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ON WEIGHTED BILINEAR INEQUALITY WITH A KERNEL

Introducing a kernel into integral bilinear Hardy-type operators represents a new direction in operator theory. In the general case, since the criterion for the validity of an integral Hardy-type inequality with a kernel in a weighted Lebesgue space is not defined, various conditions are determined on the kernel, and wider results are obtained than in the case without a kernel. This paper examines integral bilinear Hardy-type operators involving the Hardy and Hardy-Volterra operators in weighted Lebesgue spaces. We obtained criteria for the validity of integral bilinear Hardy-type inequalities with kernels in the classes $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$ for the parameter ranges $1 < \max\{p,r\} \le q < \infty$, $1 < \min\{p,r\} \le q < \max\{p,r\} < \infty$ and $1 < q < \min\{p,r\} < \infty$. An iteration technique is applied to characterize the boundedness of certain multilinear operators, reducing the problem to a linear operator case. This study contributes to the theory of integral operators in functional analysis and can be used in finding solutions to differential equations. The results obtained in this work fully describe the conditions for the validity of bilinear inequalities with a kernel over a wide range of parameters. This, in turn, lays a theoretical foundation for further research in this direction and makes it possible to extend the theory of operators with a kernel to a new class of inequalities.
Keywords: bilinear operator, Hardy-type inequality, Hardy-type operator, Lebesgue space, weight function, kernel.

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Өзек қатысқан салмақты бисызықты теңсіздік туралы

Интегралдық бисызықты Харди тәріздес операторларға өзек енгізу операторлар теориясындағы жаңа бағыт болып табылады. Жалпы жағдайда, салмақты Лебег кеңістігінде өзекпен берілген интегралдық Харди тәріздес теңсіздіктің орындалу критерийі анықталмағандықтан, өзекке әртүрлі шарттар анықталды және ядросыз жағдайға қарағанда кеңірек нәтижелер алынады. Бұл жұмыста салмақты Лебег кеңістіктерінде Харди және Харди-Вольтерра операторларын қамтитын интегралдық бисызықты Харди тәріздес операторлар зерттеледі. $\mathcal{O}_1^+$ және $\mathcal{O}_1^-$ кластарына жататын өзектері бар бисызықты интегралдық Харди тәріздес теңсіздіктердің орындылығының критерийлері параметрлердің $1 < \max\{p,r\} \le q < \infty$, $1 < \min\{p,r\} \le q < \max\{p,r\} < \infty$ және $1 < q < \min\{p,r\} < \infty$ аралықтарында алынды. Кейбір мультисызықты операторлардың шенелімділігін сипаттау үшін итерациялық әдіс қолданылады, бұл есепті сызықтық оператор жағдайына келтіреді. Бұл зерттеу функционалдық талдаудағы интегралдық операторлар теориясына үлес қосады және дифференциалдық теңдеулердің шешімдерін табуда қолданылуы мүмкін. Жұмыста алынған нәтижелер өзек қатысқан бисызықты теңсіздіктердің орындалу шарттарын параметрлердің кең ауқымында толық сипаттайды. Бұл өз кезегінде осы бағыттағы одан әрі зерттеулер үшін теориялық негіз қалап, өзекті операторлар теориясын жаңа кластағы теңсіздіктерге кеңейтуге мүмкіндік береді.
Түйін сөздер: бисызықты оператор, Харди тәріздес теңсіздік, Харди тәріздес оператор, Лебег кеңістігі, салмақ функциясы, өзек.

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О весовом билинейном неравенстве с ядром

Введение ядра в интегральные билинейные операторы типа Харди представляет собой новое направление в теории операторов. В общем случае, поскольку критерий выполнения интегрального неравенства типа Харди с ядром в весовом пространстве Лебега не определён, определяются различные условия на ядро, и получаются более широкие результаты, чем в случае без ядра. В данной работе исследуются интегральные билинейные операторы типа Харди, включающие операторы Харди и Харди-Вольтерра, в весовых пространствах Лебега. Получены критерии выполнения билинейных интегральных неравенств типа Харди с ядрами из классов $\mathcal{O}_1^+$ и $\mathcal{O}_1^-$ для диапазонов параметров $1 < \max\{p,r\} \le q < \infty$, $1 < \min\{p,r\} \le q < \max\{p,r\} < \infty$ и $1 < q < \min\{p,r\} < \infty$. Применяется итерационная техника для характеризации ограниченности некоторых мультилинейных операторов, сводящая задачу к случаю линейного оператора. Данное исследование вносит вклад в теорию интегральных операторов в функциональном анализе и может быть использовано при нахождении решений дифференциальных уравнений. Полученные в работе результаты полностью описывают условия выполнения билинейных неравенств с ядром в широком диапазоне параметров. Это, в свою очередь, закладывает теоретическую основу для дальнейших исследований в данном направлении и позволяет расширить теорию операторов с ядром на новый класс неравенств.

Ключевые слова: билинейный оператор, неравенство типа Харди, оператор типа Харди, пространство Лебега, весовая функция, ядро.

1 Introduction

Let $I = (0,\infty), 1 < p, q, r < \infty$ and $\frac{1}{p}+\frac{1}{p'} = 1$. u, v, and w are weights, i.e. positive measurable functions on I. We define the weighted Lebesgue space $L_{p,v}(I)$ as the set of all measurable functions f on I for which the norm $\|f\|_{p,v} = \left(\int_0^\infty v(t)|f(t)|^p dt\right)^{\frac{1}{p}}$ is finite.

The well-known Hardy operator

$$H_1 f(x) := \int_0^x f(s)ds, \quad x \in (0,\infty),$$

and the inequality that describes when H_1 is bounded from $L_{q,u}(I)$ to $L_{p,v}(I)$

$$\left(\int_0^\infty (H_1 f(x))^q u(x)dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{1/p}, \quad f \geqslant 0, \tag{1}$$

are now fully established for all ratios of the parameters p and q. A full and detailed history of continuous and discrete Hardy-type inequalities is available in the books [1, 2] and their equivalence with the inequalities on the cone of non-negative sequences is demonstrated in [3].

Further research in this area focused on the boundedness of operators of the form

$$K_1 f(t) := \int_0^t K(t,s)f(s)ds, \quad t \in (0,\infty), \tag{2}$$

in various function spaces. This investigation is complex. For instance, the boundedness of the Hardy-Volterra operator K_1 from a weighted Lebesgue space $L_{q,u}(I)$ to $L_{p,v}(I)$ is generally resolved for any kernels $K(t,s)$ only for cases $1 \le p \le \infty$, $q = \infty$, and $p = 1$, $1 \le q \le \infty$ [4]. For the case $1 < p, q < \infty$, the general solution is still unresolved. Consequently, the problem is typically addressed by imposing additional conditions on the kernels. V.D. Stepanov initiated the investigation of Hardy-type integral inequalities with kernels [5, 6]. Back in 1991, S. Bloom and R. Kerman [7], as well as R. Oinarov [8], independently presented a class of kernels $K(x,t)$ that satisfy the following condition

$$d^{-1}(K(t,z) + K(z,s)) \le K(t,s) \le d(K(t,z) + K(z,s)) \tag{3}$$

for $0 < s \le z \le t < \infty$, where the constant $d \ge 1$ does not depend on s, z, t. In work [9], less restrictive classes of kernels $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$, $n \le 0$ was introduced.

In recent decades, the bilinear Hardy operator B_1 has been the subject of study and is defined as follows

$$B_1(f,g)(t) := H_1 f(t) \cdot H_1 g(t), \quad t \in (0,\infty).$$

Aguilar Canestro et. al. and Krepela have established the boundedness criteria for B_1 from $L_{p,v}(I) \times L_{s,w}(I)$ to $L_{q,u}(I)$, which is equivalent to proving the bilinear weighted Hardy inequality

$$\left(\int_0^\infty u(t)(B_1(f,g)(t))^q dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty w(t)g^r(t)dt\right)^{\frac{1}{r}}\left(\int_0^\infty v(t)f^p(t)dt\right)^{\frac{1}{p}}, f, g \ge 0, \tag{4}$$

for all possible ranges of the parameters $1 < p, q, r < \infty$ (see [10,11]). In [10], the discretization technique was used. However, this standard technical method is particularly complex for characterizing the inequality (4). Subsequently, a simpler proof of the characterization of (4) was presented in [11]. This approach consists of two steps, each step treating the problem as the standard Hardy inequality (1). The same method was also applied in [12] to describe the bilinear Hardy-type inequality for decreasing functions. Recently, the validity of the discrete version of inequality (4) has been proven for all possible relation of the parameters p, q, r [13]. Ongoing research also focuses on multidimensional cases of the bilinear inequality [14], [15]. Recently, results on discrete bilinear Hardy inequality involving a matrix satisfying the discrete version of condition (3) have been obtained in [16] and [17].

