IRSTI 27.31.21 DOI: https://doi.org/10.26577/JMMCS131320263

M. Borikhanov
Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
e-mail: borikhanov@math.kz

COMPARISON PRINCIPLE FOR A NONLINEAR DIFFUSION EQUATION UNDER NONLOCAL BOUNDARY CONDITION

In this paper, we study a Cauchy problem with a nonlocal Neumann boundary condition for a nonlinear time-space fractional diffusion equation involving the Caputo fractional derivative and the fractional p -Laplacian. The principal contribution of the paper is the establishment of a comparison principle for weak subsolutions and supersolutions under suitable assumptions on the parameters β, γ, k, and ℓ. This result shows that the ordering of the initial data and the corresponding nonlinear terms is preserved throughout the evolution, despite the presence of both temporal and spatial nonlocality. The proof requires a careful treatment of the Caputo derivative, the monotonicity properties of the fractional p-Laplacian, and the nonlocal Neumann boundary condition. The comparison principle then serves as the main analytical tool for investigating the qualitative behaviour of solutions. As direct applications, we derive a boundedness result in the dissipative case and establish a sufficient criterion for finite-time blow-up by comparing a weak solution with a suitable subsolution of an associated fractional differential equation.
Key words: comparison principle, Caputo fractional derivative, fractional $p-$Laplacian, nonlocal Neumann boundary condition, blow-up.

М. Бөріханов
Математика және математикалық модельдеу институты, Алматы, Қазақстан
e-mail: borikhanov@math.kz

Бейлокал шекаралық шартымен берілген бейсызықты диффузия теңдеуі үшін салыстыру принципі

Бұл мақалада Капутоның бөлшек ретті туындысы мен бөлшек p-Лапласианды қамтитын бейсызықты уақыт-кеңістік бөлшек диффузия теңдеуі үшін бейлокал Нейман шекаралық шарты бар Коши есебі зерттеледі. Жұмыстың негізгі нәтижесі β, γ, k және ℓ параметрлеріне қойылатын тиісті шарттар кезінде әлсіз субшешімдер мен супершешімдер үшін салыстыру принципін дәлелдеу болып табылады. Алынған нәтиже уақыт пен кеңістік бойынша локал еместікке қарамастан, бастапқы деректер мен сәйкес сызықтық емес мүшелер арасындағы реттіліктің эволюция барысында сақталатынын көрсетеді. Дәлелдеу Капутоның бөлшек ретті туындысын, бөлшек p -Лапласианның монотондық қасиеттерін және бейлокал Нейман шекаралық шартын мұқият талдауды талап етеді. Дәлелденген салыстыру принципі кейін шешімдердің сапалық қасиеттерін зерттеудің негізгі аналитикалық құралы ретінде қолданылады. Тікелей қолданылулар ретінде диссипативті жағдайда әлсіз шешімдердің шенелгендігі дәлелденеді, сондай-ақ әлсіз шешімді сәйкес бөлшек дифференциалдық теңдеудің қолайлы субшешімімен салыстыру арқылы шешімнің ақырлы уақытта жарылуының жеткілікті шарты алынады.
Түйін сөздер: салыстыру приципі, Капутоның бөлшек ретті туындысы, бөлшек ретті $p-$Лапласиан, бейлокал Нейман шекаралық шарты, шешімнің жарылуы (қирауы).

М. Бориханов
Институт математики и математического моделирования, Алматы, Казахстан
e-mail: borikhanov@math.kz

Принцип сравнения для нелинейного уравнения диффузии с нелокальным граничным условием

В данной работе исследуется задача Коши с нелокальным граничным условием Неймана для нелинейного дробного уравнения диффузии по времени и пространству, содержащего дробную производную Капуто и дробный $p-$Лапласиан. Основным результатом работы является установление принципа сравнения для слабых субрешений и суперрешений при подходящих предположениях относительно параметров β, γ, k и ℓ. Полученный результат показывает, что порядок между начальными данными и соответствующими нелинейными членами сохраняется в процессе эволюции, несмотря на наличие нелокальности как по времени, так и по пространству. Доказательство требует тщательного анализа дробной производной Капуто, свойств монотонности дробного $p-$Лапласиана и нелокального граничного условия Неймана. Установленный принцип сравнения затем используется в качестве основного аналитического инструмента для исследования качественного поведения решений. В качестве непосредственных приложений доказывается ограниченность слабых решений в диссипативном случае, а также устанавливается достаточный критерий разрушения решения за конечное время посредством сравнения слабого решения с подходящим субрешением соответствующего дробного дифференциального уравнения.
Ключевые слова: принцип сравнения, дробная производная Капуто, дробный $p-$Лапласиан, нелокальное граничное условие Неймана, разрушение решения.

