ISSN 1563-0277 • eISSN 2617-4871

KazNU Science • ҚазҰУ Ғылымы • Наука КазНУ

ӘЛ-ФАРАБИ атындағы ҚАЗАҚ ҰЛТТЫҚ УНИВЕРСИТЕТІ

КАЗАХСКИЙ НАЦИОНАЛЬНЫЙ УНИВЕРСИТЕТ имени АЛЬ-ФАРАБИ

AL-FARABI KAZAKH NATIONAL UNIVERSITY

ХАБАРШЫ

МАТЕМАТИКА, МЕХАНИКА, ИНФОРМАТИКА СЕРИЯСЫ

ВЕСТНИК

СЕРИЯ МАТЕМАТИКА, МЕХАНИКА, ИНФОРМАТИКА

JOURNAL

OF MATHEMATICS, MECHANICS AND COMPUTER SCIENCE

4(120) 2023

ISSN 1563 – 0277
eISSN 2617 – 4871

ӘЛ-ФАРАБИ атындағы ҚАЗАҚ ҰЛТТЫҚ УНИВЕРСИТЕТІ

ХАБАРШЫ

Математика, механика, информатика сериясы

КАЗАХСКИЙ НАЦИОНАЛЬНЫЙ УНИВЕРСИТЕТ имени АЛЬ-ФАРАБИ

ВЕСТНИК

Серия математика, механика, информатика

AL-FARABI KAZAKH NATIONAL UNIVERSITY

Journal of Mathematics, Mechanics and Computer Science

№4 (120)

Алматы
«Қазақ университеті»
2023

Зарегистрирован в Министерстве информации и коммуникаций Республики Казахстан, свидетельство №16508-Ж от 04.05.2017 г. (Время и номер первичной постановки на учет №766 от 22.04.1992 г.). Язык издания: английский. Выходит 4 раза в год. Тематическая направленность: теоретическая и прикладная математика, механика, информатика.

Редакционная коллегия

научный редактор – *Б.Е. Кангужин, д.ф.-м.н., профессор, КазНУ им. аль-Фараби,*
заместитель научного редактора – *Д.И. Борисов, д.ф.-м.н., профессор, Институт математики с вычислительным центром Уфимского научного центра РАН, Башкирский государственный педагогический университет им. М. Акмуллы, Россия*
ответственный секретарь – *С.Е. Айтжанов, к.ф.-м.н., доцент, КазНУ им. аль-Фараби*

Гиви Пэйман (США); *Моханраджж Муругесан (Индия)*; *Эльбаз Ибрагим Мохаммед Абу Эльмагд (Египет)*; *Кабанихин С.И. (Россия)*; *Майнке М. (Германия)*; *Малышкин В.Э. (Россия)*; *Ракишева З.Б. (Казахстан)*; *Ружанский М. (Бельгия)*; *Сагитов С.М. (Швеция)*; *Сукочев Ф.А. (Австралия)*; *Тайманов И.А. (Россия)*; *Темляков В.Н. (США)*; *Шиничи Накасука (Япония)*; *Моханраджж Муругесан (Индия)*; *Гиви Пейман (США)*; *Дарибаев Беимбет Серикович (Казахстан)*; *Иманкулов Тимур Сакенович (Казахстан)*; *Сергей Горлач (Германия)*; *Триго Пауло (Португалия)*; *Минсу Хан (Южная Корея)*; *Эжилчелван Пол Девадосс (Великобритания)*; *Кантони Вирджинио (Италия)*; *Феррейра Хорхе (Бразилия)*; *Мохаммед Отман (Малайзия)*; *Альберто Кабада (Испания)*

Индексируется и участвует:

Научное издание
Вестник КазНУ. Серия "Математика, механика, информатика", № 4 (120) 2023.
Редактор – С.Е. Айтжанов. Компьютерная верстка – С.Е. Айтжанов

ИБ N 15242
Формат 60 × 84 1/8. Бумага офсетная. Печать цифровая. Объем 8,5 п.л.
Заказ N 132. Издательский дом "Қазақ университетi"
Казахского национального университета им. аль-Фараби. 050040, г. Алматы, пр.аль-Фараби, 71, КазНУ.
Отпечатано в типографии издательского дома "Қазақ университетi".

© КазНУ им. аль-Фараби, 2023

1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

IRSTI 27.31.21 DOI: https://doi.org/10.26577/JMMCS2023v120i4a1

A.J. Castro[1], L.K. Zhapsarbayeva[2*]
[1]Nazarbayev University, Kazakhstan, Astana
[2]L.N. Gumilyov Eurasian National University, Kazakhstan, Astana
[*]e-mail: leylazhk67@gmail.com

SOME LOCAL WELL POSEDNESS RESULTS IN WEIGHTED SOBOLEV SPACE $H^{1/3}$ FOR THE 3-KDV EQUATION

The paper analyses the local well posedness of the initial value problem for the k-generalized Korteweg-de Vries equation for $k = 3$ with irregular initial data. k-generalized Korteweg-de Vries equations serve as a model of magnetoacoustic waves in plasma physics, of the nonlinear propagation of pulses in optical fibers. The solvability of many dispersive nonlinear equations has been studied in weighted Sobolev spaces in order to manage the decay at infinity of the solutions. We aim to extend these researches to the k-generalized KdV with $k = 3$. For initial data in classical Sobolev spaces there are many results in the literature for several nonlinear partial differential equations. However, our main interest is to investigate the situation for initial data in Sobolev weighted spaces, which is less understood. The low regularity Sobolev results for initial value problems for this dispersive equation was established in unweighted Sobolev spaces with $s \geq 1/12$ and later further improved for $s \geq \frac{-1}{6}$. The paper improves these results for 3-KdV equation with initial data from weighted Sobolev spaces.
Key words: nonlinear equations, dispersive equations, contraction, semigroup, nonlinear propagation.

А.Х. Кастро[1], Л.Қ. Жапсарбаева[2*]
[1]Назарбаев Университеті, Қазақстан, Астана қ.
[2]Л.Н. Гумилев атындағы ЕҰУ, Қазақстан, Астана қ.
[*]e-mail: leylazhk67@gmail.com

Салмақтық Соболев кеңістігінде 3-КдФ теңдеуі үшін $H^{1/3}$ жергілікті қисындылық жайлы кейбір нәтижелер

Бұл жұмыста бастапқы деректері регулярлы емес $k = 3$ болған жағдайдағы k-жалпыланған Кортевег-де Фриз теңдеуі үшін бастапқы есептің локалдықисындылығына талдау жасалады. k-жалпыланған Кортевег-де Фриз теңдеулері плазма физикасындағы магнитоакустикалық толқындардың, және сонымен қатар оптикалық талшықтардағы импульстердің сызықты емес таралуының моделі ретінде қызмет етеді. Шешімдердің шексіздікте ыдырауын жақсырақ бақылау үшін, көптеген сызықтық емес дисперсиялық теңдеулердің шешілімділігі салмақтық Соболев кеңістігінде зерттеледі. Біздің мақсатымыз $k = 3$ болатын k-жалпыланған Кортевег-де Фриз теңдеуі үшін осы зерттеулерді жалғастыру болып табылады. Әдебиетте бастапқы деректері классикалық Соболев кеңістітерінде жататын бірқатар сызықты емес дербес дифференциалдық теңдеулер үшін көптеген нәтижелер бар. Дегенмен, біздің басты мүддеміз бастапқы деректері салмақтық Соболев кеңістігінде жатқан жағдайды зерттеу болып табылады, бұл жағдай аса түсініксіз болып табылады. Қарастырылып отырған дисперсия теңдеулері үшін бастапқы есептерүшін төменгі регулярлық Соболевтік нәтижелер салмақсыз Соболев кеңістігінде $s \geq 1/12$ мәндері үшін орнатылған және кейінірек $s \geq \frac{-1}{6}$ мәндері үшін де жақсартылған. Біз бұл нәтижелерді алынған бастапқы деректері салмақтық Соболев кеңістігінде жататын $k = 3$ болған жағдайдағы k-жалпыланған Кортевег-де Фризтеңдеуі үшін жақсартамыз.

© 2023 Al-Farabi Kazakh National University

Түйін сөздер: сызықты емес теңдеулер, дисперсиялық теңдеулер, сығу, жартылай топ, сызықтық емес таралу.

А.Х. Кастро[1], Л.К. Жапсарбаева[2*]
[1]Назарбаев Университет, Казахстан, г. Астана
[2]Евразийский национальный университет им. Л.Н. Гумилева, Казахстан, г. Астана
*e-mail: leylazhk67@gmail.com

Некоторые результаты о локальной корректности в весовом пространстве Соболева $H^{1/3}$ для уравнения 3-КдФ

В данной работе анализируется локальная корректность начальной задачи для нелинейного k-обобщенного уравнения Кортевега-де Фриза для $k = 3$ с нерегулярными начальными данными. k-обобщенные уравнения Кортевега-де Фриза служат моделью магнитоакустических волн в физике плазмы, а также нелинейного распространения импульсов в оптических волокнах. Разрешимость многих дисперсионных нелинейных уравнений изучены в весовых пространствах Соболева с целью лучшего управления распадом решений на бесконечности. Нашей целью является продолжить эти расследования для k-обобщенного уравнения Кортевега-де Фриза с $k = 3$. В литературе имеются множество результатов для ряда нелинейных уравнений в частных производных с начальными данными в классических пространствах Соболева. Однако наш основной интерес представляет исследование ситуации с начальными данными в весовых пространствах Соболева, которая остается менее понятной. Низко регулярные Соболевские результаты для рассматриваемых нелинейных дисперсионных уравнений были установлены в невесовых пространствах Соболева для значений $s \geq 1/12$ и позднее были улучшены для $s \geq \frac{-1}{6}$. В данной статье эти результаты были улучшены для уравнения 3-КдФ с начальными данными из весовых пространств Соболева.
Ключевые слова: нелинейные уравнения, дисперсионные уравнения, сжатие, полугруппа, нелинейное распространение.

1 Introduction

We investigate the Cauchy problem for the k-generalized Korteweg-de Vries equation with $k = 3$ (or briefly gKdV-3)

$$v_t + v_{xxx} + (v^4)_x = 0, \; x \in \mathbb{R}, \, t > 0 \tag{1}$$

with initial data $v(x, 0) = v_0(x)$, $x \in \mathbb{R}$, from weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)$. Equation (1) serves as a model of magnetoacoustic waves in plasma physics [13], of the nonlinear propagation of pulses in optical fibers [18].

The well-posedness of the initial value problem for the gKdV-3 equation was firstly established in the work of C. Kenig, G. Ponce and L. Vega [16] in classical Sobolev spaces with regularity $s \geq 1/12$ and later optimally improved by A. Grünrock [10] and T. Tao [23] for $s \geq -1/6$, using Bourgain's spaces techniques.

Inspired by T. Kato [12], in order for manage the decay of the solutions as $x \to \infty$, the several nonlinear dispersive equations has been investigated in weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)$ ([3–9, 21]). We aim to extend these researches to 3-KdV as we detail below.

We claim the Banach fixed point theorem to the integral equation version of the initial value problem (1), i.e.

$$v(x,t) = W(t)v_0(x) - \int_0^t W(t-\tau)(v^4)_x(x,\tau)\,d\tau, \qquad (2)$$

where $W(t)v_0(x)$ is the solution of the initial value problem for the associated linear partial differential equation, that introduced in (8) below.

2 Materials and methods

Our main result is the following theorem.

Theorem 1. *Suppose that $m \in [0, 1/6]$. For initial value v_0 from weighted Sobolev space $H^{1/3}(\mathbb{R}) \cap L^2(|x|^{2m}dx)$ there exist a unique solution v of the integral equation (2) that belongs to the weighted Sobolev space $v(\cdot,t) \in H^{1/3}(\mathbb{R}) \cap L^2(|x|^{2m}dx)$, $\quad t \in (0,T]$ for $T > 0$.*

We mentioned above the sharp Sobolev results (for $s \geq -1/6$). So, it is natural to improve the regularity s on the weighted Sobolev resuls for $0 < s < 1/3$. Indeed, in [6] we considered the situation for $s = 1/12 + \varepsilon$, employing a more delicate analysis.

Now we introduce the notations. For a constant $c > 0$ satisfying inequality $a \leq cb$, we write $a \lesssim b$. And, if $a \lesssim b$ and $b \lesssim a$, then we write $a \sim b$.

We denote by

$$\mathcal{F}(h)(\xi) := \int_{-\infty}^{\infty} \exp(-ix\xi)h(x)\,dx, \quad \xi \in \mathbb{R}$$

the Fourier transform of $h \in L^2(\mathbb{R})$ and by

$$\mathcal{F}^{-1}(h)(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ix\xi)h(\xi)\,d\xi, \quad x \in \mathbb{R}$$

its inverse Fourier transform.

Let $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$. The Sobolev space $H^s(\mathbb{R})$ can be defined by the norm

$$\|h\|_{H^s} := \Big(\int_{-\infty}^{\infty} \langle\xi\rangle^{2s}|\mathcal{F}(h)(\xi)|^2\,d\xi\Big)^{1/2},$$

where $s \in \mathbb{R}$ is the order of the Sobolev space. The inclusion $H^{s'}(\mathbb{R}) \subset H^s(\mathbb{R})$ holds for $s \leq s'$, that is,

$$\|h\|_{H^s} \lesssim \|h\|_{H^{s'}}. \qquad (3)$$

In order to treat functions defined in a space-time domain we introduce mixed norm spaces. Let $1 \leq p, q \leq \infty$. We say that $h \in L^p_x L^q_T$ if

$$\|h\|_{L^p_x L^q_T} := \Big\{\int_{-\infty}^{\infty} \Big(\int_0^T |h(x,t)|^q\,dt\Big)^{p/q} dx\Big\}^{1/p}$$

and $h \in L^q_T L^p_x$, if

$$\|h\|_{L^q_T L^p_x} := \Big\{ \int_0^T \Big(\int_{-\infty}^{\infty} |h(x,t)|^p \, dx \Big)^{q/p} dt \Big\}^{1/q}.$$

For $p = \infty$ or $q = \infty$, we have the definition involving the essential supremum.

The fractional derivative D^λ_x for $\lambda \in \mathbb{C}$ can be defined as the Fourier multiplier given by

$$\mathcal{F}(D^\lambda_x h)(\xi) := |\xi|^\lambda \mathcal{F}(h)(\xi).$$

Similarly, the operator $(1 + D^2_x)^\lambda$ is defined as follows

$$\mathcal{F}\big((1 + D^2_x)^\lambda h\big)(\xi) := (1 + |\xi|^2)^\lambda \mathcal{F}(h)(\xi).$$

Consequently, by Plancherel theorem we have

$$\|h\|_{H^s} \sim \|(1 + D^2_x)^{s/2} h\|_{L^2} \lesssim \|h\|_{L^2} + \|D^s_x h\|_{L^2}.$$

We exploit the Hilbert transform H introduced as

$$\mathcal{F}(Hh)(\xi) := -i\mathrm{sgn}(\xi)\mathcal{F}(h)(\xi).$$

Hence, D_x can be expressed via $\frac{\partial}{\partial x}$ in the following way $D_x = H\frac{\partial}{\partial x}$.

We recall the the fractional version of Leibniz rule ([16, Theorem A.8]). Let $\lambda \in (0,1)$, $\lambda_1, \lambda_2 \in [0, \lambda]$ such that $\lambda = \lambda_1 + \lambda_2$. And let $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$ with

$$1/p = 1/p_1 + 1/p_2, \quad 1/q = 1/q_1 + 1/q_2, \tag{4}$$

then

$$\|D^\lambda_x(gh) - gD^\lambda_x h - hD^\lambda_x g\|_{L^p_x L^q_T} \lesssim \|D^{\lambda_1}_x g\|_{L^{p_1}_x L^{q_1}_T} \|D^{\lambda_2}_x h\|_{L^{p_2}_x L^{q_2}_T}. \tag{5}$$

Also, $q_1 = \infty$ for $\lambda_1 = 0$.

We evoke the derivative chains rules in fractional calculus ([16, Theorem A.6])

$$\|D^\lambda_x F(h)\|_{L^p_x L^q_T} \lesssim \|F'(h)\|_{L^{p_1}_x L^{q_1}_T} \|D^\lambda_x h\|_{L^{p_2}_x L^{q_2}_T} \tag{6}$$

with $0 < \lambda < 1$, $1 < p, p_1, p_2, q, q_2 < \infty$ and $1 < q_1 \leq \infty$ such that (4).

