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ON GEOMETRIC PROPERTIES OF ℓ^p-SPACES ON UNITARY DUALS OF COMPACT GROUPS

In this paper, we investigate the Banach space geometry of the noncommutative spaces $\ell^p_{\mathrm{sch}}(\widehat{G})$ associated with the unitary dual $\widehat{G}$ of a compact group G. These spaces consist of operator-valued families whose norms are defined through the Schatten–von Neumann classes corresponding to irreducible unitary representations of G. We establish basic structural results, including Hölder-type inequalities, duality relations, and an exact complex interpolation formula. Using these tools, we prove Clarkson type inequalities and modified Clarkson inequalities with constants depending only on p. Consequently, $\ell^p_{\mathrm{sch}}(\widehat{G})$ has the Kadec–Klee property and is uniformly convex and uniformly smooth for every $1 < p < \infty$. We also derive quantitative power-type estimates for the associated moduli of convexity and smoothness, including optimal-order estimates in the appropriate ranges of p. Finally, we show that these spaces have type $\min\{2, p\}$ and cotype $\max\{2, p\}$. Thus, their geometric properties closely parallel those of the classical L^p- and ℓ^p-spaces. This provides a unified geometric description of this noncommutative scale.
Key words: compact group, ℓ^p spaces associated with compact groups, uniformly smooth Banach space, uniformly convex Banach space, Clarkson's inequality, type and cotype, duality, complex interpolation.

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Компакт топтардың унитарлық дуалдарындағы ℓ^p-кеңістіктерінің геометриялық қасиеттері туралы

Бұл жұмыста біз G компакт тобының $\widehat{G}$ унитарлық дуалымен байланысты $\ell^p_{\mathrm{sch}}(\widehat{G})$ коммутативті емес кеңістіктерінің Банах кеңістіктік геометриясын зерттейміз. Бұл кеңістіктер G тобының келтірілмейтін унитарлық кескінделулеріне сәйкес келетін Шаттен–фон Нейман кластары арқылы нормалары анықталатын оператор мәнді үйірлерден тұрады. Біз Гёльдер типтес теңсіздіктерді, дуалдылық қатынастарын және дәл комплекс интерполяция формуласын қамтитын негізгі құрылымдық нәтижелерді дәлелдейміз. Осы нәтижелерді пайдалана отырып, Кларксон типтес теңсіздіктерді және тек p параметріне тәуелді тұрақтылары бар модификацияланған Кларксон теңсіздіктерін аламыз. Нәтижесінде, әрбір $1 < p < \infty$ үшін $\ell^p_{\mathrm{sch}}(\widehat{G})$ кеңістігінің Кадец–Кли қасиетіне ие екенін, сондай-ақ оның бірқалыпты дөңес және бірқалыпты тегіс болатынын көрсетеміз. Сонымен қатар, дөңестік және тегістік модульдері үшін сандық дәрежелік бағалауларды, соның ішінде p параметрінің тиісті аралықтарындағы оңтайлы ретті бағалауларды анықтаймыз. Ақырында, бұл кеңістіктердің типі $\min\{2, p\}$, ал котипі $\max\{2, p\}$ болатынын дәлелдейміз. Осылайша, олардың геометриялық қасиеттері классикалық L^p және ℓ^p кеңістіктерінің қасиеттеріне ұқсас болады. Бұл нәтиже қарастырылып отырған коммутативті емес кеңістіктер классына біртұтас геометриялық сипаттамасын береді.
Түйін сөздер: компакт топ, компакт топтармен байланысты ℓ^p-кеңістіктер, бірқалыпты тегіс Банах кеңістігі, бірқалыпты дөңес Банах кеңістігі, Кларксон теңсіздігі, тип және котип, дуалдылық, комплекс интерполяция.

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О геометрических свойствах ℓ^p-пространств на унитарных дуалах компактных групп

В данной работе мы исследуем геометрию некоммутативных пространств $\ell^p_{\mathrm{sch}}(\widehat{G})$, связанных с унитарным дуалом $\widehat{G}$ компактной группы G. Эти пространства состоят из операторнозначных семейств, нормы которых определяются посредством классов Шаттена–фон Неймана, соответствующих неприводимым унитарным представлениям группы G. Мы устанавливаем основные структурные результаты, включая неравенства типа Гёльдера, соотношения двойственности и точную формулу комплексной интерполяции. Используя эти результаты, мы доказываем неравенства типа Кларксона, а также модифицированные неравенства Кларксона с постоянными, зависящими только от p. В качестве следствий мы показываем, что пространство $\ell^p_{\mathrm{sch}}(\widehat{G})$ обладает свойством Кадеца–Кли и является равномерно выпуклым и равномерно гладким для каждого $1 < p < \infty$. Кроме того, мы получаем количественные степенные оценки для соответствующих модулей выпуклости и гладкости, включая оценки оптимального порядка в соответствующих диапазонах значений p. Наконец, мы доказываем, что эти пространства имеют тип $\min\{2, p\}$ и котип $\max\{2, p\}$. Таким образом, их геометрические свойства близки к свойствам классических пространств L^p и ℓ^p. Полученные результаты дают единое геометрическое описание рассматриваемой некоммутативной шкалы пространств.
Ключевые слова: компактная группа, ℓ^p-пространства, связанные с компактными группами, равномерно гладкое банахово пространство, равномерно выпуклое банахово пространство, неравенство Кларксона, тип и котип, двойственность, комплексная интерполяция.

1 Introduction

The notion of uniformly convex spaces was first introduced by Clarkson [7]. A Banach space X is said to be uniformly convex if for each $0 < \varepsilon \leq 2$ there exists $\delta > 0$ such that $\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon$ implies $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$. Geometrically, this means that the midpoint of a chord of the unit sphere cannot approach to the surface of that sphere unless the length of the chord tends to zero [7].

Note that any finite dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$, and any Hilbert space $\mathcal{H}$ is uniformly convex, as follows from the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{1}$$

In the same paper [7, Section 3], Clarkson proved that the classical Lebesgue spaces $L^p(\mu)$ and ℓ^p, for $1 < p < \infty$, satisfy this property too, i.e., they are uniformly convex.

Another important notion is uniform smoothness, which is dual to uniform convexity. A Banach space X is said to be uniformly smooth if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x + y\| + \|x - y\| \leq 2 + \varepsilon\|y\|$$

whenever $x, y \in X$, $\|x\| = 1$, and $\|y\| \leq \delta$.

Equivalent formulations of uniform convexity and uniform smoothness in terms of the modulus of convexity and the modulus of smoothness, respectively, are given in Section 2.3 (Definitions 2 and 3).

In this paper, we investigate the uniform convexity and uniform smoothness of noncommutative ℓ^p-spaces on unitary dual $\widehat{G}$ of a compact group G based on the Schatten-von Neumann ideals (see, Theorem 1), simply denoted as $\ell^p_{sch}(\widehat{G})$ [13, Section 2.1.4] (see also [11, Section 2.14.2], [15]). They arise naturally in noncommutative harmonic analysis on compact groups and play a role, for instance, in the Hausdorff–Young the theorem for compact groups [11, Section 2.14.1].

Clarkson's proof of the uniform convexity and uniform smoothness of $L^p(\mu)$ (or ℓ^p) spaces relies on certain inequalities which may be regarded as L^p-analogues of the parallelogram identity (1) (see [7, Theorem 2] for more details). These inequalities are known as Clarkson inequalities. Our first aim is to establish Clarkson-type inequalities (see, [27] for similar inequalities on related ℓ^p-spaces based on Hilbert-Schmidt ideals) for the spaces $\ell^p_{sch}(\widehat{G})$ (see Proposition 8). To this end, we also prove complex interpolation (see, Subsection 3.2) and duality results (see, Subsection 3.1) for these spaces, in analogy with the classical Lebesgue spaces. These results are used in the proof of the Clarkson-type inequalities and are also of independent interest.

Another classical geometric aspect of Banach space theory is the study of type and cotype. These notions are closely related to uniform convexity and uniform smoothness (see, for example, [20, Theorem 1.e.16] and [12]). It is well known that $L^p(\mu), 1 \leq p < \infty$, has type $\min\{2,p\}$ and cotype $\max\{2,p\}$. Moreover, every Hilbert space has type 2 and cotype 2, and, conversely, a Banach space which is simultaneously of type 2 and cotype 2, is isomorphic to a Hilbert space [18, Proposition 3.1]. In this paper, we prove that the spaces $\ell^p_{sch}(\widehat{G})$ have the same type and cotype properties as the classical Lebesgue spaces, namely type $\min\{2,p\}$ and cotype $\max\{2,p\}$.

The parallelogram identity characterizes Hilbert spaces among normed spaces. Therefore, inequalities of Clarkson type may be regarded as quantitative substitutes for the parallelogram identity in non-Hilbertian Banach spaces. From this perspective, type and cotype measure, in certain sense, how far a Banach space is from Hilbert space geometry. This provides one of the motivations for studying Clarkson-type inequalities and type/cotype properties for the noncommutative spaces $\ell^p_{sch}(\widehat{G})$.

We also prove a modified version of the Clarkson type inequalities involving additional constants (see Theorem 2). Earlier versions of such inequalities for L^p spaces and Schatten-von Neumann ideals $\mathcal{S}^p$ were used in connection with a conjecture of Gross arising from quantum field theory (see [6] and [2] for more details). Further applications of inequalities of this type can be found in [21].

2 Preliminaries

In this section, we recall the necessary preliminaries. Throughout this paper a group is always assumed to be compact group (shortly, group) (see, for example, [25, Section III.7.3]).

Let G and H be the groups. By $\mathrm{Hom}(G,H)$, we denote the set of all (group) homomorphisms from G into H. When $G = H$, we shortly write $\mathrm{Hom}(G)$. The set of all bijective homomorphisms from G onto G is denoted by $\mathrm{Aut}(G)$.

Throughout this paper, by $\left(\mathcal{L}(\mathcal{H}), \|\cdot\|_{\mathcal{L}(\mathcal{H})}\right)$ and $\left(\mathcal{S}^p(\mathcal{H}), \|\cdot\|_{\mathcal{S}^p(\mathcal{H})}\right)$, $1 \leq p \leq \infty$, we denote the $*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and a Schatten-von

Neumann ideal of compact operators on $\mathcal{H}$, respectively (see, [26, Chapter 2] or [14, Chapter 3].

2.1 Strongly continuous irreducible unitary representations

Let G be a compact group and let $\mathcal{H}$ be a complex Hilbert space. A unitary representation of G on $\mathcal{H}$ is a homomorphism

$$\xi : G \to \mathcal{U}(\mathcal{H}),$$

where $\mathcal{U}(\mathcal{H})$ denotes the unitary group of $\mathcal{H}$. It is called strongly continuous if the map $x \mapsto \xi(x)v$ is continuous from G into $\mathcal{H}$ for every $v \in \mathcal{H}$.

A closed subspace $\mathcal{K} \subset \mathcal{H}$ is called ξ-invariant if $\xi(x)\mathcal{K} \subseteq \mathcal{K}$ for every $x \in G$. The representation ξ is called irreducible if its only ξ-invariant closed subspaces are $\{0\}$ and $\mathcal{H}$.

We denote by $\mathrm{Rep}(G)$ the set of all strongly continuous irreducible unitary representations of G. Two representations $\xi : G \to \mathcal{U}(\mathcal{H}_\xi)$ and $\eta : G \to \mathcal{U}(\mathcal{H}_\eta)$ are said to be equivalent if there is an invertible linear operator $A : \mathcal{H}_\xi \to \mathcal{H}_\eta$ such that

$$A\xi(x) = \eta(x)A, \qquad x \in G.$$

For further details, see [25].