1 Introduction

For fractional evolution equations, the comparison principle is an important tool for proving positivity, uniqueness, continuous dependence on initial data, a priori bounds, global boundedness, and blow-up criteria. In particular, when the nonlinear source term is monotone, the difference between a subsolution and a supersolution can be tested by its positive part. Then the monotonicity of the nonlinear operator, together with a fractional Gronwall-type inequality, allows one to conclude that the positive part vanishes. This yields the desired ordering of the two solutions. Such ideas are widely used in the theory of fractional differential equations and fractional diffusion problems; see, for example, [1–6].

In this paper, we study the initial-boundary value problem for the nonlinear time-space fractional diffusion equation

$$\begin{cases} \partial_{0|t}^{\alpha} u + (-\Delta)_p^s u = \beta |u|^{k-1} u + \gamma |u|^{\ell-2} u, \ (x,t) \in \Omega \times (0,T) \equiv \Omega_T, \\ \\ u(0,x) = u_0(x), \ x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain; $k > 0, \ell > 1$, $\beta, \gamma \in \mathbb{R}$ and $\partial_{0|t}^{\alpha}$ is the left Caputo fractional derivative of order $\alpha \in (0,1)$; $(-\Delta)_p^s$ is the fractional p-Laplacian with $s \in (0,1), p \geq 2$.

In addition, problem (1) is considered under the Neumann-type boundary condition

$$\mathcal{N}_{s,p} u(x,t) = 0, \ x \in \mathbb{R}^n \setminus \overline{\Omega}, \ 0 < t < T, \tag{2}$$

which stands for

$$\mathcal{N}_{s,p} u(x,t) := \int_{\Omega} \frac{|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))}{|x-y|^{n+sp}} dy.$$

Here, for each fixed $t \in (0,T)$ and $x \in \mathbb{R}^n \setminus \overline{\Omega}$, the variable $y \in \Omega$ is the integration variable. Thus, $\mathcal{N}_{s,p} u(x,t)$ measures the nonlocal interaction between the exterior point x and all interior points $y \in \Omega$.

A new "nonlocal normal derivative" was introduced by S. Dipierro et al. [7] for fractional Laplacian. Later, D. Mugnai eand al [8, 9] studied the nonlocal normal p-derivative, or p-Neumann boundary condition and describes the natural Neumann boundary condition in the presence of the fractional p-Laplacian, which extends the notion of nonlocal normal derivative.

In general, nonlocal Neumann boundary conditions play the role of prescribing a nonlocal flux through the exterior of the domain. In contrast with the classical local Neumann condition, which is imposed on the boundary, the nonlocal condition is naturally imposed on the whole exterior region $\mathbb{R}^n \setminus \overline{\Omega}$. This reflects the fact that fractional diffusion is governed by long-range interactions: the value of the solution at one point is influenced by values at distant points. Therefore, the boundary behavior is not described only by pointwise information on $\partial\Omega$, but by the interaction between points inside Ω and points outside Ω.

For the fractional p-Laplacian, the operator $\mathcal{N}_{s,p}u$ measures the nonlocal interaction between the exterior point $x \in \mathbb{R}^n \setminus \overline{\Omega}$ and the interior values $u(y), y \in \Omega$. The condition (2) can therefore be understood as a zero nonlocal flux condition. It is also consistent with the corresponding integration-by-parts formula, which allows the diffusion term to be written in a weak form through the nonlocal energy integral.