The goal of this paper is to discuss the following bilinear Hardy-type inequality involving the Hardy-Volterra operator and the Hardy operator:

$$\left(\int_0^\infty u(x)(H_1 g(x) \cdot K_1 f(x))^q dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty w(x)g^r(x)dx\right)^{\frac{1}{r}}\left(\int_0^\infty v(x)f^p(x)dx\right)^{\frac{1}{p}},$$

$$g \in L_{r,w}(I), f \in L_{p,v}(I), \tag{5}$$

for $f, g \ge 0$. Here, the kernel $K(x,t)$ belongs to the classes $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$, which are less restrictive than the classes of kernels that satisfy condition (3).

The conditions for the validity of a bilinear inequality of the form (5) have previously been obtained for kernels in the classes $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$. In [18], the authors applied results obtained for the quasilinear operator to characterize the conditions for the validity of the bilinear inequality (5) under the parameter case $1 < r/(r-q) < p/q$. This study considers the same parameter case but utilizes a different method. Note that this paper considers the approach introduced in [11] to characterize the inequality (5). Due to the similarity of proofs for kernels from the classes $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$, only the proof for inequality (5) involving kernel from the class $\mathcal{O}_1^-$ is provided in this paper.

2 Materials and methods

Let $\alpha := pq/(p-q)$ for $p > q$ and $\beta := rq/(r-q)$ for $r > q$. Throughout the article, the symbol $E \ll M$ means that $E \leq cM$ for insignificant constant $c > 0$. The symbol $E \approx M$ means that $E \ll M \ll E$. Furthermore, $\chi_{(a,b)}(\cdot)$ stands for the characteristic function of the interval $(a, b) \subset \mathbb{R}$.

Let's give the definitions of the classes $\mathcal{O}_1^-$ and $\mathcal{O}_1^+$.

Definition 1. Let $K(\cdot,\cdot)$ be a non-negative function, defined on the set $\{(x, s), 0 < s \leq x < \infty\}$, measurable and non-increasing in the second argument. We say that the function $K(x, s)$ belongs to the class $\mathcal{O}_1^-$ if there exist non-negative measurable functions $\mathcal{T}(\cdot)$ and $Q(\cdot,\cdot)$ and a constant $d \geq 1$ such that

$$d^{-1}(K(x,t) + \mathcal{T}(x)Q(t,s)) \leq K(x,s) \leq d(K(x,t) + \mathcal{T}(x)Q(t,s))$$

for all $t, x, s : 0 < s \leq t \leq x < \infty$.

Definition 2. Let $K(\cdot,\cdot)$ be a non-negative function, defined on the set $\{(x, s), 0 < s \leq x < \infty$, measurable and non-decreasing in the first argument. We say that the function $K(x, s)$ belongs to the class O_1^+ if there exist non-negative measurable functions $T(\cdot)$ and $R(\cdot,\cdot)$ and a constant $d \geq 1$ such that

$$d^{-1}(R(x,t)T(s) + K(t,s)) \leq K(x,s) \leq d(R(x,t)T(s) + K(t,s))$$

for all $t, x, s : 0 < s \leq t \leq x < \infty$.

Remark 1. (see [9]). The functions $R(x, s)$ and $Q(x, s)$ can be considered non-decreasing with respect to the variable x and non-increasing with respect to the variable s.

To prove the main result, we require the following lemma and remark from [19]:

Lemma 1. Let λ be a σ-finite measure on $[a, b]$, where $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. For $\gamma > 0$, the relations

$$\left(\int\limits_{[a,b]} f(x)d\lambda(x)\right)^{\gamma+1} \ll \int\limits_{[a,b]} f(t)\left(\int\limits_{[a,t]} f(x)d\lambda(x)\right)^{\gamma} d\lambda(t) \ll \left(\int\limits_{[a,b]} f(x)d\lambda(x)\right)^{\gamma+1}, f \geq 0, \tag{6}$$

$$\left(\int\limits_{[a,b]} f(x)d\lambda(x)\right)^{\gamma+1} \ll \int\limits_{[a,b]} f(t)\left(\int\limits_{[t,b]} f(x)d\lambda(x)\right)^{\gamma} d\lambda(t) \ll \left(\int\limits_{[a,b]} f(x)d\lambda(x)\right)^{\gamma+1}, f \geq 0, \tag{7}$$

hold. For $\gamma \in (-1, 0)$ and $\int_{[a,b]} f(x)d\lambda(x) < \infty$, relations (6) and (7) also hold.

Remark 2. If $\gamma \in (-1,0)$ and $\int_{[a,b]} f(x)d\lambda(x) = \infty$, then the second inequalities in (6) and (7) obviously hold.

We also use the following theorems related to the two weighted Hardy inequality (1) (see [2, 20] and [21]):

Theorem A. *Let u, v be weight functions. For $p, q \in (1,\infty)$ set*

$$C := \sup_{f\geq 0} \left(\int_0^\infty \left(\int_a^t f(s)ds\right)^q u(t)\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_0^\infty f^p(t)v(t)\right)^{-\frac{1}{p}}.$$

Then the following hold. (i) If $1 < p \leqslant q < \infty$, then

$$C \approx \sup_{0<x<\infty} \left(\int_x^\infty u(s)ds\right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(t)dt\right)^{\frac{1}{p'}}.$$

(ii) If $1 < q < p < \infty$, then

$$C \approx \left(\int_0^\infty \left(\int_x^\infty u(s)ds\right)^{\frac{\alpha}{q}} \left(\int_0^x v^{1-p'}(s)ds\right)^{\frac{\alpha}{q'}} v^{1-p'}(x)dx\right)^{\frac{1}{\alpha}}$$
$$\approx \left(\int_0^\infty \left(\int_x^\infty u(s)ds\right)^{\frac{\alpha}{p}} \left(\int_0^x v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}} u(x)dx\right)^{\frac{1}{\alpha}}.$$

Theorem B. *Let u, v be weight functions. For $p, q \in (1,\infty)$ set*

$$C := \sup_{f\geq 0} \left(\int_0^\infty \left(\int_t^\infty f(s)ds\right)^q u(t)dt\right)^{\frac{1}{q}} \left(\int_0^\infty f^p(t)v(t)dt\right)^{-\frac{1}{p}}.$$

Then the following hold. (i) If $1 < p \leqslant q < \infty$, then

$$C \approx \sup_{0<x<\infty} \left(\int_0^x u(s)ds\right)^{\frac{1}{q}} \left(\int_x^\infty v^{1-p'}(t)dt\right)^{\frac{1}{p'}}$$

(ii) If $1 < q < p < \infty$, then

$$C \approx \left(\int_0^\infty \left(\int_0^x u(s)ds\right)^{\frac{\alpha}{q}} \left(\int_x^\infty v^{1-p'}(s)ds\right)^{\frac{\alpha}{q'}} v^{1-p'}(x)dx\right)^{\frac{1}{\alpha}}$$
$$\approx \left(\int_0^\infty \left(\int_0^x u(s)ds\right)^{\frac{\alpha}{p}} \left(\int_x^\infty v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}} u(x)dx\right)^{\frac{1}{\alpha}}.$$

We also use boundedness criteria of the Hardy-Volterra operator (2) from $L_{p,v}(I)$ to $L_{q,u}(I)$. The result from [9] (Theorem 5) regarding the inequality

$$\|K_1 f\|_{q,u} \leq C\|f\|_{p,v}, \quad \forall f \in L_{p,v}(I), \tag{8}$$

with a kernel belonging to the class $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$ can be rewritten as follows:

Theorem F. Let $1 < p \le q < \infty$ and the kernel $K(x,s)$ of the operator (2) belong to the class $\mathcal{O}_1^+$. Then the inequality (8) holds if and only if one of the following conditions is satisfied:

$$F_1 = \sup_{0<z<\infty} \left(\int_z^\infty R^q(x,z)u(x)dx \right)^{\frac{1}{q}} \left(\int_0^z T^{p'}(s)v^{1-p'}(s)ds \right)^{\frac{1}{p'}} < \infty,$$

$$F_2 = \sup_{0<z<\infty} \left(\int_0^z K^{p'}(z,s)v^{1-p'}(s)ds \right)^{\frac{1}{p'}} \left(\int_z^\infty u(x)dx \right)^{\frac{1}{q}} < \infty.$$

Moreover, $\|K_1\| \approx F_1 + F_2$, where $\|K_1\|$ means the norm the of operator $\|K_1\|$ from the space $L_{p,v}(I)$ to $L_{q,u}(I)$.