The solution of IVP for Airy equation

$$\begin{cases} v_t + v_{xxx} = 0, & x \in \mathbb{R},\ t > 0, \\ v(x,0) = v_0(x), & x \in \mathbb{R} \end{cases} \tag{7}$$

can be represented as $v(x,t) = W(t)v_0(x)$, where we denote by $W(t)$ the Fourier multiplier defined as

$$\mathcal{F}(W(t)v_0)(\xi) := \exp(it\xi^3)\mathcal{F}(v_0)(\xi). \tag{8}$$

Plancherel theorem implies that

$$\|W(t)v_0\|_{L^2_x} \sim \|v_0\|_{L^2}. \tag{9}$$

Some properties of the semigroup $\{W(t)\}_{t>0}$ can be applied to prove the theorem 2. We recall the estimates for semigroup from [14, Theorem 2.4]

$$\|W(t)v_0\|_{L^8_x L^8_T} \lesssim \|v_0\|_{L^2}, \tag{10}$$

and from [22, Theorem 2]

$$\|W(t)v_0\|_{L^6_x L^\infty_T} \lesssim \|v_0\|_{H^{1/3}} \tag{11}$$

and from [16, Theorem 3.5]

$$\|\frac{\partial}{\partial x}W(t)v_0\|_{L^\infty_x L^2_T} \lesssim \|v_0\|_{L^2}. \tag{12}$$

We exploit in Section 3 bounds of the Airy semigroup, that we present in the following lemma.

Lemma 1. *Suppose that* $v_0 \in H^{1/3}(\mathbb{R})$*. Then,*

$$\|\frac{\partial}{\partial x}W(t)v_0\|_{L^{24}_x L^{8/3}_T} \lesssim \|v_0\|_{H^{1/3}}. \tag{13}$$

Proof. First we construct operators

$$A_z := D_x^{4z/3}(1 + D_x^2)^{-1/6}W(t)$$

which are analytic for $z \in \mathbb{C}$, $0 \le \operatorname{Re} z \le 1$. The estimates (11) above implies

$$\|A_{iy}v_0\|_{L^6_x L^\infty_T} = \|W(t)(1 + D_x^2)^{-1/6}D_x^{4iy/3}v_0\|_{L^6_x L^\infty_T} \lesssim \|D_x^{4iy/3}v_0\|_{L^2} = \|v_0\|_{L^2},$$

for any $y \in \mathbb{R}$ and also (12) implies that

$$\begin{aligned}\|A_{1+iy}v_0\|_{L^\infty_x L^2_T} &= \|\frac{\partial}{\partial x}W(t)D_x^{(1+4iy)/3}(1 + D_x^2)^{-1/6}Hv_0\|_{L^\infty_x L^2_T} \\ &\lesssim \|D_x^{1/3}(1 + D_x^2)^{-1/6}Hv_0\|_{L^2} \le \|Hv_0\|_{L^2} \sim \|v_0\|_{L^2}.\end{aligned}$$

Consequently, by Stein's theorem [1] for any $\theta \in (0, 1)$ and $p, q \in [1, \infty]$ such that

$$\frac{1}{p} = \frac{1-\theta}{6} + \frac{\theta}{\infty}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$$

we obtain $\|A_\theta v_0\|_{L^p_x L^q_T} \lesssim \|v_0\|_{L^2}$. Thus, we have $\|A_{3/4}v_0\|_{L^{24}_x L^{8/3}_T} \lesssim \|v_0\|_{L^2}$ for $\theta = 3/4$. It follows that (13).

We present the following bounds [16] that

$$\left\|\frac{\partial}{\partial x}\int_0^t W(t-\tau)h(\cdot,\tau)\,d\tau\right\|_{L^\infty_T L^2_x} \lesssim \|h\|_{L^1_x L^2_T} \tag{14}$$

and [9]

$$\Big\| \int_0^t W(t-\tau)h(\cdot, t\tau)\, d\tau \Big\|_{L^\infty_T L^2_x} \le \|h\|_{L^{q'}_T L^{p'}_x}, \tag{15}$$

for $p \ge 2$ and

$$\frac{1}{q} = \frac{1}{6} - \frac{1}{3p}, \quad \frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}.$$

Now we recall Fonseca-Linares-Ponce pointwise formula established in [8] which allows to commute fractional powers $|x|^m$ and the Airy semigroup $W(t)$, with the proper adjustments. Namely, the following identity

$$|x|^m W(t)v_0(x) = W(t)(|x|^m v_0)(x) + W(t)\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}(v_0)(\xi)](x) \tag{16}$$

holds for all $t > 0$, $v_0 \in H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)$, with $0 < s < 2$ and $0 < m \le s/2$, and almost any $x \in \mathbb{R}$. Also, the L^2–norm of the last term can be bound as followa

$$\|\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}(v_0)(\xi)]\|_{L^2_x} \lesssim (1+t)\big(\|v_0\|_{L^2} + \|D^{2m}_x v_0\|_{L^2}\big). \tag{17}$$

We note that only the particular case of $s = 2\lambda$ and $m = \lambda$, for $0 < \lambda < 1$ is considered in [8].

3 Proof of Theorem 2

In Section 3.1 we treat the the initial value problem for the 3-KdV in the Sobolev space. Previously we noted that the well posedness of the initial value problem for the 3-KdV in the Sobolev space $H^{1/3}(\mathbb{R})$ is already known. The local well posedness results for 3-KdV was proved in classical Sobolev spaces with $s \ge 1/12$ in [16, Theorem 2.6]. Then this result was extended up to $s \ge -1/6$ in [10, 23]. Nevertheless, the local well posedness of the IVP for 3-KdV in weighted Sobolev spaces with regularity $s \le 1/3$ is interesting open question. Inspired by Kenig-Ponce-Vega ([16, pp. 583–585]) and Fonseca-Linares-Ponce ([8, pp. 5364–5366]) works, we prove our new local well posedness result in weighted Sobolev spaces (Section 3.2).

By using the Banach fixed-point theorem to the mapping

$$\Psi(v) := W(t)v_0 - \int_0^t W(t-\tau)(v^4)_x(\cdot, \tau)\, d\tau,$$

our goal is to establish that this mapping is a contraction on a conveniently chosen subspace of $L^\infty_T H^{1/3}_x \cap L^\infty_T L^2_x(|x|^{2m}dx)$.

3.1 Unweighted case ($m = 0$)

Let $Y^\delta_T := \{v : \|v\|_{Y_T} \le \delta\}$, will be the complete metric space (with δ, $T > 0$ that are fixed) with the norm

$$\|v\|_{Y_T} := \sum_{j=1}^{6} \sigma^T_j(v), \tag{18}$$

where

$$\sigma_1^T(v) := \|v\|_{L_T^\infty H_x^{1/3}},\;\; \sigma_4^T(v) := \|D_x^{1/3} v_x\|_{L_x^\infty L_T^2},\;\; \sigma_2^T(v) := \|v_x\|_{L_x^{24} L_T^{8/3}},$$

$$\sigma_5^T(v) := \|v\|_{L_x^6 L_T^\infty},\;\; \sigma_3^T(v) := \|D_x^{1/3} v\|_{L_x^8 L_T^8},\;\; \sigma_6^T(v) := \|v_x\|_{L_x^\infty L_T^2}.$$

Step 1. First, we will prove that Ψ is well defined on Y_T^δ. Now show that $\Psi(v) \in Y_T^\delta$ for any $v \in Y_T^\delta$, that is,

$$\|\Psi(v)\|_{Y_T} = \sum_{j=1}^{6} \sigma_j^T(\Psi(v)) \le \delta. \tag{19}$$

We note that the treatment of the terms on the left hand side of (19) can be converted to $L_T^2 L_x^2$–norm of $D_x^{1/3}(v^4)_x$ and $(v^4)_x$. Leibniz rule (5), the fractional derivatives chain rule (6) and Hölder integral inequality imply that

$$\begin{aligned}
\|D_x^{1/3}(v^4)_x\|_{L_T^2 L_x^2} &\sim \|D_x^{1/3}(v^3 v_x)\|_{L_x^2 L_T^2} \\
&\le \|D_x^{1/3}(v^3 v_x) - v^3 D_x^{1/3} v_x - v_x D_x^{1/3}(v^3)\|_{L_x^2 L_T^2} \\
&\quad + \|v^3 D_x^{1/3} v_x\|_{L_x^2 L_T^2} + \|v_x D_x^{1/3}(v^3)\|_{L_x^2 L_T^2} \\
&\lesssim \|v^2\|_{L_x^3 L_T^\infty} \|D_x^{1/3} v\|_{L_x^8 L_T^8} \|v_x\|_{L_x^{24} L_T^{8/3}} + \|v\|_{L_x^6 L_T^\infty}^3 \|D_x^{1/3} v_x\|_{L_x^\infty L_T^2} \\
&= \|v\|_{L_x^6 L_T^\infty}^2 \|D_x^{1/3} v\|_{L_x^8 L_T^8} \|v_x\|_{L_x^{24} L_T^{8/3}} + \|v\|_{L_x^6 L_T^\infty}^3 \|D_x^{1/3} v_x\|_{L_x^\infty L_T^2} \\
&= \big(\sigma_5^T(v)\big)^2 \sigma_3^T(v) \sigma_2^T(v) + \big(\sigma_5^T(v)\big)^3 \sigma_4^T(v) \lesssim \|v\|_{Y_T}^4.
\end{aligned} \tag{20}$$

We observe that (20) motivates the choice of the norms σ_2^T, σ_3^T, σ_4^T and σ_5^T.

Otherwise, the necessity of the norm σ_6^T can be justified as below

$$\begin{aligned}
\|(v^4)_x\|_{L_T^2 L_x^2} &\sim \|v^3 v_x\|_{L_x^2 L_T^2} \le \|v^3\|_{L_x^2 L_T^\infty} \|v_x\|_{L_x^\infty L_T^2} = \|v\|_{L_x^6 L_T^\infty}^3 \|v_x\|_{L_x^\infty L_T^2} \\
&= \big(\sigma_5^T(v)\big)^3 \sigma_6^T(v) \le \|v\|_{Y_T}^4.
\end{aligned} \tag{21}$$

Now we will analyse the norms $\sigma_j^T(\Psi(v))$, $j = 1, \ldots, 6$, which rely on the Airy semigroup estimates and the estimates (20) and (21) that we deduced.

Plancherel formula, Minkowski inequality, (9) and Hölder integral inequality give us

$$\begin{aligned}
\sigma_1^T(\Psi(v)) &\lesssim \|W(t) v_0\|_{L_T^\infty L_x^2} + \int_0^T \|W(t-\tau)(v^4)_x(\cdot,\tau)\|_{L_T^\infty L_x^2}\, d\tau \\
&\quad + \|W(t) D_x^{1/3} v_0\|_{L_T^\infty L_x^2} + \int_0^T \|W(t-\tau) D_x^{1/3}(v^4)_x(\cdot,\tau)\|_{L_T^\infty L_x^2}\, d\tau \\
&\le \|v_0\|_{H^{1/3}} + T^{1/2} \|(v^4)_x\|_{L_T^2 L_x^2} + T^{1/2} \|D_x^{1/3}(v^4)_x\|_{L_T^2 L_x^2} \\
&\lesssim \|v_0\|_{H^{1/3}} + T^{1/2} \|v\|_{Y_T}^4.
\end{aligned} \tag{22}$$

Here we are allowed to permuted $D_x^{1/3}$ and $W(t)$ since both are Fourier multipliers.

In the same way, by exploiting Lemma 2, the estimates (10), (12), (11) and the Sobolev embedding theorem(3) we deduce

$$\sum_{j=2}^{6}\sigma_j^T(\Psi(v)) \lesssim \|v_0\|_{H^{1/3}} + \|(v^4)_x\|_{L^1_T L^2_x} + \|D_x^{1/3}(v^4)_x\|_{L^1_T L^2_x}$$
$$\lesssim \|v_0\|_{H^{1/3}} + T^{1/2}\|v\|^4_{Y_T}. \tag{23}$$

Therefore, if $v \in Y_T^\delta$, collecting (22) and (23) one gives

$$\|\Psi(v)\|_{Y_T} \le C\|v_0\|_{H^{1/3}} + CT^{1/2}\delta^4$$

for some constant $C > 0$. Consequently, taking

$$\delta := 2C\|v_0\|_{H^{1/3}} \tag{24}$$

and choosing $T > 0$ such that

$$\frac{\delta}{2} + CT^{1/2}\delta^4 \le \delta, \tag{25}$$

we get (19).

Step 2. Secondly, we will show that Ψ is a contraction on Y_T^δ. Let $v, w \in Y_T^\delta$, for δ defined in (24). Our goal is to show that

$$\|\Psi(v) - \Psi(w)\|_{Y_T} \le K\|v - w\|_{Y_T} \tag{26}$$

for some $0 < K < 1$ and T sufficiently small to specify below. We have

$$\Psi(v) - \Psi(w) = \int_0^t W(t-\tau)(v^4 - w^4)_x \, dt', \tag{27}$$

then we need to prove

$$\|(v^4 - w^4)_x\|_{L^2_T L^2_x} \lesssim \delta^3\|v - w\|_{Y_T} \tag{28}$$

and

$$\|D_x^{1/3}(v^4 - w^4)_x\|_{L^2_T L^2_x} \lesssim \delta^3\|v - w\|_{Y_T}. \tag{29}$$

Really, using the same argument as in the Step 1 and invoking (28) and (29), instead of (20) and (21), for some $C > 0$ we obtain

$$\|\Psi(v) - \Psi(w)\|_{Y_T} \le CT^{1/2}\delta^3\|v - w\|_{Y_T}. \tag{30}$$

Consequently, by taking $T > 0$ such that $CT^{1/2}\delta^3 < 1$ and (25), we conclude that (26).

Notice that

$$v^4 - w^4 = (v - w)(v^3 + v^2 w + v w^2 + w^3) \tag{31}$$

and differentiating,

$$(v^4 - w^4)_x = (v^3 + v^2 w + v w^2 + w^3)\,(v-w)_x \\ + (v-w)\big(3v^2 v_x + 2vwv_x + v^2 w_x + w^2 v_x + 2vww_x + 3w^2 w_x\big).$$

Therefore, (28) and (29) can be converted to

$$\|u_1 u_2 u_3 (u_4)_x\|_{L^2_x L^2_T} \lesssim \delta^3 \|v-w\|_{Y_T} \tag{32}$$

and

$$\|D_x^{1/3}(u_1 u_2 u_3 (u_4)_x)\|_{L^2_x L^2_T} \lesssim \delta^3 \|v-w\|_{Y_T} \tag{33}$$

for $u_1, u_2, u_3, u_4 \in \{v, w, v-w\}$ and one, and only one, of the u_j's being equal to $v-w$.

Inequality (32) can be proved by Hölder integral inequality

$$\|u_1 u_2 u_3 (u_4)_x\|_{L^2_x L^2_T} \le \|u_1 u_2 u_3\|_{L^2_x L^\infty_T} \|(u_4)_x\|_{L^\infty_x L^2_T} \le \prod_{j=1}^{3} \|u_j\|_{L^6_x L^\infty_T} \|(u_4)_x\|_{L^\infty_x L^2_T} \\ \le \prod_{j=1}^{3} \sigma_5^T(u_j)\sigma_6^T(u_4) \le \prod_{j=1}^{4} \|u_j\|_{Y_T} \le \delta^3 \|v-w\|_{Y_T}.$$

We split the proof of (33) a in a few parts. By using the same argument as in (20), we obtain

$$\|D_x^{1/3}(u_1 u_2 u_3 (u_4)_x)\|_{L^2_x L^2_T} \lesssim \|D_x^{1/3}(u_1 u_2 u_3)\|_{L_x^{24/11} L_T^8} \|(u_4)_x\|_{L_x^{24} L_T^{8/3}} \\ + \|u_1 u_2 u_3\|_{L^2_x L^\infty_T} \|D_x^{1/3}(u_4)_x\|_{L^\infty_x L^2_T} \\ = \|D_x^{1/3}(u_1 u_2 u_3)\|_{L_x^{24/11} L_T^8}\, \sigma_2^T(u_4) + \prod_{j=1}^{3} \sigma_5^T(u_j)\sigma_4^T(u_4). \tag{34}$$

Further, the Leibniz rule (5) and Hölder integral inequality give us

$$\|D_x^{1/3}(u_1 u_2 u_3)\|_{L_x^{24/11} L_T^8} \lesssim \|u_1 u_2 D_x^{1/3} u_3\|_{L_x^{24/11} L_T^8} + \|u_3 D_x^{1/3}(u_1 u_2)\|_{L_x^{24/11} L_T^8} \\ + \|u_1 u_2\|_{L^3_x L^\infty_T} \|D_x^{1/3} u_3\|_{L^8_x L^8_T} \\ \lesssim \sigma_5^T(u_3) \|D_x^{1/3}(u_1 u_2)\|_{L_x^{24/7} L_T^8} + \prod_{j=1}^{2} \sigma_5^T(u_j)\sigma_3^T(u_3) \tag{35}$$

and

$$\|D_x^{1/3}(u_1 u_2)\|_{L_x^{24/7} L_T^8} \lesssim \|u_1 D_x^{1/3} u_2\|_{L_x^{24/7} L_T^8} + \|u_2 D_x^{1/3} u_1\|_{L_x^{24/7} L_T^8} \\ + \|u_1\|_{L^6_x L^\infty_T} \|D_x^{1/3} u_2\|_{L^8_x L^8_T} \\ \lesssim \sigma_5^T(u_1)\sigma_3^T(u_2) + \sigma_5^T(u_2)\sigma_3^T(u_1). \tag{36}$$

Putting together (34)–(36) we deduce (33).

