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# MULTIPOINT BOUNDARY VALUE PROBLEMS FOR IMPULSIVE HYPERBOLIC EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

This paper investigates a multipoint boundary value problem for an impulsive hyperbolic equation with a generalized piecewise constant argument. The proposed formulation simultaneously incorporates impulsive effects, a generalized piecewise constant argument, and multipoint boundary conditions, making the solvability analysis substantially more challenging than that of previously studied models. The relevance of the study is motivated by the need to develop adequate mathematical models for processes combining continuous dynamics with discrete information updating and instantaneous control actions. To investigate the problem, Dzhumabayev's parameterization method is employed, reducing the original boundary value problem to an equivalent family of parameterized Cauchy problems together with a system of functional equations for the unknown functional parameters. Sufficient conditions for the existence and uniqueness of the solution are established under the invertibility of the constructed parameter matrix. Furthermore, a constructive iterative algorithm for the sequential construction of the solution is developed, providing a theoretical basis for its subsequent numerical implementation. The scientific novelty of the paper lies in extending the parameterization method to a new class of multipoint boundary value problems for impulsive hyperbolic equations with generalized piecewise constant arguments. The principal advantage of the proposed approach is its ability to investigate, within a unified mathematical framework, problems involving impulsive effects, generalized piecewise constant arguments, and multipoint boundary conditions simultaneously. The obtained results can be applied to mathematical modeling of automatic control systems, heat transfer processes, and technological systems involving discrete information updating and impulsive control actions.

**Keywords:** hyperbolic equation, impulsive effect, piecewise constant argument, multipoint boundary value problem, parameterization method.

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## Бөлікті-тұрақты аргументті импульсті гиперболалық теңдеулер үшін көпнүктелік шеттік есептер

Жұмыста жалпыланған бөлікті-тұрақты аргументі бар импульстік гиперболалық теңдеу үшін қойылған көпнүктелі шеттік есеп қарастырылады. Есепте импульстік әсерлерді, жалпыланған бөлікті-тұрақты аргументті және көпнүктелі шеттік шарттарды бір мезгілде қамтиды, бұл бұрын зерттелген модельдермен салыстырғанда есептің шешімділігін талдауды едәуір күрделендіреді. Зерттеудің өзектілігі үздіксіз динамикасы дискретті ақпараттың жаңартылуымен және лездік басқару әсерлерімен ұштасатын процестердің барабар математикалық модельдерін құру қажеттілігімен анықталады. Зерттеу барысында Д. С. Джұмабаевтың параметрлеу әдісі қолданылып, бастапқы шеттік есеп параметрленген Коши есептерінің эквивалентті жүйесіне және функционалдық параметрлерге қатысты функционалдық теңдеулер жүйесіне келтірілді. Құрылған параметрлер матрицасының қайтарылымдылығы негізінде шешімнің бар болуы мен бірмәнділігінің жеткілікті шарттары алынды. Сонымен қатар шешімді кезең-кезеңімен құруға мүмкіндік беретін конструктивті итерациялық алгоритм ұсы-

нылып, оның кейінгі сандық жүзеге асырылуына теориялық негіз қаланды. Жұмыстың ғылыми жаңалығы параметрлеу әдісін жалпыланған бөлікті-тұрақты аргументі бар импульстік гиперболалық теңдеулер үшін қойылған көпнүктелі шеттік есептердің жаңа класына қолдануымен анықталады. Ұсынылған тәсілдің негізгі артықшылығы импульстік әсерлерді, жалпыланған бөлікті-тұрақты аргументті және көпнүктелі шеттік шарттарды бірыңғай математикалық модель аясында қатар зерттеуге мүмкіндік беруінде. Алынған нәтижелер автоматты басқару, жылу алмасу және технологиялық процестерде кездесетін дискретті ақпарат жаңартылатын әрі импульстік басқару жүзеге асырылатын жүйелерді математикалық модельдеуде қолданылуы мүмкін.
**Түйін сөздер:** гиперболалық теңдеу, импульстік әсер, бөлікті-тұрақты аргумент, көпнүктелік шеттік есеп, параметрлеу әдісі.

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## Многоточечные краевые задачи для импульсных гиперболических уравнений с кусочно-постоянными аргументами

В работе исследуется многоточечная краевая задача для импульсного гиперболического уравнения с обобщенным частично-постоянным аргументом. Рассматриваемая постановка объединяет импульсные воздействия, обобщенный частично-постоянный аргумент и многоточечные краевые условия, что существенно усложняет анализ разрешимости задачи по сравнению с ранее исследованными моделями. Актуальность исследования обусловлена необходимостью построения адекватных математических моделей процессов, в которых непрерывная динамика сочетается с дискретным обновлением информации и мгновенными управляющими воздействиями. Для исследования используется метод параметризации Д. С. Джумабаева, позволяющий свести исходную краевую задачу к эквивалентному семейству параметризованных задач Коши и системе функциональных уравнений относительно функциональных параметров. Получены достаточные условия существования и единственности решения, основанные на обратимости построенной матрицы параметров. Разработан итерационный алгоритм последовательного построения решения, который обладает конструктивным характером и может служить основой для дальнейшей численной реализации. Научная новизна работы заключается в распространении метода параметризации на класс многоточечных краевых задач для импульсных гиперболических уравнений с обобщенным частично-постоянным аргументом. Основным преимуществом предложенного подхода является возможность исследования в рамках единой математической схемы задач, одновременно содержащих импульсные воздействия, обобщенный частично-постоянный аргумент и многоточечные краевые условия. Полученные результаты могут быть использованы при математическом моделировании процессов с дискретным обновлением информации и импульсным управлением в задачах автоматического управления, теплообмена и технологических процессов.
**Ключевые слова:** гиперболическое уравнение, импульсное воздействие, кусочно-постоянный аргумент, многоточечная краевая задача, метод параметризации.

## 1 Introduction

Many real-world processes in modern natural sciences, engineering, and economics possess a complex dynamical structure. These processes evolve continuously in time and space while simultaneously undergoing instantaneous (impulsive) changes and discrete information updates. Such hybrid phenomena cannot be adequately described within the framework of classical differential equations. In this regard, impulsive differential equations and equations with generalized piecewise constant arguments have become effective mathematical models for describing systems with continuous and discrete dynamics [1], [2].

