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OPTIMAL NEUMANN BOUNDARY CONTROL OF THE HEAT EQUATION WITH H^1 CONTROLS: EXISTENCE, OPTIMALITY CONDITIONS AND STRONG CONVEXITY

This paper investigates an optimal Neumann boundary control problem for the one-dimensional heat equation. The control input is the heat flux applied at the left boundary of the domain, while the objective is to drive the temperature at the right boundary toward a given target trajectory over a finite time interval. The admissible control set is taken as a bounded, closed, and convex subset of the Hilbert space $H^1(0, T)$, which yields enhanced regularity compared with the classical $L^2(0, T)$ framework. The existence and uniqueness of the optimal control are established via the direct method of the calculus of variations, together with the strong convexity of the cost functional. The Fréchet differentiability of the cost functional is proved, and an explicit expression for the gradient is derived through the associated adjoint problem. A necessary optimality condition is obtained in the form of a variational inequality. Furthermore, the strong convexity of the cost functional guarantees the uniqueness of the optimal control and implies that the optimality condition is also sufficient.

Keywords: optimal boundary control, heat equation, Neumann boundary condition, Fréchet differentiability, adjoint problem, strong convexity.

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H^1 басқарулары бар жылуөткізгіштік теңдеуі үшін Нейман шекаралық оптималды басқаруы: бар болуы, оптималдық шарттары және әлді дөңестік

Бұл мақалада бірөлшемді жылу теңдеуі үшін Нейман шекаралық басқарудың оптималды есебі зерттеледі. Басқару кірісі ретінде кеңістіктік облыстың сол жақ шекарасына түсірілегін жылу ағыны алынады, ал мақсат — ақырлы уақыт аралығында оң жақ шекарадағы температураны берілген мақсатты траекторияға жақындату. Мүмкін басқарулар жиыны $H^1(0, T)$ Гильберт кеңістігінің шектелген, тұйық және дөңес ішкі жиыны ретінде қабылданады, бұл классикалық $L^2(0, T)$ негізіндегі жағдаймен салыстырғанда жоғары регулярлықты қамтамасыз етеді. Оптималды басқарудың бар болуы мен жалғыздығы вариациялық есептеулердің тікелей әдісімен және шығын функционалының әлді дөңестігімен негізделеді. Шығын функционалының Фреше бойынша дифференциалдануы Лагранж көбейткіштері әдісімен дәлелденеді, ал градиенттің айқын өрнегі қосарланған есеп арқылы алынады. Қажетті оптималдық шарты вариациялық теңсіздік түрінде алынады. Сонымен қатар, шығын функционалының әлді дөңестігі оптималды басқарудың жалғыздығын қамтамасыз етеді және оптималдық шарттың жеткілікті екенін білдіреді.

Түйін сөздер: оптималды шекаралық басқару, жылу теңдеуі, Нейман шекаралық шарты, Фреше бойынша дифференциалдану, қосарланған есеп, әлді дөңестік.

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Оптимальное граничное управление Неймана для уравнения теплопроводности с H^1 -управлениями: существование, условия оптимальности и сильная выпуклость

В данной работе исследуется задача оптимального граничного управления Неймана для одномерного уравнения теплопроводности. В качестве управляющего воздействия рассматривается тепловой поток, подаваемый на левую границу пространственной области, а целью является приведение температуры на правой границе к заданной целевой траектории на конечном временном интервале. Допустимое множество управлений принимается как ограниченное, замкнутое и выпуклое подмножество гильбертова пространства $H^1(0, T)$, что обеспечивает повышенную регулярность по сравнению с классической постановкой в $L^2(0, T)$. Существование и единственность оптимального управления устанавливаются прямым методом вариационного исчисления совместно с сильной выпуклостью функционала стоимости. Дифференцируемость функционала стоимости по Фреше доказывается методом множителей Лагранжа, а явное выражение для градиента выводится через соответствующую сопряжённую задачу. Необходимое условие оптимальности получено в форме вариационного неравенства. Кроме того, сильная выпуклость функционала стоимости гарантирует единственность оптимального управления и означает, что условие оптимальности является также и достаточным.

Ключевые слова: оптимальное граничное управление, уравнение теплопроводности, граничное условие Неймана, дифференцируемость по Фреше, сопряжённая задача, сильная выпуклость.

1 Introduction

Partial differential equations (PDEs) are fundamental to the mathematical modeling of physical processes, arising naturally in fields as diverse as fluid mechanics, heat transfer, and reaction-diffusion systems. Among these, the one-dimensional heat equation stands as one of the most classical and thoroughly studied models in mathematical physics, describing the diffusion of thermal energy. Its theoretical richness and practical relevance have established it as one of the central benchmark problems in control theory. The regulation of heat conduction processes is of considerable importance in numerous engineering applications, including aerospace thermal protection systems, energy conversion devices, and chemical reactor temperature management. In many such settings, actuation is physically constrained to the boundary of the spatial domain, and the interaction with the system is most naturally characterized through the heat flux rather than the temperature itself. This motivates the study of Neumann boundary control for the heat equation, wherein the applied flux at one boundary serves as the control input. Building on this foundation, boundary control problems for the heat equation have attracted considerable attention in the literature. In his renowned book [8], which is frequently referenced in optimal control theory, Lions thoroughly explores optimal control problems for the heat equation. The book examines the problem of determining the Neumann boundary conditions from the space $L^2(0, T)$ in detail, in order to ensure that the solution of the system converges to the given target function at the final time.

In the context of parabolic equations, the heat equation has served as a prototypical model for investigating controllability and optimal control problems for distributed parameter systems. By utilizing boundary observations, Háo [3] reformulated the noncharacteristic Cauchy problem into an optimal control framework, where the unknown initial condition and the boundary heat flux are treated as control variables to be recovered. Hasanoğlu [4] considered the problem of finding unknown pair $\{h(t, x), f(t)\}$ in the equation $y_t - (a(x)y_x)_x = h(t, x)$ with conditions $y_x(t, 0) = 0$, $-a(L)y_x(t, L) = v[y(t, L) - f(t)]$ from the final overdetermination. Şener and Subaşı [13] analyzed the optimal control problem of the

Neumann boundary condition on the right-hand side for the one-dimensional heat equation $y_t - (a(x)y_x)_x = h(t, x)$ by minimizing the cost functional

$$J_\alpha(f) = \int_0^L [y(T, x; f) - g(x)]^2 dx + \alpha \|f\|_{H^1(0, T)}^2,$$

where the given target function $g(x) \in L^2(0, L)$ and $\alpha > 0$. Koç et al. [10] considered an optimal control problem in which the heat flux applied at the left boundary serves as the control function, with the objective of steering the state toward a given target function $g(t, x)$ in the L^2 -norm. The present study is closely related to [10] in its choice of the left-boundary heat flux as the control in $H^1(0, T)$ and its variational framework, which includes the existence and uniqueness of an optimal control, the Fréchet differentiability of the cost functional, the derivation of its gradient via an adjoint problem, a necessary optimality condition, and the construction of a gradient-based minimizing sequence. The principal distinction from [10] lies in the cost functional: whereas [10] aims to steer the state toward a given target over the space-time domain, the present study formulates the tracking objective in terms of the boundary trace $w(\cdot, L)$ at the right boundary of the rod. This difference leads to a distinct adjoint formulation, in which the boundary tracking error enters as Neumann boundary data at $x = L$, and the adjoint problem is analyzed in the space $V^{1,0}(\Omega)$ (Section 3). Beyond this difference, the present work establishes the existence of an optimal control by the direct method of the calculus of variations, proves strong convexity of the cost functional, and establishes the Lipschitz continuity of the gradient. These properties yield an explicit geometric convergence rate for the projected gradient sequence, providing a quantitative convergence result beyond the construction of a minimizing sequence.

