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ALGORITHMIC LEARNING OF SUBSPACES IN COMPUTABLE VECTOR SPACES

Algorithmic learning of algebraic structures connects computability theory with artificial intelligence. Classical inductive inference mainly deals with formal languages and recursive functions. In contrast, analyzing vector spaces allows us to explore the theoretical limits of automated scientific discovery. This study investigates the algorithmic learnability of subspace families in computable vector spaces over a computable field. We focus on the fundamental family of all finite-dimensional subspaces and examine whether learnability is an isomorphism invariant. By combining constructive model theory with Gold's paradigm of identification in the limit, our results identify the exact recursion-theoretic barriers to learning algebraic invariants. We use the classical model of explanatory learning in the limit from positive data (Ex-learning), where Turing machines act as learners. By applying canonical index numberings of finite sets and effective quotient space constructions, we establish two fundamental theorems. Our first result shows that if a computable vector space has a decidable linear dependence relation, the family of all its finite-dimensional subspaces is Ex-learnable. Second, we construct a computable vector space where the relation of linear dependence is undecidable. By using an effective priority-style diagonalization argument, we prove that the family of finite-dimensional subspaces in this space fails to be Ex-learnable. These results prove that algorithmic learnability is not an isomorphism invariant of an abstract algebraic structure. Instead, it is strictly governed by the algorithmic complexity (Turing degree) of the linear dependence predicate. This finding provides a theoretical foundation for automated theorem proving and neuro-symbolic AI systems.

Keywords: computable vector space, algorithmic learning theory, inductive inference, linear dependence, computable numberings.

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Есептелімді векторлық кеңістіктердегі ішкі кеңістіктерді алгоритмдік оқыту

Алгебралық құрылымдарды алгоритмдік оқыту есептелімділік теориясын жасанды интеллектпен байланыстырады. Классикалық индуктивті қорытындылау негізінен формалды тілдермен және рекурсивті функциялармен жұмыс істейді. Керісінше, векторлық кеңістіктерді талдау автоматтандырылған ғылыми жаңалықтардың теориялық шекараларын зерттеуге мүмкіндік береді. Бұл зерттеу есептелімді өріс үстіндегі есептелімді векторлық кеңістіктердегі ішкі кеңістіктер жиындарының алгоритмдік оқытылуын зерттейді. Біз барлық ақырлы өлшемді ішкі кеңістіктердің фундаменталды жиынына назар аударамыз және оқытылудың изоморфизм инварианты болып-болмайтынын тексереміз. Конструктивті модельдер теориясын Голдтың шектегі идентификациялау парадигмасымен біріктіре отырып, біздің нәтижелеріміз алгебралық инварианттарды оқытудағы нақты рекурсивті-теориялық кедергілерді анықтайды. Біз позитивті деректер бойынша шекте түсіндірмелі оқытудың (Ex-оқыту) классикалық моделін қолданамыз, мұнда Тьюринг машиналары оқытушы агенттер ретінде әрекет етеді. Шектеулі жиындардың канондық индекстік нөмірленуін және тиімді фактор-кеңістіктер конструкцияларын қолдана отырып, біз екі негізгі теореманы дәлелдейміз. Бірінші нәтижеміз көрсеткендей, егер есептелімді векторлық кеңістікте шешілімді сызықтық тәуелділік қатынасы болса, онда оның барлық ақырлы өлшемді ішкі кеңістіктер жиыны Ex-оқытылады.

Екіншіден, біз сызықтық тәуелділік қатынасы шешілімсіз болатын есептелімді векторлық кеңістік құрамыз. Приоритеттер әдісімен тиімді диагоналдау дәлелдеуін қолдана отырып, біз бұл кеңістіктегі ақырлы өлшемді ішкі кеңістіктер жиынының Ех-оқытылмайтынын дәлелдейміз. Бұл нәтижелер алгоритмдік оқытылудың абстрактты алгебралық құрылымның изоморфизм инварианты емес екенін дәлелдейді. Керісінше, ол қатаң түрде сызықтық тәуелділік предикатының алгоритмдік күрделілігімен (Тьюринг дәрежесімен) анықталады. Бұл тұжырым теоремаларды автоматты түрде дәлелдеу жүйелері мен нейросимволдық жасанды интеллект жүйелері үшін теориялық негіз қамтамасыз етеді.