The qualitative theory of differential equations with generalized piecewise constant arguments has been extensively developed in recent years. Important results on integral manifolds and almost periodic solutions were obtained by Akhmet [7, 8], while Green's functions for boundary value problems with piecewise constant arguments were constructed by Nieto and Rodriguez-Lopez [9]. Some recent results concerning related classes of differential equations and boundary value problems are presented in [17, 18]. The parameterization method proposed by Dzhumabayev [4] provides an efficient approach for investigating boundary value problems by reducing them to equivalent families of Cauchy problems and systems of functional equations.

The parameterization method has been successfully applied to hyperbolic equations with generalized piecewise constant arguments [5] and to impulsive hyperbolic equations with discrete memory [3, 6]. In addition, the asymptotic behavior of differential equations with singular impulses has been investigated in [10]. However, multipoint boundary value problems for impulsive hyperbolic equations with generalized piecewise constant arguments have received considerably less attention. The simultaneous presence of impulsive effects, generalized piecewise constant arguments, and multipoint boundary conditions significantly complicates the analysis of solvability and requires the development of new analytical approaches [11], [12], [16].

The purpose of this paper is to investigate a multipoint boundary value problem for an impulsive hyperbolic equation with a generalized piecewise constant argument. By applying Dzhumabayev's parameterization method, the original boundary value problem is reduced to an equivalent family of parameterized Cauchy problems and a system of functional equations with respect to the unknown functional parameters. Sufficient conditions for the existence and uniqueness of the solution are established, and a constructive iterative algorithm for its successive approximation is developed. The obtained results extend the applicability of the parameterization method to a new class of multipoint boundary value problems and provide a theoretical basis for their numerical implementation.

## 2 Main results

Let $\Omega = [0, T] \times [0, \omega]$. In the domain $\Omega$, we consider the following boundary value problem for an impulsive hyperbolic equation with a generalized piecewise constant argument:

$$\frac{\partial^2 u}{\partial t \partial x} = A(t,x)\frac{\partial u(t,x)}{\partial x} + B(t,x)\frac{\partial u(t,x)}{\partial t} + C(t,x)\,u(t,x) + A_0(t,x)\frac{\partial u(\gamma(t),x)}{\partial x}$$
$$+ B_0(t,x)\frac{\partial u(\gamma(t),x)}{\partial t} + C_0(t,x)\,u(\gamma(t),x) + f(t,x), \quad t \neq \theta_j, \quad j = \overline{1, N-1}, \quad (1)$$

$$P_0(x)\frac{\partial u(0,x)}{\partial x} + \sum_{s=1}^{N-1} P_s(x)\frac{\partial u(\zeta_s,x)}{\partial x} + P_N(x)\frac{\partial u(T,x)}{\partial x} = \varphi_0(x), \quad x \in [0,\omega] \quad (2)$$

$$\lim_{t\to\theta_p+0}\frac{\partial u\left(t,x\right)}{\partial x}-\lim_{t\to\theta_p-0}\frac{\partial u\left(t,x\right)}{\partial x}=\ \varphi_p\left(x\right)\quad x\in\left[0,\omega\right],\quad p=\ \overline{1,N-1} \tag{3}$$

$$u\left(t,0\right)=\psi\left(t\right),\quad t\in\left[0,T\right]. \tag{4}$$

$\gamma\left(t\right)=\zeta_s$ if $t\in\left[\theta_{s-1},\theta_s\right)$, $s=\overline{1,N}$. $\theta_{s-1}<\zeta_s<\theta_s$, $s=1,2\ldots,N$; $0=\theta_0<\theta_1<\ldots<\theta_{N-1}<\theta_N=T$; Where $u(t,x)$ is the unknown function: $A(t,x),B(t,x),C(t,x)$, $A_0(t,x)$, $B_0(t,x),C_0\left(t,x\right)$ as well as the function $f(t,x)$, are continuous in the domain $\Omega$, $P_0\left(x\right)$, $P_i\left(x\right)$, $P_N\left(x\right)$ continuous in the domain $\Omega$; the functions $\varphi_p(x)$, $\varphi_0(x)$ are continuously differentiable on the interval $[0,\omega]$, where $p=\overline{1,N-1}$ and the function $\psi(t)$ is continuously differentiable on the interval $[0,T]$.

Let $\Delta_N(\omega)$ denote the partition of the domain $\Omega$ by the lines $t=\theta_s$. $\Omega_s=[\theta_{s-1},\theta_s)\times[0,\omega]$, $s=\overline{1,N},\Omega=\bigcup_{s=1}^{N}\Omega_s$ and $PC\left(\Omega,\{\theta_j\}_{j=1}^{N-1},\ \mathbb{R}\right)$ we denote the space of piecewise continuous functions $u(t,x)$ defined on the domain $\Omega$, which may have discontinuities at the points $t=\theta_j,j=\overline{1,N-1}$. The norm in this space is defined by $\|u\|_1=\max_{1\le s\le N}\sup_{(t,x)\in\Omega_s}|u(t,x)|$. A function $u(t,x)\in PC\big(\Omega,\{\theta_j\}_{j=1}^{N-1},\mathbb{R}\big)$ is called a solution of problem (1)–(4) provided that the following conditions hold:

(i) The function $u(t,x)$ admits the partial derivatives $\frac{\partial u(t,x)}{\partial x}\in PC\big(\Omega,\{\theta_j\}_{j=1}^{N-1},\mathbb{R}\big)$, $\frac{\partial u(t,x)}{\partial t}\in PC\big(\Omega,\{\theta_j\}_{j=1}^{N-1},\mathbb{R}\big)$.

(ii) The mixed partial derivative $\frac{\partial^2 u}{\partial t\partial x}\in PC\left(\Omega,\{\theta_j\}_{j=1}^{N-1},\ \mathbb{R}\right)$, exists at every point $(t,x)\in\Omega$, with the exception of the points $(\theta_{s-1},x)$, $s=\overline{1,N}$, $x\in[0,\omega]$, where the corresponding one-sided mixed partial derivatives are assumed to exist.

(iii) The hyperbolic equation (1) is satisfied by $u(t,x)$ throughout each subdomain $(\theta_{s-1},\theta_s)\times[0,\omega]$, $s=\overline{1,N}$ and remains valid at the points $(\theta_{s-1},x)$, $s=\overline{1,N}$, $x\in[0,\omega]$, when the second-order mixed partial derivative is interpreted in the right-sided sense.