The reachability problem for parabolic boundary control problems with control constraints is studied in [1], where attainable sets and the structure of admissible controls are analyzed. The controllability and approximate controllability of the heat equation with variable coefficients on an unbounded domain are investigated in [2], where Dirichlet boundary control in $L^\infty(0, T)$ is employed, and a transformation operator approach is used to reduce the variable-coefficient problem to the constant-coefficient case. From a numerical perspective, [7] derived exact discrete solutions for an optimal boundary control problem constrained by the one-dimensional heat equation, providing a rigorous framework for benchmarking time integration methods in the optimal control context.

Motivated by these developments, the present paper investigates an optimal Neumann boundary control problem for the one-dimensional heat equation. The control variable represents the heat flux applied at the left boundary, while the temperature at the right boundary is driven toward a prescribed target trajectory over a finite time horizon. In contrast to many earlier studies where the admissible control set is taken in $L^2(0, T)$, the present work considers controls belonging to the Hilbert space $H^1(0, T)$. This choice provides greater regularity of the admissible heat flux and ensures $H^{1,1}(\Omega)$ regularity of the state. However, computing the gradient requires solving an auxiliary problem in addition to the adjoint problem. As discussed above, the present study differs from [10] in the choice of the cost functional, which tracks the temperature at the right boundary rather than the state over the entire spatial domain. Moreover, unlike [13], where the control acts at the right

boundary and the objective is to match a spatial target at the final time, the present work applies the control at the left boundary and tracks the temperature at the right boundary over the entire time interval. The analysis further establishes strong convexity of the cost functional and an explicit geometric convergence rate for the projected gradient sequence.

We now present the following heat equation with Neumann boundary conditions:

$$\begin{cases} w_t = aw_{xx} + h(t, x), & (t, x) \in \Omega := (0, T) \times (0, L), \\ w(0, x) = w_0(x), & 0 < x < L, \\ w_x(t, 0) = \nu(t), \quad w_x(t, L) = 0, & 0 < t < T. \end{cases} \quad (1)$$

In system (1), x and t denote position and time, respectively, $w(t, x)$ denotes the temperature of the rod at t and x , $w_0(x)$ denotes the initial temperature of the system, $h(t, x)$ is the heat source for the system, $\nu(t)$ is the heat flux, T is a given final time and $a > 0$ is the heat conductivity. Model (1) is a boundary value problem for a linear parabolic equation. We assume that $h(t, x)$, $w_0(x)$ are given functions that hold the following conditions:

$$w_0(x) \in H^1(0, L), \quad h(t, x) \in L_2(\Omega). \quad (2)$$

We need to specify the solution of model (1). We first recall some function spaces.

The space $L^2(\Omega)$ consists of all measurable functions in the sense of Lebesgue on Ω having a finite norm given by

$$\|f\|_{L^2(\Omega)} = \left(\int_0^T \int_0^L (f(t, x))^2 dx dt \right)^{\frac{1}{2}}.$$

The space $H^1(0, L)$ is defined as the subset of all elements f in $L^2(0, L)$ such that its generalized derivative f' has a finite L^2 norm. The scalar product and norm in this space are defined by

$$\langle f, g \rangle_{H^1(0, L)} = \int_0^L (f(x)g(x) + f'(x)g'(x)) dx$$

and

$$\|f\|_{H^1(0, L)} = \left(\langle f, f \rangle_{H^1(0, L)} \right)^{\frac{1}{2}},$$

respectively. The space $H^1(0, T)$ is defined in a similar way.

The space $H^{1,1}(\Omega)$ is defined as the subset of all elements f in $L^2(\Omega)$ such that its generalized derivatives f_t, f_x have a finite L^2 norm. The scalar product and norm in this space are defined by

$$\langle f, g \rangle_{H^{1,1}(\Omega)} = \int_0^T \int_0^L (fg + f_t g_t + f_x g_x) dx dt$$

and

$$\|f\|_{H^{1,1}(\Omega)} = \left(\langle f, f \rangle_{H^{1,1}(\Omega)} \right)^{\frac{1}{2}},$$

respectively.

Definition 1 *A function $w \in H^{1,1}(\Omega)$ is a weak solution of model (1) if $w(0, x) = w_0(x)$ and the following integral identity holds*

$$\int_0^t \int_0^L (w_\tau \varphi + a w_x \varphi_x - h \varphi) dx d\tau = -a \int_0^t \nu(\tau) \varphi(\tau, 0) d\tau, \quad \forall t \in (0, T] \quad (3)$$

for all $\varphi \in H^{1,0}(\Omega) := \{\varphi \in L^2(\Omega) : \varphi_x \in L^2(\Omega)\}$ where φ_x denotes the weak derivative of φ with respect to x , and $\varphi(\tau, 0)$ is understood in the trace sense.

In order to apply the result of [3, Theorem 2] to model (1), we identify the Neumann boundary data in the one-dimensional setting as

$$g(t, 0) := -a \nu(t), \quad g(t, L) := 0, \quad t \in (0, T).$$

Since $\nu \in H^1(0, T)$, we have

$$g(\cdot, 0) \in H^1(0, T),$$

which in the one-dimensional setting (where the boundary at $x = 0$ is a single point) coincides with the boundary space $H^{0,1}$ required in [3, Theorem 2]. Thus, the boundary data satisfy the assumptions of [3, Theorem 2], which justifies its application to (1).

Theorem 1 ([3]) *Let the condition (2) hold and $\nu \in H^1(0, T)$ be a given function. Then there exists a unique weak solution $w \in H^{1,1}(\Omega)$ of model (1), and this weak solution satisfies the following estimate*

$$\begin{aligned} \max_{0 \leq t \leq T} \|w\|_{L^2(0,L)} + \|w\|_{L^2(\Omega)} + \|w_x\|_{L^2(\Omega)} + \|w_t\|_{L^2(\Omega)} + \|w(\cdot, 0)\|_{L^2(0,T)} \\ + \|w(\cdot, L)\|_{L^2(0,T)} \leq c_1 \left(\|w_0\|_{H^1(0,L)} + \|h\|_{L^2(\Omega)} + \|\nu\|_{H^1(0,T)} \right) \end{aligned} \quad (4)$$

where c_1 is a constant independent of w .