Түйін сөздер: есептелімді векторлық кеңістік, алгоритмдік оқыту теориясы, индуктивті қорытындылау, сызықтық тәуелділік, есептелімді нөмірлеу.

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Алгоритмическое обучение подпространств в вычислимых векторных пространствах

Алгоритмическое обучение алгебраических структур связывает теорию вычислимости с искусственным интеллектом. Классический индуктивный вывод в основном имеет дело с формальными языками и рекурсивными функциями. В отличие от этого, анализ векторных пространств позволяет исследовать теоретические пределы автоматизированных научных открытий. Данное исследование изучает алгоритмическую обучаемость семейств подпространств в вычислимых векторных пространствах над вычислимым полем. Мы сосредотачиваемся на фундаментальном семействе всех конечномерных подпространств и выясняем, является ли обучаемость инвариантом изоморфизма. Объединяя теорию конструктивных моделей с парадигмой Голда об идентификации в пределе, наши результаты выявляют точные рекурсивно-теоретические барьеры для обучения алгебраическим инвариантам. Мы используем классическую модель объяснительного обучения в пределе по позитивным данным (Ех-обучение), где в качестве обучающихся агентов выступают машины Тьюринга. Применяя канонические нумерации конечных множеств и конструкции эффективных факторпространств, мы устанавливаем две фундаментальные теоремы. Наш первый результат показывает, что если вычислимое векторное пространство имеет разрешимое отношение линейной зависимости, то семейство всех его конечномерных подпространств является Ех-обучаемым. Во-вторых, мы строим вычислимое векторное пространство, в котором отношение линейной зависимости неразрешимо. Используя эффективный метод диагонализации с приоритетами, мы доказываем, что семейство конечномерных подпространств в этом пространстве не является Ех-обучаемым. Эти результаты доказывают, что алгоритмическая обучаемость не является инвариантом изоморфизма абстрактной алгебраической структуры. Напротив, она строго определяется алгоритмической сложностью (тьюринговой степенью) предиката линейной зависимости. Данный вывод предоставляет теоретическую основу для систем автоматического доказательства теорем и нейросимволического искусственного интеллекта.

Ключевые слова: вычислимое векторное пространство, теория алгоритмического обучения, индуктивный вывод, линейная зависимость, вычислимые нумерации.

1 Introduction

Algorithmic Learning Theory (ALT), initiated by E. M. Gold in his 1967 study of identification in the limit [5], investigates the capability of algorithmic learners to infer rules or structures from data. While early inductive inference research focused primarily on formal languages and recursive functions, the field has since expanded to explore deep connections with algebraic structures and mathematical logic [1, 13].

The foundational framework for examining the algorithmic properties of algebraic models relies on the theory of constructive models, established by Mal'tsev [10] and advanced by

Ershov and Goncharov [4]. In this paradigm, a structure is computable if its domain is enumerated by a Turing machine and its operations are represented by total computable functions. A central issue in this area is how the syntactic descriptions of model classes interact with their algorithmic complexity, often analyzed using the Ershov hierarchy and Turing degrees [6, 15].

Investigating the learnability of specific substructures — such as ideals in rings or subgroups of abelian groups — began with Stephan and Ventsov [16]. Their results demonstrated that the learnability of such families is governed by their topological and recursion-theoretic complexity. To formalize these learning processes, the theory of computable numberings, developed by Badaev and Goncharov [2], provides the necessary structural architecture for managing canonical hypotheses.

The learnability of subspaces in vector spaces was significantly advanced by Harizanov and Stephan [8]. They focused on recursively enumerable (r.e.) infinite-dimensional subspaces, showing that learnability is linked to properties such as the existence of low linear dependence algorithms. Additional research by Harizanov et al. [7] further highlighted how vector space automorphisms and embeddings depend on the algorithmic complexity of the basis.