Theorem G. Let $1 < p \le q < \infty$ and the kernel $K(x,s)$ of the operator (2) belong to the class $\mathcal{O}_1^-$. Then the inequality (8) holds if and only if one of the following conditions is satisfied:

$$G_1 = \sup_{0<z<\infty} \left(\int_z^\infty K^q(x,z)u(x)dx \right)^{\frac{1}{q}} \left(\int_0^z v^{1-p'}(s)ds \right)^{\frac{1}{p'}} < \infty,$$

$$G_2 = \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s)v^{1-p'}(s)ds \right)^{\frac{1}{p'}} \left(\int_z^\infty \mathcal{T}^q(x)u(x)dx \right)^{\frac{1}{q}} < \infty.$$

Moreover, $\|K_1\| \approx G_1 + G_2$, where $\|K_1\|$ means the norm the of operator $\|K_1\|$ from the space $L_{p,v}(I)$ to $L_{q,u}(I)$.

Moreover, the results for the case $1 < q < p < \infty$ read as follows (see [22]):

Theorem D. Let $1 < q < p < \infty$ and the kernel $K(x,s)$ of the operator (2) belong to the class $\mathcal{O}_1^+$. Then the inequality (8) holds if and only if one of the following conditions is satisfied:

$$D_1 = \left(\int_0^\infty \left(\int_t^\infty R^q(x,t)u(x)dx \right)^{\frac{\alpha}{q}} \left(\int_0^t T^{p'}(s)v^{1-p'}(s)ds \right)^{\frac{\alpha}{q'}} T^{p'}(t)v^{1-p'}(t)dt \right)^{\frac{1}{r}} < \infty,$$

$$D_2 = \left(\int_0^\infty \left(\int_0^t K^{p'}(t,s)v^{1-p'}(s)ds \right)^{\frac{\alpha}{p'}} \left(\int_t^\infty u(x)dx \right)^{\frac{\alpha}{p}} u(t)dt \right)^{\frac{1}{\alpha}} < \infty.$$

Moreover, $\|K_1\| \approx D_1 + D_2$, where $\|K_1\|$ means the norm of the operator $\|K_1\|$ from the space $L_{p,v}(I)$ to $L_{q,u}(I)$.

Theorem E Let $1 < q < p < \infty$ and the kernel $K(x,s)$ of the operator (2) belong to the class $\mathcal{O}_1^-$. Then the inequality (8) holds if and only if one of the following conditions is satisfied:

$$E_1 = \left(\int_0^\infty \left(\int_t^\infty K^q(x,t)u(x)dx \right)^{\frac{\alpha}{q}} \left(\int_0^t v^{1-p'}(s)ds \right)^{\frac{\alpha}{q'}} v^{1-p'}(t)dt \right)^{\frac{1}{\alpha}} < \infty,$$

$$E_2 = \left(\int_0^\infty \left(\int_0^t Q^{p'}(t,s)v^{1-p'}(s)ds \right)^{\frac{\alpha}{p'}} \left(\int_t^\infty \mathcal{T}^q(x)u(x)dx \right)^{\frac{\alpha}{p}} \mathcal{T}^q(t)u(t)dt \right)^{\frac{1}{\alpha}} < \infty.$$

Moreover, $\|K_1\| \approx E_1 + E_2$, where $\|K_1\|$ means the norm the of operator $\|K_1\|$ from the space $L_{p,v}(I)$ to $L_{q,u}(I)$.

To conclude this section, we recall the following duality property of the $L_{p,v}(I)$ spaces. If $p \in (1,\infty)$ and v is a weight, then for any f it holds that

$$\left(\int_0^\infty f^p(x)v(x)\mathrm{d}x\right)^{\frac{1}{p}} = \sup_{h\geq 0} \frac{\int_0^\infty f(x)h(x)\mathrm{d}x}{\left(\int_0^\infty h^{p'}(x)v^{1-p'}(x)\mathrm{d}x\right)^{1/p'}}. \tag{9}$$

3 The main results

3.1 The results for the case $1 < \max\{p,r\} \leq q < \infty$.

Theorem 1 *Let $1 < \max\{p,r\} \leq q < \infty$ and assume that $K(x,t)$ of the operator (2) belongs to the class $\mathcal{O}_1^-$. Then the inequality (5) holds if and only if the following quantities*

$$A_1^- = \sup_{0<t<\infty} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \sup_{0<z\leq t} \left(\int_0^z v^{1-p'}(s)ds\right)^{\frac{1}{p'}} \left(\int_t^\infty K^q(x,z)u(x)dx\right)^{\frac{1}{q}},$$

$$A_2^- = \sup_{0<t<\infty} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_t^\infty \mathcal{T}^q(x)u(x)dx\right)^{\frac{1}{q}} \left(\int_0^t Q^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$

are finite. Moreover, $C \approx \max\{A_1^-, A_2^-\}$, where C is the sharp constant in (5).

Theorem 2 *Let $1 < \max\{p,r\} \leq q < \infty$and assume that $K(x,t)$ of the operator (2) belongs to the class $\mathcal{O}_1^+$. Then the inequality (5) holds if and only if the following quantities*

$$A_1^+ = \sup_{0<t<\infty} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \sup_{0<z\leq t} \left(\int_0^z T^{p'}(s)v^{1-p'}(s)ds\right)^{\frac{1}{p'}} \left(\int_t^\infty R^q(x,z)u(x)dx\right)^{\frac{1}{q}},$$

$$A_2^+ = \sup_{0<t<\infty} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_t^\infty u(x)dx\right)^{\frac{1}{q}} \left(\int_0^t K^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$

are finite. Moreover, $C \approx \max\{A_1^+, A_2^+\}$, where C is the sharp constant in (5).

Proof of Theorem 1. Let $0 \leq f, g$ be such that $\int_0^\infty f(s)ds$, $\int_0^\infty g(s)ds < \infty$. Assume that the inequality (5) holds with a the sharp C, then

$$C = \sup_{f\geq 0}\sup_{g\geq 0} \left(\int_0^\infty u(x)\left(\int_0^x g(t)dt\right)^q \left(\int_0^x K(x,s)f(s)ds\right)^q dx\right)^{\frac{1}{q}} \|f\|_{p,v}^{-1}\|g\|_{r,w}^{-1}. \tag{10}$$

We define $\bar{u}(x) = u(x)\left(\int_0^x g(t)dt\right)^q$. First we take the supremum over $f \geq 0$, by using Theorem G, we obtain that

$$C = \sup_{g\geq 0} \|g\|_{r,w}^{-1} \sup_{f\geq 0} \|f\|_{p,v}^{-1} \left(\int_0^\infty \bar{u}(x)\left(\int_0^x K(x,s)f(s)ds\right)^q dx\right)^{\frac{1}{q}} \approx \max\{J_1, J_2\}, \tag{11}$$

where

$$J_1 = \sup_{g\geq 0} \|g\|_{r,w}^{-1} \sup_{0<z<\infty} \left(\int_z^\infty K^q(x,z)\bar{u}(x)dx \right)^{\frac{1}{q}} \left(\int_0^z v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$J_2 = \sup_{g\geq 0} \|g\|_{r,w}^{-1} \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s)v^{1-p'}(s)ds \right)^{\frac{1}{p'}} \left(\int_z^\infty \mathcal{T}^q(x)\bar{u}(x)dx \right)^{\frac{1}{q}}.$$

Now we estimate each term in (11) separately. We set characteristic function and changing order of supremum in J_1 as follows:

$$J_1 = \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s)ds \right)^{\frac{1}{p'}} \sup_{g\geq 0} \|g\|_{r,w}^{-1} \left(\int_0^\infty \chi_{(z,\infty)}(x)K^q(x,z)u(x) \left(\int_0^x g(t)dt \right)^q dx \right)^{\frac{1}{q}}.$$

Since $1 < r \leq q$, by using standard results from Theorem A (i), we get the following equivalency

$$J_1 \approx \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s)ds \right)^{\frac{1}{p'}} \sup_{0<t<\infty} \left(\int_t^\infty \chi_{(z,\infty)}(x)K^q(x,z)u(x)dx \right)^{\frac{1}{q}} \left(\int_0^t w^{1-r'}(s)ds \right)^{\frac{1}{r'}}. \tag{12}$$

Now, by dividing the range of the supremum with respect to t into two intervals we have

$$\begin{aligned} J_1 &\approx \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s)ds \right)^{\frac{1}{p'}} \max\left\{ \sup_{0<t\leq z} \left(\int_z^\infty K^q(x,z)u(x)dx \right)^{\frac{1}{q}} \left(\int_0^t w^{1-r'}(s)ds \right)^{\frac{1}{r'}}, \right. \\ &\qquad \left. \sup_{z\leq t<\infty} \left(\int_t^\infty K^q(x,z)u(x)dx \right)^{\frac{1}{q}} \left(\int_0^t w^{1-r'}(s)ds \right)^{\frac{1}{r'}} \right\} \\ &= \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s)ds \right)^{\frac{1}{p'}} \max\left\{ \left(\int_z^\infty K^q(x,z)u(x)dx \right)^{\frac{1}{q}} \left(\int_0^z w^{1-r'}(s)ds \right)^{\frac{1}{r'}}, \right. \\ &\qquad \left. \sup_{z\leq t<\infty} \left(\int_t^\infty K^q(x,z)u(x)dx \right)^{\frac{1}{q}} \left(\int_0^t w^{1-r'}(s)ds \right)^{\frac{1}{r'}} \right\} \\ &= \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s)ds \right)^{\frac{1}{p'}} \sup_{z\leq t<\infty} \left(\int_t^\infty K^q(x,z)u(x)dx \right)^{\frac{1}{q}} \left(\int_0^t w^{1-r'}(s)ds \right)^{\frac{1}{r'}}. \end{aligned}$$