Unless otherwise stated, in what follows when we speak of a solution to model (1), we shall always mean the weak solution in the sense of (3) above.

This paper is organized as follows: Section 2 introduces the problem formulation and uses the direct method of the calculus of variations to prove the existence and uniqueness of the optimal control. Section 3 introduces the adjoint problem and proves Fréchet differentiability of the cost functional. Section 4 derives the optimality conditions and proves the strong convexity of the cost functional. Section 5 constructs a gradient-based minimizing sequence and proves its convergence to the unique optimal control. Section 6 presents concluding remarks.

2 Problem Formulation and Well-Posedness

In this section, we employ the quadratic cost optimal control problem for the heat equation (1) using the framework of Lions [8]. We also prove that the optimal control problem considered has a unique optimal solution via the direct method of the calculus of variations.

Let V_{ad} be the subset of the Hilbert space $H^1(0, T)$ defined by

$$V_{ad} := \left\{ \nu \in H^1(0, T) : 0 < \underline{\nu} \leq \nu(t) \leq \bar{\nu}, \left| \frac{d\nu}{dt} \right| \leq \nu_d, \text{ for a.e. } t \in (0, T) \right\} \quad (5)$$

where $0 < \underline{\nu} \leq \bar{\nu}$ and $\nu_d \geq 0$ are given constants. This set V_{ad} is called the admissible control set. It is nonempty (it contains the constant functions with values in $[\underline{\nu}, \bar{\nu}]$), bounded and convex, and it is closed in $H^1(0, T)$: $H^1(0, T)$ is continuously embedded in $C[0, T]$, so the pointwise bounds pass to the limit, and the set $\{g \in L^2(0, T) : |g| \leq \nu_d \text{ a.e.}\}$ is closed in $L^2(0, T)$, so the bound on ν' is preserved as well. The restriction $\underline{\nu} > 0$ is a modeling choice: since $a > 0$ and $w_x(t, 0) = \nu(t)$, a positive flux corresponds to heat extraction through the left end $x = 0$, so that (5) describes a rod cooled at its left end by a device whose extraction rate lies between a minimum value and a capacity, as in the temperature management applications mentioned in the Introduction. Moreover, we assume that $h(t, x)$, $w_0(x)$ are given functions satisfying condition (2) and $\nu(t)$ is an unknown function.

In this paper, our objective is to find a control function $\nu(t)$ placed at the boundary of the system to achieve a desired temperature profile. More precisely, we seek $\nu(t) \in V_{ad}$ such that the right-boundary value $w(t, L; \nu)$ of the corresponding solution of (1) is as close as possible to a desired function $\mu(t)$ in the L^2 -norm. Since, by Theorem 1, the weak solution $w(t, x; \nu)$ of (1) and its trace $w(\cdot, L; \nu) \in L^2(0, T)$ are defined for every $\nu \in H^1(0, T)$, we define the functional $J_\alpha : H^1(0, T) \rightarrow \mathbb{R}$ on the whole space by

$$J_\alpha(\nu) = \int_0^T [w(t, L; \nu) - \mu(t)]^2 dt + \alpha \|\nu - \nu^+\|_{H^1(0, T)}^2 \quad (6)$$

where $\alpha > 0$ is a constant called the regularization parameter, $\mu(t) \in L^2(0, T)$ is the given desired function for the temperature at the right boundary $x = L$, and $\nu^+ \in H^1(0, T)$ is a given reference element. Throughout the remainder of the paper, we assume as standing hypotheses that condition (2) holds, $\mu(t) \in L^2(0, T)$ and $\nu^+ \in H^1(0, T)$. The functional J_α is called the cost functional. The optimal control problem consists in minimizing J_α over the admissible set $V_{ad} \subset H^1(0, T)$.

The present work differs from existing studies in the literature in the following respects.

Control space. Previous works, including [1, 3–5, 8], formulate the optimal control problem with the control taken from $L^2(0, T)$. In contrast, we consider admissible controls in the Hilbert space $H^1(0, T)$, which imposes additional regularity on the boundary flux: the optimal control is not only square-integrable but also has a square-integrable time derivative, thereby excluding highly oscillatory or discontinuous flux inputs.

Cost functional and target. Unlike [13], where the control acts at the right boundary and the target is a profile prescribed at the final time, and unlike [10], where the target is

prescribed over the entire space-time domain $\Omega = (0, T) \times (0, L)$, the present study considers a heat flux control applied at the left boundary with the objective of tracking a time-dependent temperature trajectory $\mu(t)$ at the right boundary $x = L$ over the entire interval $(0, T)$. In the adjoint problem, this difference is reflected in the appearance of Neumann boundary data at $x = L$, rather than a terminal-time or space-time tracking target.

Convexity and optimality. The regularization term in the cost functional ensures the strong convexity of J_α with convexity constant $\chi = \alpha$ (Lemma 3). Together with the existence result of Theorem 2, obtained via the direct method of the calculus of variations, this strong convexity ensures the uniqueness of the optimal control, and implies that the derived variational inequality is both necessary and sufficient, thereby completely characterizing ν^* . An explicit gradient formula is obtained via the adjoint problem, which provides the theoretical basis for gradient-based numerical schemes; the development and convergence analysis of such schemes are pursued in Section 5.

More precisely, the optimal boundary control problem considered can be written as follows:

$$\left\{ \begin{array}{l} \min_{\nu \in V_{ad}} J_\alpha(\nu), \\ \text{where} \\ J_\alpha(\nu) = \int_0^T [w(t, L; \nu) - \mu(t)]^2 dt + \alpha \|\nu - \nu^+\|_{H^1(0, T)}^2 \\ \text{such that} \\ w_t = aw_{xx} + h(t, x), \quad (t, x) \in \Omega, \\ w(0, x) = w_0(x), \quad 0 < x < L, \\ w_x(t, 0) = \nu(t), \quad w_x(t, L) = 0, \quad 0 < t < T. \end{array} \right. \quad (7)$$

An element $\nu^* \in V_{ad}$ that achieves the minimum value of J_α on the admissible control set, V_{ad} , is called an optimal control for the problem (7). Note that the regularization term, $\alpha \|\nu - \nu^+\|_{H^1(0, T)}^2$, guarantees strong convexity of the cost functional. Furthermore, well-posedness of model (1) is proved in [3].

The aim of this study is to find a minimizer $\nu^* \in V_{ad}$ for the cost functional (6) and to provide a characterization of such a ν^* .

The existence and uniqueness of a solution to problem (7) require some properties of the cost functional J_α . To establish the existence and uniqueness of a solution to the optimal control problem (7), we analyze the variation of the cost functional J_α . This requires understanding how variations in the control input affect the system state.