(iv) The boundary condition (2) and the initial condition (4) are satisfied by the function $u(t,x)$; specifically, condition (2) holds at $t=0$ and $t=T$, while condition (4) is satisfied at $x=0$;

(v) The impulsive conditions (3) are satisfied at $t=\theta_p$, $p=\overline{1,N}$, $x\in[0,\omega]$. At each impulse point $t=\theta_p$, the partial derivative with respect to $x$ is defined by its right-hand limit,

$$\frac{\partial u(\theta_p,x)}{\partial x}:=\lim_{t\to\theta_p+0}\frac{\partial u(t,x)}{\partial x},$$

while the corresponding left-hand limit exists. Thus, condition (3) determines the jump of $\frac{\partial u(t,x)}{\partial x}$ at $t=\theta_p$.

Introduction of Auxiliary Functions and the Framework of Dzhumabayev's Parameterization Method.

First, we introduce the new functions $v\left(t,x\right) = \frac{\partial u(t,x)}{\partial x}$, $\quad w\left(t,x\right) = \frac{\partial u(t,x)}{\partial t}$ Then the boundary value problem (1)–(4) is reduced to a family of boundary value problems for impulsive differential equations with piecewise constant arguments.

$$\frac{\partial v\left(t,x\right)}{\partial t} = A\left(t,x\right)v\left(t,x\right) + A_0\left(t,x\right)v\left(\gamma\left(t\right),x\right) + B\left(t,x\right)w\left(t,x\right) + B_0\left(t,x\right)w\left(\gamma\left(t\right),x\right) + C\left(t,x\right)u\left(t,x\right) + C_0\left(t,x\right)u\left(\gamma\left(t\right),x\right) + f\left(t,x\right),\ t \neq \theta_j,\ j = \overline{1,N-1}, \quad (5)$$

$$P_0\left(x\right)v\left(0,x\right) + \sum_{s=1}^{N-1} P_s\left(x\right)v\left(\zeta_s,x\right) + P_N\left(x\right)v\left(T,x\right) = \varphi_0\left(x\right), \qquad x \in [0,\omega] \quad (6)$$

$$\lim_{t\to\theta_p+0} v\left(t,x\right) - \lim_{t\to\theta_p-0} v\left(t,x\right) = \varphi_p\left(x\right), x \in \left[0,\omega\right], p = \overline{1,N-1}, \quad (7)$$

$$u\left(t,x\right) = \psi\left(t\right) + \int_0^x v\left(t,\xi\right)d\xi,\ w\left(t,x\right) = \dot{\psi}\left(t\right) + \int_0^x \frac{\partial v(t,\xi)}{\partial t}d\xi, (t,x) \in \Omega_s,\quad s = \overline{1,N}. \quad (8)$$

A triple of functions $v(t,x), u(t,x), w(t,x)$ is regarded as a solution of problem (5)–(8) associated with the family of impulsive differential equations with piecewise constant arguments whenever the following requirements are fulfilled: the functions satisfy $v(t,x) \in PC\left(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R}\right)$, $\frac{\partial v(t,x)}{\partial t} \in PC\left(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R}\right)$, $u(t,x) \in PC\left(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R}\right)$, $w(t,x) \in PC\left(\Omega, \{\theta_j\}_{j=1}^{N-1}, \mathbb{R}\right)$; for every subdomain $[\theta_{s-1},\theta_s) \times [0,\omega]$, $s = \overline{1,N}$ the functions satisfy the differential system (5). Moreover, the one-sided partial derivative of $v(t,x)$ with respect to t is assumed to exist at every point $(\theta_{s-1},x)$, where $x \in [0,\omega]$; the boundary condition (6) is fulfilled along the boundary lines $t=0$ and $t=T$; the functions$u(t,x)$ and $w(t,x)$ are linked with $v(t,x)$ through the integral relations (8), where the time derivative $\frac{\partial v(t,x)}{\partial t}$ is involved.

For each $s = \overline{1,N}$, let $v_s(t,x)$ denote the restriction of $v(t,x)$ to the subdomain $\Omega_s$ i.e., or equivalently $v_s(t,x) = v(t,x)$, $\quad (t,x) \in \Omega_s$. Let $C\big(\Omega, \Delta_N(\omega), \mathbb{R}^N\big)$ denote the space of vector-valued functions $v([t],x) = \big(v_1(t,x), v_2(t,x), \ldots, v_N(t,x)\big)$, where $v_s : \Omega_s \to \mathbb{R}$ and each component $v_s(t,x)$ possesses the left-hand limit $\lim\limits_{t\to\theta_s-0} v_s(t,x)$. The norm in this space is defined by $\|v\|_2 = \max_{s=\overline{1,N}} \sup_{t\in[\theta_{s-1},\theta_s)} \|v_s(t,x)\|$. Assume that $[v([t],x) = \big(v_1(t,x), v_2(t,x), \ldots, v_N(t,x)\big)$ in $C\big(\Omega, \Delta_N(\omega), \mathbb{R}^N\big)$. The components $v_s(t,x)$, $s = \overline{1,N}$, are required to satisfy the following system of differential equations:

$$\frac{\partial v_s}{\partial t} = A\left(t,x\right)v_s\left(t,x\right) + A_0\left(t,x\right)v_s(\gamma(t),x) + B\left(t,x\right)w\left(t,x\right) + B_0\left(t,x\right)w(\gamma(t),x) + C\left(t,x\right)u\left(t,x\right) + C_0\left(t,x\right)u(\gamma(t),x) + f\left(t,x\right), \quad (t,x) \in \Omega_s, \quad s = \overline{1,N} \quad (9)$$

and the following conditions are satisfied:

$$P_0(x)\,v_0(0,x)+\sum_{s=1}^{N-1}P_s(x)\,v_s(\zeta_s,x)+P_N(x)\lim_{t\to T-0}v_N(t,x)=\varphi_0(x)\,,\ x\in[0,\omega]\,,\quad(10)$$

$$v_{p+1}(\theta_p,x)-\lim_{t\to\theta_p-0}v_p(t,x)=\varphi_p(x)\,,x\in[0,\omega]\,,p=\overline{1,N-1},\quad(11)$$

$$u(t,x)=\psi(t)+\int_0^x v_s(t,\xi)\,d\xi,\quad w(t,x)=\dot{\psi}(t)+\int_0^x\frac{\partial v_s(t,\xi)}{\partial t}d\xi,(t,x)\in\Omega_s,\ \ s=\overline{1,N},\ (12)$$

in equation (9), we have taken into account that $\gamma(t)=\zeta_s,\quad t\in[\theta_{s-1},\theta_s)\,,\quad s=\overline{1,N}$.