Moreover, in [8] the authors studied the parabolic problem associated to this new class of operators, namely

$$\begin{cases} u_t(x,t) + (-\Delta)_p^s u(x,t) = 0 & \text{in } \Omega, \quad t > 0 \\ \mathcal{N}_{s,p}u(x,t) = 0 & \text{in } \mathbb{R}^n \backslash \bar{\Omega}, \quad t > 0 \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

They also proved conservation of mass and monotonicity of the associated energy, as in [7].

In recent years, parabolic equations involving the fractional p-Laplacian have attracted attention, but mostly in the setting of Dirichlet boundary conditions, see for instance [10–13].

Motivated by these developments, in the present paper we study a nonlinear time-space fractional diffusion equation involving the Caputo fractional derivative and the fractional p-Laplacian under a nonlocal Neumann boundary condition (2).

2 Preliminaries

In this section we list definitions and properties related to the fractional operators.

Definition 1 [[3], p. 91]. The $\alpha \in (0,1)$ order of left Caputo fractional derivative for $u \in C^1([0,T])$ is defined by

$$\partial_{0|t}^{\alpha} u(t) = I_{0|t}^{1-\alpha} \frac{d}{dt} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\, ds, \ \ \forall t \in (0,T],$$

where $I_{0|t}^{1-\alpha}$ is the left Riemann-Liouville fractional integral.

Property 2 [[3], p. 95-96] If $0 < \alpha < 1$, $u \in AC^1[0,T]$ or $u \in C^1[0,T]$, then

$$I_{0|t}^{\alpha}(\partial_{0|t}^{\alpha} u)(t) = u(t) - u(0) \quad \text{and} \quad \partial_{0|t}^{\alpha}(I_{0|t}^{\alpha} u)(t) = u(t)$$

hold almost everywhere on $[0,T]$. Moreover, it yields

$$\partial_{0|t}^{1-\alpha}\int_0^t \partial_{0|\tau}^{\alpha}u(\tau)d\tau = \left(I_{0|t}^{\alpha}\frac{d}{dt}I_{0|t}^{1}I_{0|t}^{1-\alpha}\frac{d}{dt}u\right)(t) = u(t)-u(0).$$

Definition 3 [[14]. Lemma 5.1] The fractional p-Laplacian operator for $s\in(0,1), p>1$ and $u\in W^{s,p}(\Omega)$, is defined by

$$(-\Delta)_p^s u(x) = C_{n,s,p}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}dy,$$

where $x\in\Omega$ is the fixed point at which the operator is evaluated, whereas $y\in\mathbb{R}^n$ is the integration variable. Moreover,

$$C_{n,s,p} = \frac{sp2^{2s-2}}{\pi^{\frac{n-1}{2}}}\frac{\Gamma(\frac{n+sp}{2})}{\Gamma(\frac{p+1}{2})\Gamma(1-s)}$$

is a normalization constant and "P.V." is an abbreviation for "in the principal value sense".
Proposition 4 [[15] Theorem 6.3]. Let u,v be any bounded C^2 functions in $\mathbb{R}^n$. Then,

$$\int_\Omega(\Delta)_p^s u dx = \int_{\mathbb{R}^n\setminus\Omega}\mathcal{N}_{s,p}udx$$

and

$$\begin{aligned}\frac{C_{n,s,p}}{2}&\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}dxdy\\ &= \int_\Omega v(-\Delta)_p^s u dx + \int_{\mathbb{R}^n\setminus\Omega} v\mathcal{N}_{s,p}udx.\end{aligned}$$

We define the inner product of the operator $(-\Delta)_p^s$ for $u,v\in W^{s,p}(\Omega)$ as

$$\langle(-\Delta)_p^s u, v\rangle = \int_\Omega\int_\Omega\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}dxdy. \tag{3}$$

3 Comparison Principle

In this section we study a comparison principle for the fractional parabolic equation.

We first define a weak subsolution and supersolution to the problem (1)-(2).