Let $\delta\nu \in H^1(0, T)$ be an increment in the control. The corresponding variation in the state is defined by $\delta w(t, x; \nu) = w(t, x; \nu + \delta\nu) - w(t, x; \nu)$. The difference problem for model (1) is written as

$$\begin{cases} \delta w_t = a \delta w_{xx}, & (t, x) \in \Omega, \\ \delta w(0, x) = 0, & 0 < x < L, \\ \delta w_x(t, 0) = \delta \nu(t), \delta w_x(t, L) = 0, & 0 < t < T. \end{cases} \quad (8)$$

Furthermore, the difference problem is of the same type as problem (1) with homogeneous initial data. So, Definition 1 gives us the definition of a weak solution to model (8) and Theorem 1 guarantees the existence and uniqueness of such a weak solution in the space $H^{1,1}(\Omega)$ since $\delta \nu \in H^1(0, T)$.

Applying estimate (4) of Theorem 1 to the difference problem (8) with $w_0 = 0$ and $h = 0$, we obtain the trace estimate

$$\|\delta w(\cdot, L; \nu)\|_{L^2(0, T)}^2 \leq c_2 \|\delta \nu\|_{H^1(0, T)}^2 \quad (9)$$

where $c_2 := c_1^2 > 0$ is independent of ν and $\delta \nu$.

The following theorem guarantees the existence of a unique minimum element ν^* for the cost functional (6).

Theorem 2 *Assume that the hypotheses of Theorem 1 are satisfied and $\nu^+ \in H^1(0, T)$ is a given element. Then the optimal control problem (7) has a unique solution $\nu^* \in V_{ad}$.*

Proof 1 *Since V_{ad} is a bounded, closed, and convex subset of the reflexive Banach space $H^1(0, T)$, it is weakly sequentially compact. Let $\{\nu_n\} \subset V_{ad}$ be a minimizing sequence for J_α . By weak compactness, there exists a subsequence $\{\nu_m\} \subset \{\nu_n\}$ such that*

$$\nu_m \rightharpoonup \nu^* \quad \text{weakly in } H^1(0, T).$$

Since V_{ad} is convex and closed, it is weakly closed, and hence $\nu^* \in V_{ad}$.

The map

$$\nu \mapsto w(\cdot, L; \nu)$$

is affine and continuous from $H^1(0, T)$ into $L^2(0, T)$, as follows from the estimate (9). Therefore, it is weakly continuous, and

$$w(\cdot, L; \nu_m) \rightharpoonup w(\cdot, L; \nu^*) \quad \text{weakly in } L^2(0, T).$$

Consequently, by the weak lower semicontinuity of the squared norm,

$$\|w(\cdot, L; \nu^*) - \mu\|_{L^2(0, T)}^2 \leq \liminf_{m \rightarrow \infty} \|w(\cdot, L; \nu_m) - \mu\|_{L^2(0, T)}^2.$$

Similarly, the squared $H^1(0, T)$ -norm is weakly lower semicontinuous. Hence

$$J_\alpha(\nu^*) \leq \liminf_{m \rightarrow \infty} J_\alpha(\nu_m) = \inf_{\nu \in V_{ad}} J_\alpha(\nu).$$

Therefore, ν^* is a minimizer of J_α on V_{ad} for every $\nu^+ \in H^1(0, T)$. Uniqueness follows from the strong convexity established in Lemma 3 below.

3 Adjoint-Based Analysis and Fréchet Differentiability

The variational approach gives an explicit expression for the Fréchet derivative of $J_\alpha(\nu)$, which in turn yields an explicit optimality condition.

Multiplying equation (1) by a function ϑ and integrating the resulting expression over Ω , then adding the result to the cost functional $J_\alpha(\nu)$, we get the following Lagrangian $L(w, \nu, \vartheta)$

$$\begin{aligned} L(w, \nu, \vartheta) = & \int_0^T [w(t, L; \nu) - \mu(t)]^2 dt + \alpha \|\nu - \nu^+\|_{H^1(0, T)}^2 \\ & + \int_0^T \int_0^L [w_t - aw_{xx} - h] \vartheta dx dt. \end{aligned} \quad (10)$$

Here, the function $\vartheta = \vartheta(t, x)$ is called the Lagrange multiplier.

The first variation of the functional (10) is defined by

$$\delta L = \lim_{\varepsilon \rightarrow 0} \frac{L(w + \varepsilon \delta w, \nu, \vartheta) - L(w, \nu, \vartheta)}{\varepsilon}.$$

By direct differentiation, the first variation of the functional (10) is given by

$$\delta L = 2 \int_0^T [w(t, L; \nu) - \mu(t)] \delta w(t, L; \nu) dt + \int_0^T \int_0^L [\delta w_t - a \delta w_{xx}] \vartheta dx dt \quad (11)$$

where $\delta w(t, x; \nu)$ is the solution to the difference problem (8). This derivation is formal, since w_{xx} need not be well-defined for a weak solution in $H^{1,1}(\Omega)$.

Using the $\delta L = 0$ stationarity condition, we get the following adjoint problem:

$$\begin{cases} \vartheta_t + a\vartheta_{xx} = 0, & (t, x) \in \Omega, \\ \vartheta(T, x) = 0, & 0 < x < L, \\ \vartheta_x(t, 0) = 0, \vartheta_x(t, L) = -\frac{2}{a}[w(t, L; \nu) - \mu(t)], & 0 < t < T. \end{cases} \quad (12)$$

If we replace t in (12) by the new variable $\tau = T - t$, then we obtain a problem of the same type as (1). We define the weak solution of the adjoint problem (12) as the function ϑ in $V^{1,0}(\Omega)$, satisfying the following integral identity ([3, 6]):

$$\begin{aligned} \int_0^T \int_0^L (\vartheta \psi_t + a \vartheta_x \psi_x) dx dt = & -2 \int_0^T [w(t, L; \nu) - \mu(t)] \psi(t, L) dt \\ & - \int_0^L \vartheta(0, x) \psi(0, x) dx \end{aligned} \quad (13)$$

$\forall \psi \in H^{1,1}(\Omega)$. Here $V^{1,0}(\Omega) := C([0, T]; L^2(0, L)) \cap L^2((0, T); H^1(0, L))$ is the Banach space of functions with the norm

$$\|\vartheta\|_{V^{1,0}(\Omega)} := \max_{0 \leq t \leq T} \|\vartheta\|_{L^2(0, L)} + \|\vartheta_x\|_{L^2(\Omega)}.$$

Remark 1 *To justify the existence of a weak solution $\vartheta \in V^{1,0}(\Omega)$ to the adjoint problem (12), we verify that the Neumann boundary data*

$$\vartheta_x(t, L) = -\frac{2}{a}[w(t, L; \nu) - \mu(t)], \quad t \in (0, T), \quad (14)$$

possesses the required regularity. Indeed, by the trace estimate of Theorem 1, the solution w of problem (1) satisfies

$$\|w(\cdot, L; \nu)\|_{L^2(0, T)} \leq c_1 \left(\|w_0\|_{H^1(0, L)} + \|h\|_{L^2(\Omega)} + \|\nu\|_{H^1(0, T)} \right) < \infty.$$

Since $\mu \in L^2(0, T)$ by assumption, it follows that

$$w(\cdot, L; \nu) - \mu(\cdot) \in L^2(0, T), \quad (15)$$

and hence the boundary data $\vartheta_x(\cdot, L) \in L^2(0, T)$. Therefore, by the classical solvability results for linear parabolic equations in $V^{1,0}(\Omega)$ with $L^2(0, T)$ Neumann boundary data ([6, Chapter III, Section 4] and Lions and Magenes [9]), as well as Theorem 1 of H ao [3] (which is distinct from Theorem 1 of the present paper), the adjoint problem (12) admits a unique weak solution $\vartheta \in V^{1,0}(\Omega)$.