Despite these advancements, the role of finite-dimensional subspaces remains fundamental to linear algebra and automated reasoning. A significant open question arises regarding the interaction between algorithmic learning and pure algebra: does the simplicity of an algebraic structure — specifically its finite dimension — guarantee that the corresponding inductive inference task is computationally robust? Furthermore, the question of whether algorithmic learnability acts as an isomorphism invariant within these classes warrants formal investigation, which we address in the present study.

Object of study: The fundamental family of all finite-dimensional subspaces within computable vector spaces over a computable field (specifically, the field of rational numbers $\mathbb{Q}$).

Subject of study: The algorithmic complexity of explanatory learning in the limit (Ex-learning) for families of finite-dimensional subspaces, and its dependence on the Turing degree of the linear dependence relation.

Aim and Objectives: The primary aim of this paper is to determine whether the algorithmic learnability of finite-dimensional subspaces is an isomorphism invariant of the vector space, or if it is strictly governed by the constructive presentation of the space. To achieve this aim, we set the following objectives: (1) to construct an algorithmic learning procedure for spaces with a decidable linear dependence predicate; (2) to design an effective priority-style diagonalization construction of a computable vector space with an undecidable linear dependence predicate; and (3) to evaluate the convergence of algorithmic learners on this family.

Hypothesis: We hypothesize that despite the topological simplicity of finite-dimensional subspaces (each being finitely generated and closed), their Ex-learnability is not an invariant of the algebraic structure itself, but is strictly dictated by the algorithmic decidability of the linear dependence relation.

2 Materials and Methods

The research relies on the rigorous mathematical apparatus of computability theory, Turing machines, constructive algebraic systems, and Gold's formal model of identification in the limit from positive data. We assume familiarity with standard notations of recursion theory as outlined by Soare [15] and Rogers [14]. Let $\omega = \{0, 1, 2, \dots\}$ denote the set of natural numbers. Throughout this paper, we consider vector spaces over a computable field F ; without loss of generality, we fix $F = \mathbb{Q}$, the field of rational numbers with its standard effective encoding.

Definition 1. A vector space V over a computable field F is called a *computable vector space* if it satisfies the following effective conditions:

1. The underlying domain $|V|$ is a computable (recursive) subset of ω .
2. The vector addition operation $+: |V| \times |V| \rightarrow |V|$ and the scalar multiplication operation $\cdot: F \times |V| \rightarrow |V|$ are total computable functions.
3. The equality relation $=$ on $|V|$ is algorithmically decidable.

While the fundamental arithmetic operations of a computable vector space are guaranteed to be computable by definition, the deeper algebraic structure — such as basis extraction and subspace generation — may exhibit immense algorithmic complexity. The pivotal recursion-theoretic parameter of a vector space is the complexity of its linear dependence relation [11].

Definition 2. A computable vector space V is said to have a *decidable linear dependence relation* if the set of all linearly dependent finite tuples of vectors in V , defined as

$$D_V = \{(v_1, v_2, \dots, v_k) \in |V|^k \mid \text{there exist } \alpha_1, \dots, \alpha_k \in F \text{ not all zero such that } \sum_{i=1}^k \alpha_i v_i = 0\},$$

is a computable (recursive) set. Otherwise, the linear dependence relation is called undecidable.

2.1 The Algorithmic Learning Model

We formalize our investigation using the classical paradigm of explanatory learning in the limit from positive data (Ex-learning), originally introduced by Gold [5] and extensively formalized for indexed families by Jain et al. [9]. Let $\mathcal{L}$ be a designated family of subspaces of a computable vector space V .

A *positive presentation* (or a *text*) for a subspace $S \in \mathcal{L}$ is an infinite sequential enumeration of vectors $\sigma = (v_0, v_1, v_2, \dots)$ from V such that the range of the sequence equals the subspace precisely: $\text{rng}(\sigma) = \{v_0, v_1, v_2, \dots\} = S$. We denote by $\sigma[n] = (v_0, v_1, \dots, v_{n-1})$ the finite initial segment of length n of the text σ .

