{z}(\bar{\zeta}-1)}{\bar{\zeta}\bar{z}+1} = \frac{z+1}{\zeta z+1}, \quad \zeta \in \partial C_1,$$

and

$$\frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} = \frac{-z+1}{\zeta z+1}, \quad \zeta \in \partial C_2,$$

$$\frac{\bar{z}(\bar{\zeta}+1)}{\zeta\bar{z}+1} = \frac{-z-1}{\zeta - z}, \quad \zeta \in \partial C_2.$$

Thus, on $\partial M \cap \partial C_1$:

$$\begin{aligned} S_M\gamma(z) &= \frac{1}{2\pi i}\int_{\partial M\cap\partial C_1} \gamma(\zeta)\left[\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1\right]\frac{d\zeta}{\zeta-1} \\ &= \frac{1}{2\pi i}\int_{\partial C_1} \Gamma(\zeta)\left[\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1\right]\frac{d\zeta}{\zeta-1}, \end{aligned}$$

where

$$\Gamma(\zeta) = \begin{cases} \gamma(\zeta), & \zeta \in \partial M \cap \partial C_1, \\ 0, & \zeta \in \partial C_1 \setminus (\partial M \cap \partial C_1). \end{cases}$$

By considering the properties of the Poisson kernel for C_1:

$$\lim_{z\to t} S_M\gamma(z) = \gamma(t).$$

Consequently, for $\zeta \in \partial S \cap \partial C_1$ up to the points $\pm i$ of the domain M, the corresponding relation holds; however, at these points, continuity is not preserved unless Γ vanishes there. Similarly, the relation for $z \in \partial M \cap \partial C_2$ is derived:

$$\frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} = \frac{z+1}{\zeta z+1}, \quad \zeta \in \partial C_1,$$

$$\frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} = \frac{-z+1}{\zeta - z}, \quad \zeta \in \partial C_1,$$

and

$$\frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} = \frac{-z+1}{\zeta z+1}, \quad \zeta \in \partial C_2.$$

Thus, on $\partial M \cap \partial C_1$*:*

$$\begin{aligned} S_M\gamma(z) &= \frac{1}{2\pi i}\int_{\partial M \cap \partial C_2} \gamma(\zeta)\left[\frac{\zeta+1}{\zeta-z} + \frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} - 1\right]\frac{d\zeta}{\zeta+1} \\ &= \frac{1}{2\pi i}\int_{\partial C_2} \Upsilon(\zeta)\left[\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1\right]\frac{d\zeta}{\zeta-1}, \end{aligned}$$

where

$$\Upsilon(\zeta) = \begin{cases} \gamma(\zeta), & \zeta \in \partial M \cap \partial C_2, \\ 0, & \zeta \in \partial C_2 \setminus (\partial M \cap \partial C_2). \end{cases}$$

Using the properties of the Poisson kernel for C_2*, we obtain the limit relation:*

$$\lim_{z \to t} S_M\gamma(z) = \gamma(t).$$

Considering this relation on $\zeta \in \partial M \cap \partial C_2$ *up to the corner points* $\pm i$ *of the domain* M*, we note that at these points, continuity, generally speaking, is broken unless the function* Υ *takes zero values there.*

To analyze the boundary behavior at points $\pm i$*, we use the representation of the constant function* 1 *in integral form*

$$\begin{aligned} 1 = &\frac{1}{2\pi i}\int_{\partial M \cap \partial C_1}\left[\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta-1)}{\zeta z+1} + \frac{\bar{z}(\bar{\zeta}-1)}{\bar{\zeta}\bar{z}+1} - 1\right]\frac{d\zeta}{\zeta-1} \\ &+ \frac{1}{2\pi i}\int_{\partial M \cap \partial C_2}\left[\frac{\zeta+1}{\zeta-z} + \frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} - 1 + \frac{z(\zeta+1)}{\zeta z+1} + \frac{\bar{z}(\bar{\zeta}+1)}{\bar{\zeta}\bar{z}+1} - 1\right]\frac{d\zeta}{\zeta+1}. \end{aligned}$$

