3-бөлім	**Раздел 3**	**Section 3**
Қолданбалы математика	**Прикладная математика**	**Applied mathematics**

IRSTI 27.25.19 DOI: https://doi.org/10.26577/JMMCS1313202611

G.E. Taugynbayeva, N.K. Bakytova*
L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
*e-mail: nazerkebakytova29@gmail.com

EXACT ORDERS OF FUNCTION RECOVERY FROM INACCURATE DATA IN THE UNIFORM METRIC AND THE LIMITING ERROR OF INACCURATE INFORMATION

This paper considers the problem of recovering multivariate functions from $SW_2^r(0,1)^s$ Sobolev classes with dominating mixed derivative in the uniform metric $L^\infty(0,1)^s$ based on inaccurate information. The aim of this work is to investigate the accuracy of recovering functions from the considered class based on a finite number of linear functionals determined by a finite amount of inaccurate information in the form of Fourier coefficients, as well as to determine the optimal error of the inaccurate information for an optimal computational aggregate and to establish its massiveness. The study is carried out in the context of the problem of Computational (Numerical) Diameter. As a result of the research, exact orders of recovery of functions from Sobolev classes with dominating mixed derivative in the uniform metric from inaccurate data are obtained, and an optimal computational aggregate is constructed. The limiting error of inaccurate information for the constructed computational aggregate is found, and a set of computational aggregates has been constructed such that their limiting errors do not exceed the limiting error of the given aggregate, that is, the massiveness of the limiting error is determined. The obtained theoretical results are confirmed by numerical experiments. All of the above constitutes the scientific novelty of this work.
Keywords: Sobolev class with dominating mixed derivative, uniform metric, function recovery from inaccurate information, limiting error, massiveness of the limiting error.

Г.Е. Таугынбаева, Н.К. Бақытова*
Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Астана, Қазақстан
*e-mail: nazerkebakytova29@gmail.com

Бірқалыпты метрикада дәл емес мәліметтер бойынша функцияны жуықтау қателігінің дәл реті және дәл емес мәліметтің шектік қателігі

Мақалада басым аралас туындылы $SW_2^r(0,1)^s$ Соболев кластарынан алынған көп айнымалылы функцияларды $L^\infty(0,1)^s$ бірқалыпты метрикасында дәл емес мәлімет бойынша жуықтау есебі қарастырылған. Жұмыстың мақсаты қарастырылып отырған кластағы функцияларды қателікпен берілген сызықтық функционалдардың ақырлы жиыны бойынша жуықтау дәлдігін табу және Фурье коэффициенті түріндегі сандық мәліметтердің ақырлы жиыны негізіндегі оптималді есептеуіш агрегат үшін дәл емес мәліметтің шектік қателігі мен оның массивтілігін анықтау болып табылады. Мақаладағы зерттеу Компьютерлік (Есептеуіш) Диаметр есебі аясында жүргізілген. Зерттеу нәтижесінде басым аралас туындысы бар Соболев кластарының функцияларын бірқалыпты метрикада дәл емес мәліметтер бойынша жуықтаудың дәл реттері алынды және оптималды есептеуіш агрегаты құрылды. Мақаладағы зерттеу Компьютерлік (Есептеуіш) Диаметр есебі аясында жүргізілген. Зерттеу нәтижесінде басым аралас туындысы бар Соболев кластарының функцияларын бірқалыпты метрикада дәл емес мәліметтер бойынша жуықтаудың дәл реттері алынды және оптималды есептеуіш агрегаты құрылды.

Құрылған есептеуіш агрегат үшін дәл емес мәліметтің шектік қателігі, сондай-ақ шекті қателіктің массивтілігі анықталды, яғни шектік қателіктері берілген агрегаттың шекті қатесінен аспайтын есептеуіш агрегаттар жиыны құрылды. Алынған теориялық нәтижелер сандық эксперименттер арқылы расталған. Жоғарыда аталғандардың барлығы осы жұмыстың ғылыми жаңалығын құрайды.
Түйін сөздер: басым аралас туындылы Соболев класы, бірқалыпты метрика, дәл емес ақпарат бойынша функцияны жуықтау, шектік қателік, шектік қателіктің массивтілігі.

Г.Е. Таугынбаева, Н.К. Бақытова*
Евразийский национальный университет им. Л.Н. Гумилёва, Астана, Казахстан
*e-mail: nazerkebakytova29@gmail.com

Точные порядки восстановления функции по неточным данным в равномерной метрике и предельная погрешность неточной информации

В статье рассматривается задача восстановления функций многих переменных из классов $SW_2^r(0,1)^s$ Соболева с доминирующей смешанной производной в $L^\infty(0,1)^s$ равномерной метрике по неточной информации. Целью работы является исследование точности восстановления функций указанного класса по конечному числу линейных функционалов, которые заданы конечным объёмом неточной информации в виде коэффициентов Фурье, а также определение предельной погрешности неточной информации для оптимального вычислительного агрегата, и в определении его массивности. В статье исследование проводится в контексте задачи Компьютерного (Вычислительного) Поперечника. В результате исследования получены точные порядки восстановления функций из классов Соболева с доминирующей смешанной производной в равномерной метрике по неточным данным и построен оптимальный вычислительный агрегат. Найдена предельная погрешность неточной информации для построенного вычислительного агрегата, а также построено множество вычислительных агрегатов, предельные погрешности которых не превышают предельную погрешность данного агрегата, то есть определена массивность предельной погрешности. Полученные теоретические результаты подтверждаются численными экспериментами. Все перечисленное составляет научную новизну данной работы.
Ключевые слова: класс Соболева с доминирующей смешанной производной, равномерная метрика, восстановление функции по неточной информации, предельная погрешность, массивность предельной погрешности.

1 Introduction

The problem of approximate reconstruction of functions from accurate and inaccurate information plays a central role in modern approximation theory, computational mathematics, and the theory of numerical methods. In cases where information about a function is given inaccurately, the problem of approximate reconstruction reduces to constructing an approximation aggregate that minimizes the error in a given function space.

In this work, the information functionals are given with error, where the magnitude of the error must be as large as possible while preserving the order of recovery corresponding to exact values of the functionals, with the simultaneous identification of a set of recovery operators among which the aggregate under study is optimal.

This entire formulation is called the *Computational (Numerical) Diameter* (C(N)D), and is divided into three sequential tasks:

C(N)D-1. Recovery from accurate information, depending on the type of functionals and algorithms for processing the numerical information obtained from them, encompasses all known approximation theory, numerical analysis, computational mathematics, and function theory – Fourier series, bases;

C(N)D-2. In the optimal computational aggregate, the values of the informational functionals can be replaced by nearby values while maintaining optimality. The search for the largest such deviations forms an independent problem of finding extremal errors – a new optimization problem.;

C(N)D-3. There exists, or does not exist, another computational aggregate with a structure analogous to that of the considered optimal computational aggregate, and perhaps even more general, but with a larger order of extremal error.

If it turns out that in C(N)D-1 there is more than one extremal computational aggregate, then a C(N)D-2 and C(N)D-3 analysis is performed for each of them, since their computational quality is determined not only by the magnitude of the limiting error but also by how well the structure of the aggregate is adapted to the specifics of the application.

The C(N)D problem is presented in full historical and mathematical depth in articles [1]–[5].

The problem of recovery from accurate information (C(N)D-1) is well known and has been developed over many decades in numerous publications, each with its own notation. These include works of A. Sard, S.M. Nikol'skii, N.S. Bakhvalov, K.I. Babenko and S.L. Sobolev, C.A. Micchelli and T.J. Rivlin, J.F. Traub, G. Wasilkowski, H. Woźniakowski, A. Pinkus, V.N. Temlyakov, N.P. Korneichuk, E. Novak, H. Woźniakowski, J. Vybiral, B. Kashin, E. Kosov, I. Limonova, and many others (see, e.g., [6]– [8]).

Publications on recovery from inaccurate information fall into three groups: the first comprising works of N. Temirgaliyev and his co-authors [1]– [5], the second – of V.M. Tikhomirov, G.G. Magaril-Il'yaev, K.Yu. Osipenko, and their co-authors [9]– [10], and the third – J. Traub, H. Woźniakowski, L. Plaskota and their co-authors [11]– [12]. All differ both in problem formulation and in terminology for essentially the same mathematical objects. The problem of recovery from inaccurate information in the context of second part of C(N)D-2 and C(N)D-3 is new.

In various formulations, the problem of approximate recovery of functions has been studied by many authors (see, e.g., [1]– [5] and the bibliography therein, and all necessary notation is introduced below).

For example, in [13], S.A. Smolyak constructed grids of the form

$$\xi = \left(\tfrac{k_1}{2^{\tau_1}}, \dots, \tfrac{k_s}{2^{\tau_s}}\right),$$

where $n - s \le \tau_1 + \dots + \tau_s \le n$ and $0 \le k_j < 2^{\tau_j}$, $(\tau_j, k_j \in \mathbb{Z}_+,\ j = 1, \dots, s)$, and the corresponding linear operators $\varphi_N(z_1, \dots, z_N; x)$ with recovery error in the metric $L^\infty(0,1)^s$ on the classes $SW_2^r(0,1)^s$ for $r > \frac{1}{2}$:

$$\sup_{f \in SW_2^r(0,1)^s} \big\| f(x) - \varphi_N(f(\xi_1), \dots, f(\xi_N); x) \big\|_{L^\infty(0,1)^s} \ll_{r,s} \frac{\ln^{r(s-1)} N}{N^{r-\frac{1}{2}}} \quad (N = 1, 2, \dots).$$

K.E. Sherniyazov [14] established that for $r > \frac{1}{2}$ the following estimate holds:

$$\frac{1}{N^{r-\frac{1}{2}}} \ll_{s,r} \delta_N\big(0; Tf = f; SW_2^r(0,1)^s; \mathcal{L}_N\big)_{L^2(0,1)^s} \ll_{s,r} \frac{\ln^{(r+\frac{1}{2})(s-1)} N}{N^{r-\frac{1}{2}}} \quad (N = 1, 2, \dots).$$

In the work of Sh.U. Azhgaliyev [15], the exact order of the information power of computational aggregates constructed from Fourier coefficients was found for the recovery of functions from Sobolev classes with dominant mixed derivative in the Hilbert and uniform metrics:

$$\delta_N\big(0; Tf = f; SW_2^r(0,1)^s; \mathcal{L}_N\big)_{L^2(0,1)^s} \asymp \frac{\ln^{r(s-1)} N}{N^r} \quad (N = 1, 2, \ldots;\ r > \tfrac{1}{2}),$$

and

$$\delta_N\big(0; Tf = f; SW_2^r(0,1)^s; \mathcal{L}_N\big)_{L^\infty(0,1)^s} \asymp \frac{\ln^{r(s-1)} N}{N^{r-\frac{1}{2}}} \quad (N = 1, 2, \ldots;\ r > 0).$$

In paper [16], the C(N)D problem for the recovery of functions from Sobolev classes with dominant mixed derivative in the Hilbert norm was completely solved. Namely, the following theorem was obtained.