An algorithmic learner is a total Turing machine $\mathcal{M}$ that maps finite initial segments of texts to hypotheses. To represent hypotheses mathematically, we utilize canonical index numberings of finite sets [3]. Let D_k denote the canonical finite set of vectors in V associated with the index $k \in \omega$ under a standard bijection between ω and the collection of all finite subsets of ω . A hypothesis output by the machine, $h_n = \mathcal{M}(\sigma[n])$, is interpreted algebraically as the finite-dimensional subspace spanned by the finite generator set D_{h_n} :

$$\text{Hypothesis } h_n \equiv \text{span}(D_{h_n}).$$

Definition 3. A family of subspaces $\mathcal{L}$ is said to be *Ex-learnable* (Ex-identifiable in the limit) if there exists a Turing machine $\mathcal{M}$ such that for every target subspace (the specific subspace to be identified by the learner) $S \in \mathcal{L}$ and for every possible positive text σ for S , the infinite sequence of conjectures generated by $\mathcal{M}$,

$$\mathcal{M}(\sigma[0]), \mathcal{M}(\sigma[1]), \mathcal{M}(\sigma[2]), \dots,$$

converges to a correct canonical index of S . Formally, there must exist a stabilization stage $n_0 \in \omega$ and a fixed index $h^* \in \omega$ such that for all $n \geq n_0$, we have $\mathcal{M}(\sigma[n]) = h^*$ and $\text{span}(D_{h^*}) = S$.

Our methodological framework constructs explicit algorithms for learning when linear dependence is computable, and employs priority-based diagonalization constructions over quotient spaces $V_\infty/\text{span}(R)$ to prove non-learnability when dependence is undecidable.

3 Results and Discussion

In this section, we present our main theorems regarding the learnability of the family of all finite-dimensional subspaces, denoted henceforth as $\mathcal{L}_{\text{fin}}(V)$. We first establish that effective algebra guarantees learnability, and then prove our negative diagonalization result.

3.1 Positive Learning Theorem

We begin by demonstrating that "standard" computable vector spaces — such as the finite-dimensional rational spaces $\mathbb{Q}^n$ or the infinite-dimensional space of rational polynomials $\mathbb{Q}[x]$ with their natural representations — permit effective identification of all their finite-dimensional subspaces.

Theorem 1. *Let V be a computable vector space over $\mathbb{Q}$ with a decidable linear dependence relation. Then the family $\mathcal{L}_{\text{fin}}(V)$ of all finite-dimensional subspaces of V is Ex-learnable.*

Proof. By Definition 2, the assumption that V has a decidable linear dependence relation implies the existence of a total algorithmic procedure $\text{IsDependent}(v, \{b_1, \dots, b_k\})$ which accepts a vector $v \in V$ and a finite set of vectors $\{b_1, \dots, b_k\} \subset V$, terminating in finite time with output **True** if $v \in \text{span}(\{b_1, \dots, b_k\})$, and **False** otherwise.

We construct an explicit algorithmic learner $\mathcal{M}$ operating sequentially on any input text $\sigma = (v_0, v_1, \dots)$ for a target subspace $S \in \mathcal{L}_{\text{fin}}(V)$ through the following inductive rules:

1. **Initialization** ($n = 0$): The learner initializes its internal basis set as empty: $B_0 = \emptyset$. It computes the canonical index k_0 such that $D_{k_0} = \emptyset$, and outputs the initial hypothesis $\mathcal{M}(\sigma[0]) = k_0$, which corresponds to the zero subspace $\{0\}$.
2. **Inductive Step** ($n \geq 0$): Suppose at stage n , upon reading the initial segment $\sigma[n]$, the learner has maintained a linearly independent basis set $B_n \subset V$. When presented with the next vector $v_n = \sigma(n)$ from the text, the machine executes the procedure $\text{IsDependent}(v_n, B_n)$.
 - If the procedure returns **True**, the vector v_n is already spanned by the current basis and provides no new algebraic information. The learner sets $B_{n+1} = B_n$.