2.2 Noncommutative ℓ^p-spaces associated with a compact group

Let G be a compact group. We denote by $\widehat{G}$ its unitary dual, that is, the set of equivalence classes of strongly continuous irreducible unitary representations of G. For $[\xi] \in \widehat{G}$, let $\xi : G \to \mathcal{U}(\mathcal{H}_\xi)$ be a representative. Since G is compact, $\mathcal{H}_\xi$ is finite dimensional. We write $d_\xi := \dim(\mathcal{H}_\xi)$. Let $\mathcal{M}(\widehat{G})$ be the space of all mappings H on $\widehat{G}$ such that $H(\xi) \in \mathcal{L}(\mathcal{H}_\xi), [\xi] \in \widehat{G}$. The space $S'(\widehat{G})$ of tempered distributions on $\widehat{G}$ is understood in the sense of [13, Section 2.1.4]; see also [11, Section 2.14.2].

Definition 1. *For $1 \le p < \infty$, the space $\ell^p_{sch}(\widehat{G})$ consists of all $H \in S'(\widehat{G})$ such that*

$$\|H\|_{\ell^p_{sch}(\widehat{G})} := \left(\sum_{[\xi]\in\widehat{G}} d_\xi \|H(\xi)\|^p_{\mathcal{S}^p(\mathcal{H}_\xi)} \right)^{1/p} < \infty.$$

For $p = \infty$, the space $\ell^\infty_{sch}(\widehat{G})$ consists of all $H \in S'(\widehat{G})$ such that

$$\|H\|_{\ell^\infty_{sch}(\widehat{G})} := \sup_{[\xi]\in\widehat{G}} \|H(\xi)\|_{\mathcal{L}(\mathcal{H}_\xi)} < \infty.$$

These spaces are Banach spaces for all $1 \le p \le \infty$; see, for example, [11, Section 2.14.2].

Remark 1. *After the first version of this work was completed, we became aware that, when $\widehat{G}$ is countable, these spaces can be realized isometrically as subspaces of classical Schatten-von Neumann spaces. Assume that $\widehat{G} = \{[\xi_n]\}_{n\ge1}$, and for each $[\xi] \in \widehat{G}$, let*

$$j_\xi : \mathcal{S}^p(\mathcal{H}_\xi) \to \mathcal{S}^p(\ell^2)$$

be an isometric embedding. For $H \in \ell^p_{sch}(\widehat{G})$, define

$$J(H) = \left(\dim(\xi_n)^{1/p} j_{\xi_n}(H(\xi_n))\right)_{n\geq 1}.$$

Then J is an isometric embedding of $\ell^p_{sch}(\widehat{G})$ into the ℓ^p-direct sum of countably many copies of $\mathcal{S}^p(\ell^2)$. Indeed,

$$\|J(H)\|^p = \sum_{n\geq 1} \dim(\xi_n)\|j_{\xi_n}(H(\xi_n))\|^p_{\mathcal{S}^p(\ell^2)} = \sum_{n\geq 1} \dim(\xi_n)\|H(\xi_n)\|^p_{\mathcal{S}^p(\mathcal{H}_{\xi_n})} = \|H\|^p_{\ell^p_{sch}(\widehat{G})}.$$

Moreover, this ℓ^p-direct sum embeds isometrically into $\mathcal{S}^p(\ell^2)$. Consequently, some of the Banach-space geometric properties considered below might also be obtained from known results on Schatten classes. Nevertheless, we keep the notation $\ell^p_{sch}(\widehat{G})$ and work directly with this space, since it is the natural framework for Fourier analysis on compact groups.

We shall use the following Hölder inequality, which follows from the Schatten-class Hölder inequality and the scalar Hölder inequality.

Proposition 1. *Let $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then, for any $H_1 \in \ell^p_{sch}(\widehat{G})$ and $H_2 \in \ell^q_{sch}(\widehat{G})$, we have $\|H_1 H_2\|_{\ell^r_{sch}(\widehat{G})} \leq \|H_1\|_{\ell^p_{sch}(\widehat{G})}\|H_2\|_{\ell^q_{sch}(\widehat{G})}$.*

For $H \in \ell^p_{sch}(\widehat{G})$, define $H^*(\xi) = H(\xi)^*, |H|(\xi) = \left(H^*(\xi)H(\xi)\right)^{1/2}$. The following fact follows directly from the definitions.

Proposition 2. *Let $H \in \ell^p_{sch}(\widehat{G})$. Then $H^* \in \ell^p_{sch}(\widehat{G})$ and $\|H^*\|_{\ell^p_{sch}(\widehat{G})} = \|H\|_{\ell^p_{sch}(\widehat{G})} = \||H|\|_{\ell^p_{sch}(\widehat{G})}$.*

2.3 Uniformly convex and uniformly smooth Banach spaces, type and co-type properties

In general, for a given Banach space, one can define the notions of its modulus of convexity and modulus of smoothness.

Definition 2. *Let $(X, \|\cdot\|_X)$ be a Banach space. Its modulus of convexity and modulus of smoothness are defined by*

$$\delta_X(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\|_X \mid x, y \in X, \quad \|x\|_X = \|y\|_X = 1, \quad \|x-y\|_X = \varepsilon\right\},$$

for $0 < \varepsilon \leq 2$, and

$$\rho_X(t) := \sup\left\{\frac{\|x+ty\|_X + \|x-ty\|_X}{2} - 1 \mid x, y \in X, \quad \|x\|_X = \|y\|_X = 1\right\}.$$

for $t > 0$, respectively.

These notions are helpful to classify uniformly convex and uniformly smooth Banach spaces, respectively.

Definition 3. *Let X be a Banach space, and $\delta_X(\varepsilon)$, $0 < \varepsilon < 2$, and $\rho_X(t)$, $t > 0$, be its modulus of convexity and modulus of smoothness, respectively. Then, the Banach space X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$, and uniformly smooth if $\lim\limits_{t\to 0} \frac{\rho_X(t)}{t} = 0$.*

Note that the notions of uniform convexity and uniform smoothness are the dual notions to each other [20, Propositition 1.e.2] (see also [8, 19]).

We now recall the definition of the type and cotype of the Banach space X. Let $\{r_j\}_{j=1}^{\infty}$ denotes the sequence of Rademacher functions (see [20, Section 1.e] or [1, Section 6.2] for more details).

Definition 4. *[20, Page 74] A Banach space X is said to be type p for some $1 < p \leq 2$ if there exists constant M such that for every finite number of vectors $\{x_j\}_{j=1}^{n}$ in X, we have*

$$\left(\int_0^1 \left\| \sum_{j=1}^{n} r_j(t) x_j \right\|^2 dt\right)^{1/2} \leq M \left(\sum_{j=1}^{n} \|x_j\|^p\right)^{1/p},$$

and X is said to be cotype q for some $q \geq 2$ if there exists constant M such that for every finite number of vectors $\{x_j\}_{j=1}^{n}$ in X, we have

$$\left(\int_0^1 \left\| \sum_{j=1}^{n} r_j(t) x_j \right\|^2 dt\right)^{1/2} \geq M \left(\sum_{j=1}^{n} \|x_j\|^q\right)^{1/q}.$$

2.4 Complex interpolation

We use the standard complex interpolation method of Calderón. For a compatible couple of complex Banach spaces (X_0, X_1), we denote by $(X_0, X_1)_\theta$, $0 < \theta < 1$, the corresponding complex interpolation space. We refer the reader to [3, 5] for details.

Let X be a Banach space, $1 \leq p \leq \infty$, and $w > 0$. We denote by $X \oplus_{p,w} X$ the space $X \times X$ equipped with the norm

$$\|(x,y)\|_{X\oplus_{p,w}X} = \begin{cases} \left(\|x\|_X^p + w\|y\|_X^p\right)^{1/p}, & 1 \leq p < \infty, \\ \max\{\|x\|_X, w\|y\|_X\}, & p = \infty. \end{cases}$$

When $w = 1$, we simply write $X \oplus_p X$.

We shall use the following interpolation formula for weighted direct sums.

Proposition 3. *[5, Section 13.6] (see also [3, Theorem 5.5.3] and [22]) Let X_1, X_2 be Banach spaces, let $1 \leq p_1, p_2 \leq \infty$, and let $w_1, w_2 > 0$. If at least one of the spaces $X_1 \oplus_{p_1,w_1} X_1$ and $X_2 \oplus_{p_2,w_2} X_2$ is reflexive, then*

$$(X_1 \oplus_{p_1,w_1} X_1, X_2 \oplus_{p_2,w_2} X_2)_\theta = (X_1, X_2)_\theta \oplus_{p,w} (X_1, X_2)_\theta,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad w = w_1^{p(1-\theta)/p_1} w_2^{p\theta/p_2}, \qquad 0 < \theta < 1.$$

3 Family of ℓ^p-spaces $\ell^p_{sch}(\widehat{G})$ based on Schatten-von Neumann ideals

3.1 Duality results

We first present a reflexivity result of the spaces $\ell^p_{sch}(\widehat{G})$, $1 < p < \infty$, which will be useful to prove the duality of these spaces.

Proposition 4. *The space $\ell^p_{sch}(\widehat{G})$, $1 < p < \infty$, is reflexive, i.e., $\left(\ell^p_{sch}(\widehat{G})\right)^{**} \cong \ell^p_{sch}(\widehat{G})$.*

Proof. Note that $\ell^p_{sch}(\widehat{G})$, $1 < p < \infty$, is uniformly convex, by Theorem 1, which will be proved in Subsection 3.4. We end the proof with the Milman-Pettis theorem which states that every uniformly convex space is reflexive. □

We now prove the following duality between spaces $\ell^p_{sch}(\widehat{G})$.

Proposition 5. *Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(i) For any $H \in \ell^p_{sch}(\widehat{G})$, we have

$$\|H\|_{\ell^p_{sch}(\widehat{G})} = \sup\{\left|\langle H, F\rangle_{\widehat{G}}\right| : \quad F \in \ell^q_{sch}(\widehat{G}), \quad \|F\|_{\ell^q_{sch}(\widehat{G})} = 1\},$$

where

$$\langle H, F\rangle_{\widehat{G}} = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\mathrm{Tr}(H(\xi)F(\xi)).$$

(ii) $\left(\ell^p_{sch}(\widehat{G})\right)^*$ is isometrially isomorphic to $\ell^q_{sch}(\widehat{G})$, i.e. $\left(\ell^p_{sch}(\widehat{G})\right)^* \cong \ell^q_{sch}(\widehat{G})$.