By changing the order of supremum in last expression we obtain

$$J_1 \approx A_1^-. \tag{13}$$

J_2 may be estimated using the same approach as J_1.

$$J_2 = \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s)v^{1-p'}(s)ds \right)^{\frac{1}{p'}} \sup_{g\geq 0} \|g\|_{r,w}^{-1} \left(\int_0^\infty \chi_{(z,\infty)}(x)\mathcal{T}^q(x)u(x) \left(\int_0^x g(t)dt \right)^q dx \right)^{\frac{1}{q}}.$$

Applying Theorem A (i), we proceed

$$J_2 \approx \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s) v^{1-p'}(s)ds\right)^{\frac{1}{p'}} \sup_{0<t<\infty} \left(\int_t^\infty \chi_{(z,\infty)}(x)\mathcal{T}^q(x)u(x)dx\right)^{\frac{1}{q}} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \tag{14}$$

$$= \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s) v^{1-p'}(s)ds\right)^{\frac{1}{p'}} \max\left\{ \sup_{0<t\le z} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_z^\infty \mathcal{T}^q(x)u(x)dx\right)^{\frac{1}{q}}, \right.$$

$$\left. \sup_{z\le t<\infty} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_t^\infty \mathcal{T}^q(x)u(x)dx\right)^{\frac{1}{q}} \right\}$$

$$= \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s) v^{1-p'}(s)ds\right)^{\frac{1}{p'}} \max\left\{ \left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_z^\infty \mathcal{T}^q(x)u(x)dx\right)^{\frac{1}{q}}, \right.$$

$$\left. \sup_{z\le t<\infty} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_t^\infty \mathcal{T}^q(x)u(x)dx\right)^{\frac{1}{q}} \right\}$$

$$= \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s) v^{1-p'}(s)ds\right)^{\frac{1}{p'}} \sup_{z\le t<\infty} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_t^\infty \mathcal{T}^q(x)u(x)dx\right)^{\frac{1}{q}}$$

$$= \sup_{0<t<\infty} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_t^\infty \mathcal{T}^q(x)u(x)dx\right)^{\frac{1}{q}} \sup_{0<z\le t} \left(\int_0^z Q^{p'}(z,s) v^{1-p'}(s)ds\right)^{\frac{1}{p'}}.$$

The estimate $Q(z,s) \le Q(t,s)$ for $z \le t$ follows from the fact $Q(z,s)$ satisfying condition (3). It allows that the last supremum respect to z can be taken t, then

$$J_2 \approx A_2^-. \tag{15}$$

It follows by (11), (13) and (15) that $C \approx \max\{A_1^-, A_2^-\}$.

Remark 3. Theorem 1 and Theorem 2 mean that the inequality (5) holds for both cases $1 < p \le r \le q < \infty$ and $1 < r < p \le q < \infty$.

3.2 The results for the case $1 < \min\{p,r\} \le q < \max\{p,r\} < \infty$.

Theorem 3 *Let $1 < p \le q < r < \infty$ and assume that $K(x,t)$ of the operator (2) belongs to the class $\mathcal{O}_1^-$. Then the inequality (5) holds if and only if the following quantities*

$$B_1^- = \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s)ds\right)^{\frac{1}{p'}} \left(\int_0^z w^{1-r'}(t)dt\right)^{\frac{1}{r'}} \left(\int_z^\infty K^q(x,z)u(x)dx\right)^{\frac{1}{q}},$$

$$B_2^- = \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s)ds\right)^{\frac{1}{p'}} \left(\int_z^\infty \left(\int_t^\infty K^q(x,z)u(x)dx\right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dt\right)^{\frac{1}{\beta}},$$

$$B_3^- = \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s)v^{1-p'}(s)ds\right)^{\frac{1}{p'}} \left(\int_0^z w^{1-r'}(t)dt\right)^{\frac{1}{r'}} \left(\int_z^\infty \mathcal{T}^q(x)u(x)dx\right)^{\frac{1}{q}},$$

$$B_4^- = \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s) v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \left(\int_z^\infty \left(\int_t^\infty \mathcal{T}^q(x) u(x) dx \right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s) ds \right)^{\frac{\beta}{q'}} w^{1-r'}(x) dt \right)^{\frac{1}{\beta}}$$

are finite. Moreover, $C \approx \max\{B_1^-, B_2^-, B_3^-, B_4^-\}$, *where* C *is the sharp constant in (5).*

Theorem 4 *Let* $1 < p \le q < r < \infty$ *and assume that* $K(x,t)$ *of the operator* (2) *belongs to the class* $\mathcal{O}_1^+$. *Then the inequality* (5) *holds if and only if the following quantities*

$$B_1^+ = \sup_{0<z<\infty} \left(\int_0^z T^{p'}(s) v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \left(\int_0^z w^{1-r'}(t) dt \right)^{\frac{1}{r'}} \left(\int_z^\infty R^q(x,z) u(x) dx \right)^{\frac{1}{q}},$$

$$B_2^+ = \sup_{0<z<\infty} \left(\int_0^z T^{p'}(s) v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \left(\int_z^\infty \left(\int_t^\infty R^q(x,z) u(x) dx \right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s) ds \right)^{\frac{\beta}{q'}} w^{1-r'}(x) dt \right)^{\frac{1}{\beta}},$$

$$B_3^+ = \sup_{0<z<\infty} \left(\int_0^z K^{p'}(z,s) v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \left(\int_0^z w^{1-r'}(t) dt \right)^{\frac{1}{r'}} \left(\int_z^\infty u(x) dx \right)^{\frac{1}{q}},$$

$$B_4^+ = \sup_{0<z<\infty} \left(\int_0^z K^{p'}(z,s) v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \left(\int_z^\infty \left(\int_t^\infty u(x) dx \right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s) ds \right)^{\frac{\beta}{q'}} w^{1-r'}(x) dt \right)^{\frac{1}{\beta}}$$

are finite. Moreover, $C \approx \max\{B_1^+, B_2^+, B_3^+, B_4^+\}$, *where* C *is the sharp constant in (5).*

Proof of Theorem 3. The proof can be started as Theorem 1. Given that $q < r$ in this theorem, Theorem A (ii) is applied to equation (12), yielding

$$J_1 \approx \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \times$$

$$\times \left(\int_0^\infty \left(\int_t^\infty \chi_{(z,\infty)}(x) K^q(x,z) u(x) dx \right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s) ds \right)^{\frac{\beta}{q'}} w^{1-r'}(t) dt \right)^{\frac{1}{\beta}}$$

$$\approx \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \left(\left(\int_0^z \left(\int_z^\infty K^q(x,z) u(x) dx \right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s) ds \right)^{\frac{\beta}{q'}} w^{1-r'}(t) dt \right)^{\frac{1}{\beta}} \right.$$

$$\left. + \left(\int_z^\infty \left(\int_t^\infty K^q(x,z) u(x) dx \right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s) ds \right)^{\frac{\beta}{q'}} w^{1-r'}(t) dt \right)^{\frac{1}{\beta}} \right).$$

By using Lemma 1 to the estimation above, we obtain

$$J_1 \approx \sup_{0<z<\infty} \left(\int_0^z v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \left(\left(\int_0^z w^{1-r'}(t) dt \right)^{\frac{1}{r'}} \left(\int_z^\infty K^q(x,z) u(x) dx \right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(\int_z^\infty \left(\int_t^\infty K^q(x,z) u(x) dx \right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s) ds \right)^{\frac{\beta}{q'}} w^{1-r'}(x) dt \right)^{\frac{1}{\beta}} \right) \approx \max\{B_1^-, B_2^-\}. \quad (16)$$

Similarly, J_2 can be proved. In this case, Theorem A (ii) applies to (14), then

$$J_2 \approx \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s) v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \times$$

$$\times \left(\int_0^\infty \left(\int_t^\infty \chi_{(z,\infty)}(x) \mathcal{T}^q(x) u(x) dx \right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s) ds \right)^{\frac{\beta}{q'}} w^{1-r'}(t) dt \right)^{\frac{1}{\beta}}$$

$$= \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,s) v^{1-p'}(s) ds \right)^{\frac{1}{p'}} \left(\left(\int_0^z w^{1-r'}(t) dt \right)^{\frac{1}{r'}} \left(\int_z^\infty \mathcal{T}^q(x) u(x) dx \right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(\int_z^\infty \left(\int_t^\infty \mathcal{T}^q(x) u(x) dx \right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s) ds \right)^{\frac{\beta}{q'}} w^{1-r'}(x) dt \right)^{\frac{1}{\beta}} \right) \approx \max\{B_3^-, B_4^-\}. \quad (17)$$

Summing up, from (11) (16) and (17) we have the estimate $C \approx \max\{B_1^-, B_2^-, B_3^-, B_4^-\}$.