Lemma 1 *Let $\nu, \delta\nu \in H^1(0, T)$ be given elements. If $w = w(t, x; \nu)$ is the solution to the problem (1) and $\vartheta(t, x; \nu)$ is the solution to the adjoint problem (12), then the following identity holds:*

$$2 \int_0^T [w(t, L; \nu) - \mu(t)] \delta w(t, L) dt = a \int_0^T \delta\nu(t) \vartheta(t, 0) dt. \quad (16)$$

Proof 2 *Substituting δw for ψ in (13) yields*

$$\int_0^T \int_0^L (\vartheta \delta w_t + a \vartheta_x \delta w_x) dx dt = -2 \int_0^T [w(t, L; \nu) - \mu(t)] \delta w(t, L) dt,$$

where the initial condition $\delta w(0, x) = 0$ has been utilized. Furthermore, by writing the weak formulation of the difference problem (8) as given in Definition 1 and taking $\varphi = \vartheta$, we obtain

$$\int_0^T \int_0^L (\delta w_t \vartheta + a \delta w_x \vartheta_x) dx dt = -a \int_0^T \delta\nu(t) \vartheta(t, 0) dt.$$

Since the left-hand sides of these two integral identities are identical, their right-hand sides must also coincide. This completes the proof.

Theorem 3 ([11]) *Let the conditions (2) be satisfied and let $\mu \in L^2(0, T)$. Then the cost functional $J_\alpha(\nu)$ defined by (6) is Fr chet differentiable on $H^1(0, T)$, and for every $\nu \in H^1(0, T)$ its Fr chet derivative is given by*

$$J'_\alpha(\nu) = \eta + 2\alpha(\nu - \nu^+) \quad (17)$$

where $\eta = \eta(t)$ is the solution of the following boundary value problem

$$\begin{cases} \eta''(t) - \eta = -a\vartheta(t, 0), & t \in (0, T), \\ \eta'(0) = 0, & \eta'(T) = 0 \end{cases} \quad (18)$$

and $\vartheta = \vartheta(t, x; \nu)$ is the solution of the adjoint problem (12) corresponding to ν . In particular, J_α is Fréchet differentiable at every point of the admissible set $V_{ad} \subset H^1(0, T)$.

Proof 3 Let $\nu \in H^1(0, T)$ be given and let $\delta\nu \in H^1(0, T)$ be an arbitrary increment. We examine the variation $\delta J_\alpha(\nu) = J_\alpha(\nu + \delta\nu) - J_\alpha(\nu)$. By the definition of the cost functional (6), we write

$$\begin{aligned} \delta J_\alpha(\nu) &= \int_0^T [w(t, L; \nu + \delta\nu) - \mu(t)]^2 dt + \alpha \|\nu + \delta\nu - \nu^+\|_{H^1(0, T)}^2 \\ &\quad - \int_0^T [w(t, L; \nu) - \mu(t)]^2 dt - \alpha \|\nu - \nu^+\|_{H^1(0, T)}^2. \end{aligned}$$

Applying the difference of squares identity and using $w(t, L; \nu + \delta\nu) = \delta w(t, L) + w(t, L; \nu)$, together with the inner product definition in $H^1(0, T)$, we obtain

$$\begin{aligned} \delta J_\alpha(\nu) &= 2 \int_0^T [w(t, L; \nu) - \mu(t)] \delta w(t, L) dt + \|\delta w(\cdot, L)\|_{L^2(0, T)}^2 \\ &\quad + 2\alpha \langle \nu - \nu^+, \delta\nu \rangle_{H^1(0, T)} + \alpha \|\delta\nu\|_{H^1(0, T)}^2. \end{aligned}$$

Applying Lemma 1 to the first term on the right-hand side, we get

$$\begin{aligned} \delta J_\alpha(\nu) &= a \int_0^T \vartheta(t, 0) \delta\nu(t) dt + \|\delta w(\cdot, L)\|_{L^2(0, T)}^2 \\ &\quad + 2\alpha \langle \nu - \nu^+, \delta\nu \rangle_{H^1(0, T)} + \alpha \|\delta\nu\|_{H^1(0, T)}^2. \end{aligned}$$

By (9), the second term satisfies $\|\delta w(\cdot, L)\|_{L^2(0, T)}^2 \leq c_2 \|\delta\nu\|_{H^1(0, T)}^2$, and therefore it is of order $O(\|\delta\nu\|_{H^1(0, T)}^2)$. Hence, we have

$$\delta J_\alpha(\nu) = a \int_0^T \vartheta(t, 0) \delta\nu(t) dt + 2\alpha \langle \nu - \nu^+, \delta\nu \rangle_{H^1(0, T)} + o(\|\delta\nu\|_{H^1(0, T)}). \quad (19)$$

In order to express (19) in the standard Fréchet form

$$\delta J_\alpha(\nu) = \langle J'_\alpha(\nu), \delta\nu \rangle_{H^1(0, T)} + o(\|\delta\nu\|_{H^1(0, T)}),$$

we need to represent the term $a \int_0^T \vartheta(t, 0) \delta\nu(t) dt$ as an $H^1(0, T)$ inner product. To this end, we introduce the auxiliary boundary value problem

$$\eta''(t) - \eta(t) = -a\vartheta(t, 0), \quad t \in (0, T), \quad \eta'(0) = 0, \quad \eta'(T) = 0. \quad (20)$$

Substituting (20) into (19) and applying integration by parts, while using the boundary conditions $\eta'(0) = \eta'(T) = 0$, we obtain

$$\begin{aligned}
\delta J_\alpha(\nu) &= \int_0^T (-\eta''(t) + \eta(t)) \delta\nu(t) dt + 2\alpha \langle \nu - \nu^+, \delta\nu \rangle_{H^1(0,T)} + o(\|\delta\nu\|_{H^1(0,T)}) \\
&= \int_0^T (\eta'(t) \delta\nu'(t) + \eta(t) \delta\nu(t)) dt + 2\alpha \langle \nu - \nu^+, \delta\nu \rangle_{H^1(0,T)} + o(\|\delta\nu\|_{H^1(0,T)}) \\
&= \langle \eta, \delta\nu \rangle_{H^1(0,T)} + 2\alpha \langle \nu - \nu^+, \delta\nu \rangle_{H^1(0,T)} + o(\|\delta\nu\|_{H^1(0,T)}) \\
&= \langle \eta + 2\alpha(\nu - \nu^+), \delta\nu \rangle_{H^1(0,T)} + o(\|\delta\nu\|_{H^1(0,T)}). \tag{21}
\end{aligned}$$

Since this expression holds for every $\delta\nu \in H^1(0,T)$ and the remainder is of order $o(\|\delta\nu\|_{H^1(0,T)})$, the cost functional $J_\alpha(\nu)$ is Fréchet differentiable at every $\nu \in H^1(0,T)$, and its Fréchet derivative is given by

$$J'_\alpha(\nu) = \eta + 2\alpha(\nu - \nu^+),$$

where $\eta(t)$ is the solution of the boundary value problem (18). Since $V_{ad} \subset H^1(0,T)$, the same formula holds at every point of V_{ad} .