Multiplying this equality by $\gamma(\pm i)$ *and subtracting it from* $S_M\gamma(z)$ *we obtain the expression for the case* $z \in \partial M \cap \partial C_1$*:*

$$S_M\gamma(z) - \gamma(\pm i) = \frac{1}{2\pi i}\int_{\partial M \cap \partial C_1} \tilde{\gamma}(\zeta)\left[\frac{\zeta-1}{\zeta-z} + \frac{\bar{\zeta}-1}{\bar{\zeta}-\bar{z}} - 1\right]\frac{d\zeta}{\zeta-1},$$

where $\tilde{\gamma}(\zeta) = \gamma(\zeta) - \gamma(\pm i)$ *and* $\tilde{\gamma}(\pm i) = 0$*,*

$$\lim_{z \to \pm i} S_M\gamma(z) = \gamma(t).$$

Similarly, for $z \in \partial M \cap \partial C_2$*,*

$$S_M\gamma(z) - \gamma(\pm i) = \frac{1}{2\pi i}\int_{\partial M \cap \partial C_2} \tilde{\gamma}(\zeta)\left[\frac{\zeta+1}{\zeta-z} + \frac{\bar{\zeta}+1}{\bar{\zeta}-\bar{z}} - 1\right]\frac{d\zeta}{\zeta+1},$$

where $\tilde{\gamma}(\zeta) = \gamma(\zeta) - \gamma(\pm i)$ *and* $\tilde{\gamma}(\pm i) = 0$*, hence*

$$\lim_{z \to \pm i} S_M\gamma(z) = \gamma(t).$$

In the next step, we consider the following Schwarz BVP and find a solution of ω the homogeneous Cauchy-Riemann equation in M.

Theorem 5 *The Schwarz BVP $\omega_{\bar z} = 0$ in M, $Re\,\omega = \gamma$ on ∂M*

$$\frac{2}{\pi i}\int_{\partial M\cap\partial C_1} Im\,\omega(\zeta)\frac{d\zeta}{\zeta-1} + \frac{2}{\pi i}\int_{\partial M\cap\partial C_2} Im\,\omega(\zeta)\frac{d\zeta}{\zeta+1} = c,$$

with given $\gamma \in C(\partial M, \mathbb{C})$, $c \in \mathbb{R}$ is uniquely solved by

$$\begin{aligned}\omega(z) = &\frac{1}{2\pi i}\int_{\partial M\cap\partial C_1}\gamma(\zeta)\left[\frac{2(\zeta-1)}{\zeta-z} - 1 + \frac{2z(\zeta-1)}{\zeta z+1} - 1\right]\frac{d\zeta}{\zeta-1}\\ &+ \frac{1}{2\pi i}\int_{\partial M\cap\partial C_2}\gamma(\zeta)\left[\frac{2(\zeta+1)}{\zeta-z} - 1 + \frac{2z(\zeta+1)}{\zeta z+1} - 1\right]\frac{d\zeta}{\zeta+1} + ic,\end{aligned}$$

where $\zeta = \xi + i\eta$.

Proof 3 *By Theorem 3:*

$$\begin{aligned}w(z) = &\frac{1}{2\pi i}\int_{\partial M\cap\partial C_1}\gamma(\zeta)\left[\frac{2(\zeta-1)}{\zeta-z} - 1 + \frac{2z(\zeta-1)}{\zeta z+1} - 1\right]\\ &+ \frac{1}{2\pi i}\int_{\partial M\cap\partial C_2}\gamma(\zeta)\left[\frac{2(\zeta+1)}{\zeta-z} - 1 + \frac{2z(\zeta+1)}{\zeta z+1} - 1\right]\\ &+ \frac{2}{\pi}\int_{\partial M\cap\partial C_1} Im\,w(\zeta)\frac{d\zeta}{\zeta-1} + \frac{2}{\pi}\int_{\partial S\cap\partial C_2} Im\,w(\zeta)\frac{d\zeta}{\zeta+1}\\ &- \frac{1}{\pi}\int_M\left\{w_{\bar\zeta}(\zeta)\left[\frac{1}{\zeta-z} + \frac{z}{\zeta z+1}\right]\right.\\ &\left.+\bar w_{\bar\zeta}(\zeta)\left[\frac{\bar z-1}{\zeta(\bar z-1)-\bar z-1} - \frac{\bar z+1}{\zeta(\bar z+1)+\bar z-1}\right]\right\}d\xi d\eta.\end{aligned}$$

Using Vekua's method [11], can be expressed as, for:

$$w(z) = \phi(z) + Tf(z),$$

where $\phi(z)$ — is arbitrary analytic function in domain and $Tf(z)$ — is Pompeiu operator. So $w(z)$ is a solution of the homogeneous Cauchy-Riemann equation and applying Theorem 6, we get:

$$\lim_{z\to\zeta} Re\,w(z) = \omega(\zeta).$$

Then this complete proof.