Next, we introduce the following functional parameters: $\mu_s(x)=v_s(\zeta_s,x),\quad s=\overline{1,N}$ and $x\in[0,\omega]$.

By making the following substitution, $v_s(t,x)=\widetilde{v}_s(t,x)+\mu_s(x)\,,\ \ (t,x)\in\Omega_s,\ \ s=\overline{1,N}$, we obtain a boundary value problem for a family of parameterized differential equations of the following form:

$$\frac{\partial\widetilde{v}_s}{\partial t}=A(t,x)\left(\widetilde{v}_s(t,x)+\mu_s(x)\right)+A_0(t,x)\,\mu_s(x)+B(t,x)\,w(t,x)+B_0(t,x)\,w(\zeta_s,x)+$$
$$C(t,x)\,u(t,x)+C_0(t,x)\,u(\zeta_s,x)+f(t,x)\,,\quad(t,x)\in\Omega_s,\quad s=\overline{1,N},\quad(13)$$

$$\widetilde{v}_s(\zeta_s,x)=0,\quad s=\overline{1,N},\quad x\in[0,\omega].\quad(14)$$

boundary condition

$$P_0(x)\big(\widetilde{v}_1(0,x)+\mu_1(x)\big)+\sum_{s=1}^{N-1}P_s(x)\big(\widetilde{v}_s(\zeta_s,x)+\mu_s(x)\big)$$
$$+P_N(x)\lim_{t\to T-0}\big(\widetilde{v}_N(t,x)+\mu_N(x)\big)=\varphi_0(x),x\in[0,\omega].\quad(15)$$

impulsive condition

$$\widetilde{v}_{p+1}(\theta_p,x)+\mu_{p+1}(x)-\lim_{t\to\theta_p-0}\widetilde{v}_p(t,x)-\mu_p(x)=\varphi_p(x)\,,\quad x\in[0,\omega]\,,p=\overline{1,N-1},\ (16)$$

$$u(t,x)=\psi(t)+\int_0^x\big(\widetilde{v}_s(t,\xi)+\mu_s(\xi)\big)\,d\xi,w(t,x)=\dot{\psi}(t)+\int_0^x\frac{\partial\widetilde{v}_s(t,\xi)}{\partial t}\,d\xi,(t,x)\in\Omega_s,s=\overline{1,N}.\quad(17)$$

A quadruple $\big\{v([t],x),\,\mu(x),\,u(t,x),\,w(t,x)\big\}$, is called a solution of the parameterized problem (13)–(17) if the following conditions are satisfied: $\widetilde{v}_s(t,x)\in C\big(\Omega,\Delta_N(\omega),\mathbb{R}\big),\qquad s=$

$\overline{1,N}$, and $\frac{\partial \widetilde{v}_s(t,x)}{\partial t} \in C\big(\Omega, \Delta_N(\omega), \mathbb{R}\big)$, $s = \overline{1,N}$. The functional parameters satisfy $\mu_s(x) \in C([0,\omega], \mathbb{R})$, $s = \overline{1,N}$. The functions $u(t,x)$, $w(t,x) \in C\big(\Omega, \Delta_N(\omega), \mathbb{R}\big)$. The quadruple $\big(\widetilde{v}([t],x),\ \mu(x),\ u(t,x),\ w(t,x)\big)$ satisfies equations (13)–(17).

Moreover, these elements satisfy the system of differential equations (13), the initial conditions (14), the boundary condition (15), and the impulsive conditions (16) for all $(t,x) \in \Omega_s$, $s = \overline{1,N}$, $(t,x) \in \Omega_s$, $s = \overline{1,N}$ on the interval $[0,\omega]$. The functions $u(t,x)$ and $w(t,x)$ are related to $v_s(t,x)$ and its partial derivative with respect to time, $\frac{\partial v_s(t,x)}{\partial t}$, through the integral equations (17) involving the functional parameters $\mu_s(x)$, $s = \overline{1,N}$. Under these assumptions, problem (13)–(17) is reduced to a family of Cauchy problems for differential equations. For fixed functions $\mu_s(x)$, $u(t,x)$, and $w(t,x)$, the family of Cauchy problems (13)–(14) admits a unique solution, which can be represented in the following form:

$$\begin{aligned}\widetilde{v}_s(t,x) &= e^{\alpha_s(t,x)} \int_{\zeta_s}^{t} e^{-\alpha_s(t,x)} \left[A(\tau,x) + A_0(\tau,x)\right] d\tau \mu_s(x) \\ &+ e^{\alpha_s(t,x)} \int_{\zeta_s}^{t} e^{-\alpha_s(t,x)} \left[B(\tau,x)\, w(\tau,x) + C(\tau,x)\, u(\tau,x)\right] d\tau \\ &+ e^{\alpha_s(t,x)} \int_{\zeta_s}^{t} e^{-\alpha_s(t,x)} \left[B_0(\tau,x)\, w(\zeta_s,x) + C_0(\tau,x)\, u(\zeta_s,x)\right] d\tau \\ &\qquad + e^{\alpha_s(t,x)} \int_{\zeta_s}^{t} e^{-\alpha_s(t,x)} f(\tau,x)\, d\tau, \quad (t,x) \in \Omega_s, \quad s = \overline{1,N}. \end{aligned} \tag{18}$$

Where $\alpha_s(t,x) = \int_{\zeta_s}^{t} A(\tau,x)\, d\tau$, $(t,x) \in \Omega_s$, $s = \overline{1,N}$.
We introduce the following notation:
$D_s(t,x) = e^{\alpha_s(t,x)} \int_{\zeta_s}^{t} e^{-\alpha_s(\tau,x)} \left[A(\tau,x) + A_0(\tau,x)\right] d\tau$,
$H_s(t,x,\omega,u) = e^{\alpha_s(t,x)} \int_{\zeta_s}^{t} e^{-\alpha_s(t,x)} \left[B(\tau,x)\, w(\tau,x) + C(\tau,x)\, u(\tau,x)\right] d\tau + e^{\alpha_s(t,x)} \int_{\zeta_s}^{t} e^{-\alpha_s(t,x)} \left[B_0(\tau,x)\, w(\zeta_s,x) + C_0(\tau,x)\, u(\zeta_s,x)\right] d\tau$,
$F_s(t,x) = e^{\alpha_s(t,x)} \int_{\zeta_s}^{t} e^{-\alpha_s(t,x)} f(\tau,x)\, d\tau$, $(t,x) \in \Omega_s$, $s = \overline{1,N}$.
Then, the representation of expression (18) can be written in the following form:

$$\widetilde{v}_s(t,x) = D_s(t,x)\, \mu_s(x) + H_s(t,x,w,u) + F_s(t,x), \quad (t,x) \in \Omega_s, \quad s = \overline{1,N}. \tag{19}$$