3.2 Weighted case $(0 < m \leq 1/6)$

Let us define the space

$$Z_T^\delta := \{v \, : \, \|v\|_{Z_T} < \delta\}$$

for some suitably taken $\delta, T > 0$, with

$$\|v\|_{Z_T} := \|v\|_{Y_T} + \||x|^m v\|_{L_T^\infty L_x^2} \tag{37}$$

and $\|v\|_{Y_T}$ introduced in (18).

Step 1a. First, we establish that Ψ is well defined on Z_T^δ. Above we examined the Y_T–norm of $\Psi(v)$. In this section we focus on $L_T^\infty L_x^2$–norm of $|x|^m\Psi(v)$. We can write

$$\begin{aligned}\||x|^m \Psi(v)\|_{L_T^\infty L_x^2} &\leq \||x|^m W(t) v_0\|_{L_T^\infty L_x^2} + \Big\| |x|^m \int_0^t W(t-\tau)(v^4)_x \, d\tau \Big\|_{L_T^\infty L_x^2} \\ &=: I + II.\end{aligned}$$

By (16), (17), Plancherel theorem (9) and (3), we control the linear term

$$\begin{aligned} I &\leq \|W(t)(|x|^m v_0)\|_{L_T^\infty L_x^2} + \|W(t)(\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}(v_0)(\xi)])\|_{L_T^\infty L_x^2} \\ &\lesssim \||x|^m v_0\|_{L^2} + (1+T)\big(\|v_0\|_{L^2} + \|D_x^{2m} v_0\|_{L^2}\big) \\ &\lesssim \||x|^m v_0\|_{L^2} + (1+T)\|v_0\|_{H^{1/3}}. \end{aligned} \tag{38}$$

Let φ be a compact support, such that that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $(-1, 1)$. Using the pointwise formula (16) and Minkowski integral inequality, we split the nonlinear term II as follows

$$\begin{aligned} II &\leq \int_0^T \Big\| W(t-\tau)\Big(|x|^m \varphi(x)(v^4)_x\Big)\Big\|_{L_T^\infty L_x^2} d\tau \\ &\quad + \Big\| \int_0^t W(t-\tau)\frac{\partial}{\partial x}[|x|^m(1-\varphi(x))v^4] \, d\tau \Big\|_{L_T^\infty L_x^2} \\ &\quad + \Big\| \int_0^t W(t-\tau)\Big(\frac{\partial}{\partial x}\big\{|x|^m(1-\varphi(x))\big\}v^4\Big) \, d\tau \Big\|_{L_T^\infty L_x^2} \\ &\quad + \int_0^T \Big\| W(t-\tau)\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}((v^4)_x))]\Big\|_{L_T^\infty L_x^2} d\tau \\ &=: II_1 + II_2 + II_3 + II_4. \end{aligned} \tag{39}$$

The estimates (9) and (21), the compact support of φ, Hölder integral inequality imply that

$$II_1 \lesssim \int_0^T \||x|^m \varphi(x)(v^4)_x\|_{L_x^2} \, d\tau \lesssim T^{1/2}\|(v^4)_x\|_{L_T^2 L_x^2} \lesssim T^{1/2}\|v\|_{Z_T}^4. \tag{40}$$

By (14) we bound II_2

$$II_2 \lesssim \||x|^m(1-\varphi(x))v^4\|_{L^1_xL^2_T} \lesssim \||x|^m v^4\|_{L^1_xL^2_T} \le \||x|^m v\|_{L^2_xL^2_T}\|v^3\|_{L^2_xL^\infty_T}$$
$$= \||x|^m v\|_{L^2_TL^2_x}\|v\|^3_{L^6_xL^\infty_T} \le T^{1/2}\||x|^m v\|_{L^\infty_TL^2_x}\big(\sigma_5^T(v)\big)^3 \le T^{1/2}\|v\|^4_{Z_T}. \tag{41}$$

By invoking the semigroup property (15) with $q=18$ and $p=3$ ($q'=18/17$ and $p'=3/2$) and Minkowski's integral inequality, we bound the following term

$$II_3 \lesssim \|\frac{\partial}{\partial x}[|x|^m(1-\varphi(x)]v^4\|_{L^{18/17}_TL^{3/2}_x} \lesssim \|v^4\|_{L^{18/17}_TL^{3/2}_x}$$
$$\lesssim T^\theta\|v\|^4_{L^\infty_TL^6_x} \le T^\theta\|v\|^4_{L^6_xL^\infty_T} = T^\theta\big(\sigma_5^T(v)\big)^4 \le T^\theta\|v\|^4_{Z_T} \tag{42}$$

for some $\theta>0$. Finally, formulas (9), (17), (20) and (21) allow to deduce

$$II_4 \sim \int_0^T \Big\|\mathcal{F}^{-1}[\Phi_{t,m}(\mathcal{F}((v^4)_x))]\Big\|_{L^\infty_TL^2_x} d\tau$$
$$\lesssim (1+T)T^{1/2}\big(\|(v^4)_x\|_{L^2_TL^2_x} + \|D_x^{1/3}(v^4)_x\|_{L^2_TL^2_x}\big)$$
$$\lesssim (1+T)T^{1/2}\|v\|^4_{Z_T}. \tag{43}$$

Finally, the bounds (23), (38)–(43) give

$$\|\Psi(v)\|_{Z_T} \le C(1+T^\theta)\|v_0\|_{H^{1/3}} + C\||x|^m v_0\|_{L^2} + CT^\theta\delta^4$$

for $v\in Z_T^\delta$, $C,\theta>0$. Therefore, if we take

$$\delta := 2C\big(\|v_0\|_{H^{1/3}} + \||x|^m v_0\|_{L^2}\big) \tag{44}$$

and T such that

$$\frac{\delta}{2} + CT^\theta\big(\|v_0\|_{H^{1/3}} + \delta^4\big) \le \delta, \tag{45}$$

then the following inequality holds

$$\|\Psi(v)\|_{Z_T} \le \delta.$$

Consequently, Ψ maps Z_T^δ into itself.

Step 2a. Now we need to prove that Ψ is a contraction on Z_T^δ. Suppose that $v,w\in Z_T^\delta$, where δ from (44) and T to determine in a moment. For some $\theta>0$ to establish the following estimate

$$\big\||x|^m\big(\Psi(v)-\Psi(w)\big)\big\|_{L^\infty_TL^2_x} \lesssim T^\theta\delta^3\|v-w\|_{Z_T} \tag{46}$$

we is the main goal of the part Step 2a. Analogously, using the same argument as in estimating the nonlinear term II in Step 1a, we bound the left hand side norm of (46).

Applying the expression (27) we change v^4 by $v^4 - w^4$. We apply (28) and (29), instead of (20) and (21), for the new factors related to II_1 and II_4 . As for II_2 and II_3 it suffices to insert

$$\||x|^m(v^4 - w^4)\|_{L^1_x L^2_T} \lesssim T^\theta \delta^3 \|v - w\|_{Z_T}$$

and

$$\|v^4 - w^4\|_{L^{18/17}_T L^{3/2}_x} \lesssim T^\theta \delta^3 \|v - w\|_{Z_T}$$

in (41) and (42), respectively.

Furthermore, by using expressions (31) the last inequalities can be rewritten as follows

$$\||x|^m(v - w)u_1u_2u_3\|_{L^1_x L^2_T} \lesssim T^\theta \delta^3 \|v - w\|_{Z_T}$$

and

$$\|(v - w)u_1u_2u_3\|_{L^{18/17}_T L^{3/2}_x} \lesssim T^\theta \delta^3 \|v - w\|_{Z_T},$$

where u_1, u_2, u_3 represent the functions v or w. Really, these inequalities can be obtained by Hölder's inequality,

$$\begin{aligned}
\||x|^m(v - w)u_1u_2u_3\|_{L^1_x L^2_T} &\leq \||x|^m(v - w)\|_{L^2_x L^2_T} \|u_1u_2u_3\|_{L^2_x L^\infty_T} \\
&\leq T^{1/2} \||x|^m(v - w)\|_{L^\infty_T L^2_x} \prod_{j=1}^{3} \|u_j\|_{L^6_x L^\infty_T} \leq T^{1/2}\delta^3 \|v - w\|_{Z_T}
\end{aligned}$$

and

$$\begin{aligned}
\|(v - w)u_1u_2u_3\|_{L^{18/17}_T L^{3/2}_x} &\leq T^{17/18} \|(v - w)u_1u_2u_3\|_{L^{3/2}_x L^\infty_T} \\
&\leq T^{17/18} \|v - w\|_{L^6_x L^\infty_T} \prod_{j=1}^{3} \|u_j\|_{L^6_x L^\infty_T} \leq T^{17/18}\delta^3 \|v - w\|_{Z_T}.
\end{aligned}$$

In summary, collecting (30) and (46) we obtain

$$\|\Psi(v) - \Psi(w)\|_{Z_T} \leq CT^\theta \delta^3 \|v - w\|_{Z_T}$$

for some $C, \theta > 0$.

Finally, we prove that Ψ is a contraction on Z^δ_T for $T > 0$ such that $CT^\theta\delta^3 < 1$ and (45).

References

[1] Benedek A., Panzone R., *The space L^p, with mixed norm*, Duke Math. J., 28 (1961): 301–324.

[2] Bourgain J., *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation*, Geom. Funct. Anal., 3 (1993): 209–262.

[3] Bustamante E., Jimenez J., Mejia J., *Cauchy problems for fifth order KDV equations in weighted Sobolev spaces*, Electron. J. Differential Equations, (2015): 1–24.

[4] Bustamante E., Jimenez J., Mejia J., *The Zakharov-Kuznetsov equation in weighted Sobolev spaces*, J. Math. Anal. Appl., 433 (2016): 149–175.

[5] Carvajal X., Gamboa P., *Persistence of solutions to nonlinear evolution equations in weighted Sobolev spaces*, Electronic Journal of Differential Equations, 169 (2010): 1—10.

[6] Castro A.J, Esfahani A., Zhapsarbayeva L., *A note on the quartic generalized Korteweg-de Vries equation in weighted Sobolev spaces*, to appear in Nonlinear Anal. (arXiv:2207.12727v2).

[7] Castro A.J., Jabbarkhanov K., Zhapsarbayeva L., *The Nonlinear Schrödinger-Airy equation in weighted Sobolev spaces*, Nonlinear Anal., 223 (2022): 113068.

[8] Fonseca G., Linares F., Ponce G., *On persistence properties in fractional weighted space*, Proc. Amer. Math. Soc., 143 (2015): 5353–5367.

[9] Ginibre J., Tsutsumi Y., *Uniqueness of solutions for the generalized Korteweg-de Vries equation*, SIAM J. Math. Anal., 20 (1989): 1388–1425.

[10] Grünrock A., *A bilinear Airy-estimate with application to gKdV-3*, Differ. Integral Equ., 18 (2005): 1333–1339.

[11] Guo Z., *Global well-posedness of Korteweg-de Vries equation in* $H^{-3/4}$, J. Math. Pur. Appl., 91 (2009): 583–597.

[12] Kato T., *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, in Studies in applied mathematics, vol. 8 of Adv. Math. Suppl. Stud., Academic Press, New York. (1983): 93–128.

[13] Kakutani T., Ono H., *Weak nonlinear hydromagnetics waves in a cold collision-free plasma*, J. Phys. Soc. Jap., 5 (1969): 1305–1318.

[14] Kenig C.E., Ponce G., Vega L., *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J., 10 (1991): 33–68.

[15] Kenig C.E., Ponce G., Vega L., *The Cauchy problem for Korteveg-de Vries in Sobolev spaces of negative indices*, Duke Math. J., 71 (1993): 1–21.

[16] Kenig C.E., Ponce G., Vega L., *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math., 46 (1993): 527–620.

[17] Kishimoto N., *Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity*, Diff. and Int. Eq., 22 (2009): 447–464.

[18] Kodama Y., *Optical solitons in a monomode fiber*, Math. Scand., 39 (1985): 597–614.

[19] Ostrovskii L.A., *Nonlinear internal waves in a rotating ocean*, Okeanologiya, 18 (1978),: 181–191.

[20] Muñoz-García A., *Cauchy problem for KdV equation in weighted Sobolev spaces*, Rev. Fac. Cienc., 8 (2019): 83–102.

[21] Nahas J., *A decay property of solutions to the k-generalized KdV equation*, Adv. Differ. Equat., 17 (2012): 833–858.

[22] Sjölin P., L^p *maximal estimates for solutions to the Schrödinger equation*, Math. Scand., 81 (1997): 35–68.

[23] Tao T., *Scattering for the quartic generalised Korteweg-de Vries equation*, J. Differ. Equ., 32 (2007): 623–651.

IRSTI 27.31.15 DOI: https://doi.org/10.26577/JMMCS2023v120i4a2

B.D. Koshanov[1,2*], M.B. Bakytbek[3], G.D. Koshanova[4], P.Zh. Kozhobekova[5], M.T. Sabirzhanov[5]

[1] Institute of Mathematics and Mathematical modeling, Kazakhstan, Almaty
[2] International University of Information Technology, Kazakhstan, Almaty
[3] M. Tynyshpaev Kazakh Academy of Transport and Communication, Kazakhstan, Almaty
[4] H.A. Yasavi International Kazakh-Turkish University, Kazakhstan, Turkestan
[5] Osh State University, Kyrgyzstan, Osh
*e-mail: koshanov@list.ru

UNIFORM ESTIMATES FOR SOLUTIONS OF A CLASS OF NONLINEAR EQUATIONS IN A FINITE-DIMENSIONAL SPACE

The need to study boundary value problems for elliptic parabolic equations is dictated by numerous practical applications in the theoretical study of the processes of hydrodynamics, electrostatics, mechanics, heat conduction, elasticity theory, quantum physics.
Let H $(dimH \geq 1)$ – a finite-dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We will study the equation of the following form

$$u + L\,(u) = g \in H,$$

where $L(\cdot)$ is a non-linear continuous transformation, g is an element of the space H, u is the required solution of the problem from H.
In this paper, we obtain two theorems on a priori estimates for solutions of nonlinear equations in a finite-dimensional Hilbert space. The work consists of four items.
The conditions of the theorems are such that they can be used in the study of a certain class of initial-boundary value problems to obtain strong a priori estimates. This is the meaning of these theorems.
Key words: finite-dimensional Hilbert space, nonlinear equations, initial-boundary value problem, weak solution, strong solution, a priori estimates of the solution.

Б.Д. Қошанов[1,2,*], М.Б. Бақытбек[3], Г.Д. Қошанова[4], П.Ж. Кожобекова[5], М.Т. Сабиржанов[5]
[1] Математика және математикалық моделдеу институты, Қазақстан, Алматы қ.
[2] Халықаралық ақпараттық технологиялар университеті, Қазақстан, Алматы қ.
[3] М. Тынышпаев атындағы Қазақ көлік және коммуникациялар академиясы, Қазақстан, Алматы қ.
[4] Қожа Ахмет Ясауи атындағы Халықаралық қазақ-түрік университеті, Қазақстан, Түркістан қ.
[5] Ош мемлекеттік университеті, Қырғызстан, Ош қ.
*e-mail: koshanov@list.ru

Сызықты емес теңдеулердің бір классы шешімдерінің бірқалыпты бағалаулары

Эллиптикалық параболалық теңдеулер үшін шекаралық есептерді зерттеу қажеттілігі гидродинамика, электростатика, механика, жылу өткізгіштік, серпімділік теориясы және кванттық физика процестерін теориялық зерттеуде көптеген практикалық қолданулардан туындайды.
Скаляр көбейтіндісі $\langle \cdot, \cdot \rangle$ және нормасы $\| \cdot \|$ бар H $(dimH \geq 1)$ – ақырлы нақты Гильберт кеңістігінде келесі түрдегі теңдеу зерттеледі

$$u + L\,(u) = g \in H,$$

мұндағы $L(\cdot)$ – сызықты емес үзіліссіз бейнелеу, g – H -тың элементі, u – H -тағы ізделінді шешімі.

