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J.O. Ruzimov
Novosibirsk State University, Novosibirsk, Russia
e-mail: z.ruzimov@g.nsu.ru

ON PUNCTUAL AND POLYNOMIAL-TIME COMPUTABLE PRESENTATIONS OF THE POLYNOMIAL RING

This paper studies the existence and properties of punctual and polynomial-time computable presentations of polynomial rings. We focus on the one-variable polynomial ring $R[x]$ over a commutative ring R with unity and establish a relatively simple criterion for the existence of a primitive recursive (p.r.) and punctual presentation of this structure. The notion of punctual computability, which requires that the atomic diagram of a structure be computable in real time, has recently attracted significant attention in computable model theory due to its natural algorithmic interpretation. We prove that a p.r. and punctual presentation of $R[x]$ exists under mild algebraic assumptions on the base ring R, thereby generalizing earlier results on computable rings. Furthermore, we investigate the polynomial-time computable (P-computable) presentation of $R[x]$ and demonstrate that such a presentation arises naturally whenever the base ring R admits a P-computable presentation. In particular, we show that the standard construction of polynomial rings preserves polynomial-time computability, providing an efficient representation of ring operations. In addition, we analyze the punctual dimension of $R[x]$ and establish a dichotomy: the punctual dimension is either exactly 1 or infinite, with no intermediate values possible. This classification reveals a sharp structural boundary and offers new insights into the interface between effective algebra and computational complexity theory.

Keywords: computability, primitive recursive structure, punctual structure, polynomial-time computable structure, polynomial ring, fields, rings of quotients.

Ж.О. Рузимов
Новосібір мемлекеттік университеті, Новосібір, Ресей
e-mail: z.ruzimov@g.nsu.ru

Көпмүшелік сақинасының пунктуалды және көпмүшелік уақытта есептелетін презентациялары туралы

Бұл жұмыста сақиналы көпмүшеліктердің пунктуалды және көпмүшелік уақытта есептелетін презентацияларының бар болуы мен қасиеттері зерттеледі. Біз бір айнымалысы бар $R[x]$ көпмүшелік сақинасына назар аударамыз, мұндағы R — бірлігі бар коммутативті сақина, және осы құрылымның примитивті рекурсивті (п.р.) және пунктуалды презентациясының бар болуы үшін салыстырмалы түрде қарапайым критерий белгілейміз. Пунктуалды есептелу ұғымы, яғни құрылымның атомдық диаграммасы нақты уақытта есептелуі талап етіледі, соңғы кездері есептелетін модельдер теориясында өзінің табиғи алгоритмдік түсіндірмесіне байланысты айтарлықтай назар аудартты. Біз $R[x]$ сақинасының п.р. және пунктуалды презентациясы R базалық сақинасына қойылатын жұмсақ алгебралық болжамдар кезінде бар болатынын дәлелдейміз, осылайша есептелетін сақиналар туралы бұрынғы белгілі нәтижелерді жалпылаймыз. Сонымен қатар, біз $R[x]$-тің көпмүшелік уақытта есептелетін (P-есептелетін) презентациясын зерттейміз және мұндай презентация базалық R сақинасы P-есептелетін презентацияға ие болған кезде табиғи түрде пайда болатынын көрсетеміз. Атап айтқанда, көпмүшелік сақиналардың стандартты құрылымы көпмүшелік есептелуді сақтайтынын және сақиналық амалдардың тиімді көрсетілімін беретінін дәлелдейміз. Сонымен қатар, біз $R[x]$-тің пунктуалды өлшемділігін талдаймыз және дихотомияны белгілейміз: пунктуалды өлшемділік не дәл 1-ге тең, не шексіздікке тең, аралық мәндер мүмкін емес. Бұл классификация өткір құрылымдық шекараны айқындайды және тиімді алгебра мен есептеу күрделілігі теориясы арасындағы байланысқа жаңа көзқарастар ұсынады.

Түйін сөздер: есептелу, примитивті рекурсивті құрылым, пунктуалды құрылым, көпмүшелік уақытта есептелетін құрылым, көпмүшелік сақина, өрістер, бөлшектер сақинасы.

Ж.О. Рузимов
Новосибирский государственный университет, Новосибирск, Россия
e-mail: z.ruzimov@g.nsu.ru