- If the procedure returns **False**, the vector v_n is linearly independent of B_n . The learner expands its internal basis: $B_{n+1} = B_n \cup \{v_n\}$.
3. **Hypothesis Output:** At stage $n + 1$, the machine computes the unique canonical index $k_{n+1} \in \omega$ corresponding to the finite set B_{n+1} (i.e., $D_{k_{n+1}} = B_{n+1}$) and outputs $\mathcal{M}(\sigma[n + 1]) = k_{n+1}$.

Convergence Analysis: Let $S \in \mathcal{L}_{\text{fin}}(V)$ be the target subspace presented by the text σ , and let $d = \dim(S)$ be its exact algebraic dimension. Since S is finite-dimensional by definition of the family, d is a finite natural number ($d < \infty$).

By construction, the learner's internal set B_n is strictly linearly independent at every stage n . Every time the algorithm encounters a vector where **IsDependent** evaluates to **False**, exactly one linearly independent vector is added to B_n , increasing the cardinality $|B_n|$ by exactly 1. Because all vectors in the text σ belong to S , any independent set formed from elements of σ can have a cardinality of at most d . Consequently, the basis update rule $B_{n+1} = B_n \cup \{v_n\}$ can be triggered at most d times across the entire infinite duration of the computation.

Since the text σ is total and enumerates all elements of S , it must eventually present a set of vectors that span S . Let $n_0 \in \omega$ be the earliest stage by which the initial segment $\sigma[n_0]$ contains a complete basis for S . By stage n_0 , the machine's internal set B_{n_0} must satisfy $\text{span}(B_{n_0}) = S$. For any subsequent stage $m \geq n_0$, any incoming vector $v_m \in S$ will inevitably satisfy $v_m \in \text{span}(B_{n_0})$. The procedure **IsDependent**(v_m, B_m) will permanently evaluate to **True**, ensuring $B_{m+1} = B_m = B_{n_0}$.

Therefore, the sequence of conjectures generated by $\mathcal{M}$ stabilizes precisely at stage n_0 on the canonical index $h^* = k_{n_0}$, where $\text{span}(D_{h^*}) = \text{span}(B_{n_0}) = S$. Thus, $\mathcal{M}$ Ex-learns $\mathcal{L}_{\text{fin}}(V)$ in the limit. $\square$

3.2 Negative Diagonalization Theorem

We now establish our main negative result, proving that the decidability of the linear dependence relation is not merely a convenient sufficient condition, but an absolute recursion-theoretic necessity for learnability.

Theorem 2. *There exists a computable vector space V over $\mathbb{Q}$ such that the family $\mathcal{L}_{\text{fin}}(V)$ of all its finite-dimensional subspaces is not Ex-learnable.*

Proof. We construct the target computable vector space V as an effective quotient space of the infinite-dimensional computable free vector space V_∞ over $\mathbb{Q}$. Let $G = \{g_0, g_1, g_2, \dots\}$ be a standard computable basis of V_∞ . Our objective is to enumerate a recursively enumerable (r.e.) set of linear relations $R \subset V_\infty$ in stages, defining the final space as the quotient structure:

$$V = V_\infty / \text{span}(R).$$

In this representation, enumerating a vector $w \in V_\infty$ into the relation set R during the construction effectively enforces the algebraic identity $w = 0$ in the quotient space V . Because R will be r.e., the quotient structure V satisfies all conditions of Definition 1, forming a valid computable vector space [11].

Let $\{\mathcal{M}_e\}_{e \in \omega}$ be a standard effective enumeration of all deterministic Turing machines, representing the complete universe of potential algorithmic learners. To guarantee that

no machine can Ex-learn $\mathcal{L}_{\text{fin}}(V)$, we must satisfy the following infinite sequence of diagonalization requirements for each $e \in \omega$:

$\mathcal{R}_e$: The machine $\mathcal{M}_e$ fails to Ex-identify the family $\mathcal{L}_{\text{fin}}(V)$.