Remark 2 The Fréchet differentiability established in Theorem 3 shows that J_α has a derivative at every point of $H^1(0,T)$. We next verify the continuity of the Fréchet derivative on $H^1(0,T)$, i.e. $J_\alpha \in C^1(H^1(0,T))$. Let $\nu_n \rightarrow \nu$ in $H^1(0,T)$ and set $\delta\nu_n := \nu_n - \nu \rightarrow 0$. By the trace estimate (9),

$$\|\delta w(\cdot, L; \nu_n)\|_{L^2(0,T)} \leq \sqrt{c_2} \|\delta\nu_n\|_{H^1(0,T)} \rightarrow 0.$$

Since the solution operator associated with the adjoint problem (12) is linear and bounded with respect to its L^2 Neumann boundary data $-\frac{2}{a}[w(t, L; \nu) - \mu(t)]$, it follows that

$$\vartheta(\cdot, 0; \nu_n) \longrightarrow \vartheta(\cdot, 0; \nu) \quad \text{in } L^2(0,T).$$

Furthermore, the auxiliary problem (18) defines a bounded linear solution operator from $L^2(0,T)$ into $H^1(0,T)$. Hence,

$$\eta(\nu_n) \longrightarrow \eta(\nu) \quad \text{in } H^1(0,T).$$

Consequently,

$$J'_\alpha(\nu_n) = \eta(\nu_n) + 2\alpha(\nu_n - \nu^+) \longrightarrow \eta(\nu) + 2\alpha(\nu - \nu^+) = J'_\alpha(\nu) \quad \text{in } H^1(0,T),$$

which proves the continuity of the mapping $\nu \mapsto J'_\alpha(\nu)$ on $H^1(0,T)$. Hence, $J_\alpha \in C^1(H^1(0,T))$; in particular, J'_α is continuous on V_{ad} .

4 Characterization of the Optimal Control and Strong Convexity

In this section, we derive a necessary condition for the optimal control $\nu^* \in V_{ad}$ of problem (7) and prove the strong convexity of the cost functional with strong convexity constant

$\chi = \alpha$. Since the cost functional J_α is Fréchet differentiable on $H^1(0, T)$, and in particular at every point of V_{ad} , by Theorem 3, the necessary optimality condition can be stated in the form of a variational inequality.

The statement of the following theorem appears in [11] without proof; a complete proof is given here.

Theorem 4 *Let the conditions (2) be satisfied and let $\nu^* \in V_{ad}$ be the optimal control for problem (7). Let $\vartheta^* = \vartheta(t, x; \nu^*)$ be the solution of the adjoint problem (12) corresponding to ν^* , and let $\eta^* = \eta^*(t)$ be the solution of the auxiliary problem (18) corresponding to $\vartheta^*(t, 0)$. Then the following variational inequality holds:*

$$\langle \eta^* + 2\alpha(\nu^* - \nu^+), \nu - \nu^* \rangle_{H^1(0, T)} \geq 0, \quad \forall \nu \in V_{ad}. \quad (22)$$

Proof 4 *Since ν^* minimizes J_α on V_{ad} and V_{ad} is a convex set, for any $\nu \in V_{ad}$ and any $\lambda \in [0, 1]$, the convex combination $\nu^* + \lambda(\nu - \nu^*) \in V_{ad}$. Therefore, by the minimality of ν^* , we have*

$$J_\alpha(\nu^* + \lambda(\nu - \nu^*)) \geq J_\alpha(\nu^*), \quad (23)$$

which gives

$$\frac{J_\alpha(\nu^* + \lambda(\nu - \nu^*)) - J_\alpha(\nu^*)}{\lambda} \geq 0. \quad (24)$$

Taking the limit as $\lambda \rightarrow 0^+$ and using the Fréchet differentiability of J_α established in Theorem 3, we obtain

$$\langle J'_\alpha(\nu^*), \nu - \nu^* \rangle_{H^1(0, T)} \geq 0, \quad \forall \nu \in V_{ad}. \quad (25)$$

Substituting the explicit expression for $J'_\alpha(\nu^*)$ given by Theorem 3, namely $J'_\alpha(\nu^*) = \eta^* + 2\alpha(\nu^* - \nu^+)$, we arrive at

$$\langle \eta^* + 2\alpha(\nu^* - \nu^+), \nu - \nu^* \rangle_{H^1(0, T)} \geq 0, \quad \forall \nu \in V_{ad}, \quad (26)$$

which completes the proof.

Remark 3 *The variational inequality (22) provides a complete characterization of the optimal control ν^* in terms of the adjoint state ϑ^* and the auxiliary function η^* . In particular, if the admissible set V_{ad} is the entire space $H^1(0, T)$, then (22) reduces to the Euler-Lagrange equation*

$$\eta^* + 2\alpha(\nu^* - \nu^+) = 0, \quad (27)$$

which, together with the state problem (1) and the adjoint problem (12), forms a coupled optimality system for the determination of ν^* .

Lemma 2 ([11]) *Let the conditions (2) be satisfied and let $\nu, \delta\nu \in H^1(0, T)$. Then the following identity holds for the derivative of the cost functional $J_\alpha(\nu)$:*

$$\langle J'_\alpha(\nu + \delta\nu) - J'_\alpha(\nu), \delta\nu \rangle_{H^1(0, T)} = 2\|\delta w(\cdot, L)\|_{L^2(0, T)}^2 + 2\alpha\|\delta\nu\|_{H^1(0, T)}^2. \quad (28)$$

Proof 5 We begin by deriving the difference problem for the adjoint problem (12). Let $\delta\nu$ be an increment of ν and let ϑ_δ denote the solution of the adjoint problem corresponding to $\nu + \delta\nu \in H^1(0, T)$. Setting $\delta\vartheta(t, x) = \vartheta(t, x; \nu + \delta\nu) - \vartheta(t, x; \nu)$ and subtracting problem (12) from the adjoint problem corresponding to $\nu + \delta\nu$, we obtain the adjoint difference problem

$$\begin{aligned} \delta\vartheta_t + a\delta\vartheta_{xx} &= 0, & (t, x) \in \Omega, \\ \delta\vartheta(T, x) &= 0, & x \in (0, L), \\ \delta\vartheta_x(t, 0) &= 0, \quad \delta\vartheta_x(t, L) = -\frac{2}{a}\delta w(t, L), & t \in (0, T) \end{aligned} \quad (29)$$

understood in the weak sense of Remark 1, i.e., $\delta\vartheta \in V^{1,0}(\Omega)$ satisfies

$$\int_0^T \int_0^L (\delta\vartheta\psi_t + a\delta\vartheta_x\psi_x) dx dt = -2 \int_0^T \delta w(t, L) \psi(t, L) dt - \int_0^L \delta\vartheta(0, x) \psi(0, x) dx \quad (30)$$

for every $\psi \in H^{1,1}(\Omega)$.