Proof. (i). Let $1 < p < \infty$, and $H \in \ell^p_{sch}(\widehat{G})$ and $F \in \ell^q_{sch}(\widehat{G})$. Then, by [14, Theorem III.8.5, p. 104], one has

$$|\langle H, F\rangle_{\widehat{G}}| \leq \sum_{[\xi]\in\widehat{G}} \dim(\xi) \|H(\xi)F(\xi)\|_{\mathcal{S}^1(\mathcal{H}_\xi)}.$$

Hence,

$$\sum_{[\xi]\in\widehat{G}} \dim(\xi) \|H(\xi)F(\xi)\|_{\mathcal{S}^1(\mathcal{H}_\xi)} \leq \sum_{[\xi]\in\widehat{G}} \dim(\xi)^{\frac{1}{p}} \|H(\xi)\|_{\mathcal{S}^p(\mathcal{H}_\xi)} \dim(\xi)^{\frac{1}{q}} \|F(\xi)\|_{\mathcal{S}^q(\mathcal{H}_\xi)}$$

$$\leq \left[\sum_{[\xi]\in\widehat{G}} \left(\dim(\xi)^{\frac{1}{p}}\right)^p \|H(\xi)\|^p_{\mathcal{S}^p(\mathcal{H}_\xi)}\right]^{\frac{1}{p}} \left[\sum_{[\xi]\in\widehat{G}} \left(\dim(\xi)^{\frac{1}{q}}\right)^q \|F(\xi)\|^q_{\mathcal{S}^q(\mathcal{H}_\xi)}\right]^{\frac{1}{q}} = \|H\|_{\ell^p_{sch}(\widehat{G})} \|F\|_{\ell^q_{sch}(\widehat{G})}.$$

If $p = 1$, then by [14, Theorem III.8.5, p. 104], one has

$$\begin{aligned}|\langle H, F\rangle_{\widehat{G}}| &\leq \sum_{[\xi]\in\widehat{G}} \dim(\xi) \|H(\xi)F(\xi)\|_{\mathcal{S}^1(\mathcal{H}_\xi)} \leq \sum_{[\xi]\in\widehat{G}} \dim(\xi) \|H(\xi)\|_{\mathcal{S}^1(\mathcal{H}_\xi)} \|F(\xi)\|_{\mathcal{L}(\mathcal{H}_\xi)} \\ &\leq \sum_{[\xi]\in\widehat{G}} \dim(\xi) \|H(\xi)\|_{\mathcal{S}^1(\mathcal{H}_\xi)} \sup_{[\xi]\in\widehat{G}} \|F(\xi)\|_{\mathcal{L}(\mathcal{H}_\xi)} \leq \sum_{[\xi]\in\widehat{G}} \dim(\xi) \|H(\xi)\|_{\mathcal{S}^1(\mathcal{H}_\xi)} \|H\|_{\ell^\infty_{sch}(\widehat{G})} \\ &= \|H\|_{\ell^1_{sch}(\widehat{G})} \|F\|_{\ell^\infty_{sch}(\widehat{G})}.\end{aligned}$$

Therefore, for all $1 \leq p < \infty$, we have $|\langle H, F\rangle_{\widehat{G}}| \leq \|H\|_{\ell^p_{sch}(\widehat{G})} \|F\|_{\ell^q_{sch}(\widehat{G})}$. Thus, we have

$$\|H\|_{\ell^p_{sch}(\widehat{G})} \geq \sup\{\left|\langle H, F\rangle_{\widehat{G}}\right| : \quad F \in \ell^q_{sch}(\widehat{G}), \quad \|F\|_{\ell^q_{sch}(\widehat{G})} = 1\}. \tag{2}$$

For $H \in \ell^p_{sch}(\widehat{G})$, construct $F \in \ell^q_{sch}(\widehat{G})$ by the equality

$$F(\xi) = \frac{|H(\xi)|^{p-1} U^*(\xi)}{\|H\|^{p-1}_{\ell^p_{sch}(\widehat{G})}}, \quad [\xi] \in \widehat{G}.$$

where $U(\xi)$ is the partial isometry from the polar decomposition of the finite-rank operators $H(\xi) = U(\xi)|H(\xi)|$, $[\xi] \in \widehat{G}$. Note that $\|F\|_{\ell^q_{sch}(\widehat{G})} = 1$. Moreover,

$$\langle H, F\rangle_{\widehat{G}} = \|H\|_{\ell^p_{sch}(\widehat{G})}, \tag{3}$$

where the fourth equality follows from the fact that the trace is unitarily invariant. Hence, (3) together with (2) end the proof of (i).

(ii). Let $F \in \ell^q_{sch}(\widehat{G})$. Then define a mapping $\Phi : \ell^q_{sch}(\widehat{G}) \mapsto \left(\ell^p_{sch}(\widehat{G})\right)^*$ in the following form (see [25, Definition 10.3.29] for more details)

$$\Phi(F)(H) = \langle F, H\rangle_{\widehat{G}} := \sum_{[\xi]\in\widehat{G}} \dim(\xi)\mathrm{Tr}(F(\xi)H(\xi)), \quad H \in \ell^p_{sch}(\widehat{G}).$$

We first show that the mapping Φ is norm preserving. By part (i), one has

$$\begin{aligned}\|\Phi(F)\|_{\left(\ell^p_{sch}(\widehat{G})\right)^*} &= \sup\left\{|\Phi(F)(H)| : \quad H \in \ell^p_{sch}(\widehat{G}), \quad \|H\|_{\ell^p_{sch}(\widehat{G})} = 1\right\} \\ &= \sup\left\{\left|\langle F, H\rangle_{\widehat{G}}\right| : \quad H \in \ell^p_{sch}(\widehat{G}), \quad \|H\|_{\ell^p_{sch}(\widehat{G})} = 1\right\} = \|F\|_{\ell^q_{sch}(\widehat{G})}, \quad \forall F \in \ell^q_{sch}(\widehat{G}).\end{aligned}$$

Finally, it is left to show that the mapping Φ is surjective. Since $\ell^q_{sch}(\widehat{G})$ is Banach space and Φ is bounded linear mapping, it follows that $\Phi(\ell^q_{sch}(\widehat{G}))$ is a closed subspace of $\left(\ell^p_{sch}(\widehat{G})\right)^*$. We now prove that $\Phi(\ell^q_{sch}(\widehat{G})) = \left(\ell^p_{sch}(\widehat{G})\right)^*$. Note that it is enough to prove that $\Phi(\ell^q_{sch}(\widehat{G}))$ is dense in $\left(\ell^p_{sch}(\widehat{G})\right)^*$. Let $H \in \left(\ell^p_{sch}(\widehat{G})\right)^{**}$ be given such that $H(\phi) = 0$, for any $\phi \in \Phi(\ell^q_{sch}(\widehat{G}))$. Hence, for any $F \in \ell^q_{sch}(\widehat{G})$, we have that $\Phi(F) = \psi \in \Phi(\ell^q_{sch}(\widehat{G}))$ for some ψ. Thus,

$$H(\Phi(F)) = 0, \quad \forall F \in \ell^q_{sch}(\widehat{G}).$$

Since $\ell^p_{sch}(\widehat{G})$ is reflexive (cf. Proposition 4), we have

$$H(\Phi(F)) = \Phi(F)(H) = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\mathrm{Tr}(F(\xi)H(\xi)) = 0, \quad \forall F \in \ell^q_{sch}(\widehat{G}).$$

Choose $F \in \ell^q_{sch}(\widehat{G})$ such that

$$F(\xi) = |H(\xi)|^{p-2} H^*(\xi), \quad \forall [\xi] \in \widehat{G}.$$

Then, we have $\|F\|^q_{\ell^q_{sch}(\widehat{G})} = \|H\|^p_{\ell^p_{sch}(\widehat{G})} < \infty$. Note that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for finite rank operators A, B [14, Section I.8.2]. Then, for a natural number k, one can show that

$$\mathrm{Tr}\left((AB)^k\right) = \mathrm{Tr}\left(A(BA)^{k-1}B\right) = \mathrm{Tr}\left((BA)^{k-1}BA\right) = \mathrm{Tr}\left((BA)^k\right).$$

Hence, $\mathrm{Tr}(p(AB)) = \mathrm{Tr}(p(BA))$ for any polynomial p defined on the compact interval which contains the spectrum of both A and B. Therefore, by Stone-Weierstrass theorem, one has $\mathrm{Tr}(f(AB)) = \mathrm{Tr}(f(BA))$, for any continuous real-valued function f on the same compact interval. Especially, the same holds for $f(t) = t^{\frac{q}{2}}$.

Continuing the proof, we have

$$\begin{aligned} 0 &= \sum_{[\xi]\in\widehat{G}} \dim(\xi)\mathrm{Tr}\left(F(\xi)H(\xi)\right) = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\mathrm{Tr}\left(|H(\xi)|^{p-2}H^*(\xi)H(\xi)\right) \\ &= \sum_{[\xi]\in\widehat{G}} \dim(\xi)\mathrm{Tr}\left(|H(\xi)|^p\right) = \|H\|^p_{\ell^p_{sch}(\widehat{G})}. \end{aligned}$$

Therefore, $H \equiv 0$. That means $\Phi(\ell^q_{sch}(\widehat{G}))$ is dense in $\left(\ell^p_{sch}(\widehat{G})\right)^*$, which ends the proof. $\square$

We also have the following duality result.

Proposition 6. *Let $w > 0$. Let $1 < p, q, r, s < \infty$ be given such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$\left(\ell^p_{sch}(\widehat{G}) \oplus_{r,w} \ell^p_{sch}(\widehat{G})\right)^* \cong \ell^q_{sch}(\widehat{G}) \oplus_{s,\frac{1}{w}} \ell^q_{sch}(\widehat{G}). \tag{4}$$

In particular,

$$\left(\ell^p_{sch}(\widehat{G}) \oplus_r \ell^p_{sch}(\widehat{G})\right)^* \cong \ell^q_{sch}(\widehat{G}) \oplus_s \ell^q_{sch}(\widehat{G}). \tag{5}$$

Proof. We only prove (4), since (5) follows from (4) by assuming $w = 1$. By Proposition 5, we know that $\left(\ell^p_{sch}(\widehat{G})\right)^* \cong \ell^q_{sch}(\widehat{G})$. Hence, we prove (4) in the following form

$$\left(\ell^p_{sch}(\widehat{G}) \oplus_{r,w} \ell^p_{sch}(\widehat{G})\right)^* \cong \left(\ell^p_{sch}(\widehat{G})\right)^* \oplus_{s,\frac{1}{w}} \left(\ell^p_{sch}(\widehat{G})\right)^*.$$

Define a map T from $\left(\ell^p_{sch}(\widehat{G})\right)^* \oplus_{s,\frac{1}{w}} \left(\ell^p_{sch}(\widehat{G})\right)^*$ into $\left(\ell^p_{sch}(\widehat{G}) \oplus_{r,w} \ell^p_{sch}(\widehat{G})\right)^*$ of the form

$$T(\varphi_1, \varphi_2)(H_1, H_2) = \varphi_1(H_1) + w^{\frac{1}{r}-\frac{1}{s}}\varphi_2(H_2)$$

for $\varphi_1, \varphi_2 \in \left(\ell^p_{sch}(\widehat{G})\right)^*$ and $H_1, H_2 \in \ell^p_{sch}(\widehat{G})$. By Hölder inequality,

$$\begin{aligned} |T(\varphi_1, \varphi_2)(H_1, H_2)| &\leq \|\varphi_1\|_{\left(\ell^p_{sch}(\widehat{G})\right)^*} \|H_1\|_{\ell^p_{sch}(\widehat{G})} + w^{-\frac{1}{s}} \|\varphi_2\|_{\left(\ell^p_{sch}(\widehat{G})\right)^*} w^{\frac{1}{r}} \|H_2\|_{\ell^p_{sch}(\widehat{G})} \\ &\leq \left(\|\varphi_1\|^s_{\left(\ell^p_{sch}(\widehat{G})\right)^*} + \left(w^{-\frac{1}{s}}\right)^s \|\varphi_2\|^s_{\left(\ell^p_{sch}(\widehat{G})\right)^*}\right)^{\frac{1}{s}} \cdot \left(\|H_1\|^r_{\ell^p_{sch}(\widehat{G})} + \left(w^{\frac{1}{r}}\right)^r \|H_2\|^r_{\ell^p_{sch}(\widehat{G})}\right)^{\frac{1}{r}}. \end{aligned}$$

$$(6)$$

Hence,

$$\|T(\varphi_1,\varphi_2)\| \le \|(\varphi_1,\varphi_2)\|_{\left(\ell^p_{sch}(\widehat{G})\right)^* \oplus_{s,\frac{1}{w}} \left(\ell^p_{sch}(\widehat{G})\right)^*}.$$