Construction: We partition the infinite basis G of V_∞ into an infinite sequence of pairwise disjoint, infinite computable subsets $G_e = \{x_{e,0}, x_{e,1}, x_{e,2}, \dots\}$ for each $e \in \omega$. Each machine $\mathcal{M}_e$ is assigned the exclusive set of generators G_e to satisfy requirement $\mathcal{R}_e$. At any stage $s \in \omega$ of the construction, we maintain an index pointer $k(e, s)$ indicating the current "active testing generator" $x_{e,k(e,s)}$ for requirement $\mathcal{R}_e$. Initially, we set $R_0 = \emptyset$ and $k(e, 0) = 0$ for all e .

Stage s : We execute the following priority instructions simultaneously for each requirement $\mathcal{R}_e$ where $e \leq s$:

1. We construct a finite test sequence $\tau_{e,s}$ of length s consisting of the active generator vector $x_{e,k(e,s)}$ repeated s times:

$$\tau_{e,s} = (x_{e,k(e,s)}, x_{e,k(e,s)}, \dots, x_{e,k(e,s)}).$$

2. We simulate the computation of Turing machine $\mathcal{M}_e$ on the finite input segment $\tau_{e,s}$ for at most s computational steps. We check whether $\mathcal{M}_e$ terminates within s steps and outputs a canonical index $H_s = \mathcal{M}_e(\tau_{e,s})$ such that the hypothesized subspace $\text{span}(D_{H_s})$ has dimension ≥ 1 in the current quotient approximation $V_\infty/\text{span}(R_s)$.
3. **Case 1 (No reaction):** If $\mathcal{M}_e(\tau_{e,s})$ does not converge within s steps, or if it outputs an index corresponding to the zero subspace $\{0\}$, the machine has not yet conjectured a non-zero subspace. We preserve the algebraic independence of the active generator: set $R_{s+1} = R_s$ and maintain the pointer $k(e, s+1) = k(e, s)$.
4. **Case 2 (Diagonalization / Spoilage):** If $\mathcal{M}_e(\tau_{e,s})$ outputs a hypothesis H_s such that $\dim(\text{span}(D_{H_s})) \geq 1$, the learner "believes" that the non-zero generator $x_{e,k(e,s)}$ generates a valid 1-dimensional subspace. We immediately invalidate this hypothesis by enumerating the active generator into the relation set:

$$R_{s+1} = R_s \cup \{x_{e,k(e,s)}\}.$$

In the quotient space V , this enumeration forces $x_{e,k(e,s)} \equiv 0$. Consequently, the text presented to $\mathcal{M}_e$ up to this point is actually an enumeration of the zero subspace $\{0\}$. The machine's conjecture H_s , having dimension ≥ 1 , is rendered strictly false. To prepare for potential future recovery by $\mathcal{M}_e$, we increment the pointer to a fresh, uncontaminated generator: $k(e, s+1) = k(e, s) + 1$.

Verification: Let $R = \bigcup_{s \in \omega} R_s$. Since at each stage s we add only finitely many vectors to R via effective procedures, R is recursively enumerable. Hence, $V = V_\infty/\text{span}(R)$ is a computable vector space. Consider any arbitrary learner $\mathcal{M}_e$. Exactly one of the following two mutually exclusive outcomes must occur during the infinite execution:

Subcase 2a (Case 2 applies infinitely often): The learner $\mathcal{M}_e$ repeatedly conjectures non-zero subspaces upon seeing repetitions of active generators. Each time it does so, we collapse

the active generator to 0 and increment $k(e, s)$. Consequently, the infinite text σ_e generated by taking the limit of the test segments is simply an infinite sequence of zeros in V : $(0, 0, 0, \dots)$, which is the unique canonical text for the zero subspace $\{0\} \in \mathcal{L}_{\text{fin}}(V)$. However, because Case 2 is triggered infinitely many times, $\mathcal{M}_e$ outputs indices of dimension ≥ 1 infinitely often. Thus, $\mathcal{M}_e$ fails to stabilize on the correct hypothesis $\{0\}$, and convergence fails.