Similarly, setting $\delta\eta(t) = \eta_\delta(t) - \eta(t)$, where η_δ denotes the solution of (18) corresponding to $\nu + \delta\nu$, and subtracting problem (18) from the auxiliary problem corresponding to $\nu + \delta\nu$, we obtain

$$\begin{aligned} \delta\eta''(t) - \delta\eta(t) &= -a\delta\vartheta(t, 0), & t \in (0, T), \\ \delta\eta'(0) &= 0, \quad \delta\eta'(T) = 0. \end{aligned} \quad (31)$$

On the other hand, the variation of the gradient of the cost functional corresponding to the increment $\delta\nu$ can be written as

$$\delta J'_\alpha(\nu) = J'_\alpha(\nu + \delta\nu) - J'_\alpha(\nu) = \delta\eta + 2\alpha\delta\nu. \quad (32)$$

Using the inner product definition in $H^1(0, T)$, we compute

$$\begin{aligned} \langle J'_\alpha(\nu + \delta\nu) - J'_\alpha(\nu), \delta\nu \rangle_{H^1(0, T)} &= \langle \delta\eta + 2\alpha\delta\nu, \delta\nu \rangle_{H^1(0, T)} \\ &= \int_0^T \{ \delta\eta' \delta\nu' + \delta\eta \delta\nu \} dt + 2\alpha \|\delta\nu\|_{H^1(0, T)}^2. \end{aligned} \quad (33)$$

Integrating the first term by parts and using $\delta\eta'(0) = \delta\eta'(T) = 0$ from (31), together with $\delta\eta''(t) - \delta\eta(t) = -a\delta\vartheta(t, 0)$, we obtain

$$\int_0^T (\delta\eta' \delta\nu' + \delta\eta \delta\nu) dt = \int_0^T (-\delta\eta''(t) + \delta\eta(t)) \delta\nu(t) dt = a \int_0^T \delta\vartheta(t, 0) \delta\nu(t) dt. \quad (34)$$

It remains to evaluate $a \int_0^T \delta\vartheta(t, 0) \delta\nu(t) dt$. We do so by a duality argument between the weak formulations of $\delta\vartheta$ and δw .

Since $\delta w(0, x) = 0$ by (8), taking $\psi = \delta w \in H^{1,1}(\Omega)$ in (30) eliminates the last term, and gives

$$\int_0^T \int_0^L (\delta\vartheta \delta w_t + a\delta\vartheta_x \delta w_x) dx dt = -2 \int_0^T [\delta w(t, L)]^2 dt. \quad (35)$$

On the other hand, δw satisfies the weak formulation of (8) (Definition 1 with $h = 0$, homogeneous initial data, and boundary flux $\delta\nu$):

$$\int_0^T \int_0^L (\delta w_t \varphi + a \delta w_x \varphi_x) dx dt = -a \int_0^T \delta\nu(t) \varphi(t, 0) dt, \quad \forall \varphi \in H^{1,0}(\Omega). \quad (36)$$

Since $\delta\vartheta \in V^{1,0}(\Omega) \subset H^{1,0}(\Omega)$, we may take $\varphi = \delta\vartheta$ in (36). The left-hand sides of (35) and (36) coincide, so

$$a \int_0^T \delta\vartheta(t, 0) \delta\nu(t) dt = 2 \int_0^T [\delta w(t, L)]^2 dt. \quad (37)$$

Combining (33), (34), and (37), we conclude

$$\langle J'_\alpha(\nu + \delta\nu) - J'_\alpha(\nu), \delta\nu \rangle_{H^1(0,T)} = 2\|\delta w(\cdot, L)\|_{L^2(0,T)}^2 + 2\alpha\|\delta\nu\|_{H^1(0,T)}^2,$$

which completes the proof.

Lemma 3 ([11]) *Let $\alpha > 0$. Then the cost functional $J_\alpha(\nu)$ is strongly convex on V_{ad} with strong convexity constant $\chi = \alpha$, that is, for all $\nu_a, \nu_b \in V_{ad}$ and all $\lambda \in [0, 1]$, the following inequality holds:*

$$J_\alpha(\lambda\nu_a + (1 - \lambda)\nu_b) \leq \lambda J_\alpha(\nu_a) + (1 - \lambda)J_\alpha(\nu_b) - \alpha\lambda(1 - \lambda)\|\nu_a - \nu_b\|_{H^1(0,T)}^2. \quad (38)$$

Proof 6 Recall that a continuously differentiable functional $J_\alpha(\nu)$ defined on a convex set V_{ad} is strongly convex with convexity constant $\chi > 0$ if and only if the following condition is satisfied for all $\nu, \nu + \delta\nu \in V_{ad}$:

$$\langle J'_\alpha(\nu + \delta\nu) - J'_\alpha(\nu), \delta\nu \rangle_{H^1(0,T)} \geq 2\chi\|\delta\nu\|_{H^1(0,T)}^2. \quad (39)$$

By Lemma 2, applied to $\nu, \nu + \delta\nu \in V_{ad} \subset H^1(0, T)$, the following identity holds:

$$\langle J'_\alpha(\nu + \delta\nu) - J'_\alpha(\nu), \delta\nu \rangle_{H^1(0,T)} = 2\|\delta w(\cdot, L)\|_{L^2(0,T)}^2 + 2\alpha\|\delta\nu\|_{H^1(0,T)}^2. \quad (40)$$

Since $\|\delta w(\cdot, L)\|_{L^2(0,T)}^2 \geq 0$, it follows immediately that

$$\langle J'_\alpha(\nu + \delta\nu) - J'_\alpha(\nu), \delta\nu \rangle_{H^1(0,T)} \geq 2\alpha\|\delta\nu\|_{H^1(0,T)}^2. \quad (41)$$

This shows that condition (39) is satisfied with $\chi = \alpha$. Consequently, the cost functional $J_\alpha(\nu)$ is strongly convex on V_{ad} with strong convexity constant $\chi = \alpha$, which completes the proof.

Remark 4 Since $\nu \mapsto w(\cdot, L; \nu)$ is affine, writing $w(\cdot, L; \nu) = S\nu + b$ for a bounded linear operator S gives $J_\alpha(\nu) = \|S\nu - (\mu - b)\|_{L^2(0,T)}^2 + \alpha\|\nu - \nu^+\|_{H^1(0,T)}^2$, a convex term plus an α -strongly convex term; this recovers $\chi = \alpha$ directly, without the adjoint-based argument of Lemma 2. We retain the adjoint-based proof above since Lemma 2 is used in Lemma 3 and, through (41), in Theorem 5.