© 2023 Al-Farabi Kazakh National University

Бұл жұмыста біз ақырлы өлшемді кеңістіктегі сызықтық емес теңдеулерді шешуге арналған априорлық бағалаулар бойынша екі теореманы аламыз. Бұл теоремалар белгілі бір шарттарда дәлелденеді, олар сызықты емес бастапқы-шеттік есептердің бір класының соңғы өлшемді жуықтауларымен қанағаттандырылатын шарттардан алынған.
Теореманың шарттары күшті априорлық бағалаулар алу үшін бастапқы-шеттік есептердің белгілі бір класын зерттеуде қолдануға болады. Бұл теоремалардың негізгі мағынасы осында.
Түйін сөздер: ақырлы Гильберт кеңістігі, сызықтық емес теңдеу, бастапқы-шеттік есеп, әлсіз шешім, күшті шешім, шешімнің априорлық бағалануы.

Б.Д. Кошанов[1,2,*], М.Б. Бакытбек[3], Г.Д. Кошанова[4], П.Ж. Кожобекова[5], М.Т. Сабиржанов[5]
[1] Институт математики и математического моделирования, Казахстан, г. Алматы
[2] Международный университет информационных технологий, Казахстан, г. Алматы
[3] Казахская академия транспорта и коммуникации им. М. Тынышпаева, Казахстан, г. Алматы
[4] Международный Казахско-турецкий университет им. Х.А. Ясави, Казахстан, г. Туркестан
[5] Ошский государственный университет, Кыргызстан, г. Ош
[*]e-mail: koshanov@list.ru

Равномерные оценки решений одного класса нелинейных уравнений в конечномерном пространстве

Необходимость исследования краевых задач для эллиптических параболических уравнений диктуется многочисленными практическими приложениями при теоретическом исследовании процессов гидродинамики, электростатики, механики, теплопроводности, теории упругости, квантовой физики.
В H – конечномерном ($dimH \geq 1$) действительном гильбертовом пространстве со скалярным произведением $\langle \cdot, \cdot \rangle$ и с нормой $\| \cdot \|$ исследуется уравнение следующего вида

$$u + L(u) = g \in H,$$

где $L(\cdot)$ – нелинейное непрерывное преобразование, g – элемент пространства H, u – искомое решение задачи из H.
В настоящей работе получены две теоремы об априорных оценках решений нелинейных уравнений в конечномерном пространстве. Эти теоремы доказаны при выполнении некоторых условий, которые заимствованы из условий которым удовлетворяют конечномерные аппроксимации одного класса нелинейных начально-краевых задач.
Условия теорем таковы, что их можно использовать при изучении определенного класса начально-краевых задач для получения сильных априорных оценок. В этом смысл этих теорем.
Ключевые слова: конечномерное гильбертовое пространство, нелинейные уравнения, начально-краевая задача, слабое решение, сильное решение, априорные оценки решения.

1 Introduction

The problem of describing the dynamics of an incompressible fluid is an urgent problem of our time.

In mid-2000, the Clay Mathematics Institute formulated several unsolved problems of mathematics in the millennium (The Millennium Prize Problems). One of the problems is the existence and smoothness of solutions to the Navier–Stokes equations for an incompressible viscous fluid [1].

Many mathematicians have worked on this problem and obtained significant results in [2]–[4]. This problem is solved in the two-dimensional case of O.A. Ladyzhenskaya in [3]. The work [4] provides a fairly complete analysis of the state of the problem and a review of the available literature.

Works [5]– [12] are devoted to the study of the solvability in the whole of equations of the Navier-Stokes type, the continuous dependence of the solution of a parabolic equation and the smoothness of the solution. In papers [13], [14] questions of the formulation and their solvability of boundary value problems for high-order quasi-hyperbolic equations were studied. The work [15], [16] is devoted to the dedication of Green's function type Dirichlet for a polyharmonic equation and the description of the correct boundary problem for the polyharmonic operator. In the works [17]– [19], studied the questions of the Fredholm solvability of the general problem Neumann for the elliptic equation of high order on the plane.

The works [20]– [22] is devoted to the study of the uniqueness of the solution of time-regular problems for some operator-differential equations of the form where the operator A is: a) an operator with a Wave Operator with Displacement, в) the Tricomi operator, c) an arbitrary self-adjoint high-order elliptic differential operator.

In the work [23] a complete proof of Theorem 2 is given in another form. This article is a continuation of the work [23].

In this article, we obtain two theorems on a priory estimates for solutions of nonlinear equations in a finite-dimensional space. These theorems are proved under certain conditions, which are borrowed from the conditions that are satisfied by finite-dimensional approximations of one class of nonlinear initial-boundary value problems.

2 Materials and methods

3 Used conditions and designations. Formulation of the main results

In this paper, we are engaged in the derivation of uniform estimates for solutions of nonlinear equations of the form

$$u + L(u) = g \in H, \tag{1}$$

where H is a finite-dimensional Hilbert space, $L(\cdot)$ is a non-linear continuous transformation, g is an element of the space H, the solution u of problem (1) is sought in H.

We aim at such finite-dimensional equations of the form (1), which are finite-dimensional approximations of infinite-dimensional problems of the form (1) in an infinite-dimensional Hilbert space. In this case, it will turn out to be very important to obtain estimates that are independent of the approximation number and allow one to pass to the limit and obtain an a priori estimate in the limit for solving the infinite-dimensional problem. The infinite-dimensional problems of the form (1) that we are aiming at are, as a rule, problems of mathematical physics written in a restricted form.

In this section, $f(u)$ will mean an operation of the form

$$f(u) = u + L(u). \tag{2}$$

If ξ is a parameter from $[0, +\infty)$ and the vector $u(\xi)$ is a vector function continuously differentiable with respect to the parameter ξ, then we assume that the vector function $L(u(\xi))$ is also continuously differentiable, as well as the expressions arising from $L(u)$ and $f(u)$.

Let's introduce the notation L_u :

$$\left(L\left(u(\xi)\right)\right)_\xi = L_{u(\xi)} u_\xi(\xi). \tag{3}$$

It is obvious that L_u (for each $u \in H$) will be a linear operator

$$L_u v = \left(L\left(u\left(\xi\right)\right)\right)_\xi \Big|_{u_\xi = v}. \tag{4}$$

We have

$$\left(f\left(u(\xi)\right)\right)_\xi = u_\xi + L_u u_\xi = \left(E + L_u\right) u_\xi.$$

Here and throughout what follows, E is the identity transformation.

Denote

$$D_u^* = E + L_u^*, \tag{5}$$

$$D_u^* f\left(u\right) = \left(E + L_u^*\right) f\left(u\right). \tag{6}$$

Let us present the conditions used U1–U4.

Condition U1. *For the transformation $L\left(\cdot\right)$ and the operators L_u, L_u^*, D_u and D_u^* conditions are met*

$$\|L(u) - L(v)\| + \|L_u^* - L_v^*\|_{H \to H} \le \psi(\|u\|)\|u - v\|, \tag{7}$$

$$\|D_v u\| + \|D_u^*\| \le \psi(\|v\|)\|u\|, \tag{8}$$

where $\psi(\cdot)$ is a continuous function on $[0, \infty)$.

Condition U2. *There are linear invertible operators T and G such that*

$$\|G\| \le 1, \ \|T\| \le 1, \ \|G^{-1}\| + \|T^{-1}\| < \infty, \tag{9}$$

and for any $u \in H$ the relations

$$\langle L(u), Tu\rangle \ge 0, \ \langle Tu, u\rangle \ge \|Gu\|^2 \ge \|Tu\|^2. \tag{10}$$

Condition U3. *If $u \in H$ is an eigenvector of the operator G^*G, then the inequality*

$$\|u\|^2 \le \left(\|f(u)\|^2 + 2\right)^m, \ m \ge 1. \tag{11}$$

Condition U4. *If $D_u^* f(u) = \lambda u$, $\lambda > 0$, then*

$$\gamma(u) :\equiv \langle D_u^* f, u\rangle \|u\|^{-2} \ge \left(\|f(u)\|^2 + 2\right)^{-m} \|u\|^{-2}. \tag{12}$$

Theorem 1. *If conditions U1 and U2 are satisfied, then for any $g \in H$ problem*

$$f(u) = g$$

has a solution $u \in H$ such that the estimate

$$\|Gu\| \le \|g\|. \tag{13}$$

Remark 1. We will see in the applications that Theorem 1 allows us to obtain the existence of a weak solution of a certain class of problems of mathematical physics written in restricted notation (integral form), for which the problem

$$u + L(u) = g$$

is a finite-dimensional approximation.

Theorem 2. *If conditions U1, U2, U3 and U4 are satisfied, then the problem*

$$u + L(u) = g$$

for any $g \in H$ has a solution satisfying the estimate

$$\|u\|^2 \leq C_0 exp\{-\{\|g\|^2\}\}, \tag{14}$$

where C_0 is a constant number independent of $g \in H$ (depending on $m-$ from condition U4).

Remark 2. This theorem allows one to obtain the existence of a strong solution to some problems of mathematical physics (written in a restricted form). Conditions U3 and U4 can be noticeably weakened, but the remaining conditions U1 and U2 are not sufficient to obtain estimate (14) from the theorem. It can be seen from the course of the proof of Theorem 2 that estimate (14) can be significantly improved. A complete proof of Theorem 2 in a slightly different form is given in [23].

4 Proof of the theorem 1

The equation $u+L(u) = g$ is scalarly multiplied by Tu. Then, using conditions U2, we obtain

$$\langle Tu, g\rangle = \langle u, Tu\rangle + \langle L(u), Tu\rangle \geq \langle u, Tu\rangle \geq \|Gu\|^2.$$

From this and condition U2 we get the estimate

$$\|Gu\|^2 \leq \langle Tu, g\rangle \leq \|Tu\|\|g\| \leq \|Gu\|\|g\|.$$

From this estimate we obtain the a priori estimate

$$\|Gu\| \leq \|g\|. \tag{15}$$

Denote

$$M = \{u : \langle Tu, u\rangle \leq 8\langle Tg, g\rangle\}. \tag{16}$$

Recall that $\langle Tu, u\rangle$ is positive (strictly!). Therefore, $\langle Tu, u\rangle$ and $\langle Tg, g\rangle$ can be taken as the squares of norms.

Let the equation $u + L(u) = g$ have no solution. Let us define the transformation $F(u)$:

$$F(u) = -\frac{u + L(u) - g}{\sqrt{\langle T(u + L(u) - g), u + L(u) - g\rangle}}\sqrt{8\langle Tg, g\rangle}. \tag{17}$$

Since the equation $u + L(u) = g$ has no solution, this transformation is continuous. But

$$\langle TF(u), F(u)\rangle \le 8\langle Tg, g\rangle.$$

Therefore, a continuous transformation takes the set M into itself. But then (since H is finite-dimensional) according to Browder's theorem, the transformation F has a fixed point, i.e.,

$$F(u_0) = u_0. \tag{18}$$

Let us act on (18) with the operator T, and then scalarly multiply the resulting equality by the vector $u_0 + L(u_0) = g$. Then using (17) we have

$$-\frac{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle}{\sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle}}\sqrt{8\langle Tg, g\rangle} =$$

$$\langle Tu_0, u_0 + L(u_0) - g\rangle$$

or

$$-\sqrt{8}\sqrt{\langle Tg, g\rangle}\sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle} = \langle u_0, T(u_0 + L(u_0) - g)\rangle. \tag{19}$$

Let us scalarly multiply equality (18) by the vector $T(u_0 + L(u_0) - g)$. Then using (17) instead of (19) we obtain

$$-\sqrt{8}\sqrt{\langle Tg, g\rangle}\sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle} = \langle Tu_0, u_0 + L(u_0) - g\rangle. \tag{20}$$

We add equalities (19) and (20), then we obtain

$$-\sqrt{8}\sqrt{\langle Tg, g\rangle}\sqrt{\langle T(u_0 + L(u_0) - g), u_0 + L(u_0) - g\rangle} =$$

$$\frac{1}{2}\left(\langle u_0, T(u_0 + L(u_0) - g)\rangle - \langle T(u_0 + L(u_0) - g), u_0\rangle\right). \tag{21}$$

Now, since $\langle Tx, x\rangle \ge \|Gx\|^2 > 0$, we can $\frac{1}{2}\left(\langle y, Tx\rangle + \langle Ty, takex\rangle\right)$ as the scalar product. Then $\langle Tx, x\rangle$ and $\langle y, Tx\rangle + \langle Ty, y\rangle-$ will be norm squares. Then, since the right side of (21) must be negative, we get

$$-\sqrt{8}\sqrt{\langle Tg, g\rangle} \ge -\langle Tu_0, u_0\rangle$$

or

$$\sqrt{8}\sqrt{\langle Tg, g\rangle} \le -\langle Tu_0, u_0\rangle.$$

This inequality contradicts the membership of u_0 in the set M from (16). Therefore, the equation $u + L(u) = g$ has a solution u, for which, due to (15), estimate (13) is satisfied. Theorem 1 is proved.

Remark 3. Note that Theorem 1 can be proved under more general assumptions than conditions U1 and U2. The above follows from the proof of the theorem. The formulation of Theorem 1 given by us is convenient for us.

5 Acknowledgments

The authors express their gratitude for the formulation of the problem and attention to the work of Academician of the NAS of the Republic of Kazakhstan Mukhtarbai Otelbaevich Otelbaev.

This work was done by grant AP 19678182 of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

References

[1] Fefferman Ch., Existence and smoothness of the Navier-Stokes equation, Clay Mathematics Institute. (2000), 1-5.

[2] Otelbaev M.O., Existence of a strong solution of the Navier-Stokes equation, Mathematical journal, 13(50), (2013), 5-104.

[3] Ladyzhenskaya O.A., Solution "in the whole" of the Navier-Stokes boundary value problem in the case of two space variables, Reports of the Academy of Sciences of the USSR, 123 (3), (1958), 427-429.

[4] Ladyzhenskaya O.A., The sixth problem of the millennium: the Navier-Stokes equations, existence and smoothness, Advances in Mathematical Sciences, 58(2), (2003), 45-78.

[5] Hopf E., Uber die Anfangswertaufgabe fur die Hydrodinamischen Grundgleichungen., Math. Nachr.(4), (1951), 213-231.

[6] Otelbaev M.O., Examples of equations of the Navier-Stokes type that are not strongly solvable in general, Math notes, 89(5), (2011), 771-779.

[7] Otelbaev M.O., Durmagambetov A.A., Seytkulov E.N., Conditions for the existence of a strong solution in the whole of one class of non-linear evolution equations in a Hilbert space, Siberian Mathematical Journal, 49(4), (2008), 855-864.

[8] Otelbaev M.O., Zhapsarbayeva L.K., Continuous dependence of the solution of a parabolic equation in Hilbert space on parameters and initial data, Differential Equations, 45(6), (2009), 818-849.

[9] Lyons J.-L., Some methods for solving nonlinear boundary value problems, Moscow: Mir (1972).

[10] Saks R.S., Cauchy problem for the Navier-Stokes equations, Fourier method, Ufa Mathematical Journal, 3(1), (2011), 53-79.

[11] Pokhozhayev S.I., Smooth solutions of the Navier-Stokes equations, Mathematical collection, 205(2), (2014), 131-144.

[12] Koshanov B.D, Otelbaev M.O., Correct Contractions stationary Navier-Stokes equations and boundary conditions for the setting pressure, AIP Conference Proceedings, 1759. http://dx.doi.org/10.1063/1.4959619.

[13] Kozhanov A.I., Koshanov B.D., Sultangazieva Zh.B., New boundary value problems for fourth-order quasi-hyperbolic equations,Siberian Electronic Mathematical Reports, 16 (2019), 1383-1409. http://dx.doi.org/10.33048/semi.2019.16.098.