Theorem A [see [16]]. Let s, $(s = 1, 2, \ldots)$ and $r > 1$ be given. Then for the sequence

$$\varepsilon_N = \frac{\ln^{r(s-1)} N}{N^{r+\frac{1}{2}}} \quad (N = 2, 3 \ldots),$$

the following relations hold (where $D_N \equiv D_N(F)_Y$ denotes a certain set of aggregates $(l^{(N)}, \varphi_N) \equiv (l_1, \ldots, l_N; \varphi_N)$):

$$\text{C(N)D-1:} \quad \delta_N(0; Tf = f; SW_2^r(0,1)^s; \mathcal{F}_N)_{L^2(0,1)^s} \asymp \frac{\ln^{r(s-1)} N}{N^r},$$

$$\text{C(N)D-2:} \quad \delta_N(\widetilde{\varepsilon}_N) \equiv \delta_N(\widetilde{\varepsilon}_N; Tf = f; SW_2^r(0,1)^s; \mathcal{F}_N)_{L^2(0,1)^s} \asymp \widetilde{\varepsilon}_N \cdot \sqrt{N},$$

and for every increasing positive sequence $\{\eta_N\}_{N=1}^\infty$ the following equality holds:

$$\text{C(N)D-3:} \quad \lim_{N\to\infty} \frac{\delta_N\Big(\eta_N \widetilde{\varepsilon}_N = \frac{\eta_N \ln^{r(s-1)} N}{N^{r+\frac{1}{2}}}; Tf = f; SW_2^r(0,1)^s; \mathcal{F}_N\Big)_{L^2(0,1)^s}}{\delta_N(0; Tf = f; SW_2^r(0,1)^s; \mathcal{F}_N)_{L^2(0,1)^s}} = +\infty.$$

2 Problem formulation

Let us present the short formulation of the **Computational (Numerical) Diameter** problem (a complete definition, discussion, comparison with known results and related considerations, and the history of the problem are given in [1]– [6]).

The following definitions serve as the starting point for each of the three components C(N)D-1, -2, and -3 (explanations of notation are given below):

$$\begin{aligned}
\delta_N\big(\varepsilon_N;\ (l^{(N)}, \varphi_N)\big)_Y \equiv \sup\Big\{ &\big\| Tf(\,\cdot\,) - \varphi_N(z_1(f), \ldots, z_N(f);\ \cdot\,)\big\|_Y \\
&: \ f \in F,\ |l_\tau(f) - z_\tau(f)| \le \varepsilon_N\ (\tau = 1, \ldots, N)\Big\} \\
= \sup\Big\{ &\big\| Tf(\,\cdot\,) - \varphi_N(l_1(f) + \gamma_N^{(1)}\varepsilon_N, \ldots, l_N(f) + \gamma_N^{(N)}\varepsilon_N;\ \cdot\,)\big\|_Y \\
&: \ f \in F,\ |\gamma_N^{(\tau)}| \le 1\ (\tau = 1, \ldots, N)\Big\}
\end{aligned}$$

and

$$\delta_N(\varepsilon_N;\, D_N)_Y \equiv \delta_N(\varepsilon_N;\, T;\, F;\, D_N)_Y = \inf_{(l^{(N)},\varphi_N)\in D_N} \delta_N\big(\varepsilon_N;\, (l^{(N)},\varphi_N)\big)_Y,$$

where ε_N is a nonnegative sequence. In the case $\varepsilon_N \equiv 0$, this is the problem of recovery from accurate information; otherwise, it is recovery from inaccurate information.

We note at once that the problem formulation of C(N)D-1 has a respectable pedigree and has served as the subject of investigation by various groups of mathematicians. In this problem, which is common to all, different notation and terminology were employed. Since, within the C(N)D framework, C(N)D-1 is extended by two further problems — C(N)D-2 and C(N)D-3 — the notation and terminology presented below are adapted to this new setting and, for the same content, may in certain cases differ from those previously used in the mathematical literature.

We assume the following to be given: a class F and a normed space Y (or $Y \equiv C$, where C denotes throughout the field of complex numbers) of functions with domains Ω_F and Ω_Y respectively, and an operator T acting from F into Y. Further, $l_1,\dots,l_N$ is a set of functionals $l^{(N)}$ (not necessarily linear), by means of which each function f from F is assigned a finite sequence $l_1(f),\dots,l_N(f)$ – the numerical information about f of volume N. An algorithm for processing the approximate *information* $z_1(f),\dots,z_N(f)$ about f, obtained from the functionals $l_1(f),\dots,l_N(f)$ with precision ε_N, is by definition a numerical function $\varphi_N(z_1,\dots,z_N;\,\cdot\,)$, which, for any fixed $(z_1,\dots,z_N;\,\cdot\,) \in C^N$ is a function of the variable $(\cdot) \in \Omega_Y$ belonging to Y (the set of all such φ_N will be denoted by $\{\varphi_N\}_Y$). Then, upon substituting $z_1 = z_1(f),\dots,z_N = z_N(f)$, the function $\varphi_N(z_1(f),\dots,z_N(f);\,\cdot\,)$ acquires the status of *a computational aggregate* for the approximate computation in the metric Y of the operator $Tf \equiv u(\,\cdot\,;f)$. Thus, each computational aggregate is determined by a pair $(l^{(N)},\varphi_N) \equiv (l_1,\dots,l_N;\varphi_N)$. By $D_N \equiv D_N(F)_Y$ we denote a given collection of pairs $(l^{(N)},\varphi_N) \equiv (l_1,\dots,l_N;\varphi_N)$.

The main types of functionals are $l^{(N)} : P_N$ – values of function f at points $l_\xi(f) = f(\xi)$, $(\xi \in \Omega_F)$; Φ_N – trigonometric Fourier coefficients

$$l_m(f) \equiv \hat f(m) := \int_{[0,1]^s} f(x)\, e^{-2\pi i (m,x)}\, dx, \quad (m \in \mathbb{Z}^s);$$

R_N – values of the Radon transform

$$Rf(\theta,\tau) = \int_{x:\langle x,\theta\rangle=\tau\cap\Omega_f} f(x)\, dx$$

and with each functional class F there is associated its own set L_N, consisting of all linear functionals $l(f)$ defined on the linear span of F. One then has the following collections of aggregates D_N:

$$\mathcal{P}_N = P_N \times \{\varphi_N\}_Y \equiv \big\{\varphi_N\big(f(\xi_1),\dots,f(\xi_N);\cdot\big) : f \in F;\ \xi_1,\dots,\xi_N \in \Omega_F,\ \varphi_N \in \{\varphi_N\}_Y\big\}$$

– all computational aggregates from point values,

$$\mathcal{F}_N = \Phi_N \times \{\varphi_N\}_Y \equiv \Big\{\varphi_N\big(\widehat f(m^{(1)}),\dots,\widehat f(m^{(N)});\cdot\big) :$$

$$f \in F,\, m^{(1)}, \ldots, m^{(N)} \in \mathbb{Z}^s,\, \varphi_N \in \{\varphi_N\}_Y\big\}$$

– all computational aggregates from trigonometric Fourier coefficients,

$$\mathcal{R}_N = R_N \times \{\varphi_N\}_Y \equiv \big\{\varphi_N\big(Rf(u_1), \ldots, Rf(u_N); x\big):$$

$$f \in F,\, u_\tau \in \left[-\frac{1}{2}; \frac{1}{2}\right]^s \quad (\tau = 1, \ldots, N),\, \varphi_N \in \{\varphi_N\}_Y\Big\}$$

– all computational aggregates from values of the Radon transform.

In view of its particular importance, we single out separately

$$\mathcal{L}_N = L_N \times \{\varphi_N\}_Y \equiv \big\{\varphi_N\big(l_1(f), \ldots, l_N(f); \cdot\big) : f \in F,\ l_1, \ldots, l_N \in L(F),\ \varphi_N \in \{\varphi_N\}_Y\big\}$$

– all computational aggregates from linear information, where $L(F)$ is the set of all possible linear functionals on the linear span of F.

We shall denote by $c(\alpha, \beta, \ldots)$ certain positive quantities, generally different in different formulas and depending only on the indicated parameters.

If $\{A_N\}_{N=1}^{\infty}$ is a sequence of positive numbers and $\{B_N\}_{N=1}^{\infty}$ is an arbitrary numerical sequence, then $B_N \ll A_N$ means that there exists $c(\alpha, \beta, \ldots)$, such that $|B_N| \le c(\alpha, \beta, \ldots) A_N$ for every positive integer N. If $\{A_N\}_{N=1}^{\infty}$ and $\{B_N\}_{N=1}^{\infty}$ are two sequences of positive numbers, then $A_N \asymp B_N$ means that simultaneously $B_N \ll A_N$ and $B_N \gg A_N$.