It follows from equation (19) that $\widetilde{v}_1(0,x), \lim\limits_{t \to T-0} \widetilde{v}_N(t,x), \widetilde{v}_{p+1}(\theta_p,x), \lim\limits_{t \to \theta_p - 0} \widetilde{v}_p(t,x), p = \overline{1,N-1}$. Substituting the obtained expressions into relations (15) and (16), we obtain the following expressions:

$$\begin{aligned} &P_0(x)\left[1 + D_1(0,x)\right] \mu_1(x) + \sum_{s=1}^{N-1} P_s(x)\, \mu_s(x) + P_N(x)\left[1 + D_N(T,x)\right] \mu_N(x) = \\ &- P_0(x)\left[F_1(0,x) + H_1(0,x,w,u)\right] - P_N(x) \lim_{t \to T-0}\left[H_N(t,x,w,u) + F_N(t,x)\right] + \varphi_0(x). \end{aligned} \tag{20}$$

$$\begin{aligned} &\left[1 + D_{p+1}(\theta_p,x)\right] \mu_{p+1}(x) - \left[D_p(\theta_p,x) + 1\right] \mu_p(x) = \\ &\varphi_p(x) + F_{p+1}(\theta_p,x) - F_p(\theta_p,x) + H_p(\theta_p,x,w,u) - H_{p+1}(\theta_p,x,w,u), p = \overline{1,N-1}. \end{aligned} \tag{21}$$

Using the coefficients of the unknown functions $\mu_s(x)$, $s=\overline{1,N}$ appearing on the left-hand side of the system of equations (20)–(21), we construct the N×N matrix $Q(x)$ in the following form:

$$Q(x)\,\mu(x) = F_*(x) + H_*(x,w,u) \tag{22}$$

$$Q(x) = \begin{pmatrix} P_0(x)\left[1+D_1(0,x)\right]+P_1(x) & \vdots & P_{N-2}(x) & P_{N-1}(x)+P_N(x)\left[1+D_N(T,x)\right] \\ -\left[1+D_1(\theta_1,x)\right] & \vdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \dots & \vdots & \dots & \dots \\ 0 & 0 & -\left[1+D_{N-1}(\theta_{N-1},x)\right] & 1+D_N(\theta_{N-1},x) \end{pmatrix},$$

We introduce the vectors $F_*(x)$, $H_*(x,w,u)$ in the following form:

$$F_*(x) = \begin{pmatrix} -P_0(x)F_1(0,x)-P_N(x)F_N(T,x)+\varphi_0(x) \\ -F_2(\theta_1,x)+F_1(\theta_1,x)+\varphi_1(x) \\ \vdots \\ -F_N(\theta_{N-1},x)+F_{N-1}(\theta_{N-1},x)+\varphi_{N-1}(x) \end{pmatrix},$$

$$H_*(x,w,u) = \begin{pmatrix} -P_0(x)H_1(0,x,w,u)-P_N(H_N(t,x,w,u) \\ H_2(\theta_1,x,w,u)-(H_1(\theta_1,x,w,u) \\ \vdots \\ H_N(\theta_{N-1},x,w,u)-(H_{N-1}(\theta_{N-1},x,w,u) \end{pmatrix}.$$

## 3 Results and Discussion. Algorithm of the Parameterization Method

If the functions $w(t,x)$ and $u(t,x)$ are known for all $(t,x)\in\Omega_s$, $s=\overline{1,N}$, then the function $\mu(x)$ is determined from the system of functional equations (22), whose components are $\mu_s(x)\in C([0,\omega],\mathbb{R})$, $s=\overline{1,N}$.

Next, by means of the integral representation (19) and the differential equations (13), we determine the function $\widetilde{v}_s(t,x)$ and its partial derivative with respect to $t$, $\frac{\partial\widetilde{v}_s}{\partial t}$, for all $(t,x)\in\Omega_s$, $s=\overline{1,N}$.

Using the obtained function $\widetilde{v}_s(t,x)$, its derivative and the functional parameters $\mu(x)$, we then determine the functions $w(t,x)$ and $u(t,x)$. To obtain a solution of the family of problems (13)–(17), we employ an iterative method.

We define the quadruple $\{\widetilde{v}^*(t,x),\mu^*(x),u^*(t,x),w^*(t,x)\}$, as the limit of the sequence of quadruples $\left\{\widetilde{v^{(k)}}([t],x),\mu^{(k)}(x),u^{(k)}(t,x),w^{(k)}(t,x)\right\}$.

Step 0. For all $x\in[0,\omega]$, we assume that there is an inverse matrix $Q(x)$ of size $(N\times N)$. Taking $u(t,x)=\psi(t), w(t,x)=\dot{\psi}(t)$ on the right-hand side of system (22), the initial approximation of the functional parameter $\mu^{(0)}(x)=(\mu_1^{(0)}(x),\mu_2^{(0)}(x),\dots,\mu_N^{(0)}(x))$ is found from the system of functional equations:

$$Q(x)\,\mu(x) = F_*(x) + H_*\left(x,\psi,\dot{\psi}\right),$$

$$\mu^{(0)}(x) = [Q(x)]^{-1}F_*(x) + [Q(x)]^{-1}H_*(t,x,\psi,\dot{\psi}), \quad x \in [0,\omega].$$

components here $\mu_s(x) \in C([0,\omega],\mathbb{R})$, $s = \overline{1,N}$ Taking the right-hand side of the differential equations (13) as $u\,(x) = \psi(t), w\,(t,x) = \dot{\psi}\,(t)\,,\ \mu_s\,(x) = {\mu_s}^{(0)}\,(x)\,,$ $s = \overline{1,N}$ and from the Cauchy theorem (13)-(14) we find $\widetilde{v}_s^{(0)}\,(t,x)\,.$

$$\widetilde{v}_s^{(0)}\,(t,x) = D_s\,(t,x)\,\mu_s^{(0)}\,(x) + H_s\left(t,x,\psi,\dot{\psi}\right) + F_s\,(t,x)\,.$$