For $T>0$, we introduce the admissible class

$$\Psi_T := \left\{\psi\in L^p(0,T;W^{s,p}(\Omega))\cap C([0,T];L^2(\Omega)), \psi\in L^\infty(\Omega_T), \partial_{0|t}^{\alpha}\psi\in L^2(\Omega_T)\right\}.$$

We refer to [16] and [17], where one can find a description of the most useful properties of the fractional Sobolev spaces $W^{s,p}(\Omega)$.
Definition 5 Let $u(x,0)\le u_0(x)$ and $\mathcal{N}_{s,p}u(x,t)\le 0, x\in\mathbb{R}^n\setminus\overline{\Omega}$. We say that $u(x,t)$ is a weak subsolution of (1)-(2) in Ω_T, if $u(x,t)\in\Psi_T$ and satisfies

$$\begin{aligned}\int_{\Omega_T}&[\partial_{0|t}^{\alpha}u]\varphi dxdt + \int_{\Omega_T}(-\Delta)_p^s u\varphi dxdt\\ &\le \beta\int_{\Omega_T}|u|^{k-1}u\varphi dxdt + \gamma\int_{\Omega_T}|u|^{\ell-2}u\varphi dxdt,\end{aligned} \tag{4}$$

for any nonnegative $\varphi \in C_c^\infty([0,T)\times\Omega)$ with $\varphi(\cdot,T)=0$.

Similarly, let $v(x,0) \geq v_0(x), \mathcal{N}_{s,p}v(x,t) \geq 0, x \in \mathbb{R}^n \setminus \overline{\Omega}$. We say that v is a weak supersolution of (1)-(2) if $v(x,t) \in \Psi_T$ and satisfies

$$\begin{aligned}
&\int_{\Omega_T} [\partial^\alpha_{0|t} v]\varphi dxdt + \int_{\Omega_T} (-\Delta)^s_p v\varphi dxdt \\
&\qquad\geq \beta \int_{\Omega_T} |v|^{k-1} v\varphi dxdt + \gamma \int_{\Omega_T} |v|^{\ell-2} v\varphi dxdt.
\end{aligned} \tag{5}$$

A function is a weak solution if it is both a weak subsolution and supersolution.

Thoerem 6 Let $s \in (0,1), p \geq 2$ and let k, ℓ, β, γ satisfy one of the following conditions:

$$\begin{aligned}
&k \geq 1, \ell \geq 2, \beta \geq 0, \gamma \geq 0;\\
&k > 0, \ell > 1, \beta \leq 0, \gamma \leq 0;\\
&k \geq 1, \ell > 1, \beta \geq 0, \gamma \leq 0;\\
&k > 0, \ell \geq 2, \beta \leq 0, \gamma \geq 0.
\end{aligned}$$

Suppose that $u, v \in \Psi_T$ are real-valued weak subsolution and weak supersolution of (1), respectively, with $u_0(x) \leq v_0(x)$ for $x \in \Omega$. Then $u \leq v$ a.e. in Ω_T.

Proof. We choose the test function

$$\varphi(x,t) = \varphi(x,\cdot) = (u-v)_+,$$

where $(u-v)_+$ is the positive part of a real quantity $(u-v)_+ = \max\{u-v, 0\}$. Consequently, it holds $\varphi(x,0)=0$.

Subtracting (5) from (4), we obtain, for all $t \in (0,T]$

$$\begin{aligned}
&\int_0^t \int_\Omega \partial^\alpha_{0|\tau}[u-v]\varphi dxd\tau + \int_0^t \int_\Omega [(-\Delta)^s_p u - (-\Delta)^s_p v]\varphi dxd\tau \\
&\qquad\leq \underbrace{\beta \int_0^t \int_\Omega (|u|^{k-1}u - |v|^{k-1}v)\varphi dxd\tau}_{\mathcal{I}_1} \\
&\qquad+ \underbrace{\gamma \int_0^t \int_\Omega (|u|^{\ell-2}u - |v|^{\ell-2}v)\varphi dxd\tau}_{\mathcal{I}_2}.
\end{aligned} \tag{6}$$