[14] Kozhanov A.I., Koshanov B.D., Sultangazieva Zh.B., Emir Kady oglu A.N., Smatova G.D., The spectral problem for nonclassical differential equations of the sixth order. Bulletin of the Karaganda University, series Mathematics, 97(1), (2020), 79-86. https://doi.org/10.31489/2020M1/79-86

[15] Koshanov B.D., Green's functions and correct restrictions of the polyharmonic operator. Journal of Mathematics, Mechanics and Computer Science, 109(1) (2021), 34-54. https://doi.org/10.26577/JMMCS.2021.v109.i1.03

[16] Koshanov B.D., Koshanova M.D. On the representation of the Green function of the Dirichlet Problem and their properties for the polyharmonic equations, AIP Conference Proceedings, 1676, (2015), http://dx.doi.org/10.1063/1.4930446

[17] Koshanov B.D., Soldatov A.P., Boundary value problem with normal derivatives for a higher order elliptic equation on the plane, Differential Equations, 52:12 (2016), 1594-1609. https://doi.org/10.1134/S0012266116120077.

[18] Koshanov B.D., Kuntuarova A.D., Equivalence of the Fredholm solvability condition for the Neumann problem to the complementarity condition, Journal of Mathematics, Mechanics and Computer Science, 111:3 (2021), 39-51. https://doi.org/10.26577/JMMCS.2021.v111.i3.04

[19] Koshanov B.D., Soldatov A.P., Generalized Neumann problem for an elliptic equation, Complex Variables and Elliptic Equations, (2022), https://doi.org/10.1080/17476933.2021.1958797

[20] Kanguzhin B.E., Koshanov B.D., Criteria for the uniqueness of the solution of a time-nonlocal problem for a high-order differential-operator equation $l(\cdot) - A$ with a shifted wave operator A, Sibirian Mathematical Journal, 63:6 (2022), 1083-1090. https://doi.org/10.1134/S0037446622060088

[21] Kanguzhin B.E., Koshanov B.D., Uniqueness Criteria for Solving a Time Nonlocal Problem for a High-Order Differential Operator Equation $l(\cdot) - A$ with a Wave Operator with Displacement, Simmetry, 14:6, 1239 (2022), https://www.mdpi.com/2073-8994/14/6/1239

[22] Kanguzhin B.E., Koshanov B.D., Solution Uniqueness Criteria in a Time Nonlocal Problem for the Operator Differential Equation $l(\cdot) - A$ with the Tricomi Operator A, Differential Equations, 59:1 (2023), 1-12. https://doi.org/10.1134/S0012266123010019

[23] Koshanov B.D., Kakharman N., Segizbayeva R.U., Sultangaziyeva Zh.B., Two theorems on estimates for solutions of one class of nonlinear equations in a finite-dimensional space, Bulletin of the Karaganda University, series Mathematics, (2022), 70-84. https://mathematics-vestnik.ksu.kz/apart/2022-107-3/7.pdf

IRSTI 27.41.19 DOI: https://doi.org/10.26577/JMMCS2023v120i4a3

A.H. Kalidolday[1*], **E.D. Nursultanov**[1,2]
[1]Institute of Mathematics and Mathematical Modeling, Kazakhstan, Almaty
[2]M.V. Lomonosov Moscow State University (Kazakhstan branch), Kazakhstan, Astana
*e-mail: aitolkynnur@gmail.com

INTERPOLATION THEOREM FOR DISCRETE NET SPACES

In this paper, we study discrete net spaces $n_{p,q}(M)$, where M is some fixed family of sets from the set of integers $\mathbb{Z}$. Note that in the case when the net M is the set of all finite subsets of integers, the space $n_{p,q}(M)$ coincides with the discrete Lorentz space $l_{p,q}(M)$. For these spaces, the classical interpolation theorems of Marcinkiewicz-Calderon are known. In this paper, we study the interpolation properties of discrete network spaces $n_{p,q}(M)$,in the case when the family of sets M is the set of all finite segments from the class of integers $\mathbb{Z}$, i.e. finite arithmetic progressions with a step equal to 1. These spaces are characterized by such properties that for monotonically nonincreasing sequences the norm in the space $n_{p,q}(M)$ coincides with the norm of the discrete Lorentz space $l_{p,q}(M)$. At the same time, in contrast to the Lorentz spaces, the given spaces $n_{p,q}(M)$ may contain sequences that do not tend to zero. The main result of this work is the proof of the interpolation theorem for these spaces with respect to the real interpolation method. It is shown that the scale of discrete net spaces $n_{p,q}(M)$ is closed with respect to the real interpolation method. As a corollary, an interpolation theorem of Marcinkiewicz type is presented. These assertions make it possible to obtain strong estimates from weak estimates.
Key words: net spaces, discrete net spaces, Marcinkiewicz type interpolation theorem.

А.Х. Калидолдай[1*], Е.Д. Нурсултанов[1,2]
[1]Математики және математикалық модельдеу институты, Қазақстан, Алматы қ.
[2]М.В. Ломоносов атындағы ММУ Қазақстан филиалы, Қазақстан, Астана қ.
*e-mail: aitolkynnur@gmail.com

Дискретті торлы кеңістіктердегі интерполяциялық теорема

Бұл жұмыста $n_{p,q}(M)$ дискретті торлы кеңістіктері зерттеледі, мұндағы M - $\mathbb{Z}$ бүтін сандар жиынынан алынған жиындардың тұрақты тобы. M торы бүтін сандардың барлық ақырлы ішкі жиындарының жиыны болған жағдайда $n_{p,q}(M)$ кеңістігі $l_{p,q}(M)$ дискретті Лоренц кеңістігімен сәйкес келетінін ескеріңіз. Бұл кеңістіктер үшін Марцинкевич-Кальдеронның классикалық интерполяциялық теоремалары белгілі. Жұмыста M жиындар тобы $\mathbb{Z}$ бүтін сандар класындағы барлық ақырлы сегменттердің жиыны, яғни қадамы 1-ге тең ақырлы арифметикалық прогрессиялар болған жағдайдағы, $n_{p,q}(M)$ дискретті торлы кеңістіктерінің интерполяциялық қасиеттері қарастырылады. Бұл кеңістіктер монотонды өспейтін тізбектер үшін $n_{p,q}(M)$ кеңістігіндегі нормасы $l_{p,q}(M)$ дискретті Лоренц кеңістігінің нормасымен сәйкес келетін қасиеттермен сипатталады. Сонымен қатар, аталмыш $n_{p,q}(M)$ кеңістіктерді? Лоренц кеңістігінен айырмашылығы бұл кеңістіктерде нөлге ұмытылмайтын тізбектер жатады. Бұл жұмыстың негізгі нәтижесі нақты интерполяциялық әдіске қатысты осы кеңістіктер үшін интерполяциялық теоремасын дәлелдеу болып табылады. Нақты интерполяциялық әдісіне қатысты $n_{p,q}(M)$ дискретті торлы кеңістіктерінің шкаласы тұйықталғаны көрсетілген. Салдар ретінде Марцинкевич типіндегі интерполяциялық теорема ұсынылған. Бұл теорема әлсіз бағалаулардан күшті бағалаулар алуға мүмкіндік береді.
Түйін сөздер: торлы кеңістіктер, дискретті торлы кеңістіктер, Марцинкевич типті интерполяциялық теоремасы.

© 2023 Al-Farabi Kazakh National University

А.Х. Калидолдай[1*], Е.Д. Нурсултанов[1,2]
[1]Институт математики и математического моделирования, Казахстан, г. Алматы
[2]Казахстанский филиал МГУ им. М. В. Ломоносова, Казахстан, г. Астана
[*]e-mail: aitolkynnur@gmail.com

Интерполяционная теорема для дискретного сетевого пространства

В работе исследуется дискретные сетевые пространства $n_{p,q}(M)$, где M - некоторое фиксированное семейство множеств из множества целых чисел $\mathbb{Z}$. Отметим, что в случае когда сеть M есть множество всех конечных подмножеств целых чисел пространства $n_{p,q}(M)$ совпадает с дискретным пространством Лоренца $l_{p,q}(M)$. Для этих пространств известны классические интерполяционные теоремы Марцинкевича-Кальдерона.
В работе изучаются интерполяционные свойства дискретных сетевых пространств $n_{p,q}(M)$, в случае когда семейство множеств M является множеств всех конечных отрезков из класса целых чисел $\mathbb{Z}$, т.е. конечных арифметических прогрессии с шагом равным 1. Данные пространства характеризуется такими свойствами, что для монотонно не возрастающих последовательности норма в пространстве $n_{p,q}(M)$ совпадает с нормой дискретного пространства Лоренца $l_{p,q}(M)$. В то же время в отличие от пространств Лоренца данные пространства $n_{p,q}(M)$ может содержат последовательности нестремящиеся к нулю. Основным результатом данной работы является доказательство интерполяционной теоремы для этих пространств относительно вещественного интерполяционного метода. Показано, что шкала дискретных сетевых пространств $n_{p,q}(M)$ замкнута относительно вещественного интерполяционного метода. Как следствие приведена интерполяционная теорема типа Марцинкевича. Данные утверждения позволяют получить из слабых оценок сильные оценки.
Ключевые слова: сетевые пространства, дискретные сетевые пространства, интерполяционная теорема типа Марцикевича.

1 Introduction

Let S be the set of all finite sets of indices from $\mathbb{Z}^n$. For a fixed set $M \subset S$ we define the space $n_{p,q}(M)$ $(0 < p, q \le \infty)$ as the set of sequences $a = \{a_m\}_{m\in\mathbb{Z}^n}$ with quasinorm for $0 < p < \infty$, $0 < q < \infty$

$$\|a\|_{n_{p,q}(M)} = \left(\sum_{k=1}^{\infty} k^{\frac{q}{p}-1}(\bar{a}_k(M))^q\right)^{\frac{1}{q}},$$

and for $q = \infty$, $0 < p \le \infty$

$$\|a\|_{n_{p,\infty}(M)} = \sup_{1\le k<\infty} k^{\frac{1}{p}}\bar{a}_k(M),$$

where

$$\bar{a}_k(M) = \sup_{\substack{e\in M\\ |e|\ge k}} \frac{1}{|e|}\left|\sum_{m\in e} a_m\right|,$$

where $|e|$ is the number of indices in e.

These spaces were introduced in [6], and they were called net spaces.

Net spaces have found important applications in various problems of harmonic analysis, operator theory and theory of stochastic processes [1–3, 7–11]. In this paper, we study the interpolation properties of these spaces. It should be noted here that net spaces are in a sense close to the discrete Morrey spaces:

$$m_p^\lambda = \left\{ a = \{a_k\}_{k\in\mathbb{Z}} : \quad \sup_{m\in\mathbb{N}} \sup_{k\in\mathbb{Z}} \frac{1}{m^\lambda} \left(\sum_{r=k}^{k+m} |a_r|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

In the case when $a = \{a_k\}_{k\in\mathbb{Z}},\ a_k \geq 0$, for $\lambda = n\left(1 - \frac{1}{p}\right)$

$$\|a\|_{n_{p,\infty}(M)} \asymp \|a\|_{m_1^\lambda}.$$

The question of interpolation of Morrey spaces was considered in the works [5, 12] and it was shown that this scale of spaces is not closed with respect to the real interpolation method.

In this paper we show that if M is the set of all segments from $\mathbb{Z}$ the scale of spaces is closed with respect to the real interpolation method, i.e. the following relation holds

$$(n_{p_0,q_0}(M), n_{p_1,q_1}(M))_{\theta,q} = n_{p,q}(M). \tag{1}$$

Given functions F and G, in this paper $F \lesssim G$ means that $F \leq c\, G$ (or $c\, F \geq G$), where c is a positive number, depending only on numerical parameters, that may be different on different occasions. Moreover, $F \asymp G$ means that $F \lesssim G$ and $G \lesssim F$.

2 Interpolation theorem

Let (A_0, A_1) be a compatible pair of Banach spaces [4]. Let

$$K(t, a; A_0, A_1) = K(t, a) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}),\ a \in A_0 + A_1,$$

be the functional Petre. For $0 < q < \infty$, $0 < \theta < 1$

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty (t^{-\theta} K(t,a))^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

and for $q = \infty$

$$(A_0, A_1)_{\theta,q} = \left\{ a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \sup_{0<t<\infty} t^{-\theta} K(t,a) < \infty \right\}.$$

Theorem 1 *Let $1 \leq p_0 < p_1 < \infty$ and $0 < q_0, q_1, q \leq \infty$. Let M be the set of all segments from $\mathbb{Z}$. Then*

$$\left(n_{p_0,q_0}(M), n_{p_1,q_1}(M)\right)_{\theta,q} = n_{p,q}(M),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\ \theta \in (0,1)$.

Proof. Let us prove first

$$(n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta,q} = n_{p,q}(M), \tag{2}$$

where $\frac{1}{p} = 1-\theta$, $\theta \in (0,1)$. Let $\tau \in \mathbb{N}$ and M be the set of all segments from $\mathbb{Z}$. Divide our entire axis into disjoint segments $\{I_k\}_{k\in\mathbb{Z}}$, where $I_k = (2^k, 2^{k+1})$ of the measure $|I_k| = \tau$. It is obvious that $\tau = 2^{k+1} - 2^k = 2^k$. Let $a = \{a_m\}_{m\in\mathbb{Z}} \in n_{p,q}(M)$, define the sequence $b_0 = \{b_n^0\}$:

$$b_n^0 = \frac{1}{|I_k|}\left|\sum_{m\in I_k} a_m\right| \quad \text{for } n \in I_k.$$

Note that $|b_n^0| \le \bar{a}_\tau(M)$ and

$$\sum_{n\in I_k}(a_n - b_n^0) = 0.$$

Let $D_\tau(a) = \{b = \{b_n\}_{n\in\mathbb{Z}} : |b_n| \le \bar{a}_\tau(M)\}$. For the Petre functional, we have the following

$$K(t, a; n_{1,\infty}(M), n_{\infty,\infty}(M)) = \inf_{a=a_0+a_1}\left(\|a_0\|_{n_{1,\infty}(M)} + t\|a_1\|_{n_{\infty,\infty}(M)}\right)$$

$$\le \inf_{b\in D_\tau(a)}\Big(\sup_{1\le k<\infty} k\overline{(a-b)}_k(M) + t\sup_{1\le k<\infty} \bar{b}_k^0(M)\Big) \le \sup_{1\le k<\infty} k\overline{(a-b^0)}_k(M) + t\bar{a}_\tau(M).$$

Consider the first term

$$\sup_{1\le k<\infty} k\overline{(a-b^0)}_k(M) \asymp \sup_{1\le k<\tau} k\overline{(a-b^0)}_k(M) + \sup_{\tau\le k<\infty} k\overline{(a-b^0)}_k(M).$$

Let I be an arbitrary segment from M such that $|I| \ge \tau$. Hence

$$\left|\sum_{n\in I}(a_n - b_n^0)\right| = \left|\sum_{I_k\subset I}\sum_{n\in I_k}(a_n - b_n^0) + \sum_{n\in I\cap I_{k_0}}(a_n - b_n^0) + \sum_{n\in I\cap I_{k_1}}(a_n - b_n^0)\right|$$

$$\le \left|\sum_{n\in I\cap I_{k_0}} a_n\right| + \left|\sum_{n\in I\cap I_{k_1}} a_n\right| + \left(|I\cap I_{k_0}| + |I\cap I_{k_1}|\right)\bar{a}_\tau(M),$$

i.e., the following estimate holds

$$\left|\sum_{n\in I}(a_n - b_n^0)\right| \le s_1\bar{a}_{s_1} + s_2\bar{a}_{s_2} + 2\tau\bar{a}_\tau(M),$$

where $s_1 = |I\cap I_{k_0}| \le \tau$, $s_2 = |I\cap I_{k_1}| \le \tau$. Hence we have

$$\left|\sum_{n\in I}(a_n - b_n^0)\right| \le 4\sup_{\tau\ge s>1} s\,\bar{a}_s(M).$$

Hence,

$$\sup_{\tau\le k<\infty} k\overline{(a-b^0)}_k = \sup_{\tau\le k<\infty} k\sup_{|I|\ge k}\frac{1}{|I|}\left|\sum_{n\in\mathbb{Z}}(a_n - b_n^0)\right| \le 4\sup_{\tau\ge s\ge 1} s\bar{a}_s(M).$$

For the first term we have

$$\sup_{1\le k<\infty} k\overline{(a-b^0)}_k \le c \sup_{\tau\ge k\ge 1} k\bar{a}_k(M).$$

Thus,

$$K(t,a;n_{1,\infty}(M),n_{\infty,\infty}(M)) \lesssim \sup_{\tau\ge k>1} k\bar{a}_k(M)+t\bar{a}_\tau(M).$$