And we define its derivative as follows:

$$\frac{\partial \widetilde{v}_s^{(0)}\,(t,x)}{\partial t} = A\,(t,x)\,\widetilde{v}_s^{(0)}\,(t,x) + (A\,(t,x) + A_0\,(t,x))\,\mu_s^{(0)}\,(x) + B\,(t,x)\,\dot{\psi}\,(t)$$
$$+ B_0\,(t,x)\,\dot{\psi}\,(\zeta_s) + C\,(t,x)\,\psi\,(t) + C_0\,(t,x)\,\psi\,(\zeta_s)\ + f\,(t,x)\,, (t,x) \in \Omega_s, \quad s = \overline{1,N.}$$

Then, from the integral equation (17), we define the functions $u^{(0)}\,(t,x)\,.$

$$u^{(0)}\,(t,x) = \psi\,(t) + \int_0^x (\widetilde{v}_s^{(0)}(t,\xi) + \mu_s^{(0)}\,(\xi))d\xi,\ (t,x) \in \Omega_s, \quad s = \overline{1,N.}$$

$$w^{(0)}\,(t,x) = \psi(\dot{t}) + \int_0^x \frac{\partial \widetilde{v}_s^{(0)}(t,\xi)}{\partial t}d\xi, (t,x) \in \Omega_s, \quad s = \overline{1,N},$$

Step 1. (22) put $u(t,x) = u^{(0)}(t,x), w\,(t,x) = \dot{w}^{(0)}(t,x)$ on the right side of the system, $\mu^{(1)}(x) = \left(\mu_1^{(1)}(x), \mu_2^{(1)}(x), \ldots, \mu_N^{(1)}(x)\right)$ of functional parameter we define the first approximation from the equation

$$Q\,(x)\,\mu(x) = F_*\,(x) + H_*\left(x, w^{(0)}, u^{(0)}\right),\ x \in [0,\omega]$$

$$\mu^{(1)}\,(x) = Q\,(x)^{-1}(F_*\,(x) + H_*\left(x, w^{(0)}, u^{(0)}\right)), \quad x\in [0,\omega].$$

Putting the right-hand side of the differential equations (13) as $u\,(t,x) = u^{(0)}(t,x), w\,(t,x) = w^{(0)}(t,x), \mu_s\,(x) = \mu_s^{(1)}\,(x)\,,$ $s = \overline{1,N}$ we find the function $\widetilde{v}_s^{(1)}(t,x)$ from the Cauchy problem (13)–(14).

$$\widetilde{v_s}^{(1)}\,(t,x) = D_s\,(t,x)\,\mu_s^{(1)}\,(x) + H_s\left(t,x,w^{(0)},u^{(0)}\right) + F_s\,(t,x)\,.$$

$$\frac{\partial \widetilde{v}_s^{(1)}}{\partial t} = A\,(t,x)\,\widetilde{v}_s^{(1)}\,(t,x) + (A\,(t,x) + A_0\,(t,x))\,\mu_s^{(1)}\,(x) + B\,(t,x)\,w^{(0)}\,(t,x) +$$
$$+ B_0\,(t,x)\,\dot{\psi}\,(\zeta_s) + C\,(t,x)\,u^{(0)}\,(t,x) + C_0\,(t,x)\,\psi\,(\zeta_s) + f\,(t,x)\,, \quad (t,x) \in \Omega_s,$$

We find the derivative. From the integral equation (17), we determine the following functions.

$$u^{(1)}\,(t,x) = \psi\,(t) + \int_0^x (\widetilde{v}_s^{(1)}\,(t,\xi) + \mu_s^{(1)}\,(\xi))d\xi, w^{(1)}\,(t,x) = \psi(\dot{t}) + \int_0^x \frac{\partial \widetilde{v}_s^{(1)}(t,\xi)}{\partial t}d\xi, (t,x) \in \Omega_s,\ s = \overline{1,N},$$

We continue in the same way to the k-step. We put $u(t,x) = u^{(k-1)}(t,x)$, $w(t,x) = w^{(k-1)}(t,x)$ on the right side of the system, and determine, $\mu^{(k)}(x) = (\mu_1^{(k)}\,(x)\,, \mu_2^{(k)}\,(x)\,, \ldots, \mu_N^{(k)}\,(x))$ from the following equation.

$$Q\,(x)\,\mu\,(x) = F_*\,(x) + H_*\left(x, w^{(k-1)}, u^{(k-1)}\right),$$

$$\mu^{(k)}(x) = \quad Q(x)^{-1}(F_*(x) + H_*\left(x, w^{(k-1)}, u^{(k-1)}\right)),\ x \in [0,\omega].$$

From here

$$\widetilde{v}_s^{(k)}(t,x) = D_s(t,x)\,\mu_s^{(k)}(x) + H_s\left(t,x,w^{(k-1)},u^{(k-1)}\right) + F_s(t,x)\,.$$

$$\frac{\partial \widetilde{v}_s^{(k)}}{\partial t} = A(t,x)\,\widetilde{v}_s^{(k)}(t,x) + (A(t,x)+A_0(t,x))\mu_s^{(k)}(x) + B(t,x)\,w^{(k-1)}(t,x) + B_0(t,x)\,\dot{\psi}(\zeta_s) + C(t,x)\,u^{(k-1)}(t,x) + C_0(t,x)\,\psi(\zeta_s) + f(t,x)\,,\quad (t,x)\in\Omega_s,\quad s=\overline{1,N},$$

$$u^{(k)}(t,x) = \psi(t) + \int_0^x (\widetilde{v}_s^{(k)}(t,\xi) + \mu_s^{(k)}(x))d\xi,\ w^{(k)}(t,x) = \dot{\psi(t)} + \int_0^x \frac{\partial \widetilde{v}_s^{(k)}(t,\xi)}{\partial t}d\xi, (t,x)\in\Omega_s,$$

$s=\overline{1,N}$ is found. Here k=1,2,...

The method allows you to find unknown functions in three stages: From the system of functional equations (22), we define the functional parameters $\mu_s(x), s=\overline{1,N}$ for all $x\in[0,\omega]$. From the family of Cauchy problems (13), (14), we find the functions $v_s(t,x)$ for all $(t,x)\in\Omega_s,\quad s=\overline{1,N}$ and its derivative $\frac{\partial v_s(t,x)}{\partial t} forall\,(t,x)\in\Omega_s,\quad s=\overline{1,N}$. 3. From the integral equations (17) we define the functions $u(t,x)\, andw\,(t,x)$ for all $(t,x)\in\Omega_s,\quad s=\overline{1,N}$. When the conditions set forth in the initial problem are satisfied, each auxiliary problem has a unique solution. To implement the algorithm, it is necessary to establish the convergence of the approximate solutions of the problem with parameters (13)–(17) to the exact solution. The following theorem determines the convergence conditions of the proposed algorithm and the conditions for the existence of a unique solution of the problem with parameters (13)–(17).