Subcase 2b (Case 2 applies only finitely many times): There exists a stabilization stage s_0 after which Case 2 is never triggered for $\mathcal{R}_e$. Let $N = k(e, s_0)$ be the final value of the index pointer. From stage s_0 onward, when fed the infinite constant text $\sigma_e = (x_{e,N}, x_{e,N}, x_{e,N}, \dots)$, the machine $\mathcal{M}_e$ either fails to converge or almost always conjectures the zero subspace $\{0\}$ (or an index where $\dim < 1$). However, because Case 2 was never triggered after stage s_0 , the vector $x_{e,N}$ is never enumerated into R . Since G_e is independent of all other generator sets in V_∞ , $x_{e,N}$ remains strictly linearly independent and non-zero in the quotient space V . The target text σ_e is therefore a valid positive presentation of the 1-dimensional subspace $S = \text{span}(\{x_{e,N}\}) \neq \{0\}$. Since $\mathcal{M}_e$ fails to conjecture an index generating S , it converges to an incorrect hypothesis.

In both subcases, requirement $\mathcal{R}_e$ is definitively satisfied. Because the enumeration of learners $\{\mathcal{M}_e\}_{e \in \omega}$ is exhaustive, no Turing machine can Ex-learn $\mathcal{L}_{\text{fin}}(V)$. $\square$

3.3 Discussion

The results established in Theorem 1 and Theorem 2 reveal fundamental insights into the nature of algorithmic identification in classical algebra, highlighting a profound dichotomy between syntactic structure and recursion-theoretic complexity.

In classical computability theory, two algebraic structures are recursively isomorphic if there exists a computable bijection between their domains that preserves all algebraic operations. In abstract algebra, any two vector spaces over $\mathbb{Q}$ having the same countable dimension are isomorphic. However, Theorem 2 proves that algorithmic learnability is *not* an isomorphism invariant. We can construct two algebraically isomorphic vector spaces — for instance, the standard space $V_1 = \mathbb{Q}^{(\omega)}$ with a decidable standard basis, and our quotient space $V_2 = V_\infty / \text{span}(R)$ from Theorem 2 — such that $\mathcal{L}_{\text{fin}}(V_1)$ is effortlessly Ex-learnable, whereas $\mathcal{L}_{\text{fin}}(V_2)$ is completely non-learnable by any algorithmic machine.

This finding contrasts sharply with the classical results of Harizanov and Stephan [8], who analyzed infinite-dimensional recursively enumerable subspaces. In the infinite-dimensional setting, the failure of learnability often arises from topological phenomena: an r.e. subspace cannot be learned if the space contains chains of infinitely many r.e. subspaces that accumulate limit points in the topology of positive presentations. Unlike the infinite-dimensional setting, finite-dimensional subspaces lack these topological obstacles. In our study, we restricted the hypothesis space strictly to $\mathcal{L}_{\text{fin}}(V)$, where every subspace is finitely generated, closed, and topologically isolated from such accumulation chains. Consequently, our negative result in Theorem 2 is particularly striking: it demonstrates that even when all topological barriers are completely eliminated, algorithmic learnability can still be fundamentally blocked by the pure recursion-theoretic limitations of the underlying vector space logic.

The fundamental barrier established by Theorem 2 has profound implications for modern data-driven artificial intelligence. Contemporary deep learning models operate purely

inductively, attempting to infer underlying invariants directly from positive examples (data streams). However, our results demonstrate that if the structural logic of the environment (such as the linear dependence relation in our quotient space) is algorithmically undecidable, no purely inductive model can reliably converge to the correct mathematical invariant. The inability of Turing machines to identify finite-dimensional subspaces in such spaces proves that data alone, regardless of its volume, is insufficient to overcome recursion-theoretic limitations.

This theoretical constraint rigorously justifies the necessity of neuro-symbolic AI architectures. As demonstrated in recent frameworks integrating machine learning with explicit logical ontologies (e.g., the neuro-symbolic systems explored by Kadir and Mosavi [12]), purely data-driven agents must be augmented with an explicit symbolic resolver. Without a symbolic module capable of grounding predictions in decidable logical rules, an AI system cannot guarantee the correct identification of algebraic structures. Consequently, the capacity of an agent to infer invariants of linear models is bounded not by the dimensionality of the data, but by the decidability of the underlying logical equality and linear dependence relations.