О пунктуальных и полиномиально вычислимых представлениях кольца многочленов

В данной работе изучаются существование и свойства пунктуальных и полиномиально вычислимых представлений колец многочленов. Мы сосредотачиваемся на кольце многочленов от одной переменной $R[x]$ над коммутативным кольцом R с единицей и устанавливаем относительно простой критерий существования примитивно рекурсивного (п.р.) и пунктуального представления этой структуры. Понятие пунктуальной вычислимости, которое требует, чтобы атомная диаграмма структуры была вычислима в реальном времени, в последнее время привлекло значительное внимание в вычислимой теории моделей благодаря своей естественной алгоритмической интерпретации. Мы доказываем, что п.р. и пунктуальное представление кольца $R[x]$ существует при мягких алгебраических предположениях на базовое кольцо R, обобщая тем самым ранее известные результаты о вычислимых кольцах. Кроме того, мы исследуем полиномиально вычислимое (P-вычислимое) представление $R[x]$ и показываем, что такое представление возникает естественным образом всякий раз, когда базовое кольцо R допускает P-вычислимое представление. В частности, мы показываем, что стандартная конструкция колец многочленов сохраняет полиномиальную вычислимость, обеспечивая эффективное представление операций кольца. Дополнительно мы анализируем пунктуальную размерность $R[x]$ и устанавливаем дихотомию: пунктуальная размерность равна либо в точности 1, либо бесконечности, без возможных промежуточных значений. Эта классификация выявляет резкую структурную границу и предлагает новые взгляды на взаимосвязь между эффективной алгеброй и теорией сложности вычислений.
Ключевые слова: вычислимость, примитивно рекурсивная структура, пунктуальная структура, полиномиально вычислимая структура, кольцо многочленов, поля, кольца частных.

1 Introduction

This paper is devoted to the construction and analysis of computable presentations of the one-variable polynomial ring $R[x]$ over a commutative ring R with unity [1–3] in the context of computability theory and constructive algebra [4–6]. The relevance of this study is due to the fact that many problems in effective algebra and computable model theory require a fixed coding of objects from a structure by natural numbers, under which algebraic operations become computable functions on the codes [4, 6].

The aim of this work is to show how the ring $R[x]$ can be realized within: (i) primitive recursive (punctual) computability and (ii) polynomial-time computability [7–9]. To achieve this goal, we address the following tasks:

- we fix an effective coding of polynomials via coding of finite sequences of coefficients;
- we prove computability (and, under appropriate assumptions on R, primitive recursiveness) of addition and multiplication on the level of codes;
- we discuss conditions under which these operations admit polynomial bounds in the length of the input codes (P-computability);

- we formulate connections with problems about the number of non-isomorphic (punctual) presentations, including the dichotomy "1 versus ∞".

Recall that the elements of the ring $R[x]$ are polynomials of the form

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad a_i \in R, \tag{1}$$

and the operations are defined in the standard way (see, e.g., [1–3]). One of the central problems is to construct a coding of the elements of $R[x]$ by natural numbers such that the algebraic operations correspond to computable functions on codes [4, 6].

To build an effective numbering, we use Gödel's β-function $\beta(x, y)$, which makes it possible to encode finite sequences of natural numbers by a single number and to effectively extract the elements of the sequence (see, e.g., [10]). The main idea is to represent the polynomial (1) by the code of the sequence of its coefficients $(a_0, a_1, \ldots, a_n)$. Then operations on polynomials (including multiplication via the Cauchy product formula for coefficients) reduce to bounded operations on finite sequences, which allows one to prove that the operations are primitive recursive, provided that the base ring R has a primitive recursive presentation.

An infinite algebraic structure $\mathcal{S}$ is *punctual* if $\mathcal{S}$ is primitive recursive (i.e., the signature predicates and functions of $\mathcal{S}$ are primitive recursive), *and* the domain of $\mathcal{S}$ is equal to ω. The notion of a punctual structure was introduced in [11]. Modern results on punctual presentations of algebraic structures develop the approach of [11], and provide structural criteria for the existence of punctual copies and for comparing them (see, e.g., [12–15]).

There are also known sufficient conditions for the existence of punctual presentations in the presence of a unique primitive recursive injective coding and failure of local finiteness [16, 17].

In computational complexity theory, a more stringent requirement is considered: the existence of a presentation in which the operations are performed in time polynomially bounded in the length of the input (the size of the code) [18–20]. For example, in [9] a P-computable presentation of the field of algebraic numbers is constructed. In the present work, similar ideas are applied to the polynomial ring $R[x]$ over a commutative integral domain R with unity, and we discuss restrictions on R needed to obtain P-computability of the operations in $R[x]$ [9].

Finally, an important motivation comes from results on the dichotomy "1 versus ∞" for algebraic structures: one studies the number of pairwise non-isomorphic computable presentations up to computable (or punctual) isomorphism [22]. If a structure does not satisfy the criterion of punctual categoricity, then its punctual dimension is equal to ∞; this result is related to computable model theory [21, 22]. We transfer this setting to the class of one-variable polynomial rings $R[x]$, and consider conditions under which the above dichotomy arises.

2 Preliminaries

We recall the main notions which will be used in the paper.

Definition 1 *Let R be a commutative ring with unity.* The ring of polynomials over R in one variable x *is the ring $R[x]$, consisting of formal sums of the form*

$$\sum_{i=0}^{n} a_i x^i, \quad \text{where } a_i \in R,\ n \in \mathbb{Z}_{\geq 0},$$

where the coefficient a_n is called the leading coefficient*, and if $a_n \neq 0$, then n is called the* degree *of the polynomial.*

Addition and multiplication operations are governed by the following rules:

- Addition:
$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i.$$
- Multiplication:
$$\left(\sum_{i=0}^{n} a_i x^i\right) \cdot \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} \left(\sum_{\substack{i+j=k \\ 0 \leq i \leq n \\ 0 \leq j \leq m}} a_i b_j \right) x^k.$$

Fact 1 If R is a commutative ring with unity (integral domain), then the ring $R[x]$ is also a commutative ring with unity (integral domain, respectively).