Within the above notation and definitions, the problem of optimal reconstruction from inaccurate information, formulated under the name "Computational (Numerical) Diameter", consists in the sequential solution of three optimization problems.

For given (fixed throughout) T, F, Y, D_N:

C(N)D-1: From accurate information, one finds the order

$$\asymp \delta_N(0;\, D_N)_Y \equiv \delta_N(0;\, T;\, F;\, D_N)_Y$$

— the information power of the set of computational aggregates $D_N \equiv D_N(F)_Y$.

C(N)D-2: A computational aggregate $(\bar{l}^{(N)}, \bar{\varphi}_N)$ from $D_N \equiv D_N(F)_Y$, supporting the order $\asymp \delta_N(0; D_N)_Y$ is constructed, for which one investigates the existence and determination of the limiting error such that $\widetilde{\varepsilon}_N \equiv \widetilde{\varepsilon}_N\left(D_N; (\bar{l}^{(N)}, \bar{\varphi}_N)\right)_Y$ – C(N)D-2 – the limiting error (corresponding to the computational aggregate $(\bar{l}^{(N)}, \bar{\varphi}_N)$), such that

$$\delta_N(0; D_N)_Y \asymp \delta_N\left(\widetilde{\varepsilon}_N; (\bar{l}^{(N)}, \bar{\varphi}_N)\right)_Y =$$

$$= \sup\left\{\|Tf(\cdot) - \bar{\varphi}_N(z_1(f), \ldots, z_N(f); \cdot)\|_Y : f \in F,\right.$$

$$\left.\left|\bar{l}_\tau(f) - z_\tau(f)\right| \le \varepsilon_N^{(\tau)} \quad (\tau = 1, \ldots, N)\right\},$$

with the simultaneous validity of

$$\forall\, \eta_N \uparrow +\infty \quad (0 < \eta_N < \eta_{N+1},\ \eta_N \to +\infty): \qquad \lim_{N \to +\infty} \frac{\delta_N\left(\eta_N \widetilde{\varepsilon}_N; (\bar{l}^{(N)}, \bar{\varphi}_N)\right)_Y}{\delta_N(0; D_N)_Y} = +\infty.$$

C(N)D-3: The massiveness of the limiting error $\widetilde{\varepsilon}_N\left(D_N;\left(\bar{l}^{(N)},\bar{\varphi}_N\right)\right)_Y$ is established: one finds as large as possible a set $D_N(\bar{l}^{(N)},\bar{\varphi}_N)$ (usually related to the structure of the original $(\bar{l}^{(N)},\bar{\varphi}_N)$) of computational aggregates $(l^{(N)},\varphi_N)\equiv\varphi_N(l_1(f),\ldots,l_N(f);\cdot)$, constructed over all possible (not necessarily linear) functionals $l_1,\ldots,l_N$, such that for each of them the following relation holds:

$$\forall\,\eta_N\uparrow+\infty\quad(0<\eta_N<\eta_{N+1},\ \eta_N\to+\infty):\qquad \lim_{N\to+\infty}\frac{\delta_N\left(\eta_N\widetilde{\varepsilon}_N;(l^{(N)},\varphi_N)\right)_Y}{\delta_N(0;D_N)_Y}=+\infty.$$

3 Necessary definitions and auxiliary statements

We present the necessary definitions used directly in the proofs. All functions considered are assumed to be defined on $\mathbb{R}^s$, 1-periodic in each of their s variables, and integrable on the cube $[0,1]^s$.

The norm of the space $L^\infty=L^\infty(0,1)^s$ will be denoted by $\|\cdot\|_{L^\infty}$ or $\|\cdot\|_\infty$ and the number of elements of the set A is denoted by $|A|$.

For a positive R by $\Gamma_R=\Gamma_R^{(s)}$ we mean the set (the so-called *hyperbolic cross*) ([17])

$$\Gamma_R=\left\{m=(m_1,\ldots,m_s)\in\mathbb{Z}^s:\ \overline{\overline{m}}=\prod_{j=1}^{s}\overline{m}_j\le R\right\},\tag{3.1}$$

where (as throughout the paper)

$$\overline{m}_j=\max\{1,|m_j|\},\qquad \overline{\overline{m}}=\overline{m}_1\cdots\overline{m}_s.$$

Definition 1 *Let s be a positive integer and $r>0$. The Sobolev class with dominant mixed smoothness $SW_2^r(0,1)^s$ is the set of all functions $f(x)=f(x_1,\ldots,x_s)$ that are 1-periodic in each variable and representable in the form*

$$f(x)=\sum_{m\in\mathbb{Z}^s}\hat{f}(m)\,e^{2\pi i\langle m,x\rangle},\qquad \sum_{m\in\mathbb{Z}^s}|\hat{f}(m)|^2(\overline{\overline{m}})^{2r}\le 1.$$

Lemma 1 (see [15]) *Let a positive integer s and a real number $r>\frac{1}{2}$ be given. Then the following relation holds:*

$$\delta_N\big(0;Tf=f;SW_2^r(0,1)^s;\mathcal{L}_N\big)_{L^\infty(0,1)^s}\asymp\frac{\ln^{r(s-1)}N}{N^{r-\frac{1}{2}}}\quad(N=2,3,\ldots).$$

Lemma 2 (see [17]) *Let a positive integer s, and $R>1$ be given. Then*

$$\sum_{\overline{m}_1\cdots\overline{m}_s\le R}1\asymp R(\ln R)^{s-1}.$$

4 Main results and their proofs

Within the C(N)D framework, the following results are obtained.

Theorem 1 (C(N)D-2 problem) *Let a positive integer s and a real number $r > 1$ be given. Then for the computational aggregate with $\{m^{(1)}, \ldots, m^{(N)}\} = \Gamma_R$, $R \equiv R_N \asymp \frac{N}{\ln^{s-1} N}$*

$$(\bar{l}_N^{(N)}, \bar{\varphi}_N) = \bar{\varphi}_N\big(\bar{l}_N^{(1)}(f) = \widehat{f}(m^{(1)}), \ldots, \bar{l}_N^{(N)}(f) = \widehat{f}(m^{(N)})\big) =$$

$$= \sum_{k=1}^{N} \widehat{f}(m^{(k)}) e^{2\pi i (m^{(k)}, x)} = \sum_{m \in \Gamma_R} \widehat{f}(m) e^{2\pi i (m,x)}$$

the quantity $\widetilde{\varepsilon}_N = \dfrac{\ln^{r(s-1)} N}{N^{r+\frac{1}{2}}}$ *is the limiting error:*

firstly,

$$\delta_N\big(0; Tf = f; SW_2^r(0,1)^s; \mathcal{L}_N\big)_{L^\infty(0,1)^s} \asymp$$

$$\asymp \delta_N\left(\widetilde{\varepsilon}_N = \frac{\ln^{r(s-1)} N}{N^{r+\frac{1}{2}}}; Tf = f; SW_2^r(0,1)^s, \mathcal{L}_N\right)_{L^\infty(0,1)^s} =$$

$$= \inf_{(l^{(N)}, \varphi_N) \in \mathcal{L}_N} \sup_{\substack{f \in SW_2^r(0,1)^s \\ |\gamma_N^{(m)}| \le 1 \\ m = 1, \ldots, N}} \left\| f(x) - \varphi_N\big(l^{(1)}(f) + \gamma_N^{(1)} \widetilde{\varepsilon}_N, \ldots, l^{(N)}(f) + \gamma_N^{(N)} \widetilde{\varepsilon}_N; x\big) \right\|_{L^\infty(0,1)^s} \asymp$$

$$\asymp \sup_{\substack{f \in SW_2^r(0,1)^s \\ |\gamma_N^{(m)}| \le 1 \\ m = 1, \ldots, N}} \left\| f(x) - \sum_{m \in \Gamma_R} (\widehat{f}(m) + \widetilde{\varepsilon}_N \gamma_N^{(m)}) e^{2\pi i (m,x)} \right\|_{L^\infty(0,1)^s} \asymp \frac{\ln^{r(s-1)} N}{N^{r-\frac{1}{2}}},$$

secondly, for every positive sequence $\{\eta_N\}_{N=1}^\infty$ increasing to $+\infty$, the following equality holds

$$\lim_{N \to \infty} \frac{\delta_N\left(\eta_N \widetilde{\varepsilon}_N = \eta_N \frac{\ln^{r(s-1)} N}{N^{r+\frac{1}{2}}}; Tf = f; SW_2^r(0,1)^s; (\bar{l}_N^{(N)}, \bar{\varphi}_N)\right)_{L^\infty(0,1)^s}}{\delta_N\big(0, Tf = f; SW_2^r(0,1)^s, \mathcal{L}_N\big)_{L^\infty(0,1)^s}} = +\infty.$$

Proof We begin with the upper bound for $\delta_N\big(\widetilde{\varepsilon}_N; Tf = f; SW_2^r(0,1)^s; \mathcal{L}_N\big)_{L^\infty(0,1)^s}$. Let a function f belongs the the class $SW_2^r(0,1)^s$. Since $r > 1$, then $f \in L^\infty(0,1)^s$ (by definition of the class) and the problem is well-posed.