Then we have

$$\|a\|_{(n_{1,\infty}(M),n_{\infty,\infty}(M))_{\theta,q}} \asymp \left(\sum_{m\in\mathbb{Z}}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}}$$

$$=\left(\sum_{m=-\infty}^{-1}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}}+\left(\sum_{m=0}^{\infty}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}}.$$

For the first term, we have

$$\left(\sum_{m=-\infty}^{-1}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}}=\left(\sum_{m=-\infty}^{-1}\left(2^{-\theta m}\inf_{a=a_0+a_1}(\|a_0\|_{n_{1,\infty}(M)}+2^m\|a_1\|_{n_{\infty,\infty}(M)})\right)^q\right)^{\frac{1}{q}}$$

$$\le\left(\sum_{m=-\infty}^{-1}\left(2^{(1-\theta)m}\|a\|_{n_{\infty,\infty}(M)}\right)^q\right)^{\frac{1}{q}}=c\|a\|_{n_{\infty,\infty}(M)}\lesssim\|a\|_{n_{p,q}(M)}.$$

For the second term, taking into account that $\tau=2^m$ and applying the above estimate for the Petre functional and Minkowski's inequality, we obtain

$$\left(\sum_{m=0}^{\infty}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}}\le\left(\sum_{m=0}^{\infty}\left(2^{-\theta m}\sup_{\tau\ge k>1}k\bar{a}_k(M)+2^m\bar{a}_\tau(M)\right)^q\right)^{\frac{1}{q}}$$

$$\le\left(\sum_{m=0}^{\infty}\left(2^{-\theta m}\sup_{\tau\ge k>1}k\bar{a}_k(M)\right)^q\right)^{\frac{1}{q}}+\left(\sum_{m=0}^{\infty}\left(2^{(1-\theta)m}\bar{a}_{2^m}(M)\right)^q\right)^{\frac{1}{q}}.$$

Considering that $k\asymp\left(\sum_{r=1}^{2^m}r^{q-1}\right)^{\frac{1}{q}}$ and changing the order of summation we get

$$\left(\sum_{m=0}^{\infty}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}}\le\left(\sum_{m=0}^{\infty}\left(2^{-\theta m}\left(\sum_{r=1}^{2^m}r^{q-1}\bar{a}_r^q(M)\right)^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}}+\|a\|_{n_{p,q}}$$

$$=\left(\sum_{r=1}^{\infty}r^{q-1}\bar{a}_r^q(M)\sum_{m=\log_2 r}^{\infty}2^{-\theta mq}\right)^{\frac{1}{q}}+\|a\|_{n_{p,q}(M)}$$

$$\asymp \left(\sum_{r=1}^{\infty} r^{(1-\theta)q-1} \bar{a}_r^q(M) \right)^{\frac{1}{q}} + \|a\|_{n_{p,q}(M)} = c\|a\|_{n_{p,q}(M)}.$$

Hence,

$$\|a\|_{(n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta,q}} \lesssim \|a\|_{n_{p,q}}.$$

So we get the embedding

$$n_{p,q}(M) \hookrightarrow (n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta,q},$$

where $\frac{1}{p} = 1 - \theta,\ \theta \in (0,1)$.

Let us now prove the reverse embedding. Let $k \in \mathbb{N}$, $a \in (n_{1,\infty}, n_{\infty,\infty})_{\theta,q}$ and $a = a_0 + a_1$ be an arbitrary representation, where $a_0 \in n_{1,\infty}(M)$ and $a_1 \in n_{\infty,\infty}(M)$. Obviously, $\bar{a}_k(M) \leq \bar{a}_k^0(M) + \bar{a}_k^1(M)$. Then, if we denote $v(t) = t$, where $t \in (1, \infty)$, then

$$\sup_{v \geq k} k\bar{a}_k(M) \leq \sup_{k>0} k\bar{a}_k^0(M) + \sup_{v(t) \geq k} k\bar{a}_k^1(M) \leq \sup_{k \geq 1} k\bar{a}_k^0(M) + t \sup_{k \geq 1} \bar{a}_k^1(M).$$

Taking into account the arbitrariness of the representation $a = a_0 + a_1$ we have

$$\sup_{v \geq k > 0} k\bar{a}_k(M) \leq K(t, a; n_{1,\infty}(M), n_{\infty,\infty}(M)).$$

Therefore, for $0 < q \leq \infty$ we have

$$\int_0^\infty \left(t^{-\theta} K(t, a; n_{1,\infty}(M), n_{\infty,\infty}(M))\right)^q \frac{dt}{t} \geq \int_1^\infty \left(t^{-\theta} \sup_{v \geq k > 0} k\bar{a}_k(M) \right)^q \frac{dt}{t}$$

$$\geq c_1 \int_1^\infty \left(t^{-\theta} \sup_{t \geq k > 0} k\bar{a}_k(M) \right)^q \frac{dt}{t} \geq c_2 \sum_{r=0}^{\infty} \left(2^{-\theta r} \sup_{2^r \geq k > 0} k\bar{a}_k(M) \right)^q$$

$$\geq c_2 \sum_{r=1}^{\infty} \left(2^{r/p} \bar{a}_{2^r}(M) \right)^q.$$

Thus we get the embedding

$$(n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta,q} \hookrightarrow n_{p,q}(M), \tag{3}$$

where $\frac{1}{p} = 1 - \theta,\ \theta \in (0,1)$.

Hence the relation (2) holds. To prove the general case, we use the reiteration theorem [4, Theorem 3.5.3].

Let $1 < p_0 < p_1 < \infty$. From (2) it follows that there are $\theta_0, \theta_1 \in (0,1)$ such that

$$\begin{aligned} (n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta_0,q_0} &= n_{p_0,q_0}(M) \\ (n_{1,\infty}(M), n_{\infty,\infty}(M))_{\theta_1,q_1} &= n_{p_1,q_1}(M), \end{aligned} \tag{4}$$

then by the reiteration theorem it follows that

$$(n_{p_0,q_0}(M), n_{p_1,q_1}(M))_{\theta,q} = (n_{1,\infty}(M), n_{\infty,\infty}(M))_{\eta,q} = n_{p,q}(M).$$

In the last equality, we took into account that $\eta = (1-\theta)\theta_0 + \theta\theta_1$.

This proves the theorem.

3 Corollary

As a corollary, an interpolation theorem of Marcinkevich type is presented.

Corollary 1 *Let* $2 \le p_0 < p_1 < \infty$, $1 \le q_0, q_1 < \infty$, $q_0 \ne q_1$, $0 < \tau, \sigma < \infty$, M *is the set of all segments from* $\mathbb{Z}$, $G = \{a = \{a_k\}_{k\in\mathbb{Z}},\ a_k \ge 0\}$. *If the following inequalities hold for a quasilinear operator*

$$\|Ta\|_{n_{q_0,\infty}(M)} \le F_0\|a\|_{n_{p_0,\sigma}(M)},\ a \in n_{p_0,\sigma}(M), \tag{5}$$

$$\|Ta\|_{n_{q_1,\infty}(M)} \le F_1\|a\|_{n_{p_1,\sigma}(M)},\ a \in n_{p_1,\sigma}(M), \tag{6}$$

then for any $a \in G \cap n_{p,\tau}$ *we have*

$$\|Ta\|_{n_{q,\tau}(M)} \le cF_0^{1-\theta}F_1^{\theta}\|a\|_{n_{p,\tau}(M)}, \tag{7}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\theta \in (0,1)$ *and the corresponding constant* c *depends only on* $p_i, q_i, \sigma, i = 0, 1$.

Proof. According to the real interpolation method [4, Theorem 3.1.2] and the inequalities (5) and (6) it follows

$$\|Ta\|_{(n_{q_0,\infty}(M),n_{q_1,\infty}(M))_{\theta,\tau}} \le F_0^{1-\theta}F_1^{\theta}\|a\|_{(n_{p_0,\sigma}(M),n_{p_1,\sigma}(M))_{\theta,\tau}}.$$

From the relation (3) we have that

$$\|Ta\|_{n_{q,\tau}(M)} \le c\|Ta\|_{(n_{q_0,\infty}(M),n_{q_1,\infty}(M))_{\theta,\tau}}.$$

From Theorem 1, taking into account that $a = \{a_k\}_{k\in\mathbb{Z}},\ a_k \ge 0$, we get

$$\|a\|_{n_{p,\tau}(M)} \asymp \|a\|_{(n_{p_0,\sigma}(M),n_{p_1,\sigma}(M))_{\theta,q}}.$$

4 Acknowledgments

The work was supported by grant funding of scientific and technical programs and projects of the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. AP14870758).

References

[1] Akylzhanov R., Ruzhansky M., "$L_p - L_q$ multipliers on locally compact groups", *J. Fun. Anal.* 278 (2020).

[2] Akylzhanov R., Ruzhansky M., "Net spaces on lattices, Hardy-Littlewood type inequalities, and their converses", *Eurasian Math. J.* 8, Issue 3 (2017) : 10-27.

[3] Akylzhanov R., Ruzhansky M., Nursultanov E.D., "Hardy-Littlewood, Hausdorff-Young-Paley inequalities, and $L_p - L_q$ Fourier multipliers on compact homogeneous manifolds", *J. Math. Anal. Appl.* 479, Issue 2 (2019) : 1519-1548.

[4] Bergh J. , Löfström J., *Interpolation Spaces. An Introduction* , (Springer, Berlin, 1976).

[5] Blasco O., Ruiz A., Vega L., ”Non interpolation in Morrey-Campanato and block spaces” , *Ann. Scuola Norm. Sup. Pisa Cl.* Sci. 4 (1999) : 31-40.

[6] Nursultanov E.D., ”Net spaces and inequalities of Hardy-Littlewood type” , *Sb. Math.* 189, no. 3 (1998) : 399-419.

[7] Nursultanov E.D., ”On the coefficients of multiple Fourier series in L_p - spaces” , *Izv. Math.* 64, no. 1 (2000) : 93-120.

[8] Nursultanov E.D., Aubakirov T.U., ”Interpolation methods for stochastic processes spaces” , *Abstr. Appl. Anal.* 2013 (2013) : 1-12.

[9] Nursultanov E.D., Kostyuchenko A.G., ”Theory of control of ”catastrophes” *Russ. Math. Surv.* 53, no. 3 (1998) : 628-629.

[10] Nursultanov E.D., Tleukhanova N.T., ”Lower and upper bounds for the norm of multipliers of multiple trigonometric Fourier series in Lebesgue spaces” , *Func. Anal. Appl.* 34, no. 2 (2000) : 151-153.

[11] Nursultanov E.D., Tikhonov S. ”Net spaces and boundedness of integral operators” , *J. Geom. Anal.* 21 (2011) : 950-981.

[12] Ruiz A., Vega L. ”Corrigenda to "Unique continuation for Schrodinger operators"and a remark on interpolation of Morrey spaces” , *Publicacions Matem‘atiques* 39 (1995) : 405-411.

IRSTI 27.31.44 DOI: https://doi.org/10.26577/JMMCS2023v120i4a4

S.A. Mambetov
Al–Farabi Kazakh National University, Kazakhstan, Almaty
Institute of Mathematics and Mathematical Modeling, Kazakhstan, Almaty
e-mail: samatmambetov09@gmail.com

A MAXIMUM PRINCIPLE FOR TIME-FRACTIONAL DIFFUSION EQUATION WITH MEMORY

One of the most beneficial techniques for studying partial differential equations of the parabolic and elliptic types is the use of the maximum and minimum principles. They enable the acquisition of specific solution attributes without the need for knowledge of the solutions' explicit representations. Despite the fact that the maximum principle for fractional differential equations has been studied since the 1970s, a particular interest in this field of study has just lately arisen.

In the present study, a maximum principle for the one-dimensional time fractional diffusion equation with memory is formulated and established. The proof of the maximal principle is based on a maximum principle for the Caputo fractional derivative. The initial boundary value problem for the time-fractional diffusion equation with memory has at most one classical solution, and the maximum principle is then used to show that this solution is continuous depends on the initial and boundary conditions.

Key words: time-fractional diffusion equation, fractional derivative, maximum principle, initial-boundary value problem.

С.А. Мамбетов
Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан, Алматы қ.
Математика және математикалық моделдеу институты, Қазақстан, Алматы қ.
e-mail: samatmambetov09@gmail.com

Жады бар уақыт бойынша бөлшек ретті диффузия теңдеуі үшін максимум қағидасы

Параболалық және эллиптикалық типтердің ішінара туындыларындағы теңдеулерді зерттеудің ең пайдалы әдістерінің бірі максимум мен минимум қағидаларын қолдану. Олар шешімдердің нақты көріністерін білуді қажет етпестен шешімнің нақты атрибуттарын алуға мүмкіндік береді. Бөлшек дифференциалдық теңдеулер үшін максимум қағидасы 1970 жылдардан бері зерттеліп келе жатқанына қарамастан, бұл зерттеу саласына ерекше қызығушылық жақында пайда болды.

Бұл зерттеу жадымен уақыт бойынша бөлшек диффузияның бір өлшемді теңдеуі үшін максимум қағидасын тұжырымдайды және белгілейді. Максимум қағидасының дәлелі сәйкесінше Капутоның бөлшек туындысы үшін максимум қағидасына негізделген. қолданба ретінде максимум қағидасы бөлшек уақыт жадымен диффузия теңдеуі үшін бастапқы-шеттік есептің бір ғана классикалық шешімі бар екенін көрсету үшін пайдаланылады және бұл шешім бастапқы және шекаралық шарттарға үздіксіз тәуелді болады.

Түйін сөздер: уақыт бойынша бөлшек ретті диффузия теңдеуі, бөлшек ретті туынды, максимум қаұидасы, бастапқы-шеттік есеп.

© 2023 Al-Farabi Kazakh National University

С.А. Мамбетов
Казахский национальный университет имени аль-Фараби, Казахстан, г. Алматы
Институт математики и математического моделирования, Казахстан, г. Алматы
e-mail: samatmambetov09@gmail.com

Принцип максимума для уравнения дробной диффузии по времени с памятью

Одним из наиболее полезных методов изучения уравнений в частных производных параболического и эллиптического типов является использование принципов максимума и минимума. Они позволяют получать конкретные атрибуты решения без необходимости знания явных представлений решений. Несмотря на то, что принцип максимума для дробно-дифференциальных уравнений изучается с 1970-х годов, особый интерес к этой области исследований возник совсем недавно.

В этом исследовании сформулирован и установлен принцип максимума для одномерного уравнения дробной диффузии во времени с памятью. Доказательство принципа максимума основано на принципе максимума для дробной производной Капуто, соответственно. В качестве приложения принцип максимума используется для демонстрации того, что существует не более одного классического решения начально-краевой задачи для уравнения диффузии с дробной временной памятью, и это решение непрерывно зависит от начальных и граничных условий.
Ключевые слова: уравнение дробной диффузии по времени, дробное производное, принцип максимума, начально-краевая задача.

Introduction and statement of problem

The maximum-minimum principles are among the best techniques for studying partial differential equations of the parabolic and elliptic types. They allow one to obtain certain properties of solutions without resorting to information about their explicit representations. Although the maximal principle for fractional differential equations has been researched since the 1970s (see [1–4]), special interest in research in this area has appeared relatively recently.

In [5] Luchko obtained a maximal principle for $\partial^{\alpha}_{0|t}$ the Caputo fractional derivative of the form:

- let $g \in C^1((0,T)) \cap ([0,T])$ attains its maximum (minimum) over $[0,T]$ at $t_0 \in (0,T]$ and $0 < \alpha < 1$, then $\partial^{\alpha}_{0|t} g(t_0) \geq 0$ $(\partial^{\alpha}_{0|t} g(t_0) \leq 0)$.

He established a maximal using Caputo, the fundamentals of the fractional diffusion equation time derivative on a bounded domain based on the aforementioned findings. The maximum principle for time-fractional diffusion equations was demonstrated using these results (see [6, 7, 9–14].