Theorem 5 *Let $1 < r \le q < p < \infty$ and assume that $K(x,t)$ of the operator (2) belongs to the class $\mathcal{O}_1^-$. Then the inequality (5) holds if and only if the following quantities*

$$M_1^- = \sup_{0<z<\infty} \left(\int_0^z w^{1-r'}(s) ds \right)^{\frac{1}{r'}} \left(\int_0^z \left(\int_z^\infty K^q(x,t) u(x) dx \right)^{\frac{\alpha}{q}} \left(\int_0^t v^{1-p'}(s) ds \right)^{\frac{\alpha}{q'}} v^{1-p'}(t) dt \right)^{\frac{1}{\alpha}},$$

$$M_2^- = \sup_{0<z<\infty} \left(\int_0^z w^{1-r'}(s) ds \right)^{\frac{1}{r'}} \left(\int_z^\infty \left(\int_t^\infty K^q(x,t) u(x) dx \right)^{\frac{\alpha}{q}} \left(\int_0^t v^{1-p'}(s) ds \right)^{\frac{\alpha}{q'}} v^{1-p'}(t) dt \right)^{\frac{1}{\alpha}},$$

$$M_3^- = \sup_{0<z<\infty} \left(\int_0^z w^{1-r'}(s) ds \right)^{\frac{1}{r'}} \left(\int_z^\infty \left(\int_t^\infty \mathcal{T}^q(x) u(x) dx \right)^{\frac{\alpha}{p}} \left(\int_0^t Q^{p'}(t,s) v^{1-p'}(s) ds \right)^{\frac{\alpha}{p'}} \mathcal{T}^q(t) u(t) dt \right)^{\frac{1}{\alpha}}$$

are finite. Moreover, $C \approx \max\{M_1^-, M_2^-, M_3^-\}$, where C is the sharp constant in (5).

Theorem 6 *Let $1 < r \le q < p < \infty$ and assume that $K(x,t)$ of the operator (2) belongs to the class $\mathcal{O}_1^+$. Then the inequality (5) holds if and only if the following quantities*

$$M_1^+ = \sup_{0<z<\infty} \left(\int_0^z w^{1-r'}(s) ds \right)^{\frac{1}{r'}} \left(\int_z^\infty \left(\int_t^\infty u(x) dx \right)^{\frac{\alpha}{p}} \left(\int_0^t K^{p'}(t,s) v^{1-p'}(s) ds \right)^{\frac{\alpha}{p'}} u(t) dt \right)^{\frac{1}{\alpha}},$$

$$M_2^+ = \sup_{0<z<\infty} \left(\int_0^z w^{1-r'}(s) ds \right)^{\frac{1}{r'}} \left(\int_0^z \left(\int_z^\infty R^q(x,t) u(x) dx \right)^{\frac{\alpha}{q}} \left(\int_0^t T^{p'}(s) v^{1-p'}(s) ds \right)^{\frac{\alpha}{q'}} T^{p'}(t) v^{1-p'}(t) dt \right)^{\frac{1}{\alpha}},$$

$$M_3^+ = \sup_{0<z<\infty} \left(\int_0^z w^{1-r'}(s) ds \right)^{\frac{1}{r'}} \left(\int_z^\infty \left(\int_t^\infty R^q(x,t) u(x) dx \right)^{\frac{\alpha}{q}} \left(\int_0^t T^{p'}(s) v^{1-p'}(s) ds \right)^{\frac{\alpha}{q'}} T^{p'}(t) v^{1-p'}(t) dt \right)^{\frac{1}{\alpha}}$$

are finite. Moreover, $C \approx \max\{M_1^+, M_2^+, M_3^+\}$, where C is the sharp constant in (5).

Proof of Theorem 5. In the proof, Remark 1 will be applied as necessary. Applying Theorem E to (10), it follows that

$$C \approx \max \left\{ \sup_{g \ge 0} \|g\|_{r,w}^{-1} \left(\int_0^\infty \left(\int_t^\infty K^q(x,t) \bar{u}(x) dx \right)^{\frac{\alpha}{q}} \left(\int_0^t v^{1-p'}(s) ds \right)^{\frac{\alpha}{q'}} v^{1-p'}(t) dt \right)^{\frac{1}{\alpha}}, \right.$$

$$\sup_{g\geq 0}\|g\|_{r,w}^{-1}\left(\int_0^\infty\left(\int_0^t Q^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}\left(\int_t^\infty \mathcal{T}^q(x)\bar{u}(x)dx\right)^{\frac{\alpha}{p}}\mathcal{T}^q(t)\bar{u}(t)dt\right)^{\frac{1}{\alpha}}\Bigg\}$$

$$=\max\{I_1,I_2\}. \tag{18}$$

Now we estimate each one in (18) separately and denote $V(t)=\left(\int_0^t v^{1-p'}(s)ds\right)^{\frac{\alpha}{q'}}v^{1-p'}(t)$. It is obvious that $\bar{p}:=\frac{\alpha}{q}>1$. Moreover, we can use (9) to I_1, we find

$$I_1=\sup_{g\geq 0}\|g\|_{r,w}^{-1}\times$$

$$\times\left(\sup_{h\geq 0}\int_0^\infty h(t)\int_t^\infty K^q(x,t)u(x)\left(\int_0^x g(s)ds\right)^q dxdt\left(\int_0^\infty h^{\bar{p}'}(s)V^{1-\bar{p}'}(s)ds\right)^{-\frac{1}{\bar{p}'}}\right)^{\frac{1}{q}}.$$

By changing order of supremum and using Fubini's theorem, we deduce

$$I_1=\sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(s)V^{1-\bar{p}'}(s)ds\right)^{-\frac{1}{\bar{p}'q}}\times$$

$$\times\sup_{g\geq 0}\|g\|_{r,w}^{-1}\left(\int_0^\infty u(x)\left(\int_0^x g(s)ds\right)^q\int_0^x K^q(x,t)h(t)dtdx\right)^{\frac{1}{q}}. \tag{19}$$

We first apply Theorem A (i), and then proceed by substituting the characteristic function.

$$I_1\approx\sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(s)V^{1-\bar{p}'}(s)ds\right)^{-\frac{1}{\bar{p}'q}}\sup_{z>0}\left(\int_z^\infty u(x)\int_0^x K^q(x,t)h(t)dtdx\right)^{\frac{1}{q}}\times$$

$$\times\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}=\sup_{z>0}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}\left(\sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(s)V^{1-\bar{p}'}(s)ds\right)^{-\frac{1}{\bar{p}'}}\times\right.$$

$$\left.\times\int_0^\infty \chi_{(z,\infty)}(x)u(x)\int_0^x K^q(x,t)h(t)dtdx\right)^{\frac{1}{q}}=\sup_{z>0}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}\times$$

$$\times\left(\sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(s)V^{1-\bar{p}'}(s)ds\right)^{-\frac{1}{\bar{p}'}}\int_0^\infty h(t)\int_t^\infty \chi_{(z,\infty)}(x)K^q(x,t)u(x)dxdt\right)^{\frac{1}{q}}.$$

We apply equation (9) once more to the previous estimate and, after updating the notation, we obtain

$$I_1\approx\sup_{z>0}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}\left(\int_0^\infty V(t)\left(\int_t^\infty \chi_{(z,\infty)}(x)K^q(x,t)u(x)dx\right)^{\frac{\alpha}{q}}dt\right)^{\frac{1}{\alpha}}$$

$$\approx \sup_{z>0}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}\left\{\left(\int_0^z V(t)\left(\int_z^\infty K^q(x,t)u(x)dx\right)^{\frac{\alpha}{q}}dt\right)^{\frac{1}{\alpha}}+\right.$$

$$\left.+\left(\int_z^\infty V(t)\left(\int_t^\infty K^q(x,t)u(x)dx\right)^{\frac{\alpha}{q}}dt\right)^{\frac{1}{\alpha}}\right\}\approx \max\{M_1^-,M_2^-\}. \tag{20}$$

Now, we consider the following quantity:

$$S=\left(\int_0^\infty\left(\int_0^t Q^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}\left(\int_t^\infty \mathcal{T}^q(x)\bar{u}(x)dx\right)^{\frac{\alpha}{p}}\mathcal{T}^q(t)\bar{u}(t)dt\right)^{\frac{1}{\alpha}}.$$