To prove the reverse inequality, fix $\varepsilon > 0$ and choose $h_1, h_2 \in \ell^p_{sch}(\widehat{G})$ such that

$$\begin{gathered} \|h_1\|_{\ell^p_{sch}(\widehat{G})} = \|h_2\|_{\ell^p_{sch}(\widehat{G})} = 1 \\ \|\varphi_1\|_{\left(\ell^p_{sch}(\widehat{G})\right)^*} \le |\varphi_1(h_1)| + \varepsilon, \quad \|\varphi_2\|_{\left(\ell^p_{sch}(\widehat{G})\right)^*} \le |\varphi_2(h_2)| + \varepsilon. \end{gathered} \tag{7}$$

Multiplying h_1 and h_2 by suitable complex numbers of absolute value one, we may assume that $\varphi_1(h_1)$ and $\varphi_2(h_2)$ are nonnegative. Set

$$f_1 = \frac{\|\varphi_1\|^{s-1}_{\left(\ell^p_{sch}(\widehat{G})\right)^*}}{\left(\|\varphi_1\|^s_{\left(\ell^p_{sch}(\widehat{G})\right)^*} + \frac{1}{w}\|\varphi_2\|^s_{\left(\ell^p_{sch}(\widehat{G})\right)^*}\right)^{\frac{1}{r}}} \cdot h_1, \quad f_2 = \frac{w^{-\frac{2}{r}}\|\varphi_2\|^{s-1}_{\left(\ell^p_{sch}(\widehat{G})\right)^*}}{\left(\|\varphi_1\|^s_{\left(\ell^p_{sch}(\widehat{G})\right)^*} + \frac{1}{w}\|\varphi_2\|^s_{\left(\ell^p_{sch}(\widehat{G})\right)^*}\right)^{\frac{1}{r}}} \cdot h_2.$$

Obviously,

$$\|(f_1,f_2)\|_{\left(\ell^p_{sch}(\widehat{G})\oplus_{r,w}\ell^p_{sch}(\widehat{G})\right)^*} = 1.$$

Evaluating $T(\varphi_1,\varphi_2)$ at (f_1,f_2) and using the choice of h_1,h_2, we obtain

$$T(\varphi_1,\varphi_2)(f_1,f_2) \ge \|(\varphi_1,\varphi_2)\|_{\left(\ell^p_{sch}(\widehat{G})\right)^* \oplus_{s,\frac{1}{w}} \left(\ell^p_{sch}(\widehat{G})\right)^*},$$

up to a term tending to zero with ε. Since $\varepsilon > 0$ is arbitrary,

$$\|T(\varphi_1,\varphi_2)\| = \|(\varphi_1,\varphi_2)\|_{\left(\ell^p_{sch}(\widehat{G})\right)^* \oplus_{s,\frac{1}{w}} \left(\ell^p_{sch}(\widehat{G})\right)^*}, \quad \forall (\varphi_1,\varphi_2) \in \left(\ell^p_{sch}(\widehat{G})\right)^* \oplus_{s,\frac{1}{w}} \left(\ell^p_{sch}(\widehat{G})\right)^*,$$

which shows that T is a norm-preserving mapping.

We now show that T is a surjective mapping. Let $\Phi \in \left(\ell^p_{sch}(\widehat{G}) \oplus_{r,w} \ell^p_{sch}(\widehat{G})\right)^*$ be a bounded linear functional. Then, for $H_1, H_2 \in \ell^p_{sch}(\widehat{G})$, one has

$$\Phi(H_1,H_2) = \Phi(H_1,0) + \Phi(0,H_2) = \Phi|_{\ell^p_{sch}(\widehat{G})\oplus_{r,w}\{0\}}(H_1,0) + w^{\frac{1}{r}-\frac{1}{s}}\Phi|_{\{0\}\oplus_{r,w}\ell^p_{sch}(\widehat{G})}(0, w^{\frac{1}{s}-\frac{1}{r}}H_2).$$

Denote by

$$\varphi_1(H_1) = \Phi|_{\ell^p_{sch}(\widehat{G})\oplus_{r,w}\{0\}}(H_1,0), \quad \varphi_2(H_2) = \Phi|_{\{0\}\oplus_{r,w}\ell^p_{sch}(\widehat{G})}(0, w^{\frac{1}{s}-\frac{1}{r}}H_2).$$

Hence, $\varphi_1 \in \left(\ell^p_{sch}(\widehat{G})\right)^*$, $\varphi_2 \in \left(\ell^p_{sch}(\widehat{G})\right)^*$ and $(\varphi_1,\varphi_2) \in \left(\ell^p_{sch}(\widehat{G})\right)^* \oplus_{s,\frac{1}{w}} \left(\ell^p_{sch}(\widehat{G})\right)^*$. The linearity of φ_1 and φ_2 follows easily from the linearity of Φ. The boundedness can be shown as follows

$$\begin{aligned} |\varphi_1(H_1)| &= \left|\Phi|_{\ell^p_{sch}(\widehat{G})\oplus_{r,w}\{0\}}(H_1,0)\right| \le \left\|\Phi|_{\ell^p_{sch}(\widehat{G})\oplus_{r,w}\{0\}}\right\|_{\left(\ell^p_{sch}(\widehat{G})\oplus_{r,w}\{0\}\right)^*} \|(H_1,0)\|_{\ell^p_{sch}(\widehat{G})\oplus_{r,w}\{0\}} \\ &\le \|\Phi\|_{\left(\ell^p_{sch}(\widehat{G})\oplus_{r,w}\ell^p_{sch}(\widehat{G})\right)^*} \|H_1\|_{\ell^p_{sch}(\widehat{G})}, \quad \forall H_1 \in \ell^p_{sch}(\widehat{G}), \end{aligned}$$

and

$$|\varphi_2(H_2)| = \left|\Phi|_{\{0\}\oplus_{r,w}\ell^p_{sch}(\widehat{G})}(0, w^{\frac{1}{s}-\frac{1}{r}}H_2)\right| \leq \left\|\Phi|_{\{0\}\oplus_{r,w}\ell^p_{sch}(\widehat{G})}\right\|_{\left(\{0\}\oplus_{r,w}\ell^p_{sch}(\widehat{G})\right)^*} \left\|(0, w^{\frac{1}{s}-\frac{1}{r}}H_2)\right\|_{\{0\}\oplus_{r,w}\ell^p_{sch}(\widehat{G})}$$
$$\leq w^{\frac{1}{s}} \|\Phi\|_{\left(\ell^p_{sch}(\widehat{G})\oplus_{r,w}\ell^p_{sch}(\widehat{G})\right)^*} \|H_2\|_{\ell^p_{sch}(\widehat{G})}, \quad \forall H_2 \in \ell^p_{sch}(\widehat{G}).$$

Lastly,

$$\|(\varphi_1,\varphi_2)\|_{\left(\ell^p_{sch}(\widehat{G})\right)^*\oplus_{s,\frac{1}{w}}\left(\ell^p_{sch}(\widehat{G})\right)^*} = \left(\|\varphi_1\|^s_{\left(\ell^p_{sch}(\widehat{G})\right)^*} + \frac{1}{w}\|\varphi_2\|^s_{\left(\ell^p_{sch}(\widehat{G})\right)^*}\right)^{\frac{1}{s}}$$
$$\leq \left(\|\Phi\|^s_{\left(\ell^p_{sch}(\widehat{G})\oplus_{r,w}\ell^p_{sch}(\widehat{G})\right)^*} + \frac{1}{w}\left(w^{\frac{1}{s}}\right)^s \|\Phi\|^s_{\left(\ell^p_{sch}(\widehat{G})\oplus_{r,w}\ell^p_{sch}(\widehat{G})\right)^*}\right)^{\frac{1}{s}} = 2^{\frac{1}{s}} \|\Phi\|_{\left(\ell^p_{sch}(\widehat{G})\oplus_{r,w}\ell^p_{sch}(\widehat{G})\right)^*} < \infty,$$

which completes the proof. □

3.2 Complex interpolation of $\ell^p_{sch}(\widehat{G})$ spaces.

In this subsection present a complex interpolation result for spaces $\ell^p_{sch}(\widehat{G})$, $1 \leq p < \infty$. For more detailed discussion on basics of complex interpolation, we refer the reader to [3, Chapter 4]

Proposition 7. *Let* $1 \leq p_0, p_1 < \infty$. *Then,* $\left(\ell^{p_0}_{sch}(\widehat{G}), \ell^{p_1}_{sch}(\widehat{G})\right)_\theta = \ell^p_{sch}(\widehat{G}), \quad 0 < \theta < 1,$ *with equal norms, where* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. Firstly, let us recall that for a given finite-rank operator K on a Hilbert space H, we understand $K^z, \mathrm{Re} z \geq 0$ as follows

$$K^z = \sum_{j=1}^N \lambda_j^z \langle \cdot, \phi_j\rangle \phi_j, \quad \lambda^z = e^{z\ln(\lambda)},$$

where $K = \sum_{j=1}^N \lambda_j \langle \cdot, \phi_j\rangle\phi_j$ is the spectral representation of K.

Note that it is sufficient to check that $\left(\ell^{p_0}_{sch}(\widehat{G}), \ell^{p_1}_{sch}(\widehat{G})\right)_\theta$ can be identified with $\ell^p_{sch}(\widehat{G})$. First suppose that $H \in \ell^p_{sch}(\widehat{G})$ and without loss of generality assume that $\|H\|_{\ell^p_{sch}(\widehat{G})} = 1$. Define a function $f \in \mathcal{F} := \mathcal{F}(\ell^{p_0}_{sch}(\widehat{G}), \ell^{p_1}_{sch}(\widehat{G}))$ from the strip $\{z : 0 \leq \mathrm{Re} z \leq 1\}$ into $\ell^{p_0}_{sch}(\widehat{G}) + \ell^{p_1}_{sch}(\widehat{G})$ by the setting

$$f(z)(\xi) = |H^*(\xi)|^{\frac{p}{p(z)}-1} H(\xi), \quad \forall [\xi] \in \widehat{G},$$

where

$$\frac{p}{p(z)} = \frac{p(1-z)}{p_0} + \frac{pz}{p_1},$$

with convention that $\frac{0}{0} := 0$. It is obvious that $p(\theta) = p$ and $f(\theta)(\xi) = H(\xi)$, $\forall [\xi] \in \widehat{G}$, i.e. $f(\theta) = H$. Moreover, $\|f\|_{\mathcal{F}} = 1$. Therefore, for a given $H \in \ell^p_{sch}(\widehat{G})$ there exists $f \in \mathcal{F}$ such that $f(\theta) = H$ and $\|f\|_{\mathcal{F}} = 1$, which implies by the definition that $H \in \left(\ell^{p_0}_{sch}(\widehat{G}), \ell^{p_1}_{sch}(\widehat{G})\right)_\theta$ and $\|H\|_{\left(\ell^{p_0}_{sch}(\widehat{G}), \ell^{p_1}_{sch}(\widehat{G})\right)_\theta} \leq 1 = \|H\|_{\ell^{p}_{sch}(\widehat{G})}$.