We consider the following time-fractional diffusion equation with memory

$$\partial^{\alpha}_{0|t} u(x,t) = \frac{\partial^2}{\partial x^2} I^{\beta}_{0|t} u(x,t) + F(x,t) \text{ in } (0,a) \times (0,T], \tag{1}$$

supplemented with the initial and boundary conditions

$$\begin{cases} u(x,0) = \phi(x) \text{ on } [0,a], \\ u(0,t) = \psi_1(t), u(a,t) = \psi_2(t) \text{ for } 0 \leq t < T, \end{cases} \tag{2}$$

since a and T are real numbers that are positive, the functions F, ϕ, ψ_1 and ψ_2 are continuous in a way that $\phi(0) = \psi_1(0)$ and $\phi(a) = \psi_2(0)$. Here, $I_{0|t}^{\beta}$ is the Riemann-Liouville fractional integral of order $\beta > 0$, defined as (see [15, P. 69])

$$I_{0|t}^{\beta}u(x,t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(x,s)\,ds, \;\; t \in (0,T],$$

and the operator $\partial_{0|t}^{\alpha}$ is the left Caputo fractional derivative with $\alpha \in (0,1)$, given by (see [15, P. 92])

$$\partial_{0|t}^{\alpha}u(x,t) = I_{0|t}^{1-\alpha}\frac{\partial}{\partial t}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial}{\partial s}u(x,s)\,ds.$$

The purpose of this article is to study the maximum principle of linear fractional diffusion equation (1).

If $\alpha \to 1$, $\beta \to 0$ then eq. (1) corresponds to the well-known heat equation. The subdiffusion equation is the equation of the type (1) with fractional derivatives with respect to the time variable [17]. The slow diffusion is described by this equation.

Below we present some well-known properties of fractional operators

Lemma 1 *[15, Lemma 2.21] If $0 < \mu < 1$ for $v(t) \in C[0,T]$, then*

$$\partial_{0|t}^{\mu}[I_{0|t}^{\mu}v(t)] = v(t),$$

holds true.

Lemma 2 *[16, Proposition 2.3] Let $v(t) \in C([0,T])$. If $\alpha + \beta < 1$, then*

$$\partial_{0|t}^{\alpha+\beta}[I_{0|t}^{\beta}v(t)] = \partial_{0|t}^{\alpha}v(t).$$

Lemma 3 *[8] (a) Let $v(t) \in C^1([0,T])$ attains its maximum at $t_0 \in (0,T)$,*

$$\partial_{0|t}^{\alpha}v(t_0) \geq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}[v(t_0) - v(0)] \geq 0,$$

for all $0 < \alpha < 1$.

(b) Let $v(t) \in C^1([0,T])$ attain its minimum at $t_0 \in (0,T)$,

$$\partial_{0|t}^{\alpha}v(t_0) \leq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}[v(t_0) - v(0)] \leq 0,$$

for all $0 < \alpha < 1$.

1 Main results

The main results of this article are presented in this section.

The existence of $u_t(x,t)$ is implied by the solutions to problems (1) and (2). Therefore, if $t>0$, then $\partial^{\alpha}_{0|t}$ occurs for any $0<\alpha<1$. This means that a solution $u(x,t)$ of the problem (1) and (2) in the region $[0,a]\times[0,T]$ is a (classical) solution in $C([0,a]\times[0,T])\cap C^{2,1}((0,a)\times(0,T])$.

Theorem 1 *Let $\alpha+\beta<1$. If $u(x,t)$ satisfies* (1),

$$u(x,0)=\phi(x)\geq 0,\; x\in[0,a],$$

$$u(0,t)=0=u(a,t),\; t\in[0,T),$$

and

$$F(x,t)\geq 0, (x,t)\in(0,a)\times(0,T],$$

then

$$u(x,t)\geq 0 \;\; for \;\; (x,t)\in(0,a)\times(0,T].$$

Let us define the function

$$v(x,t)=u(x,t)+\epsilon t^{\gamma},$$

where $\epsilon>0$ and $\alpha<\gamma$.

From (2), we obtain $v(0,t)=v(a,t)=\epsilon t^{\gamma}>0$ for $t>0$, and $v(x,0)=\phi(x)$ for $x\in[0,a]$. Since

$$\partial^{\alpha}_{0|t}v(x,t)=\partial^{\alpha}_{0|t}u(x,t)+\partial^{\alpha}_{0|t}[\epsilon t^{\gamma}]=\partial^{\alpha}_{0|t}u(x,t)+\frac{\epsilon\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}$$

and

$$I^{\beta}_{0|t}v(x,t)=I^{\beta}_{0|t}u(x,t)+I^{\beta}_{0|t}[\epsilon t^{\gamma}]=I^{\beta}_{0|t}u(x,t)+\frac{\epsilon\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\gamma+\alpha},$$

it follows that

$$\frac{\partial^2}{\partial x^2}I^{\beta}_{0|t}v(x,t)=\frac{\partial^2}{\partial x^2}I^{\beta}_{0|t}u(x,t).$$

Consequently, the function $v(x,t)$ satisfies the equation

$$\partial^{\alpha}_{0|t}v(x,t)=\frac{\partial^2}{\partial x^2}I^{\beta}_{0|t}v(x,t)+F(x,t)+\frac{\epsilon\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}\;\text{ in }\;(0,a)\times(0,T],$$

with the initial-boundary conditions

$$v(x,0)=\phi(x)\quad x\in[0,a]$$

$$v(a,t)=v(0,t)=\epsilon t^{\gamma}>0,\quad t>0.$$

Assume that there is some point $(x,t) \in [0,a] \times [0,T]$ such that $v(x,t) < 0$. Since

$$v(x,t) \geq 0 \ \text{ for } \ (x,t) \in \{0,a\} \times [0,T] \cup [0,a] \times 0,$$

there is a point $(x_0,t_0) \in (0,a) \times (0,T)$ such that $v(x_0,t_0)$ is the negative minimum of $v(x,t)$ over $(0,a) \times (0,T)$. In view of Lemma 3 (b), we have

$$\partial^{\alpha}_{0|t} v(x_0,t_0) \leq \frac{t_0^{-\alpha}}{\Gamma(1-\alpha)}[v(x_0,t_0) - v(x_0,0)] < 0.$$

Let us define $w(x,t) = I^{\beta}_{0|t} v(x,t)$. From Lemma 1 we conclude that

$$\partial^{\beta}_{0|t} \omega(x,t) = \partial^{\beta}_{0|t}[I^{\beta}_{0|t} v(x,t)] = v(x,t).$$

Using $v(x,t)$ is bounded in $[0,a] \times [0,T]$, we obtain

$$\omega(x,t) = I^{\beta}_{0|t} v(x,t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(x,s)\, ds \to 0 \text{ as } t \to 0.$$

Noting $\alpha + \beta < 1$ from Lemma 2, we have

$$\partial^{\alpha+\beta}_{0|t} \omega(x,t) = \partial^{\alpha+\beta}_{0|t}[I^{\beta}_{0|t} v(x,t)] = \partial^{\alpha}_{0|t} \omega(x,t).$$

At this point, we obtain for $\omega(x,t)$ the next initial and boundary conditions

$$\omega(x,t) = I^{\beta}_{0|t} v(x,t) = I^{\beta}_{0|t} u(x,t) + \frac{\epsilon\Gamma(1+\gamma)\, t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} > 0 \ \text{ as } \ t \to 0^+$$

and

$$I^{\beta}_{0|t} v(0,t) = I^{\beta}_{0|t} v(a,t) = I^{\beta}_{0|t} \epsilon t^{\gamma} = \frac{\epsilon\Gamma(1+\gamma)\, t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} > 0 \ \text{ for } \ t > 0.$$

Hence the function $\omega(x,t)$ satisfies the problem

$$\begin{cases} \partial^{\alpha+\beta}_{0|t} \omega(x,t) = \frac{\partial^2}{\partial x^2}\omega(x,t) + F(x,t) + \frac{\epsilon\Gamma(1+\gamma)t^{\gamma-\alpha}}{\Gamma(\gamma-\alpha)} \ \text{ in } \ (0,a) \times (0,T], \\ \omega(x,0) > 0 \ \text{ on } \ [0,a], \\ \omega(0,t) > 0, \omega(a,t) > 0 \ \text{ for } \ 0 < t \leq T. \end{cases}$$

From Lemma 3 (b), we have $\omega(x_0,t_0) < 0$. Due to $\omega(x,t) > 0$ on the boundary, there exists $(x_1,t_1) \in (0,a) \times (0,T]$ such that $\omega(x_1,t_1)$ is the negative minimum of $\omega(x,t)$ in $[0,a] \times [0,T]$. From Lemma (2) (b), it is evident that

$$\partial^{\alpha+\beta}_{0|t} \omega(x_1,t_1)] \leq \frac{t_1^{\alpha}}{\Gamma(1-\alpha)}[\omega(x_1,t_1) - \omega(x_1,0)] < 0.$$

Since $\omega(x_1,t_1)$ is a local minimum, it yields that $\dfrac{\partial^2}{\partial x^2}\omega(x_1,t_1) \geq 0$.
Therefore at the point (x_1,t_1), we obtain

$$0 > \partial_{0|t}^{\alpha+\beta}\omega(x_1,t_1) = \frac{\partial^2}{\partial x^2}\omega(x_1,t_1) + F(x_1,t_1) + \frac{\epsilon\Gamma(1+\gamma)\,t_1^{\gamma-\alpha}}{\Gamma(\gamma-\alpha)} > 0.$$

This contradiction demonstrates that $v(x,t) \geq 0$ on $[0,a]\times[0,T]$, and it follows that $u(x,t) \geq -\epsilon t^{\gamma}$ on $[0,a]\times[0,T]$ for any $\epsilon > 0$. Because of given ϵ is arbitrary, then $u(x,t) \geq 0$ on $[0,a]\times[0,T]$, which concludes the proof.

The outcome is comparable for the negativity of the solution $u(x,t)$ by considering $\phi(x) \leq 0$ and $F(x,t) \leq 0$.

Theorem 2 *Suppose that $\alpha+\beta<1$. If $u(x,t)$ satisfies* (1)*,*

$$u(x,0) = \phi(x) \leq 0,\ x \in [0,a],$$

$$u(0,t) = 0 = u(a,t),\ t \in [0,T)$$

and

$$F(x,t) \leq 0, (x,t) \in [0,a]\times(0,T],$$

then

$$u(x,t) \leq 0, (x,t) \in [0,a]\times[0,T].$$

The results in Theorems 1 and 2 can be extended to obtain the next two theorems.

Theorem 3 *Let $\alpha+\beta<1$. Suppose $u(x,t)$ satisfies* (1)*,*

$$u(x,0) = \phi(x),\ x \in [0,a],$$

$$u(0,t) = g_1(t), u(a,t) = g_2(t),\ t \in [0,T),$$

where $g_1(t)$ and $g_2(t)$ are given real numbers. If

$$F(x,t) \geq 0, (x,t) \in [0,a]\times[0,T],$$

then

$$u(x,t) \geq \min_{[0,a]}\{g_1, g_2, \phi(x)\}\ \textit{for}\ (x,t) \in [0,a]\times[0,T].$$

Let

$$M = \min_{[0,a]}\{g_1, g_2, \phi(x)\}$$

and

$$\bar{u}(x,t) = u(x,t) - M.$$

Then, $\bar{u}(0,t) = g_1 - M \geq 0$, $\bar{u}(a,t) = g_2 - M \geq 0$ for $t \in [0,T]$, and $\bar{u}(x,0) = \phi(x) - M \geq 0$ for $x \in [0,a]$, Since

$$\partial_{0|t}^{\alpha}\bar{u} = \partial_{0|t}^{\alpha}u,\ \ \frac{\partial^2}{\partial x^2}I_{0|t}^{\beta}\bar{u}(x,t) = \frac{\partial^2}{\partial x^2}I_{0|t}^{\beta}u(x,t),$$

then $\bar{u}(x,t)$ is satisfies (1), respectively. Hence, it emerges from a argument similar to the proof of Theorem 2 that

$$\bar{u}(x,t) \geq 0\ \text{ on }\ [0,a]\times[0,T].$$

Consequently,

$$u(x,t) = \min_{[0,a]}(g_1, g_2, \phi(x)) \text{ for } (x,t) \in [0,a] \times [0,T].$$

The proof is completed.

Using $\bar{u}(x,t) = -u(x,t)$, a similar proof to that of Theorem 3 gives the next conclusion.

Theorem 4 *Let $\alpha + \beta < 1$. Suppose that $u(x,t)$ satisfies* (1),

$$u(x,0) = \phi(x), x \in [0,a],$$

$$u(0,t) = g_1, u(a,t) = g_2,$$

where g_1 and g_2 are given real numbers. If

$$F(x,t) \leq 0, (x,t) \in [0,a] \times [0,T],$$

then

$$u(x,t) \leq \max_{[0,a]}(g_1, g_2, \phi(x)) \text{ for } (x,t) \in [0,a] \times [0,T].$$

The heat equations weak maximum principle is similar to Theorems 3 and 4.

The fractional variant is backed by the weak maximum principle a solution's uniqueness, as in the classical case.

Theorem 5 *Let $\alpha + \beta < 1$. The problem* (1)-(2) *has at most one solution.*

Suppose that $u_1(x,t)$ and $u_2(x,t)$ two solutions of (1)-(2). Hence,

$$\partial^{\alpha}_{0|t}(u_1(x,t) - u_2(x,t)) = \frac{\partial^2}{\partial x^2} I^{\beta}_{0|t}(u_1(x,t) - u_2(x,t)),$$

with zero initial and zero boundary conditions for $u_1(x,t) - u_2(x,t)$.
In view of Theorems 4 and 5, we have

$$u_1(x,t) - u_2(x,t) = 0 \text{ on } [0,a] \times [0,T],$$

which completes the proof. A solution $u(x,t)$ of (1)-(2) depends constantly on the initial data $\phi(x)$, according to the Theorems 4 and 5.

Theorem 6 *Assume that $\alpha + \beta < 1$. Let $u(x,t)$ and $\bar{u}(x,t)$ are the solutions of the problem* (1)-(2) *with the initial condition $\phi(x)$ and $\bar{\phi}(x)$, respectively. If*

$$\max_{[0,a]}\{|\phi(x) - \bar{\phi}(x)|\} \leq \epsilon,$$

then

$$|u(x,t) - \bar{u}(x,t)| \leq \epsilon.$$

The function

$$v(x,t) = u(x,t) - \bar{u}(x,t)$$

satisfies the problem

$$\partial^{\alpha}_{0|t} v(x,t) = \frac{\partial^2}{\partial x^2} I^{\beta}_{0|t} v(x,t),$$

with the initial data

$$v(x,0) = \phi(x) - \bar{\phi}(x)$$

and zero boundary conditions. Then, in view of Theorems 4 and 5

$$|v(x,t)| \leq \max_{[0,a]} \{|\phi(x) - \bar{\phi}(x)|\}$$

the desired result follows.

Conclusion

A maximum principle is formulated and established in this paper for the one-dimensional time fractional diffusion equation with memory. A maximum principle for the Caputo fractional derivative serves as the foundation for the proof of the maximal principle. There is only one classical solution to the initial-boundary value problem for the time-fractional diffusion equation with memory, and the maximum principle is then used to demonstrate that this solution is continuous and depends on the initial and boundary conditions.

Funding

This research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP14869090).

References

[1] A. M. Nakhushev, "Inverse problems for degenerate equations and Volterra integral equations of the third kind Progr. *Differ. Uravn.* 10 (1974), 100–111 (in Russian).

[2] M. S. Salakhitdinov, A. Khasanov, "The extremum principle for a generalized operator of integro-differentiation of fractional order *Dokl. Akad. Nauk UzSSR.* 11 (1988), 3–4 (in Russian).

[3] V. A. Nakhusheva, "The extremum principle for a nonlocal parabolic equation and the mixed problem for the generalized wave equation *Dokl. Adyg. (Cherkes.) Mezhdun. Akad. Nauk.* 2 (1996), 26–28 (in Russian).

[4] B. D. Kadirkulov, B. K. Turmetov, "On a generalization of the heat equation *Uzbek. Mat. Zh.* (2006), 40–45 (in Russian).

[5] Y. Luchko, "Maximum principle for the generalized time-fractional diffusion equation *Journal of Mathematical Analysis and Applications.* 351 (2009), 218–223.

[6] Y. Luchko, "Some uniqueness and existence results for the initial boundary-value problems for the generalized time-fractional diffusion equation *Computers and Mathematics with Applications.* 59 (2010), 1766–1772.

[7] Y. Luchko, "Initial-boundary-value problems for the generalized multiterm time-fractional diffusion equation *Journal of Mathematical Analysis and Applications.* 374 (2011), 538–548.

[8] M. Al-Refai, "On the fractional derivatives at extreme points *Electronic Journal of Qualitative Theory of Differential Equations.* 55 (2012), 1–5.

[9] M. Al-Refai, Y. Luchko, "Maximum principle for the fractional diffusion equations with the Riemann-Liouville fractional derivative and its applications *Fractional Calculus and Applied Analysis.* 17: 2 (2014), 483–498.