S can be rewritten as

$$S\approx\left(\int_0^\infty\left(\int_0^t Q^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}d\left(-\left(\int_t^\infty \mathcal{T}^q(x)\bar{u}(x)dx\right)^{\frac{\alpha}{q}}\right)\right)^{\frac{1}{\alpha}}$$

$$=\left(\int_0^\infty\int_0^t d\left(\int_0^z Q^{p'}(z,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}d\left(-\left(\int_t^\infty \mathcal{T}^q(x)\bar{u}(x)dx\right)^{\frac{\alpha}{q}}\right)\right)^{\frac{1}{\alpha}}$$

$$=\left(\int_0^\infty\int_z^\infty d\left(-\left(\int_t^\infty \mathcal{T}^q(x)\bar{u}(x)dx\right)^{\frac{\alpha}{q}}\right)d\left(\int_0^z Q^{p'}(z,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}\right)^{\frac{1}{\alpha}}$$

$$=\left(\int_0^\infty\left(\int_z^\infty \mathcal{T}^q(x)\bar{u}(x)dx\right)^{\frac{\alpha}{q}}d\left(\int_0^z Q^{p'}(z,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}\right)^{\frac{1}{\alpha}}.$$

Let $\mu(z)=\left(\int_0^z Q^{p'}(z,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}$, then

$$S\approx\left[\left(\int_0^\infty\left(\int_z^\infty \mathcal{T}^q(x)\bar{u}(x)dx\right)^{\frac{\alpha}{q}}d\mu(z)\right)^{\frac{q}{\alpha}}\right]^{\frac{1}{q}}$$

From the last estimate we rewrite I_2 as follows:

$$I_2\approx\sup_{g\geq0}\|g\|_{r,w}^{-1}\left[\left(\int_0^\infty\left(\int_z^\infty \mathcal{T}^q(x)u(x)\left(\int_0^x g(s)ds\right)^q dx\right)^{\frac{\alpha}{q}}d\mu(z)\right)^{\frac{q}{\alpha}}\right]^{\frac{1}{q}}.$$

Since $\bar{p}=\frac{\alpha}{q}>1$, we apply (9) to previous expression, we find

$$I_2\approx\sup_{g\geq0}\|g\|_{r,w}^{-1}\sup_{h\geq0}\left(\int_0^\infty h^{\bar{p}'}(t)d\mu(t)\right)^{-\frac{1}{\bar{p}'q}}\left(\int_0^\infty h(t)\int_t^\infty \mathcal{T}^q(x)u(x)\left(\int_0^x g(s)ds\right)^q dxd\mu(t)\right)^{\frac{1}{q}}$$

$$= \sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(t)d\mu(t)\right)^{-\frac{1}{\bar{p}'q}} \sup_{g\geq 0}\|g\|_{r,w}^{-1}\left(\int_0^\infty\left(\int_0^x g(s)ds\right)^q \mathcal{T}^q(x)u(x)\int_0^x h(t)d\mu(t)dx\right)^{\frac{1}{q}}, \tag{21}$$

by Theorem A (i)

$$I_2 \approx \sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(t)d\mu(t)\right)^{-\frac{1}{\bar{p}'q}} \sup_{z>0}\left(\int_z^\infty \mathcal{T}^q(x)u(x)\int_0^x h(t)d\mu(t)dx\right)^{\frac{1}{q}}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}} =$$

$$\sup_{z>0}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}\left(\sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(t)d\mu(t)\right)^{-\frac{1}{\bar{p}'}}\int_0^\infty h(t)\int_t^\infty \chi_{(z,\infty)}(x)\mathcal{T}^q(x)u(x)dxd\mu(t)\right)^{\frac{1}{q}}.$$

By applying (9) again, we colculate the following way

$$I_2 \approx \sup_{z>0}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}\left(\int_0^\infty\left(\int_t^\infty \chi_{(z,\infty)}(x)\mathcal{T}^q(x)u(x)dx\right)^{\frac{\alpha}{q}}d\mu(t)\right)^{\frac{1}{\alpha}}$$

$$\approx \sup_{z>0}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}\left(\int_0^\infty\int_t^\infty \chi_{(z,\infty)}(x)\mathcal{T}^q(x)u(x)\left(\int_x^\infty \chi_{(z,\infty)}(s)\mathcal{T}^q(s)u(s)ds\right)^{\frac{\alpha}{p}}dxd\mu(t)\right)^{\frac{1}{\alpha}}$$

$$= \sup_{z>0}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}\left(\int_0^\infty \chi_{(z,\infty)}(x)\mathcal{T}^q(x)u(x)\left(\int_x^\infty \chi_{(z,\infty)}(s)\mathcal{T}^q(s)u(s)ds\right)^{\frac{\alpha}{p}}\int_0^x d\mu(t)dx\right)^{\frac{1}{\alpha}}$$

$$\approx \sup_{z>0}\left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}}\left(\int_z^\infty \mathcal{T}^q(x)u(x)\left(\int_x^\infty \mathcal{T}^q(s)u(s)dx\right)^{\frac{\alpha}{p}}\int_0^x d\left(\int_0^t Q^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}dx\right)^{\frac{1}{\alpha}}$$

$$= M_3^-. \tag{22}$$

According to (18), (20) and (22), it follows $C \approx \max\{M_1^-, M_2^-, M_3^-\}$.

3.3 The results for the case $1 < q < \min\{p, r\} < \infty$.

Theorem 7 *Let $1 < q < \min\{p, r\} < \infty$ and assume that $K(x,t)$ of the operator (2) belongs to the class $\mathcal{O}_1^-$.*
(i) If $1 < p \leqslant \beta$, then the inequality (5) holds if and only if the following quantities

$$F_1^- = \sup_{0<z<\infty}\left(\int_0^z w^{1-r'}(t)dt\right)^{\frac{1}{r'}}\left(\int_z^\infty\left(\int_x^\infty K^q(t,x)u(t)dt\right)^{\frac{\alpha}{q}}\left(\int_0^x v^{1-p'}(s)ds\right)^{\frac{\alpha}{q'}}v^{1-p'}(x)dx\right)^{\frac{1}{\alpha}},$$

$$F_2^- = \sup_{0<z<\infty}\left(\int_0^z v^{1-p'}(t)dt\right)^{\frac{1}{p'}}\left(\int_z^\infty\left(\int_x^\infty K^q(t,z)u(t)dt\right)^{\frac{\beta}{q}}\left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}}w^{1-r'}(x)dx\right)^{\frac{1}{\beta}},$$

$$F_3^- = \sup_{0<z<\infty} \left(\int_0^z Q^{p'}(z,t)v^{1-p'}(t)dt\right)^{\frac{1}{p'}} \left(\int_z^\infty \left(\int_x^\infty \mathcal{T}^q(t)u(t)dt\right)^{\frac{\beta}{q}} \left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx\right)^{\frac{1}{\beta}},$$

$$F_4^- = \sup_{0<z<\infty} \left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_z^\infty \left(\int_x^\infty \mathcal{T}^q(t)u(t)dt\right)^{\frac{\alpha}{q}} d\left(\int_0^x Q^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}\right)^{\frac{1}{\alpha}}$$

are finite.

(ii) If $1<\beta<p$ then the inequality (5) holds if and only if the following quantities

$$\bar{F}_1^- = \left[\int_0^\infty \left(\int_x^\infty \left(\int_t^\infty K^q(s,t)u(s)ds\right)^{\frac{\alpha}{q}} \left(\int_0^t v^{1-p'}(s)ds\right)^{\frac{\alpha}{q'}} v^{1-p'}(t)dt\right)^{\frac{r}{r-\alpha}} \left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{r(\alpha-1)}{r-\alpha}} w^{1-r'}(x)dx\right]^{\frac{r-\alpha}{\alpha r}},$$

$$\bar{F}_2^- = \left[\int_0^\infty \left(\int_t^\infty \left(\int_x^\infty K^q(s,t)u(s)ds\right)^{\frac{\beta}{q}} \left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx\right)^{\frac{p}{p-\beta}} \left(\int_0^t v^{1-p'}(s)ds\right)^{\frac{p(\beta-1)}{p-\beta}} v^{1-p'}(t)dt\right]^{\frac{p-\beta}{p\beta}},$$

$$\bar{F}_3^- = \left[\int_0^\infty \left(\int_t^\infty \left(\int_x^\infty \mathcal{T}^q(s)u(s)ds\right)^{\frac{\beta}{q}} \left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx\right)^{\frac{\beta}{p-\beta}} \left(\int_0^t Q^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\beta(p-1)}{p-\beta}} \times\right.$$

$$\left.\left(\int_t^\infty \mathcal{T}^q(s)u(s)ds\right)^{\frac{\beta}{q}} \left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(t)dt\right]^{\frac{p-\beta}{p\beta}},$$

$$\bar{F}_4^- = \left[\int_0^\infty \left(\int_z^\infty \left(\int_t^\infty \mathcal{T}^q(s)u(s)ds\right)^{\frac{\alpha}{q}} d\left(\int_0^t Q^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}\right)^{\frac{r}{r-\alpha}} \left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{r(\alpha-1)}{r-\alpha}} w^{1-r'}(z)dz\right]^{\frac{r-\alpha}{\alpha r}}$$

are finite. Moreover, $C \approx \max\{F_1^-, F_2^-, F_3^-, F_4^-\}$ in case (i) and $C \approx \max\{\bar{F}_1^-, \bar{F}_2^-, \bar{F}_3^-, \bar{F}_4^-\}$ in case (ii), where C is the sharp constant in (5).