By (3), the last term on the left-hand side of inequality (6) can be written as

$$\begin{aligned}
&\int_0^t \int_\Omega [(-\Delta)^s_p u - (-\Delta)^s_p v]\varphi dxd\tau \\
&= \int_0^t \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} dxdyd\tau \\
&- \int_0^t \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} dxdyd\tau \\
&= \int_0^t \int_\Omega \int_\Omega \frac{\mathcal{A}(u,v)(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} dxdyd\tau,
\end{aligned} \tag{7}$$

with

$$\mathcal{A}(u,v) = |u(x)-u(y)|^{p-2}(u(x)-u(y)) - |v(x)-v(y)|^{p-2}(v(x)-v(y)). \tag{8}$$

We shall use the following elementary truncation property.
Lemma 7 *[Truncation monotonicity]*. Let $p>1$ and let $J_p(r) = |r|^{p-2}r$. Then, for any real numbers a,b,r,s satisfying

$$a-b = r-s,$$

one has

$$\big(J_p(a) - J_p(b)\big)(r_+ - s_+) \geq 0,$$

where $r_+ = \max\{r,0\}$ and $s_+ = \max\{s,0\}$.
Proof. Since $J_p(r) = |r|^{p-2}r$ is nondecreasing on $\mathbb{R}$, we have

$$\big(J_p(a) - J_p(b)\big)(a-b) \geq 0.$$

Hence $J_p(a) - J_p(b)$ has the same sign as $a-b$. Since $a-b=r-s$, it has the same sign as $r-s$. On the other hand, the map $r \mapsto r_+$ is also nondecreasing, and therefore

$$(r-s)(r_+ - s_+) \geq 0.$$

Consequently, $r_+ - s_+$ has the same sign as $r-s$, or is equal to zero. Thus

$$\big(J_p(a) - J_p(b)\big)(r_+ - s_+) \geq 0,$$

which completes the proof.

Now, let

$$w = u - v, \quad \varphi = w_+.$$

For fixed $x, y \in \Omega$, set

$$a = u(x) - u(y), \qquad b = v(x) - v(y),$$

and

$$r = w(x) = u(x) - v(x), \qquad s = w(y) = u(y) - v(y).$$

Then

$$a - b = w(x) - w(y) = r - s,$$

and

$$\varphi(x) - \varphi(y) = w_+(x) - w_+(y) = r_+ - s_+.$$

Therefore, by the truncation monotonicity lemma,

$$\Big(|u(x)-u(y)|^{p-2}(u(x)-u(y)) - |v(x)-v(y)|^{p-2}(v(x)-v(y))\Big) \\ \times \Big((u-v)_+(x) - (u-v)_+(y)\Big) \geq 0.$$

Hence

$$\int_0^t \int_\Omega \int_\Omega \frac{A(u,v)(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} \, dx\, dy\, d\tau \geq 0.$$

We denote

$$F_k(r) = |r|^{k-1}r, \qquad G_\ell(r) = |r|^{\ell-2}r.$$

Since the functions F_k and G_ℓ are nondecreasing in the admissible ranges of the exponents, we have

$$\big(F_k(u) - F_k(v)\big)w_+ \geq 0, \qquad \big(G_\ell(u) - G_\ell(v)\big)w_+ \geq 0$$

almost everywhere in Ω_T.
Therefore, if $\beta \leq 0$, then

$$\beta\big(F_k(u) - F_k(v)\big)w_+ \leq 0.$$

Similarly, if $\gamma \leq 0$, then

$$\gamma\big(G_\ell(u) - G_\ell(v)\big)w_+ \leq 0.$$

Thus the negative-coefficient terms are favorable and can be dropped.

For the positive-coefficient part, we use the local Lipschitz estimates

$$|F_k(u) - F_k(v)| \leq L_k|u - v|, \qquad k \geq 1,$$

and

$$|G_\ell(u) - G_\ell(v)| \leq L_\ell|u - v|, \qquad \ell \geq 2,$$

where the coefficients L_k and L_ℓ are defined by

$$L_k = C(k)\max\left\{\|u\|^{k-1}_{L^\infty(\Omega_T)}, \|v\|^{k-1}_{L^\infty(\Omega_T)}\right\},$$

$$L_\ell = C(\ell)\max\left\{\|u\|^{\ell-2}_{L^\infty(\Omega_T)}, \|v\|^{\ell-2}_{L^\infty(\Omega_T)}\right\}.$$