**Theorem 1** *Assume that the functions*

1. $A(t,x),\quad B(t,x),\quad C(t,x),\quad A_0(t,x),\quad B_0(t,x),\quad C_0(t,x),\quad f(t,x)$ *belong to* $C(\Omega,\mathbb{R})$, *the functions* $P_0(x),\quad P_i(x),\quad P_N(x)$ *belong to space* $C([0,\omega],\mathbb{R}),\quad \varphi_p(x),\quad \varphi_0(x)\in C^1([0,\omega],\mathbb{R}),\quad p=\overline{1,N-1},\quad \psi(t)\in C^1([0,T],\mathbb{R})$;
2. *if the* $N\times N$ *matrix* $Q(x)$ *is invertible for all* $x\in[0,\omega]$. *Then the parameterized problem* (13)–(17) *has a unique solution and that solution is found using the algorithm presented above.*

From the equivalent problems (1)–(4) and (13)–(17), we obtain the following.

**Theorem 2** *Suppose that the assumptions imposed on the coefficients of the equation and on the functions appearing in the boundary and impulsive conditions are satisfied. If the* $N\times N$ *matrix* $Q(x)$ *is invertible for every* $x\in[0,\omega]$, *then the boundary value problem* (1)–(4) *for the impulsive hyperbolic equation with a generalized piecewise constant argument has a unique solution.*

The proof scheme of Theorems 1 and 2 is implemented according to the proof scheme of the theorems of the article [5], which is shown below.

Example and Conclusion.

To illustrate the developed theory, let us consider an example that naturally leads to a hyperbolic equation with a generalized piecewise constant argument and impulsive effects. Consider an automatic temperature control system for an industrial furnace. In a metallurgical plant, a

long furnace is used to heat metal billets. The temperature of the metal varies continuously along the length of the furnace; however, the information from the sensors is received only at predetermined time instants. Consequently, the current heating regime depends on the previously measured temperature. In addition, at prescribed moments, the heating elements are switched on for a short period of time, causing impulsive changes in the heating process. The mathematical model of this process is described by equation (1), where the coefficients have the following physical interpretation: $A = 0.5$ characterizes the heat propagation rate; $B = 0.3$ represents the intensity of convective heat transfer; $C = -0.2$ denotes the heat loss coefficient; $A_0 = 0.4$ accounts for the influence of the previously measured temperature. At time $t = 4$, an auxiliary heating element is activated, producing an instantaneous change in the heating regime.
The unknown function $u(t, x)$ represents the temperature of the metal billet at the spatial point $x$ and time $t$.
This problem is considered as an example of the practical application of hyperbolic equations with part-constant arguments and momentum terms. As an illustration of the obtained results, let us consider the following problem.

$$\frac{\partial^2 u}{\partial t \partial x} = 0.5\frac{\partial u\,(t,x)}{\partial x}+0.3\frac{\partial u\,(t,x)}{\partial t}-0.2u\,(t,x)+0.4\frac{\partial u\,(\gamma\,(t)\,,x)}{\partial x}+0.5u\,(\gamma\,(t)\,,x)+5e^{-t}x, \quad t \neq 4; \tag{23}$$

$$\frac{\partial u\,(0,x)}{\partial x} + 2\frac{\partial u\,(2,x)}{\partial x} - \frac{\partial u\,(6,x)}{\partial x} = 0.5x + 15, x \in [0, 10]\,, \tag{24}$$

$$\lim_{t\to 4+0} \frac{\partial u\,(t,x)}{\partial x} - \lim_{t\to 4-0} \frac{\partial u\,(t,x)}{\partial x} = 20e^{-0.1x}, x \in [0, 10]\,, \tag{25}$$

$$u\,(t, 0) = 100. \quad t \in [0, 8] \tag{26}$$

First, we introduce new functions $v(t, x) = \frac{\partial u(t,x)}{\partial x}, \quad w(t, x) = \frac{\partial u(t,x)}{\partial t}$. Problem is transformed into a boundary value problem for a family of impulsive differential equations with a constant partial argument. The function $v([t], x)$ lies in the space $C(\Omega, \Delta_2(\omega), R^2)$, and its elements $v_s(t, x)$, $s = 1, 2;$ satisfy the following system of differential equations:

$$\frac{\partial v_1\,(t,x)}{\partial t} = 0.5v_1\,(t,x) + 0.3w\,(t,x) - 0.2u\,(t,x) + 0.4v_1\,(2,x) + 0.5u\,(2,x) + 5e^{-t}x,$$

$(t, x) \in \Omega_1 = [0, 4) \times [0, 10]$

$$\frac{\partial v_2\,(t,x)}{\partial t} = 0.5v_2\,(t,x) + 0.3w\,(t,x) - 0.2u\,(t,x) + 0.4v_2\,(2,x) + 0.5u\,(2,x) + 5e^{-t}x,$$

$(t, x) \in \Omega_1 = [4, 8) \times [0, 10]\,;$

$$v_1\,(0,x) + 2v_1\,(2,x) - v_2\,(6,x) = 0.5x + 15, x \in [0, 10]\,,$$

$$\lim_{t\to 4+0} v_1\,(t,x) - \lim_{t\to 4-0} v_2\,(t,x) = 20e^{-0.1x}, x \in [0, 10]\,,$$

$$u\,(t, 0) = 100. \quad u_s\,(t,x) = 100 + \int_0^x v_s\,(t,\xi)\,d\xi, \qquad w_s\,(t,x) = \int_0^x \frac{\partial v_s(t,\xi)}{\partial t} d\xi.$$