Now suppose that $H \in \left(\ell^{p_0}_{sch}(\widehat{G}), \ell^{p_1}_{sch}(\widehat{G})\right)_\theta$ and without loss of generality assume that $\|H\|_{\left(\ell^{p_0}_{sch}(\widehat{G}), \ell^{p_1}_{sch}(\widehat{G})\right)_\theta} = 1$. Let $\varepsilon > 0$ be given. Then choose $f \in \mathcal{F}$ such that $\|f\|_{\mathcal{F}} < 1 + \varepsilon$ and $f(\theta) = H$. In order to prove that $\|H\|_{\ell^{p}_{sch}(\widehat{G})} \leq 1$, by Proposition 5 (i), it is enough to show that $|\langle H, F\rangle_{\widehat{G}}| \leq 1$ for any $F \in \ell^{q}_{sch}(\widehat{G})$ with $\|F\|_{\ell^{q}_{sch}(\widehat{G})} = 1$, where q is the conjugate index of p.

Consider $F \in \ell^{q}_{sch}(\widehat{G})$ with $\|F\|_{\ell^{q}_{sch}(\widehat{G})} = 1$ and define a function g on a strip $\{z : 0 \leq \mathrm{Re} z \leq 1\}$ into $\ell^{q_0}_{sch}(\widehat{G}) + \ell^{q_1}_{sch}(\widehat{G})$ in the following form

$$g(z)(\xi) = |F^*(\xi)|^{\frac{q}{q(z)}-1}\, F(\xi), \quad \forall[\xi] \in \widehat{G},$$

where q_0, q_1 are conjugate indices of p_0, p_1, respectively, and

$$\frac{q}{q(z)} = \frac{q(1-z)}{q_0} + \frac{qz}{q_1},$$

with convention that $\frac{0}{0} := 0$. We obviously have that $g(\theta) = F$ and the fact that $\|F\|_{\ell^{q}_{sch}(\widehat{G})} = 1$ together with similar calculations as in the first part of the proof imply that

$$\|g(it)\|_{\ell^{q_0}_{sch}(\widehat{G})} = \|g(1+it)\|_{\ell^{q_1}_{sch}(\widehat{G})} = 1, \quad t \in \mathbb{R}. \tag{8}$$

Hence, $\|g\|_{\mathcal{F}} = 1$.

Let $h(z) = \langle f(z), g(z)\rangle_{\widehat{G}}$, (see Proposition 5 for the definition of the given bracket) which is a function from the strip $\{z : 0 \leq \mathrm{Re} z \leq 1\}$ into $\mathbb{R}$. Then, by Hölder's inequality in Schatten-von Neumann ideals, one has $|h(it)| < 1+\varepsilon, \quad t \in \mathbb{R}$. Similarly, we have $|h(1+it)| < 1 + \varepsilon, \quad t \in \mathbb{R}$. Therefore,

$$|h(\theta + it)| = \left|\langle f(\theta+it), g(\theta+it)\rangle_{\widehat{G}}\right| < 1 + \varepsilon, \quad t \in \mathbb{R}.$$

In particular,

$$h(\theta) = \left|\langle f(\theta), g(\theta)\rangle_{\widehat{G}}\right| = \left|\langle H, F\rangle_{\widehat{G}}\right| < 1 + \varepsilon.$$

Since F is arbitrary, it follows that $\|H\|_{\ell^{p}_{sch}(\widehat{G})} < 1 + \varepsilon = \|H\|_{\left(\ell^{p_0}_{sch}(\widehat{G}), \ell^{p_1}_{sch}(\widehat{G})\right)_\theta} + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\|H\|_{\ell^{p}_{sch}(\widehat{G})} \leq \|H\|_{\left(\ell^{p_0}_{sch}(\widehat{G}), \ell^{p_1}_{sch}(\widehat{G})\right)_\theta}$, completing the proof. □

3.3 Clarkson's inequalities and Kadec-Klee property

We now prove the following Clarkson inequalities in $\ell^{p}_{sch}(\widehat{G})$ using the simple interpolation method which was first developed by Boas [4] and M. Klaus [26, p. 15]. These inequalities can further be used to show that the spaces $\ell^{p}_{sch}(\widehat{G})$, $1 < p < \infty$, satisfy the Kadec-Klee property.

Proposition 8. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $H_1, H_2 \in \ell^{p}_{sch}(\widehat{G})$, one has the following inequalities:*

(i) If $1 < p \le 2$, then

$$\left(\left\|\frac{H_1+H_2}{2}\right\|^q_{\ell^p_{sch}(\widehat{G})} + \left\|\frac{H_1-H_2}{2}\right\|^q_{\ell^p_{sch}(\widehat{G})}\right)^{1/q} \le \left(\frac{1}{2}\left(\|H_1\|^p_{\ell^p_{sch}(\widehat{G})} + \|H_2\|^p_{\ell^p_{sch}(\widehat{G})}\right)\right)^{1/p};$$

(ii) If $2 \le p < \infty$, then

$$\left(\left\|\frac{H_1+H_2}{2}\right\|^p_{\ell^p_{sch}(\widehat{G})} + \left\|\frac{H_1-H_2}{2}\right\|^p_{\ell^p_{sch}(\widehat{G})}\right)^{1/p} \le \left(\frac{1}{2}\left(\|H_1\|^q_{\ell^p_{sch}(\widehat{G})} + \|H_2\|^q_{\ell^p_{sch}(\widehat{G})}\right)\right)^{1/q}.$$

Proof. (i). We first define the map T from $\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})$ into $\ell^p_{sch}(\widehat{G}) \oplus_q \ell^p_{sch}(\widehat{G})$ by setting

$$T(H_1, H_2)(\xi, \xi) := \left(\frac{H_1(\xi)+H_2(\xi)}{2}, \frac{H_1(\xi)-H_2(\xi)}{2}\right).$$

Note that in order to prove (i), we have to show that

$$\|T\|_{\ell^p_{sch}(\widehat{G})\oplus_p\ell^p_{sch}(\widehat{G})\to\ell^p_{sch}(\widehat{G})\oplus_q\ell^p_{sch}(\widehat{G})} \le 2^{-1/p}.$$

By Proposition 7, one has

$$\ell^p_{sch}(\widehat{G}) = \left(\ell^1_{sch}(\widehat{G}), \ell^2_{sch}(\widehat{G})\right)_\theta, \quad \frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}. \tag{9}$$

Hence, by Proposition 3 and (9), we have

$$\begin{aligned} \ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G}) &= \left(\ell^1_{sch}(\widehat{G}), \ell^2_{sch}(\widehat{G})\right)_\theta \oplus_p \left(\ell^1_{sch}(\widehat{G}), \ell^2_{sch}(\widehat{G})\right)_\theta \\ &= \left(\ell^1_{sch}(\widehat{G}) \oplus_1 \ell^1_{sch}(\widehat{G}), \ell^2_{sch}(\widehat{G}) \oplus_2 \ell^2_{sch}(\widehat{G})\right)_\theta. \end{aligned} \tag{10}$$

Note that

$$\frac{1}{q} = 1 - \frac{1}{p} = 1 - \frac{1-\theta}{1} - \frac{\theta}{2} = \frac{1-\theta}{\infty} + \frac{\theta}{2}.$$

Hence, again by Proposition 3 and (9), we have

$$\begin{aligned} \ell^p_{sch}(\widehat{G}) \oplus_q \ell^p_{sch}(\widehat{G}) &= \left(\ell^1_{sch}(\widehat{G}), \ell^2_{sch}(\widehat{G})\right)_\theta \oplus_q \left(\ell^1_{sch}(\widehat{G}), \ell^2_{sch}(\widehat{G})\right)_\theta \\ &= \left(\ell^1_{sch}(\widehat{G}) \oplus_\infty \ell^1_{sch}(\widehat{G}), \ell^2_{sch}(\widehat{G}) \oplus_2 \ell^2_{sch}(\widehat{G})\right)_\theta. \end{aligned} \tag{11}$$

Consider the operators T_1 from $\ell^1_{sch}(\widehat{G}) \oplus_1 \ell^1_{sch}(\widehat{G})$ into $\ell^1_{sch}(\widehat{G}) \oplus_\infty \ell^1_{sch}(\widehat{G})$ and T_2 from $\ell^2_{sch}(\widehat{G}) \oplus_2 \ell^2_{sch}(\widehat{G})$ into $\ell^2_{sch}(\widehat{G}) \oplus_2 \ell^2_{sch}(\widehat{G})$, which are defined by the same action as an operator T above in their respective domains.

Then, one has

$$\begin{aligned} \|T_1(H_1, H_2)\|_{\ell^1_{sch}(\widehat{G})\oplus_\infty\ell^1_{sch}(\widehat{G})} &= \max\left\{\left\|\frac{H_1(\xi)+H_2(\xi)}{2}\right\|_{\ell^1_{sch}(\widehat{G})}, \left\|\frac{H_1(\xi)-H_2(\xi)}{2}\right\|_{\ell^1_{sch}(\widehat{G})}\right\} \\ &\le \frac{1}{2}\left(\|H_1(\xi)\|_{\ell^1_{sch}(\widehat{G})} + \|H_2(\xi)\|_{\ell^1_{sch}(\widehat{G})}\right) = 2^{-1}\|(H_1, H_2)\|_{\ell^1_{sch}(\widehat{G})\oplus_1\ell^1_{sch}(\widehat{G})}. \end{aligned}$$

Hence, $\|T_1\|_{\ell^1_{sch}(\widehat{G})\oplus_1\ell^1_{sch}(\widehat{G})\to\ell^1_{sch}(\widehat{G})\oplus_\infty\ell^1_{sch}(\widehat{G})} \leq 2^{-1}$.

On the other hand, we have

$$\|T_2(H_1,H_2)\|_{\ell^2_{sch}(\widehat{G})\oplus_2\ell^2_{sch}(\widehat{G})} = \left(\left\|\frac{H_1(\xi)+H_2(\xi)}{2}\right\|^2_{\ell^2_{sch}(\widehat{G})} + \left\|\frac{H_1(\xi)-H_2(\xi)}{2}\right\|^2_{\ell^2_{sch}(\widehat{G})}\right)^{1/2}$$
$$= 2^{-1/2}\left(\|H_1(\xi)\|^2_{\ell^2_{sch}(\widehat{G})} + \|H_2(\xi)\|^2_{\ell^2_{sch}(\widehat{G})}\right)^{1/2} = 2^{-1/2}\,\|(H_1,H_2)\|_{\ell^2_{sch}(\widehat{G})\oplus_2\ell^2_{sch}(\widehat{G})}.$$

Hence,

$$\|T_2\|_{\ell^2_{sch}(\widehat{G})\oplus_2\ell^2_{sch}(\widehat{G})\to\ell^2_{sch}(\widehat{G})\oplus_2\ell^2_{sch}(\widehat{G})} \leq 2^{-1/2}. \tag{12}$$

By (10) and (11), note again that $\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})$ and $\ell^p_{sch}(\widehat{G}) \oplus_q \ell^p_{sch}(\widehat{G})$ are pair of interpolation spaces of exponent θ for $\ell^1_{sch}(\widehat{G}) \oplus_1 \ell^1_{sch}(\widehat{G})$ and $\ell^2_{sch}(\widehat{G}) \oplus_2 \ell^2_{sch}(\widehat{G})$, respectively, $\ell^1_{sch}(\widehat{G})\oplus_\infty\ell^1_{sch}(\widehat{G})$ and $\ell^2_{sch}(\widehat{G})\oplus_2\ell^2_{sch}(\widehat{G})$. Therefore, by [3, Corollary 5.5.4] and the inequalities (**??**), (12), it follows that T is bounded from $\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})$ into $\ell^p_{sch}(\widehat{G}) \oplus_q \ell^p_{sch}(\widehat{G})$ for every $1 < p < 2$ with norm

$$\|T\|_{\ell^p_{sch}(\widehat{G})\oplus_p\ell^p_{sch}(\widehat{G})\to\ell^p_{sch}(\widehat{G})\oplus_q\ell^p_{sch}(\widehat{G})} \leq \|T_1\|^{1-\theta}_{\ell^1_{sch}(\widehat{G})\oplus_1\ell^1_{sch}(\widehat{G})\to\ell^1_{sch}(\widehat{G})\oplus_\infty\ell^1_{sch}(\widehat{G})}\,\|T_2\|^{\theta}_{\ell^2_{sch}(\widehat{G})\oplus_2\ell^2_{sch}(\widehat{G})\to\ell^2_{sch}(\widehat{G})\oplus_2\ell^2_{sch}(\widehat{G})}$$
$$\leq \left(2^{-1}\right)^{1-\theta}\left(2^{-1/2}\right)^{\theta} = 2^{-1+\theta-\theta/2} = 2^{-1+\theta/2} = 2^{-1/p},$$

which ends the proof of (i).