Recall that the function

$$c(x, y) = \frac{(x + y)(x + y + 1)}{2} + x$$

is a primitive recursive bijection from $\mathbb{N}^2$ onto $\mathbb{N}$. The primitive recursive functions $l(x)$ and $r(x)$ satisfy $c(l(x), r(x)) = x$ for all $x \in \mathbb{N}$.

We introduce the Gödel function following [10].

Definition 2 (see, e.g., [10]) *We define the function*

$$\beta(x, y) \rightleftharpoons \text{rest}\left(l(x), 1 + (y + 1) \cdot r(x)\right),$$

and call it the Gödel function*. It allows us to obtain a numbering of all finite ordered sets of natural numbers.*

Proposition 1 (see, e.g., [10]) *For any $n \geq 0$ and any $a_0, a_1, \cdots, a_n \in \mathbb{N}$, there exists x such that:*

$$\begin{aligned} &\beta(x, 0) = a_0, \\ &\beta(x, 1) = a_1, \\ &\quad \cdots \\ &\beta(x, n) = a_n. \end{aligned}$$

Definition 3 *Let $\mathcal{A} = (A, L^{\mathcal{A}})$ be a structure of a finite signature L. The structure $\mathcal{A}$ is called primitive recursive if $A \subseteq \omega$ is a primitive recursive set and all operations and relations from $L^{\mathcal{A}}$ are primitive recursive.*

3 Primitive recursive presentations

Our first theorem considers the question of primitive recursive presentability of the ring of polynomials $R[x]$ in one variable.

Theorem 1 *Let R be a primitive recursive, commutative ring with unity. Then the ring of polynomials $R[x]$ in one variable over R also has a primitive recursive presentation.*

Proof 1 *We define the set*

$$\widetilde{R}_0^2 = \{(a_0, a_1, \ldots, a_n) \mid a_0 + a_1 x + \cdots + a_n x^n \in R[x],\ a_n \neq 0,\ a_i \in R\}.$$

This set is primitive recursive, since the base set R is primitive recursive.

Without loss of generality, we may assume that the number 0 is zero in the ring R (i.e., $0 = 0_R$), and 1 is unity ($1 = 1_R$). We also assign (Gödel-style) codes to zero and one: the code of zero is $g(0,1) = c(0,1)$. For the remaining pairs (x, y) we will choose from $\widetilde{R}_0^2$.

Construction. *Consider the set*

$$C = \{c(x,n) \mid \beta(x,n) \neq 0 \wedge (\forall i \leq n)(\beta(x,i) \in R)\},$$

which is also primitive recursive. We view the set C as the set of codes for $\widetilde{R}_0^2$:

$$C = \{c(x,n) \mid \beta(x,0) = a_0,\ \beta(x,1) = a_1,\ \ldots,\ \beta(x,n) = a_n,\ a_i \in R, \\ a_n \neq 0,\ 0 \leq i \leq n\}.$$

We define an equivalence relation Θ on the set C:

$$(u,v) \in \Theta \iff \begin{cases} r(u) = r(v) = n, \text{ and} \\ (\forall i \leq n)(\beta(l(u),i) = \beta(l(v),i)), \text{ and} \\ \beta(l(u),n) \neq 0_R. \end{cases}$$

We can represent the quotient set C/Θ as follows. In each Θ-equivalence class, we choose its canonical representative using a primitive recursive function:

$$\rho(u) = \mu v \leq u\ [r(u) = r(v) \wedge (\forall i \leq r(u))(\beta(l(u),i) = \beta(l(v),i) \wedge \beta(l(u),r(u)) \neq 0)].$$

The function ρ selects the smallest number from an equivalence class $[u]_\Theta$.

We discuss our presentation of the ring of polynomials $R[x]$ with one variable. Suppose that $c(k,l) \in C$ is the code of a polynomial $p_{k,l}(x)$ from $R[x]$. Respectively, $c(n,m)$ is the code of a polynomial $p_{n,m}(x)$.

Definition of addition. *We have*

$$c(n,m) \oplus_{R[x]} c(k,l) = c(h(n,m,k,l),\ H(n,m,k,l)),$$

where $H(n,m,k,l)$ computes the degree of the polynomial $p_{n,m}(x) + p_{k,l}(x)$:

- *if $m \neq l$, then $H(n,m,k,l) = \max\{m,l\}$,*

- *if* $m = l$, *then* $H(n, m, k, l) = \min\{i \mid \beta(n, m - i) \oplus_R \beta(k, l - i) \neq 0_R\}$.