Consequently, we obtain

$$\beta\int_0^t\int_\Omega \big(F_k(u) - F_k(v)\big)w_+\,dx\,d\tau \leq \beta_+ L_k\int_0^t\int_\Omega w_+^2\,dx\,d\tau,$$

and

$$\gamma\int_0^t\int_\Omega \big(G_\ell(u) - G_\ell(v)\big)w_+\,dx\,d\tau \leq \gamma_+ L_\ell\int_0^t\int_\Omega w_+^2\,dx\,d\tau.$$

Therefore, we have

$$\beta\mathcal{I}_1 + \gamma\mathcal{I}_2 \leq (\beta_+ L_k + \gamma_+ L_\ell)\int_0^t\int_\Omega w_+^2\,dx\,d\tau.$$

Then using the inequality [18, Lemma 1],

$$\frac{1}{2}\partial^\alpha_{0|t}(w_+^2)(t) \leq w_+(t)\partial^\alpha_{0|t}w(t)$$

under the condition $w_+(0) = 0$, our final inequality becomes simpler

$$\frac{1}{2}\int_0^t\int_\Omega \partial^\alpha_{0|\tau}w_+^2 dxd\tau \leq (\beta_+ L_k + \gamma_+ L_\ell)\int_0^t\int_\Omega w_+^2 dxd\tau.$$

Define
$$Y(t) = \int_\Omega w_+^2(x,t)dx,$$
which gives
$$\int_0^t \partial_{0|\tau}^\alpha Y(\tau)d\tau \le C\int_0^t Y(\tau)d\tau,$$
where $C = 2\,(\beta_+ L_k + \gamma_+ L_\ell)$.
Applying the left-hand Caputo operator $\partial_{0|t}^{1-\alpha}$ on the both sides of the last estimates and taking account Property 2,
$$\partial_{0|t}^{1-\alpha}\int_0^t \partial_{0|\tau}^\alpha Y(\tau)d\tau = \left(I_{0|t}^\alpha \frac{d}{dt} I_{0|t}^1 I_{0|t}^{1-\alpha}\frac{d}{dt}Y\right)(t) = Y(t) - Y(0)$$
with $Y(0) = 0$, we can show that $Y(t)$ satisfies a fractional integral inequality of the form
$$Y(t) \le C\int_0^t (t-\tau)^{\alpha-1} Y(\tau)d\tau.$$
Hence, in view of the weakly singular Gronwall inequality (see [19], Lemma 7.1.1 and [20], Lemma 6, p. 33) we deduce that $Y(t) = 0$, which verifies
$$w_+ = 0.$$
Finally, we deduce that
$$u \le v \quad \text{a.e. in} \quad \Omega_T,$$
which completes the proof.

4 Applications of the comparison principle

In this section, we give a priori boundedness result in the dissipative case. The second one gives a sufficient condition for finite-time blow-up by comparison with a spatially constant subsolution.

4.1 A priori boundedness in the dissipative case

Assume that $\beta \le 0, \gamma \le 0$, and let u be a weak solution of problem (1)-(2) with nonnegative bounded initial data
$$0 \le u_0(x) \le M \quad \text{for a.e.} x \in \Omega,$$
where $M > 0$ is a constant.
First, the zero function
$$\underline{u}(x,t) = 0$$
is a weak subsolution of (1)-(2). Since
$$\underline{u}(x,0) = 0 \le u_0(x),$$

the comparison principle implies

$$0 \leq u(x,t) \qquad \text{a.e. in } \Omega_T.$$

Next, define the constant function

$$\overline{u}(x,t) = M.$$

Since $\overline{u}$ is independent of the space variable, we have

$$\partial_{0|t}^{\alpha} \overline{u} = (-\Delta)_p^s \overline{u} = \mathcal{N}_{s,p} \overline{u} = 0.$$

Since $\beta \leq 0, \gamma \leq 0$, and $M > 0$, we get

$$0 \geq \beta M^k + \gamma M^{\ell-1}.$$

Therefore,

$$\partial_{0|t}^{\alpha} \overline{u} + (-\Delta)_p^s \overline{u} = 0 \geq \beta \overline{u}^k + \gamma \overline{u}^{\ell-1}.$$

Thus $\overline{u}$ is a weak supersolution of (1)-(2). Since

$$u_0(x) \leq M = \overline{u}(x,0),$$

the comparison principle gives

$$u(x,t) \leq M \quad \text{a.e. in } \Omega_T.$$

Consequently,

$$0 \leq u(x,t) \leq M \quad \text{a.e. in } \Omega_T.$$

Hence the solution remains uniformly bounded on every time interval $[0,T]$.