(ii). Define the map S from $\ell^p_{sch}(\widehat{G}) \oplus_q \ell^p_{sch}(\widehat{G})$ into $\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})$ by setting

$$S(H_1,H_2)(\xi,\xi) := \left(\frac{H_1(\xi)+H_2(\xi)}{2}, \frac{H_1(\xi)-H_2(\xi)}{2}\right).$$

Note that it is enough to show that

$$\|S\|_{\ell^p_{sch}(\widehat{G})\oplus_q\ell^p_{sch}(\widehat{G})\to\ell^p_{sch}(\widehat{G})\oplus_p\ell^p_{sch}(\widehat{G})} \leq 2^{-1/q}. \tag{13}$$

Let $(H_1,H_2) \in \ell^p_{sch}(\widehat{G}) \oplus_q \ell^p_{sch}(\widehat{G})$ and $(F_1,F_2) \in \ell^q_{sch}(\widehat{G}) \oplus_p \ell^q_{sch}(\widehat{G})$, then (see, [25, Definition 10.3.29] for more information on duality bracket)

$$\langle S(H_1,H_2),(F_1,F_2)\rangle_{\widehat{G}} = \left\langle (H_1,H_2), \left(\frac{F_1+F_2}{2}, \frac{F_1-F_2}{2},\right)\right\rangle_{\widehat{G}}.$$

Therefore, the Banach space adjoint S^* from $\left(\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})\right)^*$ into $\left(\ell^p_{sch}(\widehat{G}) \oplus_q \ell^p_{sch}(\widehat{G})\right)^*$ is also defined by the formula

$$S^*(F_1,F_2)(\xi,\xi) := \left(\frac{F_1(\xi)+F_2(\xi)}{2}, \frac{F_1(\xi)-F_2(\xi)}{2}\right).$$

Moreover, by Proposition 6, one has

$$\begin{aligned}\left(\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})\right)^* &= \ell^q_{sch}(\widehat{G}) \oplus_q \ell^q_{sch}(\widehat{G}),\\ \left(\ell^p_{sch}(\widehat{G}) \oplus_q \ell^p_{sch}(\widehat{G})\right)^* &= \ell^q_{sch}(\widehat{G}) \oplus_p \ell^q_{sch}(\widehat{G}).\end{aligned} \tag{14}$$

Since $1 < q \le 2$, by (14) and by (i) (where p and q are interchanged), it follows that the operator S^* acts from $\ell^q_{sch}(\widehat{G}) \oplus_q \ell^q_{sch}(\widehat{G})$ into $\ell^q_{sch}(\widehat{G}) \oplus_p \ell^q_{sch}(\widehat{G})$ with norm

$$\|S^*\|_{\ell^q_{sch}(\widehat{G})\oplus_q\ell^q_{sch}(\widehat{G})\to\ell^q_{sch}(\widehat{G})\oplus_p\ell^q_{sch}(\widehat{G})} \le 2^{-1/q}.$$

Hence, by the norm equality of an operator from Banach space to another and its adjoint on dual spaces [24, Theorem 4.10], we finally have that

$$\begin{aligned}\|S\|_{\ell^p_{sch}(\widehat{G})\oplus_q\ell^p_{sch}(\widehat{G})\to\ell^p_{sch}(\widehat{G})\oplus_p\ell^p_{sch}(\widehat{G})} &= \|S^*\|_{\left(\ell^p_{sch}(\widehat{G})\oplus_p\ell^p_{sch}(\widehat{G})\right)^*\to\left(\ell^p_{sch}(\widehat{G})\oplus_q\ell^p_{sch}(\widehat{G})\right)^*}\\ &= \|S^*\|_{\ell^q_{sch}(\widehat{G})\oplus_q\ell^q_{sch}(\widehat{G})\to\ell^q_{sch}(\widehat{G})\oplus_p\ell^q_{sch}(\widehat{G})} \le 2^{-1/q},\end{aligned}$$

which shows that (13) holds, and ends the proof. $\square$

As a consequence of the Clarkson inequality given in Propositition 8, one can prove that the spaces $\ell^p_{sch}(\widehat{G})$ have the following Kadec-Klee property for $1 < p < \infty$ [26, Addenda, p. 130].

Corollary 1. *Let $1 < p < \infty$ and $H, H_n \in \ell^p_{sch}(\widehat{G})$, $n \ge 1$. If*

$$\|H_n\|_{\ell^p_{sch}(\widehat{G})} \to \|H\|_{\ell^p_{sch}(\widehat{G})}, \quad n \to \infty,$$

and if $H_n \to H$, in the weak topology $\sigma(\ell^p_{sch}(\widehat{G}), \ell^q_{sch}(\widehat{G}))$, where q is the conjugate index of p, then

$$\|H_n - H\|_{\ell^p_{sch}(\widehat{G})} \to 0, \quad n \to \infty.$$

3.4 Geometric properties of $\ell^p_{sch}(\widehat{G})$

In this subsection, we consider various geometric properties such as uniform smoothness, uniform convexity as well as type and cotype properties of the family of ℓ^p-spaces $\ell^p_{sch}(\widehat{G})$ based on the Schatten-von Neumann ideals, which is defined in Definition 1.

As in the theory of classical l^p-spaces, the spaces $\ell^p_{sch}(\widehat{G})$ are uniformly convex and uniformly smooth, which is stated in the following

Theorem 1. *The space $\ell^p_{sch}(\widehat{G})$ is uniformly convex and uniformly smooth for $1 < p < \infty$.*

To prove Theorem 1, we first present the following lemma on lower (resp. upper) estimates for the modulus of convexity (resp. smoothness) similar to the case of classical ℓ^p-spaces. The proof follows from Proposition 8, and is therefore omitted.

Lemma 1. *Let $1 < p < \infty$, $0 < \varepsilon \le 2$ and $t > 0$. Let q be the conjugate index of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) If $1 < p \le 2$, then

$$\delta_{\ell^p_{sch}(\widehat{G})}(\varepsilon) \ge \frac{\varepsilon^q}{q \cdot 2^q}, \quad \rho_{\ell^p_{sch}(\widehat{G})}(t) \le \frac{t^p}{p}.$$

(ii) If $2 < p < \infty$, then

$$\delta_{\ell^p_{sch}(\widehat{G})}(\varepsilon) \ge \frac{\varepsilon^p}{p \cdot 2^p}, \quad \rho_{\ell^p_{sch}(\widehat{G})}(t) \le \frac{t^q}{q}.$$

We have the following modification of Prosposition 8 similar to [23, Theorem 5.3], which further helps us to provide optimal rate estimates on the modulus of convexity and the modulus of smoothness when $1 < p < 2$ and $2 < p < \infty$, respectively.

Theorem 2. *(i) If $2 \le p < \infty$, then for any $H_1, H_2 \in \ell^p_{sch}(\widehat{G})$, one has*

$$\left(\frac{1}{2}\left(\|H_1+H_2\|^p_{\ell^p_{sch}(\widehat{G})}+\|H_1-H_2\|^p_{\ell^p_{sch}(\widehat{G})}\right)\right)^{1/p} \le \left(\|H_1\|^2_{\ell^p_{sch}(\widehat{G})}+C_p\|H_2\|^2_{\ell^p_{sch}(\widehat{G})}\right)^{1/2},$$

where C_p is a positive constant depending only on p and $C_p \le 2p-1$.

(ii) If $1 < p \le 2$, then for any $H_1, H_2 \in \ell^p_{sch}(\widehat{G})$, one has

$$\left(\|H_1\|^2_{\ell^p_{sch}(\widehat{G})}+c_p\|H_2\|^2_{\ell^p_{sch}(\widehat{G})}\right)^{1/2} \le \left(\frac{1}{2}\left(\|H_1+H_2\|^p_{\ell^p_{sch}(\widehat{G})}+\|H_1-H_2\|^p_{\ell^p_{sch}(\widehat{G})}\right)\right)^{1/p},$$

where the constant c_p depends only on p and $c_p \ge \frac{p-1}{p+1}$.

Proof. (i). Let $2 \le p < \infty$. We first show that if (i) is valid for p, then it is also valid for $2p$. Assume that (i) holds for p and $H_1, H_2 \in \ell^{2p}_{sch}(\widehat{G})$. Consider $A = H_1^*H_1 + H_2^*H_2$ and $B = H_1^*H_2 + H_2^*H_1$. Hence, it follows from the Hölder inequality (see Proposition 1) that $A, B \in \ell^p_{sch}(\widehat{G})$.

Since $\|x\|_{\mathcal{S}^p(\mathcal{H})} = (\mathrm{Tr}(|x|^p))^{1/p}$, $x \in \mathcal{S}^p(\mathcal{H})$, we have $\|H_1+H_2\|^{2p}_{\ell^{2p}_{sch}(\widehat{G})} = \|A+B\|^p_{\ell^p_{sch}(\widehat{G})}$. Similarly, $\|H_1-H_2\|^{2p}_{\ell^{2p}_{sch}(\widehat{G})} = \|A-B\|^p_{\ell^p_{sch}(\widehat{G})}$. Therefore,

$$\begin{aligned}\frac{1}{2}\left(\|H_1+H_2\|^{2p}_{\ell^{2p}_{sch}(\widehat{G})}+\|H_1-H_2\|^{2p}_{\ell^{2p}_{sch}(\widehat{G})}\right) &= \frac{1}{2}\left(\|A-B\|^p_{\ell^p_{sch}(\widehat{G})}+\|A-B\|^p_{\ell^p_{sch}(\widehat{G})}\right)\\ &\le \left(\|A\|^2_{\ell^p_{sch}(\widehat{G})}+C_p\|B\|^2_{\ell^p_{sch}(\widehat{G})}\right)^{\frac{p}{2}},\end{aligned} \tag{15}$$

where the last inequality follows from (i) that was assumed to be true for p. Note that

$$\begin{aligned}\|A\|^2_{\ell^p_{sch}(\widehat{G})} &= \|H_1^*H_1+H_2^*H_2\|^2_{\ell^p_{sch}(\widehat{G})} \le \left(\|H_1^*H_1\|_{\ell^p_{sch}(\widehat{G})}+\|H_2^*H_2\|_{\ell^p_{sch}(\widehat{G})}\right)^2\\ &\le \left(\|H_1^*\|_{\ell^{2p}_{sch}(\widehat{G})}\|H_1\|_{\ell^{2p}_{sch}(\widehat{G})}+\|H_2^*\|_{\ell^{2p}_{sch}(\widehat{G})}\|H_2\|_{\ell^{2p}_{sch}(\widehat{G})}\right)^2 \overset{\text{Prop.2}}{=} \left(\|H_1\|^2_{\ell^{2p}_{sch}(\widehat{G})}+\|H_2\|^2_{\ell^{2p}_{sch}(\widehat{G})}\right)^2,\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\|B\|^2_{\ell^p_{sch}(\widehat{G})} = \|H_1^*H_2+H_2^*H_1\|^2_{\ell^p_{sch}(\widehat{G})} \le \left(\|H_1^*H_2\|_{\ell^p_{sch}(\widehat{G})}+\|H_2^*H_1\|_{\ell^p_{sch}(\widehat{G})}\right)^2\\ &\le \left(\|H_1^*\|_{\ell^{2p}_{sch}(\widehat{G})}\|H_2\|_{\ell^{2p}_{sch}(\widehat{G})}+\|H_2^*\|_{\ell^{2p}_{sch}(\widehat{G})}\|H_1\|_{\ell^{2p}_{sch}(\widehat{G})}\right)^2 \overset{\text{Prop.2}}{=} 4\|H_1\|^2_{\ell^{2p}_{sch}(\widehat{G})}\|H_2\|^2_{\ell^{2p}_{sch}(\widehat{G})}.\end{aligned} \tag{17}$$