Definition of multiplication. *We have*

$$c(n, m) \odot_{R[x]} c(k, l) = c(h'(n, m, k, l),\ H'(n, m, k, l)),$$

where $H'(n, m, k, l)$ *calculates the degree of the polynomial* $p_{n,m}(x) \cdot p_{k,l}(x)$*:*

$$H'(n, m, k, l) = m + l,$$

and the product coefficients are determined by the formula:

$$\beta(h'(n, m, k, l), s) = \bigoplus_{i+j=s} \beta(l(n), i) \odot_R \beta(l(k), j),$$

where $0 \leq i \leq m$, $0 \leq j \leq l$, *and* $0 \leq s \leq m + l$.

Formal verification. *We need to find primitive recursive functions* $f_\oplus$, $f_\odot$ *such that*

$$f_\oplus(n, m) = n \oplus_{R[x]} m, \quad f_\odot(n, m) = n \odot_{R[x]} m.$$

For $y \leq \min\{r(n), r(m)\}$ *and for* $\min\{r(n), r(m)\} < y \leq \max\{r(n), r(m)\}$, *let*

$$\beta(l(n), y) = \beta(f_\oplus(n, m), y)$$

in the case $r(n) < r(m)$.

Recall the definition of the function β*:*

$$\beta(n, m) = rest(l(n),\ 1 + (m + 1) \cdot r(n)),$$

Using this definition, it is not difficult to determine the laws of primitive recursive functions $h(n, m, k, l)$, $H(n, m, k, l)$, *and then* $f_\oplus$, $f_\odot$.

The set

$$R'_0 = \{n \mid \rho(n) = n\}$$

is primitive recursive. The operations $f_\oplus$ *and* $f_\odot$ *are primitive recursive for a numbering system* β *such that*

$$n = c(x, n') \in \widetilde{R}^2_0, \quad m = c(y, m') \in \widetilde{R}^2_0$$

are the smallest in their equivalence classes.

The function ρ *defines the set* $\{n \mid \rho(n) = n\}$, *selecting the element with the smallest number from each equivalence class. This set is the basis for the ring* $R[x]$.

The case of addition. *First, we determine the degree of the sum of polynomials with codes* n *and* m*:*

$$d(n, m) = \begin{cases} r(n) - i, & \text{if } r(n) = r(m),\ i = \mu j \leq r(m) : \text{ such that} \\ & \beta(l(n), r(n) - i) \oplus_R \beta(l(m), r(m) - i) \neq 0_R, \\ r(m), & \text{if } r(n) < r(m), \\ r(n), & \text{if } r(m) < r(n). \end{cases}$$

Now define $f_{\oplus}(n,m)$. *Let*

$$x = l(n), \quad n' = r(n), \quad y = l(m), \quad m' = r(m).$$

We need to find x_0, y_0 *such that the code* $c(x_0, y_0)$ *defines polynomials*

$$p(x) = \sum_{i=0}^{r(n)} a_i x^i, \quad q(x) = \sum_{i=0}^{r(m)} b_i x^i,$$

where $p(x)$ *has code* $n = c(l(n), r(n))$, *and* $q(x)$ *has code* $m = c(l(m), r(m))$.

First, we define y_0 *as the degree of the polynomial* $p(x) + q(x)$.

- *If* $m' > n'$, *then for* $i \leq m'$, *the coefficients are equal to* $a_i \oplus_R b_i \Leftrightarrow \beta(x_0, i)$.
- *If* $n' \geq i > m'$, *then the coefficients are* $a_i = \beta(x_0, i)$.
- *If* $m' > n'$, *then the coefficients are* $b_i = \beta(x_0, i)$.

The degree satisfies the following rules:

- *if* $n' > m'$, *then* $y_0 = n'$,
- *if* $m' > n'$, *then* $y_0 = m'$,
- *if* $m' = n'$, *then the degree decreases when* $a_{n'} \oplus_R b_{m'} = 0_R$, *and then* $y_0 = n' - 1$.

We define the function

$$g(x, n', y, m', i) = \begin{cases} \beta(x,i) \oplus_R \beta(y,i), & \text{if } i \leq \min\{n', m'\}, \\ \beta(y,i), & \text{if } n' < i \leq m', \\ \beta(x,i), & \text{if } m' < i \leq n'. \end{cases}$$

Now define

$$b(n,m) = \left(\sum_{i=1}^{d(n,m)} g(l(n), r(n), l(m), r(m), i) + n + m + 1 \right)!.$$

Next, we take

$$m(n,m,i) = b(n,m)(i+1) + 1.$$

In order to solve systems of the form

$$M(n,m,i) = (i+1) \cdot b(n,m) + 1,$$

we are looking for the smallest $z \leq (M(n,m,i) - m(n,m,i))^4$ *such that*

$$l(z) \cdot M(n,m,i) - r(z) \cdot m(n,m,i) = 1,$$

and we assume that

$$u_i(n,m) = l(z).$$

This gives the desired u_i *and* v_i.