Set $N = |\Gamma_R|$. Define linear functionals $l_1(f) = \widehat{f}(m^{(1)}), \ldots, l_N(f) = \widehat{f}(m^{(N)})$, where $\{m^{(1)}, \ldots, m^{(N)}\}$ is some ordering of the set Γ_R and function $\varphi_N(z_1, \ldots, z_N, x)$ as follows:

$$\bar{\varphi}_N(z_1, \ldots, z_N, x) = \sum_{k=1}^{N} z_k e^{2\pi i (m^{(k)}, x)}.$$

Then for the computational aggregate

$$\bar{\varphi}_N\big(\bar{l}_N^{(1)}(f) + \gamma_N^{(1)} \widetilde{\varepsilon}_N, \ldots, l_N^{(N)}(f) + \gamma_N^{(N)} \widetilde{\varepsilon}_N; x\big)_{L^\infty(0,1)^s} =$$

$$= \sum_{k=1}^{N} \left(\widehat{f}(m^{(k)}) + \widetilde{\varepsilon}_N \gamma_N^{(k)}\right) e^{2\pi i (m^{(k)}, x)},$$

constructed from inaccurate data, we have

$$\Big\|f(x)-\bar{\varphi}_N\big(\bar{l}_N^{(1)}(f)+\gamma_N^{(1)}\widetilde{\varepsilon}_N,\dots,\bar{l}_N^{(N)}(f)+\gamma_N^{(N)}\widetilde{\varepsilon}_N;x\big)\Big\|_{L^\infty(0,1)^s}=$$

$$=\left\|f(x)-\sum_{m\in\Gamma_R}\left(\widehat{f}(m)+\widetilde{\varepsilon}_N\gamma_N^{(m)}\right)e^{2\pi i(m,x)}\right\|_{L^\infty(0,1)^s}\le I_1+I_2, \tag{4.1}$$

where

$$I_1=\Big\|f(x)-\sum_{m\in\Gamma_R}\hat{f}(m)\,e^{2\pi i(m,x)}\Big\|_{L^\infty(0,1)^s}=\Big\|\sum_{m\in\mathbb{Z}^s\setminus\Gamma_R}\hat{f}(m)\,e^{2\pi i(m,x)}\Big\|_{L^\infty(0,1)^s}$$

and

$$I_2=\tilde{\varepsilon}_N\Big\|\sum_{m\in\Gamma_R}\gamma_N^{(m)}\,e^{2\pi i(m,x)}\Big\|_{L^\infty(0,1)^s}.$$

From Lemma 1 it follows that

$$I_1=\Big\|\sum_{m\in\mathbb{Z}^s\setminus\Gamma_R}\hat{f}(m)\,e^{2\pi i(m,x)}\Big\|_{L^\infty(0,1)^s}\ll\frac{\ln^{r(s-1)}N}{N^{r-\frac{1}{2}}}\quad(N=2,3,\dots).$$

We estimate I_2 from above. By Lemma 2 and the inequalities $|\gamma_N^{(m)}|\le 1$ $(m=1,\dots,N)$, we have

$$I_2=\tilde{\varepsilon}_N\Big\|\sum_{m\in\Gamma_R}\gamma_N^{(m)}\,e^{2\pi i(m,x)}\Big\|_{L^\infty(0,1)^s}\le\tilde{\varepsilon}_N\sum_{m\in\Gamma_R}|\gamma_N^{(m)}|\le$$

$$\le\tilde{\varepsilon}_N\sum_{m\in\Gamma_R}1=\tilde{\varepsilon}_N|\Gamma_R|=\tilde{\varepsilon}_N N=\frac{\ln^{r(s-1)}N}{N^{r+\frac{1}{2}}}\cdot N=\frac{\ln^{r(s-1)}N}{N^{r-\frac{1}{2}}}.$$

Moreover, since inequality (4.1) holds for all $f\in SW_2^r(0,1)^s$, we have

$$\delta_N\Big(\widetilde{\varepsilon}_N=\frac{\ln^{r(s-1)}N}{N^{r+\frac{1}{2}}};Tf=f;SW_2^r(0,1)^s,\mathcal{L}_N\Big)_{L^\infty(0,1)^s}\ll$$

$$\ll\sup_{\substack{f\in SW_2^r(0,1)^s\\|\gamma_N^{(m)}|\le 1\\m=1,\dots,N}}\Big\|f(x)-\bar{\varphi}_N\big(\bar{l}_N^{(1)}(f)+\gamma_N^{(1)}\widetilde{\varepsilon}_N,\dots,\bar{l}_N^{(N)}(f)+\gamma_N^{(N)}\widetilde{\varepsilon}_N;x\big)\Big\|_{L^\infty(0,1)^s}\ll$$

$$\ll\frac{\ln^{r(s-1)}N}{N^{r-\frac{1}{2}}}.$$

Then, together with Lemma 1,

$$\frac{\ln^{r(s-1)}N}{N^{r-\frac{1}{2}}}\ll\delta_N\big(0;Tf=f;SW_2^r(0,1)^s;\mathcal{L}_N\big)_{L^\infty(0,1)^s}\ll$$

$$\ll \delta_N \left(\widetilde{\varepsilon}_N = \frac{\ln^{r(s-1)} N}{N^{r+\frac{1}{2}}}; Tf = f; SW_2^r(0,1)^s, \mathcal{L}_N \right)_{L^\infty(0,1)^s} =$$

$$= \inf_{(l^{(N)},\varphi_N)\in\mathcal{L}_N} \sup_{\substack{f\in SW_2^r(0,1)^s \\ |\gamma_N^{(m)}|\le 1 \\ m=1,\dots,N}} \left\| f(x) - \varphi_N\big(l_N^{(1)}(f) + \gamma_N^{(1)}\widetilde{\varepsilon}_N, \dots, l_N^{(N)}(f) + \gamma_N^{(N)}\widetilde{\varepsilon}_N; x\big) \right\|_{L^\infty(0,1)^s}.$$

As a result,

$$\delta_N \left(\widetilde{\varepsilon}_N = \frac{\ln^{r(s-1)} N}{N^{r+\frac{1}{2}}}; Tf = f; SW_2^r(0,1)^s, \mathcal{L}_N \right)_{L^\infty(0,1)^s} \asymp \frac{\ln^{r(s-1)} N}{N^{r-\frac{1}{2}}}.$$

We proceed to the second part of the proof. It is necessary to prove that for every increasing positive sequence $\{\eta_N\}_{N=1}^\infty$ the following equality holds:

$$\overline{\lim_{N\to\infty}} \frac{\delta_N \left(\eta_N \widetilde{\varepsilon}_N = \frac{\ln^{r(s-1)} N}{N^{r+\frac{1}{2}}}; Tf = f; SW_2^r(0,1)^s; (\bar{l}^{(N)}, \bar{\varphi}_N) \right)_{L^\infty(0,1)^s}}{\delta_N(0, Tf = f; SW_2^r(0,1)^s; \mathcal{L}_N)_{L^\infty(0,1)^s}} = +\infty.$$

Let a positive integer $N \ge 2$, and a set of functionals $l^{(N)}: \, l_1(f) = \hat{f}(m^{(1)}), \dots, l_N(f) = \hat{f}(m^{(N)})$, where $\{m^{(1)}, \dots, m^{(N)}\} \subset \mathbb{Z}^s$, be given. For $R \asymp \frac{N}{\ln^{s-1} N}$ we set

$$\tilde{g}_N(x) = \sum_{m\in\Gamma_R} \tau_N^{(m)} e^{2\pi i(m,x)}$$

and

$$\tau_N^{(m)} = \min\Big\{ \frac{1}{\sqrt{N}(\overline{\overline{m}})^r}, \, \eta_N^* \tilde{\varepsilon}_N \Big\}, \quad \eta_N^* = \min\{\eta_N, \, R^{r/2}\}. \tag{4.2}$$

We show that $\tilde{g}_N$ belongs to the class $SW_2^r(0,1)^s$. To this end, we verify the inequality

$$\|\tilde{g}_N\|_{SW_2^r(0,1)^s} = \Big(\sum_{m\in\mathbb{Z}^s} (\tau_N^{(m)})^2 (\overline{\overline{m}})^{2r} \Big)^{1/2} \le 1.$$

By Lemma 2, $|\Gamma_R| = N$, so

$$\|g_N\|_{SW_2^r(0,1)^s} = \left(\sum_{m\in\Gamma_R} (\tau_N^{(m)})^2 (\overline{\overline{m}})^{2r} \right)^{1/2} \le \left(\sum_{m\in\Gamma_R} \frac{1}{N(\overline{\overline{m}})^{2r}} \overline{\overline{m}}^{2r} \right)^{1/2} =$$

$$= \left(\sum_{m\in\Gamma_R} \frac{1}{N} \right)^{1/2} = \left(\frac{1}{N} \cdot |\Gamma_R| \right)^{1/2} = \left(\frac{1}{N} \cdot N \right)^{1/2} = 1.$$

We estimate the norm $\|\tilde{g}_N(x)\|_{L^\infty(0,1)^s}$ from below:

$$\|\tilde{g}_N(x)\|_{L^\infty(0,1)^s} = \Big\| \sum_{m\in\Gamma_R} \tau_N^{(m)} e^{2\pi i(m,x)} \Big\|_{L^\infty(0,1)^s} = \sum_{m\in\Gamma_R} \tau_N^{(m)}. \tag{4.3}$$

Let us split the sum over A_N and B_N, where $\Gamma_R = A_N \cup B_N$, $A_N = \left\{m : \frac{1}{\sqrt{N}(\overline{\overline{m}})^r} \leq \eta_N^* \tilde{\varepsilon}_N\right\}$, $B_N = \left\{m : \frac{1}{\sqrt{N}(\overline{\overline{m}})^r} > \eta_N^* \tilde{\varepsilon}_N\right\}$.