Theorem 8 *Let $1<q<\min\{p,r\}<\infty$ and assume that $K(x,t)$ of the operator (2) belongs to the class $\mathcal{O}_1^+$.*

(i) If $1<p\leqslant\beta$, then the inequality (5) holds if and only if the following quantities

$$F_1^+ = \sup_{0<z<\infty} \left(\int_0^z w^{1-r'}(t)dt\right)^{\frac{1}{r'}} \left(\int_z^\infty \left(\int_x^\infty R^q(t,x)u(t)dt\right)^{\frac{\alpha}{q}} \left(\int_0^x T^{p'}(s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{q'}} T^{p'}(x)v^{1-p'}(x)dx\right)^{\frac{1}{\alpha}},$$

$$F_2^+ = \sup_{0<z<\infty} \left(\int_0^z T^{p'}(t)v^{1-p'}(t)dt\right)^{\frac{1}{p'}} \left(\int_z^\infty \left(\int_x^\infty R^q(t,z)u(t)dt\right)^{\frac{\beta}{q}} \left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx\right)^{\frac{1}{\beta}},$$

$$F_3^+ = \sup_{0<z<\infty} \left(\int_0^z K^{p'}(z,t)v^{1-p'}(t)dt\right)^{\frac{1}{p'}} \left(\int_z^\infty \left(\int_x^\infty u(t)dt\right)^{\frac{\beta}{q}} \left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx\right)^{\frac{1}{\beta}},$$

$$F_4^+ = \sup_{0<z<\infty} \left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{1}{r'}} \left(\int_z^\infty \left(\int_x^\infty u(t)dt\right)^{\frac{\alpha}{q}} d\left(\int_0^x K^{p'}(x,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}\right)^{\frac{1}{\alpha}}$$

are finite.

(ii) If $1<\beta<p$ then the inequality (5) holds if and only if the following quantities

$$\bar{F}_1^+ = \left[\int_0^\infty \left(\int_x^\infty \left(\int_t^\infty R^q(s,t)u(s)ds\right)^{\frac{\alpha}{q}} \left(\int_0^t T^{p'}(s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{q'}} T^{p'}(t)v^{1-p'}(t)dt\right)^{\frac{r}{r-\alpha}} \left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{r(\alpha-1)}{r-\alpha}} w^{1-r'}(x)dx\right]^{\frac{r-\alpha}{\alpha r}},$$

$$\bar{F}_2^+ = \left[\int\limits_0^\infty \left(\int\limits_t^\infty \left(\int\limits_x^\infty R^q(s,t)u(s)ds\right)^{\frac{\beta}{q}} \left(\int\limits_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx\right)^{\frac{p}{p-\beta}} \left(\int\limits_0^t T^{p'}(s)v^{1-p'}(s)ds\right)^{\frac{p(\beta-1)}{p-\beta}} T^{p'}(t)v^{1-p'}(t)dt\right]^{\frac{p-\beta}{p\beta}},$$

$$\bar{F}_3^+ = \left[\int\limits_0^\infty \left(\int\limits_t^\infty \left(\int\limits_x^\infty u(s)ds\right)^{\frac{\beta}{q}} \left(\int\limits_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx\right)^{\frac{\beta}{p-\beta}} \left(\int\limits_0^t K^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\beta(p-1)}{p-\beta}} \left(\int\limits_t^\infty u(s)ds\right)^{\frac{\beta}{q}} \times \right.$$

$$\left.\left(\int\limits_0^t w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(t)dt\right]^{\frac{p-\beta}{p\beta}},$$

$$\bar{F}_4^+ = \left[\int\limits_0^\infty \left(\int\limits_z^\infty \left(\int\limits_t^\infty u(s)ds\right)^{\frac{\alpha}{q}} d\left(\int\limits_0^t K^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}}\right)^{\frac{r}{r-\alpha}} \left(\int\limits_0^z w^{1-r'}(s)ds\right)^{\frac{r(\alpha-1)}{r-\alpha}} w^{1-r'}(z)dz\right]^{\frac{r-\alpha}{\alpha r}}$$

are finite. Moreover, $C \approx \max\{F_1^+, F_2^+, F_3^+, F_4^+\}$ *in case (i) and* $C \approx \max\{\bar{F}_1^+, \bar{F}_2^+, \bar{F}_3^+, \bar{F}_4^+\}$ *in case (ii), where* C *is the sharp constant in (5).*

Proof of Theorem 7. The proof of this theorem begins in the same way as Theorem 5. By applying Theorem A (ii) to (19), it follows that

$$I_1 \approx \sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(s)V^{1-\bar{p}'}(s)ds\right)^{-\frac{1}{\bar{p}'q}} \times$$

$$\left(\int_0^\infty \left(\int_z^\infty u(x)\int_0^x K^q(x,t)h(t)dtdx\right)^{\frac{\beta}{q}} \left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(z)dz\right)^{\frac{1}{\beta}}.$$

We define $W(z) = \left(\int_0^z w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(z)$. Next, we divide the range of the integral for two parts, then we conclude

$$I_1 \approx \sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(s)V^{1-\bar{p}'}(s)ds\right)^{-\frac{1}{\bar{p}'q}} \left[\left(\int_0^\infty \left(\int_z^\infty u(x)\int_0^z K^q(x,t)h(t)dtdx\right)^{\frac{\beta}{q}} W(z)dz\right)^{\frac{1}{\beta}} + \right.$$

$$\left.\left(\int_0^\infty \left(\int_z^\infty u(x)\int_z^x K^q(x,t)h(t)dtdx\right)^{\frac{\beta}{q}} W(z)dz\right)^{\frac{1}{\beta}}\right]$$

$$\approx \max\left\{\sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(s)V^{1-\bar{p}'}(s)ds\right)^{-\frac{1}{\bar{p}'q}} \left(\int_0^\infty \left(\int_z^\infty h(t)\int_t^\infty K^q(x,t)u(x)dxdt\right)^{\frac{\beta}{q}} W(z)dz\right)^{\frac{1}{\beta}},\right.$$

$$\left.\sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(s)V^{1-\bar{p}'}(s)ds\right)^{-\frac{1}{\bar{p}'q}} \left(\int_0^\infty \left(\int_0^z h(t)\int_z^\infty K^q(x,t)u(x)dxdt\right)^{\frac{\beta}{q}} W(z)dz\right)^{\frac{1}{\beta}}\right\}$$

$$= \max\{I_{11}, I_{12}\}. \tag{23}$$

Here, we have two possible cases:
i) $1 < \bar{p}' \leqslant \frac{\beta}{q} \Leftrightarrow 1 < p \leqslant \beta,;$ ii) $1 < \frac{\beta}{q} < \bar{p}' \Leftrightarrow 1 < \beta < p.$

In case (i), we apply Theorem B (i) to I_{11}.

$$I_{11} \approx \left[\sup_{0<z<\infty} \left(\int_z^\infty \left(\int_x^\infty K^q(t,x)u(t)dt \right)^{\bar{p}} V^{(1-\bar{p}')(1-\bar{p})}(x)dx \right)^{\frac{1}{\bar{p}}} \left(\int_0^z W(t)dt \right)^{\frac{q}{\beta}} \right]^{\frac{1}{q}} \approx$$

$$\sup_{0<z<\infty} \left(\int_z^\infty \left(\int_x^\infty K^q(t,x)u(t)dt \right)^{\frac{\alpha}{q}} \left(\int_0^x v^{1-p'}(s)ds \right)^{\frac{\alpha}{q'}} v^{1-p'}(x)dx \right)^{\frac{1}{\alpha}} \left(\int_0^z w^{1-r'}(t)dt \right)^{\frac{1}{r'}} = F_1^-. \tag{24}$$

In case (ii), Theorem B (ii) can be used to estimate I_{11}.