Corollary 1. Any $w \in C^1(M;\mathbb{C}) \cap C(\overline{M};\mathbb{C})$ can be represented as

$$\begin{aligned}w(z) = &\frac{1}{2\pi i}\int_{\partial M}\mathrm{Re}\,w(\zeta)\,\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\\ &- \frac{1}{2\pi}\int_M\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{w_{\bar\zeta}(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right)d\xi d\eta + i\,\mathrm{Im}\,w(0).\end{aligned} \tag{13}$$

Theorem 6 *The Schwarz BVP $\omega_{\bar z} = f$ in M, $Re\,\omega = \gamma$ on ∂M, $Im\,\omega(0) = 0$*

$$\frac{2}{\pi i}\int_{\partial M\cap\partial C_1} Im\,\omega(\zeta)\,\frac{d\zeta}{\zeta-1} + \frac{2}{\pi i}\int_{\partial M\cap\partial C_2} Im\,\omega(\zeta)\,\frac{d\zeta}{\zeta+1} = c,$$

with given $f \in L_1(M,\mathbb{C})$, $\gamma \in C(\partial M,\mathbb{C})$, $c \in \mathbb{R}$ is uniquely solved by

$$\begin{aligned}\omega(z) = {} & \frac{1}{2\pi i}\int_{\partial M\cap\partial C_1}\gamma(\zeta)\left[\frac{2(\zeta-1)}{\zeta-z} - 1 + \frac{2z(\zeta-1)}{\zeta z+1} - 1\right]\frac{d\zeta}{\zeta-1}\\ & + \frac{1}{2\pi i}\int_{\partial M\cap\partial C_2}\gamma(\zeta)\left[\frac{2(\zeta+1)}{\zeta-z} - 1 + \frac{2z(\zeta+1)}{\zeta z+1} - 1\right]\frac{d\zeta}{\zeta+1} + ic\\ & - \frac{1}{2\pi}\int\limits_M\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right]d\xi d\eta,\end{aligned}$$

where $\zeta = \xi + i\eta$.

Proof 4

$$\begin{aligned}\omega(z) = {} & \frac{1}{2\pi i}\int_{\partial M\cap\partial C_1}\gamma(\zeta)\left[\frac{2(\zeta-1)}{\zeta-z} - 1 + \frac{2z(\zeta-1)}{\zeta z+1} - 1\right]\frac{d\zeta}{\zeta-1}\\ & - \frac{1}{2\pi}\int\limits_{M\cap C_1}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right]d\xi d\eta\\ & + \frac{1}{2\pi i}\int_{\partial M\cap\partial C_2}\gamma(\zeta)\left[\frac{2(\zeta+1)}{\zeta-z} - 1 + \frac{2z(\zeta+1)}{\zeta z+1} - 1\right]\frac{d\zeta}{\zeta+1} + ic\\ & - \frac{1}{2\pi}\int\limits_{M\cap C_2}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right]d\xi d\eta\\ = {} & \frac{1}{2\pi i}\int_{\partial M\cap\partial C_1}\gamma(\zeta)\left[\frac{2(\zeta-1)}{\zeta-z} - 1 + \frac{2z(\zeta-1)}{\zeta z+1} - 1\right]\frac{d\zeta}{\zeta-1}\\ & + \frac{1}{2\pi i}\int_{\partial M\cap\partial C_2}\gamma(\zeta)\left[\frac{2(\zeta+1)}{\zeta-z} - 1 + \frac{2z(\zeta+1)}{\zeta z+1} - 1\right]\frac{d\zeta}{\zeta+1} + ic\\ & - \frac{1}{2\pi}\int\limits_M\left[\frac{f(\zeta)}{\zeta}\,\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\,\frac{1+z\bar\zeta}{1-z\bar\zeta}\right]d\xi\,d\eta.\end{aligned}$$

This representation is directly obtained from formula (13) *under the assumption that a solution ω exists.*

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Authors contribution

B.K.: methodology; writing original draft preparation; supervision; project administration. N.A.: writing review and editing. B.K.: writing review and editing; supervision. All authors have read and agreed to the published version of the manuscript.

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