Now define

$$x(n,m) = \sum_{i=0}^{d(n,m)} g(l(n), r(n), l(m), r(m), i) \cdot u_i(n,m).$$

We have found the expressions that we were looking for: $x_0 = x(n,m)$, $y_0 = d(n,m)$. *Thus,*

$$c(x(n,m), d(n,m)) = n \oplus_R m,$$

and the desired function for addition is recovered:

$$f_{\oplus}(n,m) = c(x_0, y_0) = c(x(n,m), d(n,m)).$$

The case of multiplication. *Similarly, we define the multiplication function* $f_{\odot}$. *The degree of the product of polynomials with codes* n *and* m *is equal to:*

$$d'(n,m) = r(n) + r(m).$$

The coefficients of the product are determined by the formula:

$$\gamma(s) = \bigoplus_{i+j=s} \beta(l(n), i) \odot_R \beta(l(m), j), \quad 0 \le s \le d'(n,m).$$

We define the function

$$g'(x, n', y, m', s) = \bigoplus_{\substack{i+j=s \\ 0 \le i \le n' \\ 0 \le j \le m'}} \beta(x, i) \odot_R \beta(y, j).$$

Now we determine

$$b'(n,m) = \left(\sum_{s=0}^{d'(n,m)} g'(l(n), r(n), l(m), r(m), s) + n + m + 1 \right)!.$$

Next, we take

$$m'(n,m,s) = b'(n,m)(s+1) + 1.$$

To solve systems of the form

$$M'(n,m,s) = (s+1) \cdot b'(n,m) + 1,$$

we are looking for the smallest $z' \le (M'(n,m,s) - m'(n,m,s))^4$ *such that*

$$l(z') \cdot M'(n,m,s) - r(z') \cdot m'(n,m,s) = 1,$$

and we assume

$$u'_s(n,m) = l(z').$$

We define

$$x'(n,m) = \sum_{s=0}^{d'(n,m)} g'(l(n), r(n), l(m), r(m), s) \cdot u'_s(n,m).$$

Then the desired function for multiplication is:

$$f_{\odot}(n,m) = c(x'(n,m), d'(n,m)).$$

Therefore, we have constructed primitive recursive functions $f_{\oplus}$ and $f_{\odot}$ that define the operations in the ring $R[x]$. Theorem 1 is proven.

4 Punctual presentations

Recall that a countable structure is *punctual* if its domain is equal to $\mathbb{N}$, and the signature operations and predicates of the structure are primitive recursive.

We will use the following notions and results.

Definition 4 (see [17]) *Let $A \subseteq \omega$. We say that a function $f : A \to \omega$ is* primitive recursive *(*p.r.*) if there exists a total p.r.f. $f_0 : \omega \to \omega$ such that $f = f_0|_A$.*

Definition 5 ([17]) *Let $\mathcal{A}$, $\mathcal{B}$ be two structures of a signature L with primitive recursive domains $A, B \subseteq \omega$, respectively. We say that they are* ***primitively recursively isomorphic*** *(denoted by $\mathcal{A} \cong_{pr} \mathcal{B}$) if there exists an isomorphism $f : \mathcal{A} \to \mathcal{B}$ such that both f and f^{-1} are p.r.f.*

Definition 6 ([11]) *A punctual structure $\mathcal{S}$ is* punctually categorical *if every punctual structure $\mathcal{A}$ isomorphic to $\mathcal{S}$ satisfies $\mathcal{A} \cong_{pr} \mathcal{S}$.*

Lemma 1 (see, e.g., [17]) *If $\mathcal{A}$ is a primitive recursive structure with domain $A \subseteq \omega$, and there exists a primitive recursive injection $f : \omega \to A$, then $\mathcal{A} \cong_{\mathrm{pr}} \mathcal{B}$, where $\mathcal{B}$ is a punctual structure.*

Lemma 1 implies the following:

Lemma 2 (Lemma 2 in [16]) *If $\mathcal{A}$ is a primitive recursive structure that is not locally finite (that is, $\mathcal{A}$ has a finitely generated, infinite substructure), then $\mathcal{A}$ is isomorphic to a punctual structure.*

The main result of this section is the following:

Proposition 2 *If R is a primitive recursive, commutative ring with unity such that R is not locally finite, then the ring of polynomials $R[x]$ in one variable over R has a punctual presentation.*

Proof 2 *Since R is not locally finite, the ring $R[x]$ is also not locally finite. By Theorem 1, $R[x]$ has a primitive recursive presentation. By applying Lemma 2, we get a punctual isomorphic copy of $R[x]$. Proposition 2 is proven.*

5 Polynomial-time computable presentations

This section is devoted to the polynomial-time computable (P-computable) presentations of the ring of polynomials in one variable.

We give the necessary definitions and known results.