Let an element m belongs B_N, then from the condition

$$\frac{1}{\sqrt{N}\,\overline{\overline{m}}^r} > \eta_N^* \widetilde{\varepsilon}_N$$

we have

$$\sqrt{N}\,\overline{\overline{m}}^r < \frac{1}{\eta_N^* \widetilde{\varepsilon}_N}$$

from which, using

$$\widetilde{\varepsilon}_N = \frac{(\ln N)^{r(s-1)}}{N^{r+\frac{1}{2}}}$$

we obtain

$$\overline{\overline{m}}^r < \frac{1}{\sqrt{N}\eta_N^* \widetilde{\varepsilon}_N} = \frac{1}{\sqrt{N}\eta_N^* \dfrac{(\ln N)^{r(s-1)}}{N^{r+\frac{1}{2}}}}$$

$$= \frac{1}{\eta_N^* \dfrac{(\ln N)^{r(s-1)}}{N^r}} = \frac{1}{\eta_N^* \left(\dfrac{\ln^{s-1} N}{N}\right)^r} = \left(\frac{N}{(\eta_N^*)^{1/r} \ln^{s-1} N}\right)^r,$$

that is

$$\overline{\overline{m}}(\eta_N^*)^{1/r} < R,$$

hence the set B_N is a hyperbolic cross and

$$B_N = \Gamma_{R(\eta_N^*)^{-1/r}} \subset \Gamma_R.$$

Then, applying Lemma 2 and the definitions (4.2), we continue the estimate (4.3):

$$\|\widetilde{g}_N(x)\|_{L^\infty(0,1)^s} = \sum_{m \in \Gamma_R} \tau_N^{(m)} \geq \sum_{m \in B_N} \tau_N^{(m)} = \widetilde{\varepsilon}_N \eta_N^* \sum_{m \in B_N} 1 =$$

$$= \widetilde{\varepsilon}_N \eta_N^* \cdot R(\eta_N^*)^{-1/r} \left(\ln\Big(R(\eta_N^*)^{-1/r}\Big)\right)^{s-1} = \widetilde{\varepsilon}_N \eta_N^* R(\eta_N^*)^{-1/r} \left(\ln R - \frac{1}{r}\ln \eta_N^*\right)^{s-1} \asymp$$

$$\asymp \widetilde{\varepsilon}_N \eta_N^* R(\eta_N^*)^{-1/r}(\ln R)^{s-1} = \widetilde{\varepsilon}_N (\eta_N^*)^{1-\frac{1}{r}} R(\ln R)^{s-1} \asymp$$

$$\asymp N\widetilde{\varepsilon}_N (\eta_N^*)^{1-\frac{1}{r}} = \delta_N(0)(\eta_N^*)^{1-\frac{1}{r}}.$$

For this function $\widetilde{g}_N(m) = \begin{cases} \tau_N^{(m)}, & \text{if } m \in \Gamma_R, \\ 0, & \text{if } m \notin \Gamma_R, \end{cases}$ then by (4.2) $|l_k(\widetilde{g})| \leq \eta_N^* \widetilde{\varepsilon}_N \leq \eta_N \widetilde{\varepsilon}_N$.

Let us set

$$\gamma_N^{(k)} = \begin{cases} -\dfrac{l_k(\widetilde{g}_N)}{\eta_N \widetilde{\varepsilon}_N}, & \text{if } m^{(k)} \in \Gamma_R, \\ 0, & \text{if } m^{(k)} \notin \Gamma_R. \end{cases}$$

then $l_k(\tilde{g}_N) + \tilde{\varepsilon}_N \eta_N \gamma_N^{(k)} = 0$ for all $k = 1, \ldots, N$ we have

$$\Big\| \sum_{m \in \Gamma_R} \tau_N^{(m)}\, e^{2\pi i(m,x)} - \sum_{k=1}^{N} \big(\hat{g}(m^{(k)}) + \tilde{\varepsilon}_N \eta_N \gamma_N^{(k)}\big) e^{2\pi i(m,x)} \Big\|_{L^\infty(0,1)^s} =$$

$$= \Big\| \sum_{m \in \Gamma_R} \tau_N^{(m)}\, e^{2\pi i(m,x)} \Big\|_{L^\infty(0,1)^s} \gg \delta_N(0)(\eta_N^*)^{1-\frac{1}{r}}.$$

From this,

$$\overline{\lim_{n\to\infty}} \frac{\delta_N\big(\widetilde{\varepsilon}_N \eta_N^*; Tf = f; SW_2^r(0,1)^s; (\bar{l}^N, \bar{\varphi}_N)\big)_{L^\infty(0,1)^s}}{\delta_N\big(0; Tf = f; SW_2^r(0,1)^s, \mathcal{L}_N\big)_{L^\infty(0,1)^s}} \gg \overline{\lim_{n\to\infty}} (\eta_N^*)^{1-\frac{1}{r}} = +\infty,$$

which proves that $\tilde{\varepsilon}_N$ is the limiting error. Theorem 1 is completely proved.

Theorem 2 (C(N)D-3 problem) *Let s $(s = 1, 2, \ldots)$ and $r > 1$ be given. Then the limiting error of the computational aggregate*

$$(l^{(N)}, \varphi_N) = \varphi_N\big(l_N^{(1)}(f) = \hat{f}(m^{(1)}), \ldots, l_N^{(N)}(f) = \hat{f}(m^{(N)});\, x\big),$$

constructed from Fourier coefficients with arbitrary spectra $\{m^{(1)}, \ldots, m^{(N)}\} \in \mathbb{Z}^s$, does not exceed the limiting error of the computational aggregate

$$\bar{\varphi}_N\big(\bar{l}_N^{(1)}(f) = \widehat{f}(m^{(1)}), \ldots, \bar{l}_N^{(N)}(f) = \widehat{f}(m^{(N)}); x\big) = \sum_{m \in \Gamma_R} \widehat{f}(m) e^{2\pi i(m,x)},$$

that is, for every increasing positive sequence $\{\eta_N\}_{N=1}^{\infty}$ the following equality holds:

$$\overline{\lim_{N\to\infty}} \frac{\delta_N\big(\widetilde{\varepsilon}_N \eta_N; Tf = f; SW_2^r(0,1)^s, (l^{(N)}, \varphi_N)\big)_{L^\infty(0,1)^s}}{\delta_N\big(0; Tf = f; SW_2^r(0,1)^s, \mathcal{L}_N\big)_{L^\infty(0,1)^s}} = +\infty.$$

Proof Let N and the set

$$\{m^{(1)}, \ldots, m^{(N)}\} \subset \mathbb{Z}^s$$

be given. Repeating the arguments in the proof of Theorem 1, and replacing the specific computational aggregate

$$\bar{\varphi}_N\big(\bar{l}_N^{(1)}(f) = \widehat{f}(m^{(1)}), \ldots, \bar{l}_N^{(N)}(f) = \widehat{f}(m^{(N)}); x\big) = \sum_{m \in \Gamma_R} \widehat{f}(m) e^{2\pi i(m,x)},$$

by an arbitrary computational aggregate

$$(l^{(N)},\varphi_N)=\varphi_N\big(l_N^{(1)}(f)=\widehat{f}(m^{(1)}),\dots,l_N^{(N)}(f)=\widehat{f}(m^{(N)});x\big),$$

we obtain the estimate

$$\sup_{\substack{f\in SW_2^r(0,1)^s\\ |\gamma_N^{(\tau)}|\le 1\\ \tau=1,\dots,N}}\Big\|f(x)-\varphi_N\big(l_1(f)+\gamma_N^{(1)}\widetilde{\varepsilon}_N\eta_N,\dots,l_N(f)+\gamma_N^{(N)}\widetilde{\varepsilon}_N\eta_N;x\big)\Big\|_{L^\infty(0,1)^s}\gg$$

$$\gg\frac{1}{2}\left(\|\widetilde{g}_N(x)-\varphi_N(0,\dots,0;x)\|_{L^\infty(0,1)^s}+\|-\widetilde{g}_N(x)-\varphi_N(0,\dots,0;x)\|_{L^\infty(0,1)^s}\right)=$$

$$=\frac{1}{2}\left(\|\widetilde{g}_N(x)-\varphi_N(0,\dots,0;x)\|_{L^\infty(0,1)^s}+\|\widetilde{g}_N(x)+\varphi_N(0,\dots,0;x)\|_{L^\infty(0,1)^s}\right)\ge\|\widetilde{g}_N(x)\|_{L^\infty(0,1)^s}.$$

From this,

$$\frac{\delta_N\left(\eta_N\widetilde{\varepsilon}_N=\eta_N\frac{(\ln N)^{r(s-1)}}{N^{r+\frac12}};SW_2^r(0,1)^s;(l^{(N)},\varphi_N)\right)_{L^\infty(0,1)^s}}{\delta_N\big(0;Tf=f;SW_2^r(0,1)^s,\mathcal{L}_N\big)_{L^\infty(0,1)^s}}\gg$$

$$\gg\frac{\|g_N\|_{L^\infty(0,1)^s}}{\frac{(\ln N)^{r(s-1)}}{N^{r-\frac12}}}\gg\frac{\eta_N^*\frac{(\ln N)^{r(s-1)}}{N^{r-\frac12}}}{\frac{(\ln N)^{r(s-1)}}{N^{r-\frac12}}}=\eta_N^*.$$

As $N\to+\infty$

$$\varlimsup_{N\to\infty}\frac{\delta_N\big(\widetilde{\varepsilon}_N\eta_N;Tf=f;SW_2^r(0,1)^s,(l^{(N)},\varphi_N)\big)_{L^\infty(0,1)^s}}{\delta_N\big(0;Tf=f;SW_2^r(0,1)^s,\mathcal{L}_N\big)_{L^\infty(0,1)^s}}=+\infty.$$

Theorem 2 is completely proved.

5 Numerical Experiments for $SW_2^r(0,1)^s$ with parameters $s=2$, $r=2$

The numerical experiment was carried out for the Sobolev class with dominating mixed derivative $SW_2^r(0,1)^s$ with parameters $s=2$, $r=2$. For each R the number of harmonics $N=|\Gamma_R|$ in the hyperbolic cross Γ_R was computed, together with the recovery error from accurate information $E_N=\|f-\bar{\varphi}_N(f)\|_{L^\infty(0,1)^2}$, where

$$\bar{\varphi}_N(f;\,x_1,x_2)=\sum_{(m_1,m_2)\in\Gamma_R}\hat{f}(m_1,m_2)\,e^{2\pi i(m_1x_1+m_2x_2)}$$

is the computational aggregate from accurate information, and E_N^ε is the recovery error from inaccurate information.