4 Conclusion

In this paper, we investigated the algorithmic learnability of finite-dimensional subspaces within computable vector spaces, applying Gold’s paradigm of identification in the limit from positive data.

We provided a theoretical characterization of learnability for this family. We proved that if a computable vector space possesses a decidable linear dependence relation, the family of its finite-dimensional subspaces is Ex-learnable via an effective basis-expansion algorithm. Conversely, using a priority-based diagonalization argument over quotient vector spaces, we constructed computable vector spaces where these subspaces cannot be algorithmically identified by any Turing machine.

These results show that algorithmic learnability is not an isomorphism invariant of the abstract algebraic structure. Instead, the feasibility of inductive inference depends entirely on the recursion-theoretic complexity — specifically the Turing degree of unsolvability — of the linear dependence relation.

Our findings have practical implications for automated reasoning and AI systems operating on mathematical structures. Specifically, they suggest that for an automated agent to successfully identify algebraic invariants from data, the underlying representation must ensure the decidability of basic linear dependence. Future research will extend this analysis to bounded learning models, such as anomaly-bounded learning (Ex_n) and PAC-learning, as well as to the learnability of submodules over computable Noetherian rings.

References

- [1] Angluin, Dana, and Carl H. Smith. “Inductive Inference: Theory and Methods.” *ACM Computing Surveys* 15, no. 3 (1983): 237–269.

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- [2] Badaev, Serikzhan A., and Sergey S. Goncharov. “Rogers Semilattices of Families of Arithmetic Sets.” *Algebra and Logic* 40, no. 5 (2001): 283–291.
 - [3] Ershov, Yuri L. “Theory of Numberings.” In *Handbook of Computability Theory*, edited by E. R. Griffor, 473–503. Amsterdam: Elsevier, 1999.
 - [4] Ershov, Yuri L., and Sergey S. Goncharov. *Constructive Models*. New York: Consultants Bureau, 2000.
 - [5] Gold, E. Mark. “Language Identification in the Limit.” *Information and Control* 10, no. 5 (1967): 447–474.
 - [6] Goncharov, Sergey S. “Countable Parametric Classes of Algebraic Systems.” *Algebra and Logic* 36, no. 6 (1997): 352–368.
 - [7] Harizanov, Valentina S. “Pure Computable Model Theory.” In *Handbook of Recursive Mathematics*, edited by Yu. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel, 3–114. Amsterdam: Elsevier, 1998.
 - [8] Harizanov, Valentina, and Frank Stephan. “On the Learnability of Vector Spaces.” *Journal of Computer and System Sciences* 73, no. 1 (2007): 109–122.
 - [9] Jain, Sanjay, Daniel Osherson, James Royer, and Arun Sharma. *Systems That Learn: An Introduction to Learning Theory*. 2nd ed. Cambridge, MA: MIT Press, 1999.
 - [10] Mal’tsev, Anatoly I. *The Metamathematics of Algebraic Systems: Collected Papers*. Vol. 66. Amsterdam: North-Holland Publishing Company, 1971.
 - [11] Metakides, George, and Anil Nerode. “Recursively Enumerable Vector Spaces.” *Annals of Mathematical Logic* 11, no. 2 (1977): 147–171.
 - [12] Kadir, M. A., and A. Mosavi. *NEURON: A Neuro-Symbolic System for Grounded Clinical Explainability*. arXiv preprint arXiv:2605.01189, 2026.
 - [13] Osherson, Daniel N., Michael Stob, and Scott Weinstein. *Systems That Learn: An Introduction to Learning Theory for Cognitive and Computer Scientists*. Cambridge, MA: MIT Press, 1986.
 - [14] Rogers, Hartley, Jr. *Theory of Recursive Functions and Effective Computability*. Cambridge, MA: MIT Press, 1987.
 - [15] Soare, Robert I. *Turing Computability: Theory and Applications*. Berlin: Springer-Verlag, 2016.
 - [16] Stephan, Frank, and Yuri Ventsov. “Learning Algebraic Structures.” *Theoretical Computer Science* 268, no. 2 (2001): 221–273.

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