Definition 7 ([20]) *A structure $\mathcal{A} = (A, \ldots)$ in a finite signature L is called* polynomial-time computable *(or P-computable) if there exists a finite alphabet Σ such that $A \subseteq \Sigma^*$, and the set A and all functions and predicates belonging to the signature of the structure A are computable in polynomial time.*

Definition 8 ([20]) *Let B be an abstract structure. If there exists a structure A that is P-computable, and $B \cong A$, then A is called a P-computable presentation of the structure B.*

In [19], it is proved that the field of rational numbers $(\mathbb{Q}, +, \cdot)$ and any finitely generated Abelian group are isomorphic to P-computable structures.

Definition 7 applied to rings yields:

Definition 9 *A ring R is* polynomial-time computable *if the set of elements R can be represented as a subset $R \subseteq \Sigma^*$, where this set is P-computable, and all basic operations and relations on R (in particular, addition, multiplication, and equality) are computable in polynomial time.*

Definition 10 ([19]) *Two P-computable structures A and B are called* polynomially isomorphic *(or P-computably isomorphic) if there exists an isomorphism $f : A \to B$ such that the functions f and f^{-1} are computable in polynomial time. Notation: $A \cong_p B$.*

The following general results on P-computable commutative algebra are known.

Lemma 3 ([18]) *If an arbitrary structure B is computable in polynomial time, then any finitely generated substructure A of it also has a representation that is computable in polynomial time.*

Theorem 2 ([18]) *Let $\mathcal{A} = (A, +, \cdot, -, ^{-1}, 0, 1)$ be a field and $\overline{e} = e_1, \ldots, e_n$ be a finite set of generators for $\mathcal{A}$. Then the following conditions are equivalent:*

(a) $\mathcal{A}$ has a P-computable presentation;

(b) there exists an algorithm that, given the binary code of the polynomial $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, checks the condition $P(e_1, \ldots, e_n) = 0$ in time $O(2^{m^c})$, where

$$m = \max\{\deg(P), L(a) \mid a - \textit{coefficient } P\},$$

and c is a constant.

Corollary 1 ([18])) *Let $\mathcal{A} = (A, +, \cdot, -, ^{-1}, 0, 1)$ be a finitely generated field. Then $\mathcal{A}$ has a P-computable presentation.*

Theorem 3 ([23]) *Let $\mathcal{A}$ be a P-computable structure of a finite signature L, $\mathcal{B}$ be a computable structure, and let $f : \mathcal{B} \to \mathcal{A}$ be a computable isomorphic embedding. Then $\mathcal{B}$ is computably isomorphic to some P-computable structure.*

Corollary 2 ([23]) *Let $\mathcal{A}$ be a P-computable structure of finite signature L and $E \subseteq A$ be an enumerable set. Then the substructure $\mathcal{A}(E)$ generated in $\mathcal{A}$ by the set E also has a P-computable representation.*

Here we prove the following:

Theorem 4 *Let R be a commutative ring with unity, which has a polynomial-time computable presentation. Then the ring of polynomials $R[x]$ in one variable over R also has a polynomial-time computable presentation.*

Proof 3 *Let R be a P-computable commutative ring with unity. We consider the representation of polynomials over R as lists of coefficients. Each polynomial $p \in R[x]$ can be uniquely represented as a finite list of coefficients:*

$$p = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \ \mapsto \ \langle a_0, a_1, \ldots, a_n \rangle,$$

where $a_n \neq 0$ (if $p \neq 0$).

We define the set of all such lists:

$$E = \{\langle a_0, a_1, \ldots, a_n \rangle \mid n \in \mathbb{N},\ a_i \in R,\ a_n \neq 0\} \cup \{\langle\rangle\},$$

where $\langle\rangle$ corresponds to the zero polynomial, and this set is countable.

Step 1: Constructing a computable structure B. *We define the structure $B = (E, \oplus, \odot, 0_B, 1_B)$, where:*

- *$\oplus$ and $\odot$ are the operations of addition and multiplication of polynomials applied to lists of coefficients;*
- *$0_B = \langle\rangle$ (zero polynomial);*
- *$1_B = \langle 1_R \rangle$ (unit polynomial).*

Since the operations in R are P-computable, the operations $\oplus$, $\odot$ on E are computable (and even P-computable, since they require only a finite number of operations in R, each of which is performed in polynomial time). Checking equality on E reduces to an element-by-element comparison of coefficients, which also runs in polynomial time.

Thus, B is a computable structure isomorphic to $R[x]$.

Step 2: Application of Theorem 3. *Now consider a P-computable structure R (as a ring). Construct a* computable isomorphic embedding

$$f : B \hookrightarrow R^{\omega},$$

where R^{ω} is the direct sum of a countable number of instances of R,

$$R^{\omega} = \bigoplus_{i \in \mathbb{N}} R = \{(a_0, a_1, a_2, \ldots) \mid a_i \in R, \text{ and only a finite number of } a_i \neq 0\}.$$

This structure is also P-computable, since R is P-computable and the construction of the direct sum preserves P-computability.

Specifically, for the polynomial $p = \langle a_0, \dots, a_n \rangle$, we define:

$$f(p) = (a_0, a_1, \dots, a_n, 0, 0, \dots) \in R^\omega.$$

The mapping f is injective, preserves operations, and is computable (since the list of coefficients is finite). Therefore, f is a computable isomorphic embedding of B into the P-computable structure R^ω.