According to Theorem 1 (C(N)D-2), the limiting error equals

$$\tilde{\varepsilon}_N=\frac{\ln^{r(s-1)}N}{N^{r+\frac12}}=\frac{\ln^2 N}{N^{\frac52}},\quad(s=2,\ r=2).$$

As a test function we take the boundary function

$$f(x_1,x_2) = x_1^2 x_2^2 (1-x_1)^2 (1-x_2)^2, \qquad (x_1,x_2) \in [0,1]^2.$$

The function vanishes on the boundary of the square. Let us verify that $f \in SW_2^2(0,1)^2$.

Set

$$g(x) = x^2(1-x)^2, \qquad f(x_1,x_2) = g(x_1)g(x_2).$$

Due to this multiplicative representation, the Fourier coefficients factor into a product:

$$\widehat{f}(m_1,m_2) = \widehat{g}(m_1)\widehat{g}(m_2), \qquad \text{where} \qquad \widehat{g}(m) = \int_0^1 x^2(1-x)^2 e^{-2\pi i m x}\,dx.$$

Let us verify the condition from Definition 1 By virtue of the factorization we have

$$S = \left(\sum_{m\in\mathbb{Z}} |\widehat{g}(m)|^2 \max\{1,|m|\}^4 \right)^2.$$

Using the formulas for $\widehat{g}(m)$ obtained above, we get

$$\sum_{m\in\mathbb{Z}} |\widehat{g}(m)|^2 \max\{1,|m|\}^4 = \frac{1}{900} + \frac{9}{4\pi^8} \sum_{m=1}^{\infty} \frac{2}{m^4} = \frac{1}{900} + \frac{1}{20\pi^4} < 1.$$

Consequently, $S = \left(\frac{1}{900} + \frac{1}{20\pi^4}\right)^2 < 1$, and therefore $f \in SW_2^2(0,1)^2$.

Figure 1 shows the three-dimensional surface of the function $f(x_1,x_2)$.

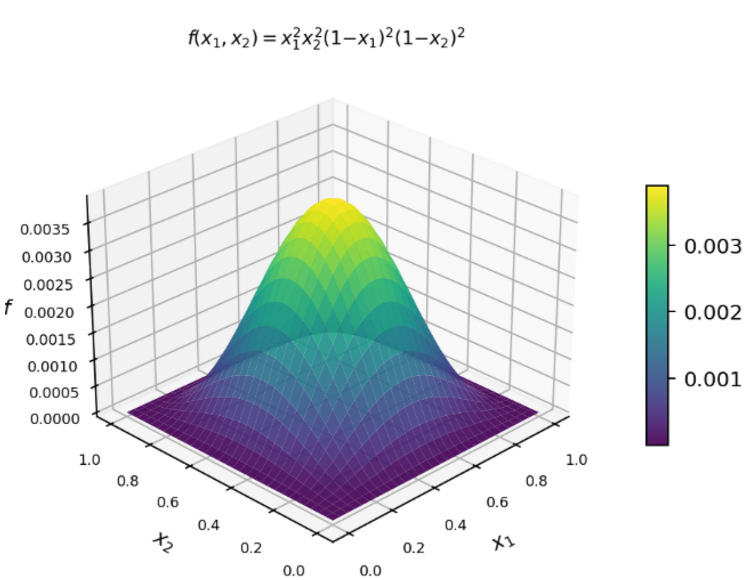

Figure 1: Three-dimensional surface of the function $f(x_1,x_2)$.

To implement the numerical experiments for function recovery from $SW_2^r(0,1)^s$ it is necessary to compute exactly the number of elements N in the set of linear functionals. The hyperbolic cross Γ_R is used as the information set, defined as (see (3.1)):

$$\Gamma_R = \big\{ m = (m_1,m_2) \in \mathbb{Z}^2 : \max\{1,|m_1|\} \cdot \max\{1,|m_2|\} \le R \big\}.$$

The cardinality $N \asymp RlnR$ determines the dimension of the approximation space. For a fixed R, N is computed by stepwise summation of integer points.

In the numerical experiment, hyperbolic crosses were constructed for parameter values $R = 5, 7, 10, 15, 20, 30, 40, 50, 70, 100, 150, 200, 300, 500$.

We compute N (for $R = 50$).The set Γ_R is symmetric with respect to the coordinate axes. Therefore, we separate the central axis ($m_1 = 0$) and the two lateral half-planes ($m_1 > 0$ and $m_1 < 0$).

This confirms that for $R = 50$ the linear information L_N consists of 1029 functionals. The following figure illustrates the geometric structure of the hyperbolic cross in the index space.

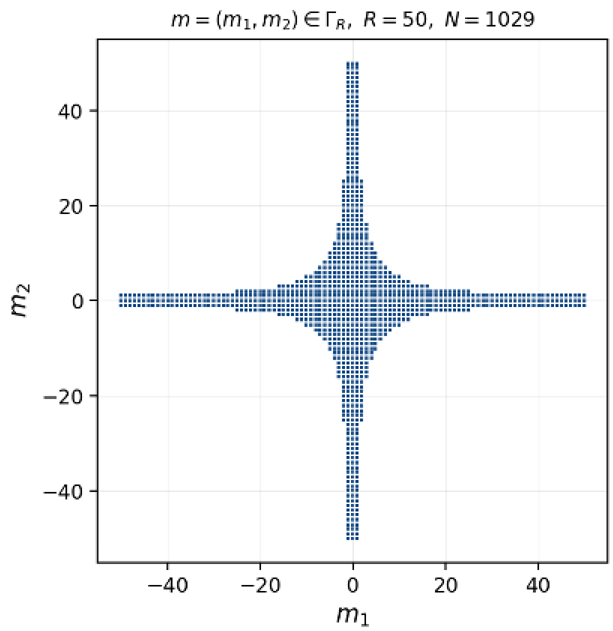

Figure 2: Hyperbolic cross for $R = 50,\ N = 1029$.

Recovery of the function f is carried out by the computational aggregate

$$\bar{\varphi}_N(f;\, x_1, x_2) = \sum_{(m_1,m_2)\in\Gamma_R} \hat{f}(m_1, m_2)\, e^{2\pi i(m_1x_1+m_2x_2)}.$$

The recovery error from accurate information:

$$E_N = \|f - \bar{\varphi}_N(f)\|_{L^\infty(0,1)^2}.$$

To simulate inaccurate information, the Fourier coefficients are given with error $\tilde{\varepsilon}_N$, for one of the cases when $\gamma_N^{(m)} = 1$, i.e.:

$$z_{m_1,m_2} = \hat{f}(m_1, m_2) + \tilde{\varepsilon}_N.$$

The computational aggregate for recovery from inaccurate information:

$$\tilde{\bar{\varphi}}_N(f;\, x_1, x_2) = \sum_{(m_1,m_2)\in\Gamma_R} z_{m_1,m_2}\, e^{2\pi i(m_1x_1+m_2x_2)},$$

and the recovery error

$$E_N^{\varepsilon} = \|f - \tilde{\bar{\varphi}}_N(f)\|_{L^\infty(0,1)^2}.$$

The experiment used the limiting error level $\tilde{\varepsilon}_N = \dfrac{\ln^2 N}{N^{5/2}}$, corresponding to the theoretical relation $\tilde{\varepsilon}_N \cdot N \asymp \delta_N(0)$, at which the contribution of the information error is asymptotically equivalent to the main recovery error.

Figures 3 and 4 show the three-dimensional error surfaces $|f(x) - \bar{\varphi}_N(f(x))|$ and $|f(x) - \tilde{\bar{\varphi}}_N(f(x))|$ for $R = 50,\ N = 1029$.

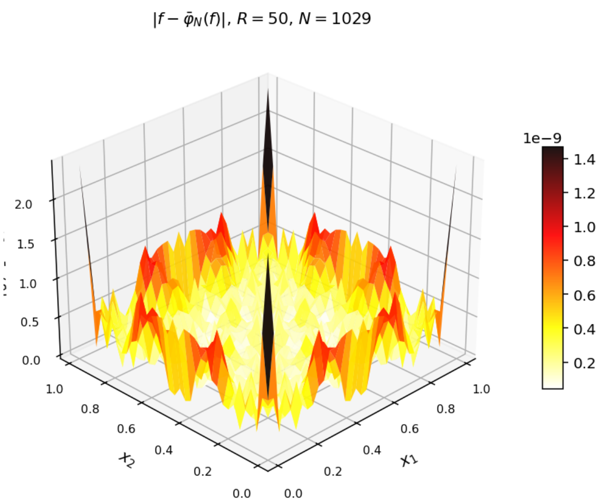

Figure 3: Error surface $|f(x) - \bar{\varphi}_N(f;x)|$ for $R = 50,\ N = 1029$.

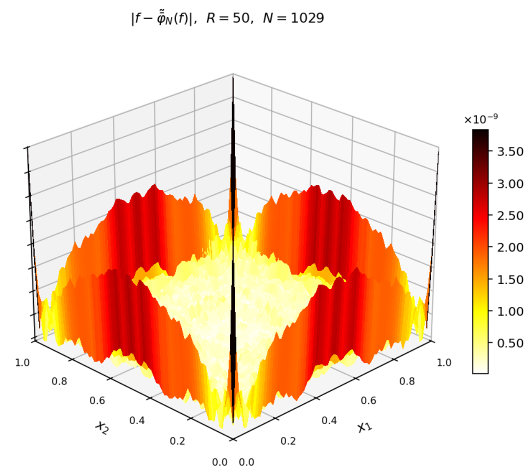

Figure 4: Error surface $|f(x) - \tilde{\bar{\varphi}}_N(f;x)|$ for $R = 50,\ N = 1029$.