$$I_{11} \approx \left[\int_0^\infty \left(\int_x^\infty \left(\int_t^\infty K^q(s,t)u(s)ds \right)^{\bar{p}} V^{(1-\bar{p}')(1-\bar{p})}(t)dt \right)^{\frac{\beta/q(\bar{p}'-1)}{(\bar{p}'-\beta/q)}} \left(\int_0^x W(s)ds \right)^{\frac{\beta/q}{\bar{p}'-\beta/q}} \times \right.$$

$$\left. W(x)dx \right]^{\frac{\bar{p}'-\beta/q}{\bar{p}'\beta/q}\frac{1}{q}} \approx \left[\int_0^\infty \left(\int_x^\infty \left(\int_t^\infty K^q(s,t)u(s)ds \right)^{\frac{\alpha}{q}} \left(\int_0^t v^{1-p'}(s)ds \right)^{\frac{\alpha}{q'}} v^{1-p'}(t)dt \right)^{\frac{r}{r-\alpha}} \times \right.$$

$$\left. \left(\int_0^x w^{1-r'}(s)ds \right)^{\frac{r(\alpha-1)}{r-\alpha}} w^{1-r'}(x)dx \right]^{\frac{r-\alpha}{\alpha r}} = \bar{F}_1^-. \tag{25}$$

There exists a kernel $\bar{K}(z,t) := \int_z^\infty K^q(x,t)u(x)dx$ in I_{12}. Given the relation $0 \le t \le s \le z$, it can be deduced that $0 \le t \le s \le x$. According to Definition 1, the kernel $\bar{K}(z,t)$ satisfies

$$\bar{K}(z,t) = \int_z^\infty K^q(x,t)u(x)dx \approx \int_z^\infty K^q(x,s)u(x)dx + Q^q(s,t) \int_z^\infty \mathcal{T}^q(x)u(x)dx$$
$$= \bar{K}(z,s) + \bar{Q}(s,t)\bar{U}(z).$$

It indicates that the the kernel $\bar{K}(z,t)$ satisfies condition of class $\mathcal{O}_1^-$. Therefore, I_{12} can be estimated using Theorem G in the following way for case (i):

$$I_{12} \approx \max \left\{ \left[\sup_{0<z<\infty} \left(\int_z^\infty \left(\int_x^\infty K^q(t,z)u(t)dt \right)^{\frac{\beta}{q}} \left(\int_0^x w^{1-r'}(s)ds \right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx \right)^{\frac{1}{\beta}} \right. \right.$$

$$\left(\int_0^z v^{1-p'}(t)dt \right)^{\frac{1}{p'}}, \sup_{0<z<\infty} \left(\int_z^\infty \left(\int_x^\infty \mathcal{T}^q(s)u(s)ds \right)^{\frac{\beta}{q}} \left(\int_0^x w^{1-r'}(s)ds \right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx \right)^{\frac{1}{\beta}}$$

$$\left. \left. \left(\int_0^z \left(\int_0^t v^{1-p'}(s)ds \right)^{\frac{\alpha}{q'}} Q^\alpha(z,t) v^{1-p'}(t)dt \right)^{\frac{1}{\alpha}} \right] \right\} = \max\{F_2^-, F_5^-\}. \tag{26}$$

In case (ii), by Theorem E

$$I_{12} \approx \max\left\{\left[\int_0^\infty \left(\int_t^\infty \left(\int_x^\infty K^q(s,t)u(s)ds\right)^{\frac{\beta}{q}} \left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx\right)^{\frac{p}{p-\beta}} \times\right.\right.$$

$$\left.\left(\int_0^t v^{1-p'}(s)ds\right)^{\frac{p(\beta-1)}{p-\beta}} v^{1-p'}(t)dt\right]^{\frac{p-\beta}{p\beta}}, \left[\int_0^\infty \left(\int_0^t \left(\int_0^s v^{1-p'}(s)ds\right)^{\frac{\alpha}{q'}} Q^\alpha(t,s)v^{1-p'}(s)ds\right)^{\frac{r}{r-\alpha}} \times\right.$$

$$\left(\int_t^\infty \left(\int_x^\infty \mathcal{T}^q(s)u(s)ds\right)^{\frac{\beta}{q}} \left(\int_0^x w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(x)dx\right)^{\frac{\beta}{p-\beta}} \left(\int_t^\infty \mathcal{T}^q(s)u(s)ds\right)^{\frac{\beta}{q}} \times$$

$$\left.\left.\left(\int_0^t w^{1-r'}(s)ds\right)^{\frac{\beta}{q'}} w^{1-r'}(t)dt\right]^{\frac{p-\beta}{p\beta}}\right\} = \max\{\bar{F}_2^-, \bar{F}_5^-\}. \tag{27}$$

Next we estimate I_2 in (21). According to Theorem A (ii), we obtain

$$I_2 \approx \sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(t)d\mu(t)\right)^{-\frac{1}{\bar{p}'q}} \left(\int_0^\infty \left(\int_x^\infty \mathcal{T}^q(t)u(t)\int_0^t h(s)d\mu(s)dt\right)^{\frac{\beta}{q}} W(x)dx\right)^{\frac{1}{\beta}}$$

$$\approx \max\left\{\left[\sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(t)d\mu(t)\right)^{-\frac{1}{\bar{p}'}} \left(\int_0^\infty \left(\int_0^x h(s)d\mu(s)\right)^{\frac{\beta}{q}} \left(\int_x^\infty \mathcal{T}^q(t)u(t)dt\right)^{\frac{\beta}{q}} W(x)dx\right)^{\frac{q}{\beta}}\right]^{\frac{1}{q}},\right.$$

$$\left.\left[\sup_{h\geq 0}\left(\int_0^\infty h^{\bar{p}'}(t)d\mu(t)\right)^{-\frac{1}{\bar{p}'}} \left(\int_0^\infty \left(\int_x^\infty h(s)\int_s^\infty \mathcal{T}^q(t)u(t)dtd\mu(s)\right)^{\frac{\beta}{q}} W(x)dx\right)^{\frac{q}{\beta}}\right]^{\frac{1}{q}}\right\}$$

$$= \max\{I_{21}, I_{22}\}. \tag{28}$$

Here, both case (i) and case (ii) will be also considered. In case (i) we apply Theorem A (i) and Theorem B (i) to I_{21} and I_{22}, respectively.

$$I_{21} \approx \left[\sup_{0<z<\infty}\left(\int_0^z d\mu(t)\right)^{\frac{1}{\bar{p}}} \left(\int_z^\infty \left(\int_x^\infty \mathcal{T}^q(t)u(t)dt\right)^{\frac{\beta}{q}} W(x)dx\right)^{\frac{q}{\beta}}\right]^{\frac{1}{q}} = F_3^- \tag{29}$$

$$I_{22} \approx \left[\sup_{0<z<\infty}\left(\int_0^z W(s)ds\right)^{\frac{q}{\beta}} \left(\int_z^\infty \left(\int_x^\infty \mathcal{T}^q(t)u(t)dt\right)^{\bar{p}} d\mu(x)\right)^{\frac{1}{\bar{p}}}\right]^{\frac{1}{q}} \approx F_4^-. \tag{30}$$

In case (ii), application of Theorem A (ii) to I_{21} and Theorem B (ii) to I_{22} yields that

$$\max\{I_{21}, I_{22}\} \approx \max\{\bar{F}_3^-, \bar{F}_4^-\}. \tag{31}$$

The inequality

$$\int_0^t \left(\int_0^s v^{1-p'}(z)dz\right)^{\frac{\alpha}{q'}} Q^\alpha(t,s)v^{1-p'}(t)dt \le \left(\int_0^t Q^{p'}(t,s)v^{1-p'}(s)ds\right)^{\frac{\alpha}{p'}} \tag{32}$$

follows from the fact that $\alpha = p' + p'\frac{\alpha}{q'}$ and $Q(t,s) \le Q(t,z)$ for $0 < z \le s \le t < \infty$, as well as the application of Lemma 1.

The inequality (32) allows us to compare $\bar{F}_3^- \le \bar{F}_5^-$ and $F_3^- \le F_5^-$. Therefore, the letter and (18),(19), (23)-(31) give that $C \approx \max\{F_1^-, F_2^-, F_3^-, F_4^-\}$ in case (i),

$C \approx \max\{\bar{F}_1^-, \bar{F}_2^-, \bar{F}_3^-, \bar{F}_4^-\}$ in case (ii).

4 Conclusion

Hardy-type operators are typically first examined without a kernel and subsequently with a kernel that satisfies specific conditions, that is, wider results are obtained than in the case without a kernel. The criteria boundedness of operator for general kernel classes are particularly valuable. The inequality (5) in [11] was previously studied for bilinear operators without a kernel. The present work extends the research initiated in [11], introducing a new direction in operator theory. At the same time, we applied the method presented in this work to derive the main results. Specifically, necessary and sufficient conditions for the validity of the integral bilinear inequality (5), involving a kernel operator from the classes $\mathcal{O}_1^-$ and $\mathcal{O}_1^+$, are obtained. Future investigations will address bilinear integral operators with kernels from wider classes than $\mathcal{O}_1^+$ and $\mathcal{O}_1^-$, considering various cases for the parameters p, q, and r.

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Authors contribution

All authors contributed equally to writing, reviewing, and editing the manuscript and approved the final version.

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