Encoding elements of R^ω with binary strings: *Let each element $a \in R$ be encoded by a binary string* $\mathrm{bin}(a)$. *Then the element $\vec{a} = (a_0, a_1, \dots, a_k, 0, 0, \dots) \in R^\omega$ is encoded by a string of the form*

$$\mathrm{bin}(\vec{a}) = \mathrm{bin}(n_1) * \mathrm{bin}(m_1) * \cdots * \mathrm{bin}(n_t) * \mathrm{bin}(m_t),$$

where n_i is the position index, m_i is the code of the element $a_{n_i} \in R$, and only nonzero elements or all positions up to a given degree k are encoded. In the case of polynomials, it is sufficient to encode

$$\mathrm{bin}(a_0) * \mathrm{bin}(a_1) * \cdots * \mathrm{bin}(a_n).$$

We define the structure

$$C = \{\mathrm{bin}(\vec{a}) \mid \vec{a} \in R^\omega\},$$

with operations $\oplus_C$ and $\odot_C$ defined via coordinate operations in R^ω. Since R is P-computable, C is a P-computable structure isomorphic to R^ω.

Now we have:

- *$A = R^\omega$ (or C) is a P-computable structure;*
- *B is a computable structure isomorphic to $R[x]$;*
- *$f : B \to R^\omega$ is a computable isomorphic embedding.*

By Theorem 4, B is computably isomorphic to some P-computable structure C'.

Since $B \cong R[x]$, then $R[x]$ also has a P-computable representation (via C'). Theorem 4 is proven.

6 Punctual dimensions

We list some known results on punctual dimensions.

Theorem 5 ([11]) *1. An equivalence structure S is punctually categorical if and only if it either S has the form $F \cup E$, where F is finite and E contains only classes of size* 1, *or S has a finite number of classes, of which no more than one is infinite.*

2. A linear order is punctually categorical if and only if it is finite.

3. *A Boolean algebra is punctually categorical if and only if it is finite.*

4. *A torsion-free Abelian group is punctually categorical if and only if it is the trivial group* 0.

5. *An Abelian p-group is punctually categorical if and only if it has the form* $F \oplus \mathbb{V}$, *where* $p\mathbb{V} = 0$ *and* F *is finite.*

Theorem 6 ([22]) *For any computable structure from the following classes, the punctual dimension is either* 1 *or* ∞*:*

1. *Equivalence structures,*

2. *Linear orders,*

3. *Torsion-free Abelian groups,*

4. *Abelian p-groups,*

5. *Boolean algebras.*

We prove the following:

Theorem 7 *For the ring of polynomials* $R[x]$ *in one variable over a non-locally-finite, commutative ring with unity* R, *the punctual dimension is either* 1 *or* ∞.

Proof 4 *Let* R *be a primitive recursive commutative ring with unity. By Proposition 2, if* R *is not locally finite, then* $R[x]$ *has a punctual representation.*

For convenience, we will consider the specific case $R = \mathbb{Z}$, *since* $\mathbb{Z}[x]$ *satisfies all the necessary properties (it is even an integral domain, not locally finite, and has a punctual representation). All arguments can be transferred to an arbitrary* R *with the specified properties.*

Similar to the proof of Theorem 3 from [22], we construct a countable family of punctual presentations $U_1, U_2, \ldots, U_n, \ldots$ *of the ring* $\mathbb{Z}[x]$ *satisfying the following requirements:*

- R^e_{ij}*: the p.r. function* $p_e : U_i \to U_j$ *is not an isomorphism (for* $i \neq j$*);*
- Q_i*:* U_i *is a punctual copy of* $\mathbb{Z}[x]$.

We fix a primitive recursive numbering of the requirements R^e_{ij}. *The requirements will be satisfied sequentially.*

Construction. *At every stage* s, *each copy* U_i *will either be isomorphic to* $\mathbb{Z}[x]$ *or have the form*

$$U_i = \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[x],$$

where $\mathbb{Q}$ *is the field of rational numbers, considered as a* $\mathbb{Z}$*-module. Such an extension of coefficients (scalars) corresponds to the localisation of* $\mathbb{Z}[x]$ *by the multiplicative system* $\mathbb{Z}\setminus\{0\}$ *and gives the ring of polynomials* $\mathbb{Q}[x]$. *This construction is standard (see, for example, textbooks on commutative algebra [1–3]).*

Note that $\mathbb{Q}$ is primitive recursive and even punctual. Therefore, by Proposition 2, $\mathbb{Q}[x]$ has a punctual presentation.