Table 1: Numerical data for E_N and $E_N^{\varepsilon_i}$ as a function of R.

R	$N = \|\Gamma_R\|$	E_N	$\varepsilon_1 = \tilde{\varepsilon}_N = \frac{\ln^2 N}{N^{5/2}}$	$E_N^{\varepsilon_1}$	$\varepsilon_2 = \tilde{\varepsilon}_N \ln N$	$E_N^{\varepsilon_2}$	$E_N^{\varepsilon_1}/E_N$	$E_N^{\varepsilon_2}/E_N$
5	61	2.941×10^{-6}	5.815×10^{-4}	2.905×10^{-6}	2.390×10^{-3}	6.946×10^{-3}	0.988	2361
7	93	1.119×10^{-6}	2.463×10^{-4}	1.158×10^{-6}	1.116×10^{-3}	3.559×10^{-3}	1.035	3181
10	149	3.884×10^{-7}	9.240×10^{-5}	4.143×10^{-7}	4.624×10^{-4}	1.499×10^{-3}	1.067	3859
15	241	1.503×10^{-7}	3.336×10^{-5}	1.545×10^{-7}	1.830×10^{-4}	6.351×10^{-4}	1.028	4225
20	345	6.312×10^{-8}	1.545×10^{-5}	6.326×10^{-8}	9.026×10^{-5}	4.448×10^{-4}	1.002	7047
30	565	1.633×10^{-8}	5.292×10^{-6}	1.654×10^{-8}	3.353×10^{-5}	1.844×10^{-4}	1.013	11293
40	793	6.369×10^{-9}	2.517×10^{-6}	6.438×10^{-9}	1.680×10^{-5}	1.204×10^{-4}	1.011	18906
50	1029	1.933×10^{-9}	1.417×10^{-6}	1.957×10^{-9}	9.825×10^{-6}	6.520×10^{-5}	1.012	33726
70	1529	7.573×10^{-10}	5.881×10^{-7}	7.722×10^{-10}	4.312×10^{-6}	2.923×10^{-5}	1.020	38593
100	2329	2.408×10^{-10}	2.296×10^{-7}	2.415×10^{-10}	1.780×10^{-6}	1.524×10^{-5}	1.003	63290
150	3721	4.503×10^{-11}	8.003×10^{-8}	4.528×10^{-11}	6.580×10^{-7}	6.437×10^{-6}	1.006	142957
200	5193	2.248×10^{-11}	3.766×10^{-8}	2.252×10^{-11}	3.222×10^{-7}	3.497×10^{-6}	1.002	155545
300	8269	7.406×10^{-12}	1.309×10^{-8}	7.440×10^{-12}	1.180×10^{-7}	1.627×10^{-6}	1.005	219756
500	14761	1.100×10^{-12}	3.481×10^{-9}	1.104×10^{-12}	3.342×10^{-8}	6.481×10^{-7}	1.003	589019

In the first column of Table 1, the parameter R of the hyperbolic cross Γ_R is given; in the second column, the number of elements $N = |\Gamma_R|$, varying from 61 to 14761. The third column contains the exact recovery error $E_N = \|f - \bar{\varphi}_N(f)\|_{L^\infty(0,1)^2}$. The fourth column contains the theoretical limiting error $\varepsilon_1 = \frac{\ln^2 N}{N^{5/2}}$ established by Theorem 1 for $s = 2$, $r = 2$. The fifth column contains the recovery error from inaccurate information $E_N^{\varepsilon_1}$ for $\varepsilon_1 = \frac{\ln^2 N}{N^{5/2}}$. The sixth column contains the increased noise level $\varepsilon_2 = \frac{\ln^3 N}{N^{5/2}}$, and the seventh column contains the corresponding recovery error $E_N^{\varepsilon_2}$. The eighth and ninth columns contain the ratios $E_N^{\varepsilon_i}/E_N$ $(i = 1, 2)$ for each level of inaccuracy, respectively.

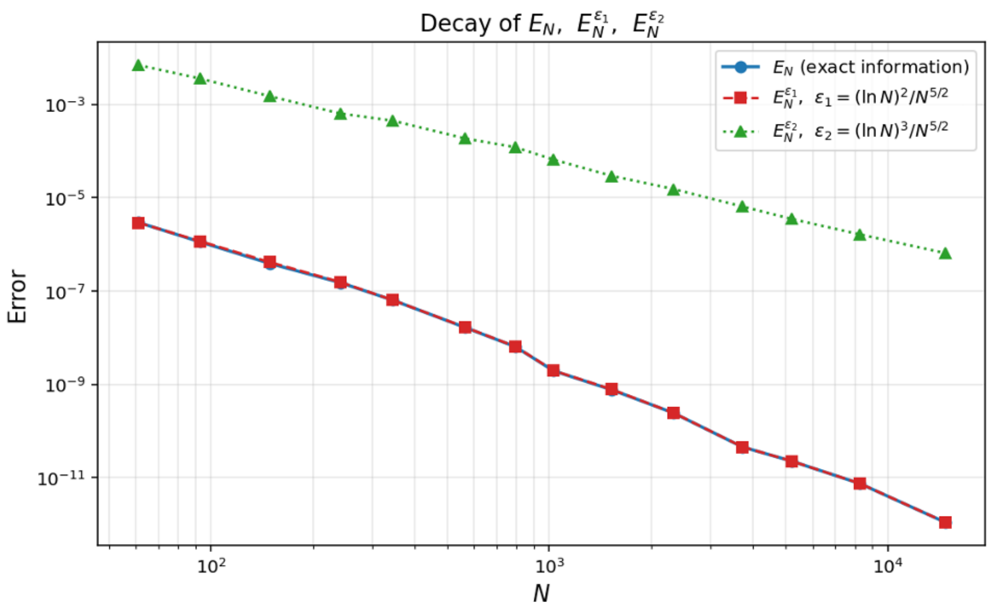

Figure 5: Decay of errors E_N, $E_N^{\varepsilon_1}$ and $E_N^{\varepsilon_2}$ as a function of N.

Figure 5 presents three curves: the recovery error from accurate information E_N (blue), the recovery error from inaccurate information $E_N^{\varepsilon_1}$ (red, $\varepsilon_1 = \frac{(\ln N)^2}{N^{5/2}}$), and the error $E_N^{\varepsilon_2}$ (green, $\varepsilon_2 = \frac{(\ln N)^3}{N^{5/2}}$). All three errors decrease monotonically as $N \to \infty$, which confirms the convergence of the computational aggregate predicted by Theorem 1. The curve E_N (blue) decreases most rapidly — from 2.941×10^{-6} at $N = 61$ to 1.100×10^{-12} at $N = 14761$, which corresponds to a decrease of more than 6 orders of magnitude. The curve $E_N^{\varepsilon_1}$ (red) decreases almost parallel to the curve E_N and lies very close to it (there is a small difference between them, but it is practically indistinguishable in the presented graph), which clearly demonstrates the preservation of the optimal recovery order at the noise level $\varepsilon_1 = \tilde{\varepsilon}_N$. This confirms the first part of Theorem 1 (C(N)D-2):

$$\delta_N(0) \asymp \delta_N(\tilde{\varepsilon}_N) \asymp \frac{\ln^{r(s-1)} N}{N^{r-\frac{1}{2}}}.$$

The curve $E_N^{\varepsilon_2}$ (green, $\varepsilon_2 = \frac{(\ln N)^3}{N^{5/2}}$) is located significantly above the two previous curves and decreases more slowly. The distance between the curves $E_N^{\varepsilon_2}$ and E_N increases as N grows, which reflects the increasing deterioration of the recovery accuracy. This confirms the

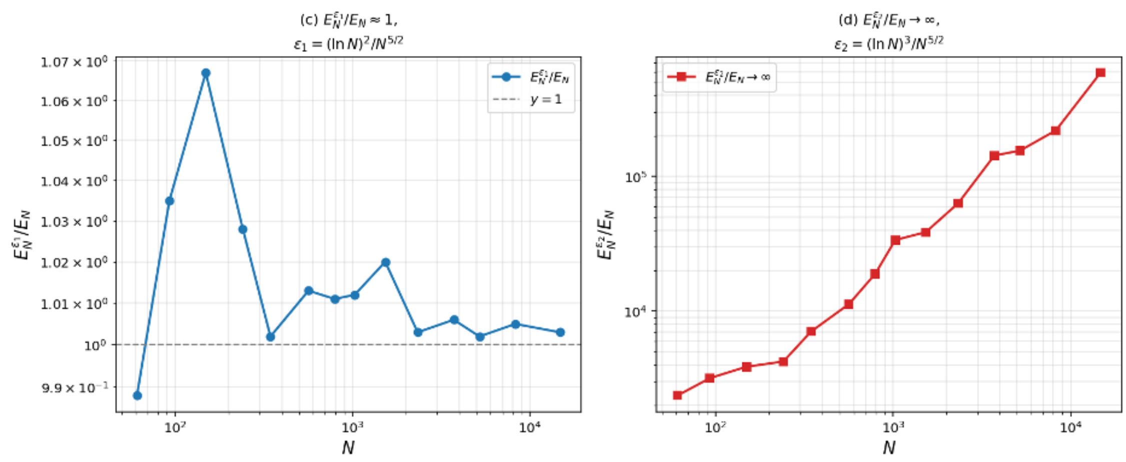

Figure 6: Ratio $E_N^{\varepsilon_i}/E_N$ at two noise levels $\varepsilon_1 = \tilde{\varepsilon}_N$ and $\varepsilon_2 = \ln N \cdot \tilde{\varepsilon}_N$.

second part of Theorem 1 (C(N)D-2): when the limiting noise level is exceeded by a factor of η_N tending arbitrarily slowly to infinity, the recovery accuracy deteriorates irreversibly.