Hence, combining (16) and (17) together with (15), we have

$$\begin{aligned}
&\frac{1}{2}\Big(\|H_1+H_2\|^{2p}_{\ell^{2p}_{sch}(\widehat{G})}+\|H_1-H_2\|^{2p}_{\ell^{2p}_{sch}(\widehat{G})}\Big)\\
&\le\left(\left(\|H_1\|^2_{\ell^{2p}_{sch}(\widehat{G})}+\|H_2\|^2_{\ell^{2p}_{sch}(\widehat{G})}\right)^2+4C_p\,\|H_1\|^2_{\ell^{2p}_{sch}(\widehat{G})}\,\|H_2\|^2_{\ell^{2p}_{sch}(\widehat{G})}\right)^{\frac{p}{2}}\\
&=\left(\|H_1\|^4_{\ell^{2p}_{sch}(\widehat{G})}+\|H_2\|^4_{\ell^{2p}_{sch}(\widehat{G})}+(4C_p+2)\,\|H_1\|^2_{\ell^{2p}_{sch}(\widehat{G})}\,\|H_2\|^2_{\ell^{2p}_{sch}(\widehat{G})}\right)^{\frac{p}{2}}\\
&\le\left(\|H_1\|^4_{\ell^{2p}_{sch}(\widehat{G})}+(1+2C_p)^2\,\|H_2\|^4_{\ell^{2p}_{sch}(\widehat{G})}+(4C_p+2)\,\|H_1\|^2_{\ell^{2p}_{sch}(\widehat{G})}\,\|H_2\|^2_{\ell^{2p}_{sch}(\widehat{G})}\right)^{\frac{p}{2}}\\
&\le\left(\left(\|H_1\|^2_{\ell^{2p}_{sch}(\widehat{G})}+(1+2C_p)\,\|H_2\|^2_{\ell^{2p}_{sch}(\widehat{G})}\right)^2\right)^{\frac{p}{2}}=\left(\|H_1\|^2_{\ell^{2p}_{sch}(\widehat{G})}+(1+2C_p)\,\|H_2\|^2_{\ell^{2p}_{sch}(\widehat{G})}\right)^p.
\end{aligned}$$

Therefore, we proved an analogue of (i) for $2p$ in the form

$$\left(\frac{1}{2}\Big(\|H_1+H_2\|^{2p}_{\ell^{2p}_{sch}(\widehat{G})}+\|H_1-H_2\|^{2p}_{\ell^{2p}_{sch}(\widehat{G})}\Big)\right)^{1/2p}\le\left(\|H_1\|^2_{\ell^{2p}_{sch}(\widehat{G})}+(1+2C_p)\,\|H_2\|^2_{\ell^{2p}_{sch}(\widehat{G})}\right)^{\frac{1}{2}},$$

and obviously $C_{2p}\le 2C_p+1$. Note that $C_2=1$ by the parallelogram identity. Hence, it is easy to prove by induction that $C_{2^n}\le 2C_{2^{n-1}}+1\le 2\left(2^{n-1}-1\right)+1=2^n-1$.

Now assume that $2\le p<\infty$, then there exists $n\in\mathbb{N}$ such that $2^n\le p<2^{n+1}$. Choose $\theta\in(0,1)$ such that $1/p=(1-\theta)/2^n+\theta/2^{n+1}$. Since (i) holds for 2^n , it follows that an operator T_1 from $\ell^{2^n}_{sch}(\widehat{G})\oplus_{2,2^n-1}\ell^{2^n}_{sch}(\widehat{G})$ into $\ell^{2^n}_{sch}(\widehat{G})\oplus_{2^n}\ell^{2^n}_{sch}(\widehat{G})$ defined as

$$T_1(H_1,H_2)(\xi,\xi):=(H_1(\xi)+H_2(\xi),H_1(\xi)-H_2(\xi))$$

is bounded with norm

$$\|T_1\|_{\ell^{2^n}_{sch}(\widehat{G})\oplus_{2,2^n-1}\ell^{2^n}_{sch}(\widehat{G})\to\ell^{2^n}_{sch}(\widehat{G})\oplus_{2^n}\ell^{2^n}_{sch}(\widehat{G})}\le 2^{1/2^n}. \tag{18}$$

Similarly, since (i) holds for 2^{n+1}, it follows that an operator T_2 from $\ell^{2^{n+1}}_{sch}(\widehat{G})\oplus_{2,2^{n+1}-1}\ell^{2^{n+1}}_{sch}(\widehat{G})$ into $\ell^{2^{n+1}}_{sch}(\widehat{G})\oplus_{2^{n+1}}\ell^{2^{n+1}}_{sch}(\widehat{G})$ defined by the same action as T_1 is bounded with norm

$$\|T_2\|_{\ell^{2^{n+1}}_{sch}(\widehat{G})\oplus_{2,2^{n+1}-1}\ell^{2^{n+1}}_{sch}(\widehat{G})\to\ell^{2^{n+1}}_{sch}(\widehat{G})\oplus_{2^{n+1}}\ell^{2^{n+1}}_{sch}(\widehat{G})}\le 2^{\frac{1}{2^{n+1}}}. \tag{19}$$

On the other hand, by Proposition 3, one has

$$\left(\ell^{2^n}_{sch}(\widehat{G})\oplus_{2,2^n-1}\ell^{2^n}_{sch}(\widehat{G}),\ell^{2^{n+1}}_{sch}(\widehat{G})\oplus_{2,2^{n+1}-1}\ell^{2^{n+1}}_{sch}(\widehat{G})\right)_\theta=\ell^p_{sch}(\widehat{G})\oplus_{2,C_p}\ell^p_{sch}(\widehat{G}), \tag{20}$$

where $C_p=(2^n-1)^{1-\theta}(2^{n+1}-1)^\theta$. Moreover, by Proposition 3, one has

$$\left(\ell^{2^n}_{sch}(\widehat{G})\oplus_{2^n}\ell^{2^n}_{sch}(\widehat{G}),\ell^{2^{n+1}}_{sch}(\widehat{G})\oplus_{2^{n+1}}\ell^{2^{n+1}}_{sch}(\widehat{G})\right)_\theta=\ell^p_{sch}(\widehat{G})\oplus_p\ell^p_{sch}(\widehat{G}). \tag{21}$$

Therefore, by (20), (21) and the general interpolation result [3, Corollary 5.5.4] together with (18), (19), we have that an operator T from $\ell^p_{sch}(\widehat{G})\oplus_{2,C_p}\ell^p_{sch}(\widehat{G})$ into $\ell^p_{sch}(\widehat{G})\oplus_p\ell^p_{sch}(\widehat{G})$ defined with same action as operator T_1 and T_2 is bounded for any $2\le p<\infty$ with norm

$$\|T\|_{\ell^p_{sch}(\widehat{G})\oplus_{2,C_p}\ell^p_{sch}(\widehat{G})\to\ell^p_{sch}(\widehat{G})\oplus_p\ell^p_{sch}(\widehat{G})}\le\left(2^{1/2^n}\right)^{1-\theta}\left(2^{1/2^{n+1}}\right)^\theta=2^{1/p},$$

which proves (i). Moreover,

$$C_p = (2^n - 1)^{1-\theta}(2^{n+1} - 1)^{\theta} \leq (2^{n+1} - 1)^{1-\theta}(2^{n+1} - 1)^{\theta} = 2^{n+1} - 1 \leq 2p - 1.$$

(ii). First note, by replacing H_1 and H_2 with $H_1 + H_2$ and $H_1 - H_2$, respectively, (ii) is equivalent to the following

$$\left(\|H_1 + H_2\|^2_{\ell^p_{sch}(\widehat{G})} + c_p \|H_1 - H_2\|^2_{\ell^p_{sch}(\widehat{G})}\right)^{1/2} \leq 2^{1/q} \left(\|H_1\|^p_{\ell^p_{sch}(\widehat{G})} + \|H_2\|^p_{\ell^p_{sch}(\widehat{G})}\right)^{1/p}, \quad (22)$$

In order to prove (ii), we prove (22).

Define the map S from $\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})$ into $\ell^p_{sch}(\widehat{G}) \oplus_{2,c_p} \ell^p_{sch}(\widehat{G})$ by setting

$$S(H_1, H_2)(\xi, \xi) := (H_1(\xi) + H_2(\xi), H_1(\xi) - H_2(\xi)).$$

Note that it is enough to show that

$$\|S\|_{\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G}) \to \ell^p_{sch}(\widehat{G}) \oplus_{2,c_p} \ell^p_{sch}(\widehat{G})} \leq 2^{1/q}. \quad (23)$$

By the similar argument as in the proof of Proposition 8,(i), it is easy to show that the adjoint operator S^* from $\left(\ell^p_{sch}(\widehat{G}) \oplus_{2,c_p} \ell^p_{sch}(\widehat{G})\right)^*$ into $\left(\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})\right)^*$ is defined as

$$S(H_1, H_2)(\xi, \xi) = (H_1(\xi) + H_2(\xi), H_1(\xi) - H_2(\xi)).$$

Furthermore, by Proposition 6, one has

$$\left(\ell^p_{sch}(\widehat{G}) \oplus_{2,c_p} \ell^p_{sch}(\widehat{G})\right)^* = \ell^q_{sch}(\widehat{G}) \oplus_{2,\frac{1}{c_p}} \ell^q_{sch}(\widehat{G}),$$
$$\left(\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})\right)^* = \ell^q_{sch}(\widehat{G}) \oplus_q \ell^q_{sch}(\widehat{G}).$$

By (i), we have

$$\|S\|_{\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G}) \to \ell^p_{sch}(\widehat{G}) \oplus_{2,c_p} \ell^p_{sch}(\widehat{G})} = \|S^*\|_{\left(\ell^p_{sch}(\widehat{G}) \oplus_{2,c_p} \ell^p_{sch}(\widehat{G})\right)^* \to \left(\ell^p_{sch}(\widehat{G}) \oplus_p \ell^p_{sch}(\widehat{G})\right)^*}$$
$$= \|S^*\|_{\ell^q_{sch}(\widehat{G}) \oplus_{2,\frac{1}{c_p}} \ell^q_{sch}(\widehat{G}) \to \ell^q_{sch}(\widehat{G}) \oplus_q \ell^q_{sch}(\widehat{G})} = \|S^*\|_{\ell^q_{sch}(\widehat{G}) \oplus_{2,C_q} \ell^q_{sch}(\widehat{G}) \to \ell^q_{sch}(\widehat{G}) \oplus_q \ell^q_{sch}(\widehat{G})} \leq 2^{1/q},$$

which proves (23). Moreover,

$$c_p = \frac{1}{C_q} \geq \frac{1}{2q - 1} \geq \frac{p - 1}{p + 1},$$

completing the proof. □

Theorem 2 provides the following optimal rate estimates for $\delta_{\ell^p_{sch}(\widehat{G})}(\varepsilon)$ when $1 < p < 2$ and for $\rho_{\ell^p_{sch}(\widehat{G})}(t)$ when $2 < p < \infty$.