Remark 5 The strong convexity of J_α established in Lemma 3 implies that the variational inequality (22) has a unique solution $\nu^* \in V_{ad}$. Moreover, since J_α is strongly convex, the necessary optimality condition (22) is also sufficient, that is, any $\nu^* \in V_{ad}$ satisfying (22) is the unique global minimizer of J_α on V_{ad} .

5 A Gradient-Based Minimizing Sequence

Using the existence and uniqueness result of Theorem 2, together with the strong convexity established in Lemma 3, we construct a minimizing sequence using the projected gradient method of [14], as applied to a related boundary control problem in [12]. Starting from any $\nu_0 \in V_{ad}$, define

$$\nu_{n+1} = P_{V_{ad}}(\nu_n - \beta_n J'_\alpha(\nu_n)), \quad n = 0, 1, 2, \dots, \quad (42)$$

where $P_{V_{ad}}$ denotes the metric projection onto the closed convex set V_{ad} and $J'_\alpha(\nu_n)$ is given by (17).

Since the control-to-observation mapping $\nu \mapsto w(\cdot, L; \nu)$ is affine and continuous, while the adjoint and auxiliary solution operators are linear and bounded (Remark 2), the mapping

$$\nu \mapsto J'_\alpha(\nu) = \eta(\nu) + 2\alpha(\nu - \nu^+)$$

is affine and Lipschitz continuous on $H^1(0, T)$. Hence, there exists a constant $L_J > 0$ such that

$$\|J'_\alpha(\nu_a) - J'_\alpha(\nu_b)\|_{H^1(0, T)} \leq L_J \|\nu_a - \nu_b\|_{H^1(0, T)}$$

for all $\nu_a, \nu_b \in H^1(0, T)$, and in particular for all $\nu_a, \nu_b \in V_{ad}$.

Theorem 5 *Let $\nu_0 \in V_{ad}$ be an initial point. The sequence $\{\nu_n\}$ defined by (42) converges to the unique minimizer ν^* of $J_\alpha(\nu)$ on V_{ad} for $\beta_n = \beta \in (0, 4\alpha/L_J^2)$. Moreover,*

$$\|\nu_n - \nu^*\|_{H^1(0, T)} \leq q^n \|\nu_0 - \nu^*\|_{H^1(0, T)}, \quad n = 0, 1, 2, \dots, \quad (43)$$

where $q := \sqrt{1 - 4\alpha\beta + \beta^2 L_J^2} \in (0, 1)$.

Proof 7 Define $A : V_{ad} \rightarrow V_{ad}$ by $A\nu := P_{V_{ad}}(\nu - \beta J'_\alpha(\nu))$. Since $P_{V_{ad}}$ is nonexpansive, for all $\nu_a, \nu_b \in V_{ad}$,

$$\|A\nu_a - A\nu_b\|_{H^1(0, T)}^2 \leq \|(\nu_a - \nu_b) - \beta(J'_\alpha(\nu_a) - J'_\alpha(\nu_b))\|_{H^1(0, T)}^2.$$

Expanding the right-hand side and using the strong convexity inequality $\langle J'_\alpha(\nu_a) - J'_\alpha(\nu_b), \nu_a - \nu_b \rangle_{H^1(0, T)} \geq 2\alpha \|\nu_a - \nu_b\|_{H^1(0, T)}^2$ (Lemma 3) together with the Lipschitz continuity of J'_α , we obtain

$$\|A\nu_a - A\nu_b\|_{H^1(0, T)}^2 \leq (1 - 4\alpha\beta + \beta^2 L_J^2) \|\nu_a - \nu_b\|_{H^1(0, T)}^2,$$

so $\|A\nu_a - A\nu_b\|_{H^1(0, T)} \leq q \|\nu_a - \nu_b\|_{H^1(0, T)}$ with $q = \sqrt{1 - 4\alpha\beta + \beta^2 L_J^2} \in (0, 1)$ for $\beta \in (0, 4\alpha/L_J^2)$. Hence A is a contraction on V_{ad} , and by the contraction mapping principle the sequence $\nu_{n+1} = A\nu_n$ converges to the unique fixed point $\tilde{\nu}$ satisfying $\tilde{\nu} = P_{V_{ad}}(\tilde{\nu} - \beta J'_\alpha(\tilde{\nu}))$. By the variational characterization of the metric projection $P_{V_{ad}}$ in $H^1(0, T)$, this fixed-point equation is equivalent to $\langle (\tilde{\nu} - \beta J'_\alpha(\tilde{\nu})) - \tilde{\nu}, \nu - \tilde{\nu} \rangle_{H^1(0, T)} \leq 0$ for all $\nu \in V_{ad}$. Dividing by $-\beta < 0$ yields $\langle J'_\alpha(\tilde{\nu}), \nu - \tilde{\nu} \rangle_{H^1(0, T)} \geq 0$ for all $\nu \in V_{ad}$, which is precisely (25). By Remark 5, (25) uniquely characterizes the minimizer ν^* of J_α on V_{ad} ; thus $\tilde{\nu} = \nu^*$ and estimate (43) holds. Note that $P_{V_{ad}}$ is the metric projection in $H^1(0, T)$, which has no closed form because V_{ad} constrains both ν and ν' , and since L_J is not estimated, the admissible step size is not explicit.

6 Conclusions

In this paper, we have investigated an optimal Neumann boundary control problem for the one-dimensional heat equation, where the control input is the heat flux applied at the left boundary and the objective is to steer the temperature at the right boundary toward a prescribed target trajectory over the time interval $(0, T)$. The admissible control set has been chosen as a bounded, closed, and convex subset of the Hilbert space $H^1(0, T)$, providing enhanced regularity compared with the classical L^2 framework.

The existence and uniqueness of the optimal control have been established via the direct method of the calculus of variations. In addition, the Fréchet differentiability of the cost functional has been proved, and an explicit expression for its gradient has been derived through the associated adjoint problem. Based on these results, a necessary optimality condition has been obtained in the form of a variational inequality involving the gradient of the cost functional and the adjoint state.

Furthermore, it has been shown that the cost functional J_α is strongly convex on the admissible control set V_{ad} with convexity constant $\chi = \alpha$. This property not only provides an alternative justification for the uniqueness of the optimal control but also implies that the derived variational inequality is both necessary and sufficient, thereby completely characterizing the unique optimal control ν^* .

Finally, the explicit gradient formula and the strong convexity of the cost functional have been used in Section 5 to construct a projected gradient sequence and to prove its convergence to the unique optimal control at a geometric rate (Theorem 5). Future research may focus on a full numerical implementation of this scheme and on extending the present analysis to nonlinear models or multidimensional heat conduction problems.

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Authors contribution

Y.A.: conceived the study, formulated the problem, developed the theoretical framework, supervised the research, contributed to the mathematical analysis, and critically revised the manuscript. **T.K.:** carried out the mathematical analysis and prepared the original draft. All authors have read and agreed to the published version of the manuscript.

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