As can be seen from Figure 6, the ratio $E_N^{\varepsilon_1}/E_N$ for $\varepsilon_1 = \frac{(\ln N)^2}{N^{5/2}}$ remains approximately constant for all N from 61 to 14761: the values vary within the range 0.988–1.067, i.e. they are in fact close to 1. This result quantitatively confirms the first part of Theorem 1: $\varepsilon_1 = \frac{(\ln N)^2}{N^{5/2}}$ is the limiting error $\tilde{\varepsilon}_N$, at which the optimal recovery order is preserved, $E_N^{\varepsilon_1} \asymp E_N$.

The ratio $E_N^{\varepsilon_2}/E_N$ for the error $\varepsilon_2 = \frac{(\ln N)^3}{N^{5/2}}$ increases as N grows: from 2361 at $N = 61$ to 589019 at $N = 14761$. This confirms the second part of Theorem 1: at the noise level ε_2, which exceeds the limiting error by a factor of $\ln N$, the optimal recovery order is violated:

$$\lim_{N\to\infty} \frac{\delta_N\big(\eta_N \widetilde{\varepsilon}_N; (\bar{l}_N^{(N)}, \bar{\varphi}_N)\big)_{L^\infty(0,1)^s}}{\delta_N(0; L_N)_{L^\infty(0,1)^s}} = +\infty.$$

The obtained numerical results confirm theoretical estimates for $\gamma_N^{(m)} = 1$ and the existence of the limiting error $\tilde{\varepsilon}_N = \frac{(\ln N)^2}{N^{5/2}}$. The numerical experiments demonstrate that when the information error does not exceed the limiting level, the optimal order of recovery is preserved, whereas exceeding this threshold leads to a substantial deterioration in the recovery accuracy. This confirms the stability properties of the computational aggregate predicted by the theoretical results and illustrates the fundamental role of the limiting error in recovery problems for the classes $SW_2^r(0,1)^s$.

The numerical experiments were implemented in the Python programming language using the NumPy and Matplotlib libraries. The Fourier coefficients were computed analytically using a recursive formula.

6 Conclusion

In the present work, we considered the problem of approximate recovery of multivariate functions from Sobolev classes with dominating mixed derivative $SW_2^r(0,1)^s$ in the uniform metric $L^\infty(0,1)^s$, in the context of the Computational (Numerical) Diameter problem, which includes a collection of problems: recovery from accurate and inaccurate information, determination of the limiting error for a specific optimal computational aggregate, as well as the construction of a class of computational aggregates whose limiting errors do not exceed a prescribed level.

In Theorem 1, a two-sided estimate of the recovery error under inaccurate information is obtained, which preserves the order $\asymp \frac{\ln^{r(s-1)} N}{N^{r-\frac{1}{2}}}$, coinciding with the order in the case of accurate data. Also, in Theorem 1, the limiting error of inaccurate information was obtained for a specific computational aggregate constructed from Fourier coefficients with spectra from the hyperbolic cross Γ_R, which preserves the order of recovery for accurate information and loses it when multiplied by any increasing positive sequence $\{\eta_N\}_{N=1}^{\infty}$.

In Theorem 2 (problem C(N)D-3), the massiveness of the limiting error was studied. It was shown that any computational aggregate constructed from an arbitrary finite spectrum of Fourier coefficients does not have a limiting error exceeding ε_N.

The theoretical results were confirmed by a numerical experiment for the class $SW_2^2(0,1)^2$.

Authors contribution

G.T.: methodology, supervision. G.T. and N.B.: writing - original draft preparation. G.T. and N.B.: editing after review. All authors have read and agreed to the published version of the manuscript.

References

[1] Temirgaliyev, N., and A. Zhubanysheva. “Computational (Numerical) Diameter in a Context of General Theory of a Recovery.” *Russian Mathematics (Izvestiya VUZ. Matematika)* 63, no. 1 (2019): 79–86. https://doi.org/10.3103/S1066369X19010109.

[2] Temirgaliyev, N., and A. Zhubanysheva. “Approximation Theory, Computational Mathematics and Numerical Analysis in New Conception of Computational (Numerical) Diameter.” *Bulletin of L.N. Gumilyov Eurasian National University. Mathematics, Computer Science, Mechanics Series* 124, no. 3 (2018): 8–88.

[3] Taugynbayeva, G., Sh. Azhgaliyev, A. Zhubanysheva, and N. Temirgaliyev. “Full C(N)D-Study of Computational Capabilities of Lagrange Polynomials.” *Mathematics and Computers in Simulation* 227 (2024): 1–25. https://doi.org/10.1016/j.matcom.2024.07.032.

[4] Zhubanysheva, A., G. Taugynbayeva, N. Nauryzbayev, and A. Shomanova. “Function-Theoretic and Probabilistic Approaches to the Problem of Recovering Functions from Korobov Classes in the Lebesgue Metric.” *Mathematics* 13, no. 21 (2025): 3363. https://doi.org/10.3390/math13213363.

[5] Temirgaliyev, N., Sh. Abikenova, Sh. Azhgaliev, Ye. Nurmoldin, G. Taugynbayeva, and A. Zhubanysheva. “Equivalence of Computed Tomography Problem with the Problem of Recovery of Functions by Finite Convolutions in a Scheme of Computational (Numerical)

Diameter." *Russian Mathematics (Izvestiya VUZ. Matematika)* 67, no. 12 (2023): 86–91. https://doi.org/10.3103/S1066369X23120095.

[6] Kashin, B., E. Kosov, I. Limonova, and V. Temlyakov. "Sampling Discretization and Related Problems." *Journal of Complexity* 71 (2022): 101653. https://doi.org/10.1016/j.jco.2022.101653.

[7] Temlyakov, V. *Multivariate Approximation*. Cambridge: Cambridge University Press, 2018.

[8] Novak, E., and H. Woźniakowski. *Tractability of Multivariate Problems. Vol. 1: Linear Information*. Zurich: European Mathematical Society Publishing House, 2008. https://doi.org/10.4171/026.

[9] Magaril-Il'yaev, G., K. Osipenko, and V. Tikhomirov. "Optimal Recovery and Extremum Theory." *Doklady Mathematics* 64, no. 1 (2001): 93–96.

[10] Magaril-Il'yaev, G., and K. Osipenko. "Optimal Recovery of Functions and Their Derivatives from Fourier Coefficients Prescribed with an Error." *Sbornik: Mathematics* 193, no. 3 (2002): 387–407. https://doi.org/10.1070/SM2002v193n03ABEH000637.

[11] Traub, J., G. Wasilkowski, and H. Woźniakowski. *Information-Based Complexity*. New York: Academic Press, 1988.

[12] Plaskota, L. *Noisy Information and Computational Complexity*. Cambridge: Cambridge University Press, 1996.

[13] Smolyak, S. "Quadrature and Interpolation Formulas on Tensor Products of Certain Classes of Functions." *Doklady of the USSR Academy of Sciences* 148, no. 5 (1963): 1042–1045.

[14] Sherniyazov, K. *Approximate Reconstruction of Functions and Solutions of Heat Conductivity Equations with Distribution Functions of Initial Temperatures from Classes E, SW and B*. PhD diss., Al-Farabi Kazakh National University, Almaty, Kazakhstan, 1998.

[15] Azhgaliyev, Sh., and N. Temirgaliyev. "On the Informational Power of Linear Functionals." *Mathematical Notes* 73, no. 6 (2003): 759–768. https://doi.org/10.1023/A:1024037410770.

[16] Temirgaliyev, N., K. E. Sherniyazov, and M. E. Berikhanova. "Exact Orders of Computational Errors in Problems of Function Recovery and Discretization of Solutions of the Klein–Gordon Equation by Fourier Coefficients." *Proceedings of the Steklov Institute of Mathematics* 282, suppl. 1 (2013): S165–S191. https://doi.org/10.1134/S0081543813070109.

[17] Korobov, N. M. *Number-Theoretic Methods in Approximate Analysis*. Moscow: Fizmatgiz, 1963.

Information about authors:

Taugynbayeva Galiya Erbolovna – PhD, Associate Professor (acting) of the Department of Fundamental Mathematics, L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, email: taugynbayeva_gye@enu.kz).

Bakytova Nazerke Kenzhebolatkyzy (correspondent author) – 2nd year Master's student, specialty 7M05402 — Mathematics, L.N. Gumilyov Eurasian National University (Astana, Kazakhstan, email: nazerkebakytova29@gmail.com).

Авторлар туралы мәлімет:

Таугынбаева Галия Ерболовна — PhD, Іргелі математика кафедрасының доцент міндетін атқарушысы, Л.Н. Гумилев атындагы Еуразия ұлттық университеті (Астана, Қазақстан, электрондық пошта: taugynbayeva_gye@enu.kz).

Бақытова Назерке Кенжеболатқызы — Л.Н. Гумилев атындагы Еуразия ұлттық университеті, 7M05402 – Математика мамандыгының 2-курс магистранты (Астана, Қазақстан, электрондық пошта: nazerkebakytova29@gmail.com).

Сведения об авторах:

Таугынбаева Галия Ерболовна – PhD, и.о. доцента кафедры фундаментальной математики, Евразийский национальный университет им. Л.Н. Гумилёва (Астана, Казахстан, электронная почта: taugynbayeva_gye@enu.kz).

Бакытова Назерке Кенжеболатовна – магистрант 2 курса специальности 7M05402 — Математика, Евразийский национальный университет им. Л.Н. Гумилёва (Астана, Казахстан, электронная почта: nazerkebakytova29@gmail.com).

Revised: June 16, 2026
Accepted: September 14, 2026