Corollary 2. *(i) If $1 < p \leq 2$, then for any $0 < \varepsilon < 2$, one has*

$$\delta_{\ell^p_{sch}(\widehat{G})}(\varepsilon) \geq \frac{c_p}{8}\varepsilon^2.$$

(ii) If $2 \le p < \infty$, then for any $t > 0$, one has

$$\rho_{\ell^p_{sch}(\widehat{G})}(t) \le \frac{C_p}{2} t^2.$$

Proof. (i). Let $1 < p \le 2$. Then, by (ii) of Theorem 2, one has

$$\left(\|K_1\|^2_{\ell^p(\widehat{G})} + c_p \|K_2\|^2_{\ell^p_{sch}(\widehat{G})}\right)^{1/2} \le \left(\frac{1}{2}\left(\|K_1 + K_2\|^p_{\ell^p_{sch}(\widehat{G})} + \|K_1 - K_2\|^p_{\ell^p_{sch}(\widehat{G})}\right)\right)^{1/p}$$

for $K_1, K_2 \in \ell^p_{sch}(\widehat{G})$. Denote $H_1 = K_1 + K_2 \in \ell^p_{sch}(\widehat{G})$ and $H_2 = K_1 - K_2 \in \ell^p_{sch}(\widehat{G})$. Hence, the last inequality can be rewritten as

$$\left(\left\|\frac{H_1 + H_2}{2}\right\|^2_{\ell^p_{sch}(\widehat{G})} + c_p \left\|\frac{H_1 - H_2}{2}\right\|^2_{\ell^p_{sch}(\widehat{G})}\right)^{1/2} \le \left(\frac{1}{2}\left(\|H_1\|^p_{\ell^p_{sch}(\widehat{G})} + \|H_2\|^p_{\ell^p_{sch}(\widehat{G})}\right)\right)^{1/p}.$$

Assuming that $\|H_1\|_{\ell^p_{sch}(\widehat{G})} = \|H_2\|_{\ell^p_{sch}(\widehat{G})} = 1$ and $\|H_1 - H_2\|_{\ell^p_{sch}(\widehat{G})} = \varepsilon$, it follows that

$$\left(\left\|\frac{H_1 + H_2}{2}\right\|^2_{\ell^p_{sch}(\widehat{G})} + \frac{c_p}{4}\varepsilon^2\right)^{1/2} \le 1.$$

Therefore,

$$\frac{c_p}{4}\varepsilon^2 \le 1 - \left\|\frac{H_1 + H_2}{2}\right\|^2_{\ell^p_{sch}(\widehat{G})} = \left(1 - \left\|\frac{H_1 + H_2}{2}\right\|_{\ell^p_{sch}(\widehat{G})}\right)\left(1 + \left\|\frac{H_1 + H_2}{2}\right\|_{\ell^p_{sch}(\widehat{G})}\right)$$

$$\le \left(1 - \left\|\frac{H_1 + H_2}{2}\right\|_{\ell^p_{sch}(\widehat{G})}\right)\left(1 + \frac{\|H_1\|_{\ell^p_{sch}(\widehat{G})} + \|H_2\|_{\ell^p_{sch}(\widehat{G})}}{2}\right) = 2\left(1 - \left\|\frac{H_1 + H_2}{2}\right\|_{\ell^p_{sch}(\widehat{G})}\right).$$

Hence, taking infimum over all $H_1, H_2 \in \ell^p_{sch}(\widehat{G})$ with norm one and $\|H_1 - H_2\|_{\ell^p_{sch}(\widehat{G})} = \varepsilon$, we finally have

$$\delta_{\ell^p_{sch}(\widehat{G})} \ge \frac{c_p}{8}\varepsilon^2$$

(ii). Let $2 \le p < \infty$ and $H_1, H_2 \in \ell^p_{sch}(\widehat{G})$. Define a function $f(x) = x^{1/p}$, $x \ge 0$. Assume that $x_1 = \|H_1 + tH_2\|^p_{\ell^p_{sch}(\widehat{G})}$ and $x_2 = \|H_1 - tH_2\|^p_{\ell^p_{sch}(\widehat{G})}$. Hence, since f is a concave function, it follows that

$$\begin{aligned}\frac{\|H_1 + tH_2\|_{\ell^p_{sch}(\widehat{G})} + \|H_1 - tH_2\|_{\ell^p_{sch}(\widehat{G})}}{2} &= \frac{f(x_1) + f(x_2)}{2} \\ \le f\left(\frac{x_1 + x_2}{2}\right) &= \left(\frac{\|H_1 + tH_2\|^p_{\ell^p_{sch}(\widehat{G})} + \|H_1 - tH_2\|^p_{\ell^p_{sch}(\widehat{G})}}{2}\right)^{1/p}.\end{aligned} \tag{24}$$

Moreover, by Theorem 2 (i), one has

$$\left(\frac{\|H_1 + tH_2\|^p_{\ell^p_{sch}(\widehat{G})} + \|H_1 - tH_2\|^p_{\ell^p_{sch}(\widehat{G})}}{2}\right)^{1/p} \le \left(\|H_1\|^2_{\ell^p_{sch}(\widehat{G})} + C_p \|tH_2\|^2_{\ell^p_{sch}(\widehat{G})}\right)^{1/2}. \tag{25}$$

Assuming $\|H_1\|_{\ell^p_{sch}(\widehat{G})} = \|H_2\|_{\ell^p_{sch}(\widehat{G})} = 1$, and combining (24) together with (25), we have

$$\frac{\|H_1 + tH_2\|_{\ell^p_{sch}(\widehat{G})} + \|H_1 - tH_2\|_{\ell^p_{sch}(\widehat{G})}}{2} - 1 \leq \left(1 + C_p t^2\right)^{1/2} - 1 \leq 1 + \frac{C_p}{2} t^2 - 1 = \frac{C_p}{2} t^2,$$

where the last inequality follows from the Bernoulli's inequality.

Therefore, taking supremum over all $H_1, H_2 \in \ell^p_{sch}(\widehat{G})$ with norm one, we finally have that

$$\rho_{\ell^p_{sch}(\widehat{G})}(t) \leq \frac{C_p}{2} t^2,$$

completing the proof. □

One of the applications of above given estimates in Lemma 1 and Corollary 2 is the following noncommutative extension of a classical theorem of Orlicz concerning unconditionally convergent series (see, for example, [20, Theorem 1.e.11]).

Let X be a Banach space and $\{x_n\}_{n\geq 1}$ be a sequence of elements from X. We recall that a series $\sum_{j=1}^{\infty} x_j$ is unconditionally convergent in X, if $\sum_{j=1}^{\infty} \theta_j x_j$ converges in X for all choices of signs $\theta_j = \pm 1$, $j \geq 1$.

Corollary 3. *If $\sum\limits_{j=1}^{\infty} H_j$ is an unconditionally convergent series in $\ell^p_{sch}(\widehat{G})$, $1 < p < \infty$, then*

$$\sum_{j=1}^{\infty} \|H_j\|^{max\{p,2\}}_{\ell^p_{sch}(\widehat{G})} < \infty.$$

Proof. Let us first recall the theorem of Kadec [17, Theorem 1], which says that if $(X, \|\cdot\|_X)$ is a uniformly convex Banach space with modulus of convexity $\delta_X(\cdot)$ and if $\sum\limits_{j=1}^{\infty} x_j$ is unconditionally convergent, then $\sum_{j=1}^{\infty} \delta_X\left(\|x_j\|_X\right) < \infty$.

By Theorem 1, we have that $\ell^p_{sch}(\widehat{G})$ is uniformly convex for $1 < p < \infty$. Therefore, by Kadec's Theorem, one has

$$\sum_{j=1}^{\infty} \delta_{\ell^p_{sch}(\widehat{G})}\left(\|H_j\|_{\ell^p_{sch}(\widehat{G})}\right) < \infty$$

for $\{H_j\}_{j\geq 1} \subset \ell^p_{sch}(\widehat{G})$. Hence, by Corollary 2, one has

$$\sum_{j=1}^{\infty} \frac{c_p}{8} \|H_j\|^2_{\ell^p_{sch}(\widehat{G})} \leq \sum_{j=1}^{\infty} \delta_{\ell^p_{sch}(\widehat{G})}\left(\|H_j\|_{\ell^p_{sch}(\widehat{G})}\right) < \infty, \tag{26}$$

for $1 < p \leq 2$, where c_p is the constant from Theorem 2 (ii). Moreover, by Lemma 1 (ii), one has

$$\sum_{j=1}^{\infty} \frac{1}{p \cdot 2^p} \|H_j\|^p_{\ell^p_{sch}(\widehat{G})} \leq \sum_{j=1}^{\infty} \delta_{\ell^p_{sch}(\widehat{G})}\left(\|H_j\|_{\ell^p_{sch}(\widehat{G})}\right) < \infty, \tag{27}$$

for $2 < p < \infty$. Therefore, combining (26) and (27), we finally have $\sum_{j=1}^{\infty} \|H_j\|^{\max\{2,p\}}_{\ell^p_{sch}(\widehat{G})} < \infty$, for $1 < p < \infty$. □

Finally, by Proposition 8 and Theorem 2, we can identify the type and cotype (cf. Definition 4) of the Banach space $\ell^p_{sch}(\widehat{G})$ for $1 < p < \infty$. Note that the result is analogous with the result for classical ℓ^p-spaces. The proof is based on the same argument as [9, Theorem 7.14.17] and is therefore omitted.

Theorem 3. *Let $1 < p < \infty$. Then $\ell^p_{sch}(\widehat{G})$ is of type $\min\{2,p\}$ and cotype $\max\{2,p\}$. In particular, let $\{H_j\}_{j=1}^n \subset \ell^p_{sch}(\widehat{G})$ be a finite sequence. If $1 < p \leq 2$, then*

$$\sqrt{c_p}\left(\sum_{j=1}^n \|H_j\|^2_{\ell^p_{sch}(\widehat{G})}\right)^{\frac{1}{2}} \leq \left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)H_j\right\|^2_{\ell^p_{sch}(\widehat{G})} dt\right)^{1/2} \leq \left(\sum_{j=1}^n \|H_j\|^p_{\ell^p_{sch}(\widehat{G})}\right)^{\frac{1}{p}}, \quad (28)$$

where c_p is the constant given in Theorem 2 (ii). If $2 \leq p < \infty$, then

$$\left(\sum_{j=1}^n \|H_j\|^p_{\ell^p_{sch}(\widehat{G})}\right)^{\frac{1}{p}} \leq \left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)H_j\right\|^2_{\ell^p_{sch}(\widehat{G})} dt\right)^{1/2} \leq \sqrt{C_p}\left(\sum_{j=1}^n \|H_j\|^2_{\ell^p_{sch}(\widehat{G})}\right)^{\frac{1}{2}}, \quad (29)$$

where C_p is the constant given in Theorem 2 (i).

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