4.2 A finite-time blow-up solution

Now let $\beta > 0, \gamma = 0, k > 1$. Take the constant initial value $u_0(x) = A > 0$. Again we look for a solution in the form

$$\underline{u}(x,t) = z(t).$$

Then $z(t)$ satisfies

$$\partial_{0|t}^{\alpha} z(t) = \beta z^k(t), \qquad z(0) = A.$$

Since $\beta > 0$ and $A > 0$, the problem admits finite-time blow-up solution; see, for example, [21, 22]. Hence, there exists $T_b < \infty$ such that

$$\lim_{t \to T_b^-} z(t) = +\infty.$$

Therefore, from

$$z(t) = \underline{u}(x,t) \leq u(x,t),$$

we get also

$$\lim_{t \to T_b^-} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

Thus u is a finite-time blow-up solution of problem (1)-(2).

5 Conclusions

In this paper, we studied a nonlinear time-space fractional diffusion equation involving the Caputo fractional derivative and the fractional p-Laplacian under a nonlocal Neumann boundary condition. The main result is a comparison principle for weak subsolutions and supersolutions under suitable assumptions on the parameters β, γ, k, and ℓ.

The proof is based on three main parts. First, we use the monotonicity of the map $r \mapsto |r|^{p-2}r$ to handle the fractional p-Laplacian term. Second, we apply a truncation argument with the positive part $(u-v)_+$, which allows us to compare subsolutions and supersolutions. Third, the memory effect coming from the Caputo fractional derivative is treated by a fractional Gronwall-type inequality. These arguments show that if the initial ordering is satisfied, then this ordering is preserved for all later times.

As applications of the comparison principle, we obtained an a priori boundedness result in the dissipative case and a finite-time blow-up criterion by comparison with a fractional ordinary differential subsolution. These applications illustrate how the comparison principle can be used to study qualitative properties of solutions, such as positivity, boundedness, and blow-up behavior.

The results show that comparison methods remain effective for nonlinear fractional diffusion problems with nonlocal Neumann boundary conditions, provided that the nonlocal diffusion term, the nonlinear source terms, and the Caputo memory term are treated in a compatible way.

6 Acknowledgments

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP26195417).

References

[1] Diethelm, Kai. 2010. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*. Lecture Notes in Mathematics 2004. Berlin: Springer.

[2] Lakshmikantham, V., and A. S. Vatsala. 2008. "General Uniqueness and Monotone Iterative Technique for Fractional Differential Equations."*Applied Mathematics Letters* 21: 828–834.

[3] Kilbas, Anatoly A., Hari M. Srivastava, and Juan J. Trujillo. 2006. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies 204. Amsterdam: Elsevier.

[4] Smadiyeva, A. 2022. "Initial-boundary value problem for the time-fractional degenerate diffusion equation."*Journal of Mathematics Mechanics and Computer Science* 113 (1): 33–42.

[5] Ye, H., J. Gao, and Y. Ding. 2007. "A Generalized Gronwall Inequality and Its Application to a Fractional Differential Equation."*Journal of Mathematical Analysis and Applications* 328: 1075–1081.

[6] Mambetov, S. 2023. "A maximum principle for time-fractional diffusion equation with memory."*Journal of Mathematics Mechanics and Computer Science* 120 (4): 32–40.

[7] Dipierro, Serena, Xavier Ros-Oton, and Enrico Valdinoci. 2017. "Nonlocal Problems with Neumann Boundary Conditions."*Revista Matemática Iberoamericana* 33: 377–416.

[8] Mugnai, Duccio, and Elisabetta Proietti Lippi. 2019. "Neumann Fractional p-Laplacian: Eigenvalues and Existence Results."*Nonlinear Analysis* 188: 455–474.