Diagonalisation. *We fix a non-zero polynomial $p(x) \in \mathbb{Z}[x]$. We initially put $p(x)$ into all U_i. For the requirement R^e_{ij}, we temporarily extend U_i to $\mathbb{Q}[x]$ by adding a new element of the form $q \cdot p(x)$, where $q \in \mathbb{Q}$, and the value of q is not initially fixed. This element is temporarily not identified with any element from $\mathbb{Z}[x]$.*

We compute $p_e(p(x))$ and $p_e(q \cdot p(x))$ in U_j. Until these values converge, we continue to construct U_i as $\mathbb{Q}[x]$, simultaneously extending the isomorphisms $\phi_k : U_k \to \mathbb{Z}[x]$ to the already defined elements.

When the values $p_e(p(x))$ and $p_e(q \cdot p(x))$ are computed, we check whether the images $\phi_j(p_e(p(x)))$ and $\phi_j(p_e(q \cdot p(x)))$ belong to the ring $\mathbb{Z}[x]$ (as a subset of $\mathbb{Q}[x]$). There are two possible cases here:

1. *Both images lie in $\mathbb{Z}[x]$. Then, since $\mathbb{Z}[x]$ is an integral domain, we can consider their ratio. If $\phi_j(p_e(q \cdot p(x))) = r \cdot \phi_j(p_e(p(x)))$ for some $r \in \mathbb{Q}$, then we choose q such that $q \neq r$ (for example, $q = r + 1$). Then we identify the added element $q \cdot p(x)$ with the element $M \cdot p(x)$ in $\mathbb{Z}[x]$, where M is an integer not equal to r. This violates the preservation of scalar multiplication under the mapping p_e, therefore p_e cannot be an isomorphism.*

2. *At least one of the images does not belong to $\mathbb{Z}[x]$. Then, after identifying $q \cdot p(x)$ with some element $M \cdot p(x) \in \mathbb{Z}[x]$, the mapping p_e translates the element from $\mathbb{Z}[x]$ to an element not lying in $\mathbb{Z}[x]$, which also contradicts the fact that p_e must be an isomorphism between copies of $\mathbb{Z}[x]$.*

After fulfilling the requirement R^e_{ij}, we return U_i to the representation $\mathbb{Z}[x]$ by factoring according to the corresponding relation (i.e., identifying $q \cdot p(x)$ with the selected element $M \cdot p(x)$). This process is primitive recursive, since all decisions and choices are made in finite time based on already computed values.

Verification. *By construction, each U_i is a punctual copy of $\mathbb{Z}[x]$, and for every pair $i \neq j$ and every e, the requirement R^e_{ij} is satisfied, i.e., p_e is not an isomorphism between U_i and U_j. Consequently, the obtained punctual representations are pairwise not punctually isomorphic. Since an infinite number of such presentations can be constructed, the punctual dimension of $\mathbb{Z}[x]$ (and therefore of $R[x]$ for a suitable R) is equal to ∞.*

According to Theorem 3 from [22], if the computable structure of one of the considered classes (equivalence structures, linear orders, Boolean algebras, Abelian groups) does not satisfy the criterion of punctual categoricality established in Theorem 2 from [22], then its punctual dimension is infinite. Moreover, in [22], a hypothesis is put forward and confirmed that for structures of these classes, the dichotomy "1 versus ∞" holds, analogous to the well-known dichotomy in computable model theory: a structure is either punctually categorical (has punctual dimension 1) or has punctual dimension ∞.

This leads to the natural assumption that if the ring R is a trivial ring (a zero ring, where $R = \{0\}$), then its punctual dimension is equal to 1 (i.e., R satisfies the criteria for punctual categoricity analogous to Theorem 2 from ([22]).

Indeed, according to Theorem 2 from ([22]), the trivial Abelian group $\{0\}$ is punctually categorical and has punctual dimension 1. Since the additive group of the ring R in this case

is precisely the trivial group $G = \{0\}$, the ring R itself as an algebraic structure inherits this property: any two of its punctual representations are punctually isomorphic.

Let us generalise this result to the ring of polynomials $R[x]$. If $R = \{0\}$, then the only possible polynomial over R is the zero polynomial, i.e. $R[x] = \{0\}$. Thus, $R[x]$ is also a trivial (single-element) ring. Being a finite structure, it is punctually categorical, and therefore its punctual dimension is 1*.*

In other cases, when R is a non-trivial commutative ring with unity, the ring $R[x]$ is infinite and has a rich algebraic structure. By analogy with the proofs for classical classes in ([22]) and with what was proven above for $R[x]$, such a structure does not satisfy the punctual categorical criterion, and therefore its punctual dimension is infinite. Theorem 7 is proven.

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Information about author:

Ruzimov Javokhir Otabekovich – postgraduate student at Novosibirsk State University (Novosibirsk, Russia, e-mail: z.ruzimov@g.nsu.ru).

Автор туралы мәлімет:

Рузимов Жавохир Отабекович – Новосібір мемлекеттік университетінің аспиранты (Новосібір, Ресей, электрондық пошта: z.ruzimov@g.nsu.ru).

Информация об авторе:

Рузимов Джавохир Отабекович – аспирант Новосибирского государственного университета (Новосибирск, Россия, электронная почта: z.ruzimov@g.nsu.ru).

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