$\theta_{s-1} < \zeta_s < \theta_s, s = 1, 2; \zeta_1 = 2; \zeta_2 = 6$. We introduce functional parameters as follows: $\mu_1 x = v_1(\zeta_1, x), \mu_2(x) = v_2(\zeta_2, x)$. By making the following substitution: $v_s(t,x) = \widetilde{v_s}(t,x) + \mu_s(x), \quad (t,x) \in \Omega_s, \quad s = \overline{1,2}$ , we obtain a problem for a family of parametric differential equations of the following form:

$$\frac{\partial v_1(t,x)}{\partial t} = 0.5\left(\widetilde{v_1}(t,x) + \mu_1(x)\right) + 0.3w(t,x) - 0.2u(t,x) + 0.4\left(\widetilde{v_1}(2,x) + \mu_1(x)\right)$$
$$+ 0.5u(2,x) + 5e^{-t}x, \quad (t,x) \in \Omega_1 = [0,4) \times [0,10];$$

$$\frac{\partial v_2(t,x)}{\partial t} = 0.5\left(\widetilde{v_2}(t,x) + \mu_2(x)\right) + 0.3w(t,x) - 0.2u(t,x) + 0.4\left(\widetilde{v_2}(t,x) + \mu_2(x)\right) +$$
$$0.5u(2,x) + 5e^{-t}x, \quad (t,x) \in \Omega_2 = [4,8) \times [0,10];$$

$$\widetilde{v_1}(0,x) + \mu_1(x) + 2(\widetilde{v_1}(2,x) + \mu_1(x) - (\widetilde{v_2}(6,x) + \mu_2(x)) = 0.5x + 15, x \in [0,10]$$

$$\lim_{t\to 4+0}(\widetilde{v_2}(t,x) + \mu_2(x)) - \lim_{t\to 4-0}(\widetilde{v_1}(t,x) + \mu_1(x)) = 20e^{-0.1x}, x \in [0,10],$$

$$u(t,0) = 100.$$

$$u_s(t,x) = 100 + \int_0^x v_s(t,\xi)\,d\xi, \quad w_s(t,x) = \int_0^x \frac{\partial v_s(t,\xi)}{\partial t} d\xi . \theta_{s-1} < \zeta_s < \theta_s, \quad s = 1,2; \ \zeta_1 = 2; \ \zeta_2 = 6.$$

$$\alpha_1(t,x) = \int_2^t 0.5d\tau = \frac{t}{2} - 1, \ (t,x) \in \Omega_1; \quad \alpha_2(t,x) = \int_6^t 0.5d\tau = \frac{t}{2} - 3, \ (t,x) \in \Omega_2.$$

For fixed $\mu_s(x), u(t,x), w(t,x)$ the solution of the family of Cauchy problems (13)–(14) is unique and has the following form:

$$\widetilde{v_1}(t,x) = e^{\frac{t}{2}-1}\int_2^t e^{-(\frac{\tau}{2}-1)}(-0.9)\,d\tau\mu_1(x) + e^{\frac{t}{2}-1}\int_2^t e^{-(\frac{\tau}{2}-1)}(-0,2*100 + 0.5*100)\,d\tau$$
$$+ e^{\frac{t}{2}-1}\int_2^t e^{-(\frac{\tau}{2}-1)} 5e^{-\tau}x d\tau,$$

$$\widetilde{v_2}(t,x) = e^{\frac{t}{2}-3}\int_6^t e^{-(\frac{\tau}{2}-3)}(-0.9)\,d\tau\mu_2(x) + e^{\frac{t}{2}-3}\int_6^t e^{-(\frac{\tau}{2}-3)}(-0,2*100 + 0.5*100)\,d\tau$$
$$+ e^{\frac{t}{2}-3}\int_6^t e^{-(\frac{\tau}{2}-3)} 5e^{-\tau}x d\tau.$$

$$D_1(t,x) = e^{\frac{t}{2}-1}\int_2^t e^{-(\frac{\tau}{2}-1)}(-0.9)\,d\tau\mu_1(x), \quad D_2(t,x) = e^{\frac{t}{2}-3}\int_6^t e^{-(\frac{\tau}{2}-3)}(-0.9)\,d\tau\mu_2(x)$$

$$H_1(t,x,w,u) = e^{\frac{t}{2}-1}\int_2^t e^{-(\frac{\tau}{2}-1)}(-0,2*100 + 0.5*100)\,d\tau$$

$$H_2(t,x,w,u) = e^{\frac{t}{2}-3}\int_6^t e^{-(\frac{\tau}{2}-3)}(-0,2*100 + 0.5*100)\,d\tau$$

$$F_1(t,x) = e^{\frac{t}{2}-1}\int_2^t e^{-(\frac{\tau}{2}-1)} 5e^{-\tau}x d\tau, \quad F_2(t,x) = e^{\frac{t}{2}-3}\int_6^t e^{-(\frac{\tau}{2}-3)} 5e^{-\tau}x d\tau.$$

Substituting the found expressions into relations (20) and (21), we find the following limits and values:

$$Q(x) = \begin{bmatrix} 1 + D_1(0,x) + 2 & -1 \\ -\left(1 + D_1(4,x)\right) & 1 + D_2(4,x) \end{bmatrix}, \qquad \det Q(x) \neq 0.$$

It is clear that the determinant of the matrix $Q\,(x)$ is nonzero. Therefore, the matrix $Q\,(x)$ has an inverse and, according to the theorem given above, the problem under consideration has a solution, which can be expressed uniquely in the following form.

$$u_1\,(t,x) = -0.2xe^{\frac{-3t}{2}+3} + 82.8e^{-0.1x} + \left(0.07x^2 + 4.7x - 82.8\right)e^{\frac{t}{2}-1} + 0.06x^2 - 35.4x + 136.8 - 36.8e^{-0.1x},$$

$$u_2\,(t,x) = -0.04xe^{\frac{-3t}{2}+9} + 154.1e^{-0.1x} - \left(3.6x^2 + 138.3x + 154.1\right)e^{\frac{t}{2}-1} + 1.6x^2 + 28.1x + 168.5 - 68.5e^{-0.1x}.$$

## 4 Conclusion

In conclusion, the presented results extend the applicability of Dzhumabayev's parameterization method to a new class of multipoint boundary value problems for impulsive hyperbolic equations with generalized piecewise constant arguments.

The advantages of the parameterization method demonstrated by our findings include: there is no need to construct the Green's function; the multipoint boundary conditions are taken into account directly; the impulsive conditions are incorporated without modifying the general framework of the method; the resulting algorithm admits straightforward software implementation.

Furthermore, these results demonstrate the constructive nature of the proposed algorithm and its applicability to the numerical investigation of applied problems. Following the numerical schemes developed for related evolution equations (see, e.g., [13]- [15]), all computations for the proposed algorithm can be efficiently implemented in MATLAB, Maple, or Python.

Future work will focus on extending the proposed method to more general classes of equations. Additionally, the convergence, stability, and error analysis of the proposed numerical algorithm will be investigated, alongside its application to practical mathematical models.

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