[9] Mugnai, Duccio, Kanishka Perera, and Elisabetta Proietti Lippi. 2022. "A Priori Estimates for the Fractional p-Laplacian with Nonlocal Neumann Boundary Conditions and Applications."*Communications on Pure and Applied Analysis* 21: 275–292.

[10] Abdellaoui, Brahim, Abdelkader Attar, Rachid Bentifour, and Ireneo Peral. 2018. "On Fractional p-Laplacian Parabolic Problem with General Data."*Annali di Matematica Pura ed Applicata* 197 (2): 329–356.

[11] Borikhanov, M. B., Michael Ruzhansky, and B. T. Torebek. 2023. "Qualitative Properties of Solutions to a Nonlinear Time-Space Fractional Diffusion Equation."*Fractional Calculus and Applied Analysis* 26: 111–146.

[12] Giacomoni, Julián, and Snehashish Tiwari. 2018. "Existence and Global Behavior of Solutions to Fractional p-Laplacian Parabolic Problems."*Electronic Journal of Differential Equations* 2018 (20): Article 44.

[13] Vázquez, Juan Luis. 2016. "The Dirichlet Problem for the Fractional p-Laplacian Evolution Equation."*Journal of Differential Equations* 260: 6038–6056.

[14] del Teso, Felipe, David Gómez-Castro, and Juan Luis Vázquez. 2021. "Three Representations of the Fractional p-Laplacian: Semigroup, Extension and Balakrishnan Formulas."*Fractional Calculus and Applied Analysis* 24 (4): 966–1002. https://doi.org/10.1515/fca-2021-0042.

[15] Barrios, Begoña, Lorenzo Montoro, Ireneo Peral, and Fernando Soria. 2020. "Neumann Conditions for the Higher Order s-Fractional Laplacian $(-\Delta)^s u$ with $s > 1$."*Nonlinear Analysis* 193: 111368.

[16] Di Nezza, Eleonora, Giampiero Palatucci, and Enrico Valdinoci. 2012. "Hitchhiker's Guide to the Fractional Sobolev Spaces."*Bulletin des Sciences Mathématiques* 136 (5): 521–573.

[17] Brasco, Lorenzo, Erik Lindgren, and Enea Parini. 2014. "The Fractional Cheeger Problem."*Interfaces and Free Boundaries* 16: 419–458.

[18] Alsaedi, Ahmed, Bashir Ahmad, and Mokhtar Kirane. 2017. "A Survey of Useful Inequalities in Fractional Calculus."*Fractional Calculus and Applied Analysis* 20 (3): 574–594. https://doi.org/10.1515/fca-2017-0031.

[19] Henry, Daniel. 1981. *Geometric Theory of Semilinear Parabolic Equations.* Lecture Notes in Mathematics 840. Berlin: Springer.

[20] Haraux, Alain. 1981. *Nonlinear Evolution Equations: Global Behavior of Solutions.* Lecture Notes in Mathematics 841. Berlin: Springer.

[21] Hnaien, Dorsaf, K. Ferdaous, and L. Rafika. 2014. "Blowing-Up Solutions and Global Solutions to a Fractional Differential Equation."*Fractional Differential Calculus* 4: 45–53.

[22] Laskri, Y., and N.-E. Tatar. 2010. "The Critical Exponent for an Ordinary Fractional Differential Problem."*Computers & Mathematics with Applications* 59: 1266–1270.

Information about author:

Meiirkhan Borikhanov – PhD, Senior researcher at Institute of Mathematics and Mathematical Modeling (Almaty, Kazakhstan, email: borikhanov@math.kz).

Автор туралы мәлімет:

Мейірхан Бөріханов — PhD, Математика және математикалық моделдеу институтының аға ғылыми қызметкері (Алматы, Қазақстан, электрондық пошта: borikhanov@math.kz)

Информация об авторе:

Мейирхан Бориханов — PhD, старший научный сотрудник Института математики и математического моделирования (Алматы, Казахстан, электронная почта: borikhanov@math.kz).

Revised: July 02, 2026
Accepted: July 31, 2026