[10] C. Y. Chan, H. T .Liu, "A maximum principle for fractional diffusion equations *Quarterly of Applied Mathematics.* 74: 3 (2016), 421–427.

[11] Z. Liu, S. Zeng, Y. Bai, "Maximum principles for multi-term space-time variable-order fractional diffusion equations and their applications *Fractional Calculus and Applied Analysis.* 19: 1 (2016), 188–211.

[12] Y. Luchko, M. Yamamoto, "On the maximum principle for a time-fractional diffusion equation. Fractional Calculus and Applied Analysis. 20: 5 (2017), 1131–1145.

[13] H. Ye, F. Liu, V. Anh, I. Turner, "Maximum principle and numerical method for the multi-term time-space Riesz-Caputo fractional differential equations *Applied Mathematics and Computation.* 227 (2014), 531–540.

[14] M. Kirane, B. T. Torebek, "Maximum principle for space and time-space fractional partial differential equations *Z. Anal. Anwend.* 40 (2021), 277–301.

[15] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, "Theory and applications of fractional differential equations *North-Holland Math. Stud. 204, Elsevier Science B.V., Amsterdam,* (2006).

[16] B. Jin, "Fractional differential equations—an approach via fractional derivatives *Applied Mathematical Sciences. 206, Springer, Cham,* (2021).

[17] V.V.Uchaikin, "Fractional Derivatives for Physicists and Engineers *V. 1, Background and Theory. V. 2, Application, Springer,* (2013).

IRSTI 27.31.15 DOI: https://doi.org/10.26577/JMMCS2023v120i4a5

E.A. Oyekan[1*], **A.O. Lasode**[2], **T.A. Olatunji**[3]
[1]Olusegun Agagu University of Science and Technology, Okitipupa, Nigeria
[2]University of Ilorin, Ilorin, Nigeria
[3]University of Bolton, BL3 5AB, UK
*e-mail: ea.oyekan@oaustech.edu.ng

INITIAL BOUNDS FOR ANALYTIC FUNCTION CLASSES CHARACTERIZED BY CERTAIN SPECIAL FUNCTIONS AND BELL NUMBERS

Over the last few years, Geometric Function Theory (GFT) as one of the most prime branch of complex analysis has gained a considerable and an impressive attention from many researchers, largely because it deals with the study of the geometric properties of analytic functions and their numerous applications in various fields of mathematics such as in special functions, probability distributions, and fractional calculus. The investigations in this paper are on two new classes of analytic functions defined in the unit disk $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$ and denoted by $\chi\mathcal{S}_q(b,\mathcal{K})$ and $\chi\mathcal{T}_q(b,\mathcal{K})$. Function f in the classes satisfy the conditions $f(0) = f'(0) - 1 = 0$, hence can be of series type $f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in \mathcal{E}$. The definition of the two new classes of analytic functions embed some well-known special functions such as the Galuê-type Struve function, modified error function and a starlike function whose coefficients are Bell's numbers while some involving mathematical principles are the q-derivative, inequalities, convolution and subordination. The main results from these classes are however, the upper estimates of some initial bounds such as $|a_n|$ $(n = 2, 3, 4)$ and the Fekete-Szegö functional $|a_3 - \phi a_2^2|$ $(\phi \in \mathbb{C})$ of functions $f \in \chi\mathcal{S}_q(b,\mathcal{K})$ and $f \in \chi\mathcal{T}_q(b,\mathcal{K})$.
Key words: analytic function, Schwarz function, Galuê-type Struve function, modified error function, Bell's numbers, coefficient estimate, Fekete-Szegö problem, subordination, convolution, q-derivative.

Э.А. Оекан[1*], А.О. Ласоде[2], Т.А. Олатунджи[3]
[1]Олусегун Агагу ғылым және технология университеті, Нигерия, Окитипупа
[2]Илорин университеті, Нигерия, Илорин қ.
[3]Болтон университеті, Ұлыбритания, Болтон қ.
*e-mail: ea.oyekan@oaustech.edu.ng

Кейбір арнайы функциялармен және сандармен сипатталатын аналитикалық функциялар кластарының бастапқы шекаралары

Соңғы бірнеше жылда геометриялық функциялар теориясы (ГФТ) кешенді талдаудың ең басты саласының бірі ретінде көптеген зерттеушілер тарапынан айтарлықтай және әсерлі назарға ие болды, өйткені ол аналитикалық функциялардың геометриялық қасиеттерін зерттеумен айналысады және арнайы функциялар, ықтималдық үлестірімдер және бөлшек есептеу сияқты математиканың әртүрлі салаларындағы көптеген қолданбаларды зерттеумен айналысады. Бұл мақалада $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$ бірлік шеңберінде анықталған аналитикалық функциялардың екі жаңа класын қарастырады және $\chi\mathcal{S}_q(b,\mathcal{K})$ және $\chi\mathcal{T}_q(b,\mathcal{K})$ арқылы белгіленеді. f функциясы $f(0) = f'(0) - 1 = 0$ шарттарын қанағаттандырады, сондықтан $f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in \mathcal{E}$. қатардың типі түрінде болуы мүмкін. Аналитикалық функциялардың екі жаңа класының анықтамалары Галуэ типті Струве функциясын, өзгертілген қате функциясын, коэффициенттері Белл сандары, ал кейбір математикалық принциптері q-туынды болып табылатын жұлдыз функциясын, теңсіздіктерді, конвульсия және бағыну болып табылатын белгілі арнайы функцияларды қамтиды.

© 2023 Al-Farabi Kazakh National University

Дегенмен, бұл кластардың негізгі нәтижелері кейбір бастапқы шекаралардың жоғарғы бағалаулары болып табылады, яғни $|a_n|$ $(n = 2, 3, 4)$ және $f \in \chi\mathcal{S}_q(b, \mathcal{K})$ және $f \in \chi\mathcal{T}_q(b, \mathcal{K})$ функцияларының Фекете-Сеге функционалы $|a_3 - \phi a_2^2|$ $(\phi \in \mathbb{C})$.
Түйін сөздер: Аналитикалық функция, Шварц функциясы, Галуэ типті Струве функциясы, модификацияланған қате функциясы, Белл сандары, коэффициентті баға, Фекете-Сеге есебі, бағыну, ұйірткі, q-туынды.

E.A. Oyekan[1*], A.O. Lasode[2], T.A. Olatunji[3]
[1]Университет науки и технологий Олусегуна Агагу, Нигерия, г. Окитипуп
[2]Университет Илорина, Нигерия, г. Илорин
[3]Университет Болтона, Великобритания, г. Болтон
*e-mail: ea.oyekan@oaustech.edu.ng

Начальные границы классов аналитических функций, характеризующих определенных специальных функций и номеров

За последние несколько лет геометрическая теория функций (ГТФ) как одна из наиболее важных отраслей комплексного анализа привлекла значительное и впечатляющее внимание многих исследователей, главным образом потому, что она занимается изучением геометрических свойств аналитических функций и их многочисленные приложения в различных областях математики, таких как специальные функции, распределения вероятностей и дробное исчисление. В данной статье исследуются два новых класса аналитических функций, определенных в единичном круге $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$ и обозначается $\chi\mathcal{S}_q(b, \mathcal{K})$ и $\chi\mathcal{T}_q(b, \mathcal{K})$. Функция f в классах удовлетворяет следующим условиям $f(0) = f'(0) - 1 = 0$, следовательно, может иметь тип ряда $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, $z \in \mathcal{E}$. В определения двух новых классов аналитических функций включены некоторые хорошо известные специальные функции, такие как функция Струве типа Галуэ, модифицированная функция ошибок и звездообразная функция, коэффициентами которой являются числа Белла, а некоторыми математическими принципами являются q-производная, неравенства, свертка и подчинение. Однако основными результатами этих классов являются верхние оценки некоторых начальных границ, таких как $|a_n|$ $(n = 2, 3, 4)$ и функционал Фекете-Сеге $|a_3 - \phi a_2^2|$ $(\phi \in \mathbb{C})$ функций $f \in \chi\mathcal{S}_q(b, \mathcal{K})$ и $f \in \chi\mathcal{T}_q(b, \mathcal{K})$.
Ключевые слова: Аналитическая функция, функция Шварца, функция Струве типа Галуэ, модифицированная функция ошибок, числа Белла, коэффициентная оценка, задача Фекете-Сегё, подчинение, свертка, q-производная.

1 Introduction and Definitions

Geometric Function Theory (GFT) is one of the most fascinating branches of complex analysis and it has gained a considerable attention from many researchers in pure mathematics. GFT deals with the study of the geometric properties of analytic functions with numerous applications in various fields of mathematics such as in the use of special functions, probability distributions, and fractional calculus.

Let $\mathcal{A}$ represent the set of normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

with the conditions $f(0) = 0 = f'(0) - 1$ and $z \in \mathcal{E} := \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S}$, a subset of $\mathcal{A}$ be the set of analytic and univalent functions in $\mathcal{E}$.

The foundation of coefficient problems in the theory of univalent functions is traceable to Bieberbach conjecture or coefficient conjecture (see [7]) of 1916 where he conjuncted that

$|a_n| \leqslant n,\ \forall n \in \mathbb{N}$. In [7], Duren emphasized that *coefficient problem* is the determination of that part of the $(k-1)$-dimensional complex plane, occupied by the points $(a_2, a_3, a_4, \ldots, a_k)$ for function". In 1985, Branges [5] verified that the conjuncture was actually true and this affirmation subsequently elevated the theory to one of the growing areas of possible research. Some well-known subclasses of class $\mathcal{S}$ are therefore the classes of starlike, convex, close-to-convex, close-to-star and spirallike functions. In addition, the coefficient bounds, generalizations and the coefficient properties of several of the subclasses of class $\mathcal{S}$ have also been sought. In fact, the nature and properties of these subclasses which are largely based on the geometries of their domains are continuously been studied with no end at sight.

In this paper, represented by ∇ is the set of analytic functions of the form

$$w(z) = \sum_{n=1}^{\infty} w_n z^n \quad (z \in \mathcal{E}). \tag{2}$$

Set ∇ is known as the set of Schwarz functions and it is normalized by the conditions $w(0) = 0$, $|w(z)| < 1$ and $z \in \mathcal{E}$. Likewise, if $h_1, h_2 \in \mathcal{A}$, then $h_1 \prec h_2$ if $h_1(z) = h_2(w(z))$ for $z \in \mathcal{E}$. Should h_2 be univalent in $\mathcal{E}$, then $h_1(z) \prec h_2(z)$ if and only if $h_1(0) = h_2(0)$ and $h_1(\mathcal{E}) \subset h_2(\mathcal{E})$. The symbol '$\prec$' means subordination. Also, let

$$h_1 = z + \sum_{n=2}^{\infty} a_n z^n, h_1 = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A},\ \text{then } h_1 \star h_2 := z + \sum_{n=2}^{\infty} (a_n \times b_n) z^n$$

where the symbol '$\star$' means convolution or Hadamard product.

The sequence $\{\eta_n\}_0^\infty$ of numbers

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots$$

was introduced by Bell [3, 4]. The Bell's numbers are generated as a result of observing the number of possible partitions of a set. In view of the Bell's number, Kumar et al. [15] established the function

$$\mathcal{K}(z) = e^{e^z - 1} = \sum_{n=0}^{\infty} \frac{\eta_n}{n!} z^n = 1 + z + z^2 + \frac{5}{6} z^3 + \frac{5}{8} z^4 + \cdots, \quad z \in \mathcal{E}. \tag{3}$$

and it was proved that function $\mathcal{K}(z)$ is starlike with respect to 1. This starlikeness property prompted the our interest to further investigate this function.

The Galuê-type Struve function (GTSF) was introduced in [16] and defined by

$$\alpha\mathcal{W}_{p,b,c,\xi}^{\lambda,\mu}(z) = z + \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(\lambda n + \mu)\Gamma(\alpha n + \frac{p}{\xi} + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1} \quad (z \in \mathcal{E}), \tag{4}$$

where $\alpha \in \mathbb{N}$, $z, p, b, c \in \mathbb{C}$, $\lambda > 0$, $\xi > 0$ and μ is an arbitrary parameter. It is evident that when $\lambda = \alpha = 1$, $\mu = \frac{3}{2}$ and $\xi = 1$ in (4), then we have the generalized Struve function (see [17]) defined by

$$\mathcal{H}_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n + \frac{3}{2})\Gamma(n + p + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1} \quad (z \in \mathcal{E}), \tag{5}$$

where $z, p, b, c \in \mathbb{C}$. Using (4), consider the function

$$\mathcal{U}_{p,b,c,\xi}(z) = 2^p \sqrt{\pi}\, \Gamma\Big(\frac{p}{\xi} + \frac{b+2}{2}\Big)\, z^{\frac{-(p+1)}{2}}\, \alpha \mathcal{W}^{\lambda,\mu}_{p,b,c,\xi}(\sqrt{z}) \quad (z \in \mathcal{E}). \tag{6}$$

Using the Pochhammer (or Appell) symbol defined in terms of Euler's gamma function, Oyekan [19] presented the relation

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma(\gamma+1)\ldots(\gamma+n-1)$$

so that from (6) we have

$$\mathcal{V}_{p,b,c,\xi}(z) = z\mathcal{U}_{p,b,c,\xi}(z) = z + \sum_{n=2}^{\infty} \Big(\frac{(\frac{-c}{4})^n}{(\mu)_{\lambda(n-1)}(\gamma)_{\alpha(n-1)}}\Big) z^n \quad (z \in \mathcal{E}). \tag{7}$$

Using the convolution principle, Oyekan [19] defined the function

$$\mathcal{L}^{\lambda,\mu,\xi}_{p,b,c}(z) = (f \star \mathcal{V}_{p,b,c,\xi})(z) = z + \sum_{n=2}^{\infty} \Big(\frac{(\frac{-c}{4})^n}{(\mu)_{\lambda(n-1)}(\gamma)_{\alpha(n-1)}}\Big) a_n z^n \quad (z \in \mathcal{E}), \tag{8}$$

where $p, b, c \in \mathbb{C}$, $\gamma = \frac{p}{\xi} + \frac{b+2}{2} \neq 0, -1, -2, \cdots$, $\alpha \in \mathbb{N}$, $\lambda, \xi > 0$ and μ is an arbitrary parameter. Function $\mathcal{V}$ in (7) is the normalized form of Galuê-type Struve function and is analytic in $\mathbb{C}$, while (8) is the simplified version.

A special function that occurs in probability, statistics, material science, and partial differential equation is the *error function*. The error function is use in quantum mechanics to eliminate the probability of observing a particle in a specified region. The error function

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} z^{n+1}}{(2n+1)!} \quad (z \in \mathcal{E}) \tag{9}$$

was reported in [1] and for additional information see [6, 8]. In particular, Ramachandra et al. [25] made a slight modification to (9) and came up with the function

$$Erf(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!}\, z^n \quad (z \in \mathcal{E}). \tag{10}$$

where the function $Erf(z)$ was used to define a class of analytic functions and solved some coefficient problems.

Using the convolution concept, and in view of (8) and (10), we can deduce the function

$$\mathcal{G}(z) = (\mathcal{L}^{\lambda,\mu,\xi}_{p,b,c} \star Erf)(z) = z + \sum_{n=2}^{\infty} \frac{(-\frac{c}{4})^{n-1}}{(2n-1)(n-1)!(\mu)_{\lambda(n-1)}(\gamma)_{\alpha(n-1)}}\, a_n z^n. \tag{11}$$

The quantum derivative (q-derivative or Jackson's derivative) operator (see [9]) for function f in (1) is defined by

$$\left.\begin{aligned} &\mathcal{D}_q f(0) = f'(0) = 1 \quad (z = 0) \quad \text{if it exists,} \\ &\mathcal{D}_q f(z) = \tfrac{f(z)-f(qz)}{z(1-q)} = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad (z \neq 0), \\ &\mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z)) = \sum_{n=2}^{\infty} [n]_q [n-1]_q a_n z^{n-2} \end{aligned}\right\} \tag{12}$$

such that the q-number n is defined by

$$[n]_q = \begin{cases} 0 & for \quad n = 0 \\ 1 & for \quad n = 1 \\ 1+q & for \quad n = 2 \\ \frac{1-q^n}{1-q} = \sum\limits_{n=0}^{n-1} q^n & for \quad n \in \mathbb{R} \end{cases} \tag{13}$$

and $\lim\limits_{q\uparrow 1}[n]_q = n$. The q-derivative is the q-analogue of the classical derivative of functions where it plays a significant role in defining many q-operators in various areas of q-analysis. For some historical details, properties, applications, and some results on some subclasses of analytic functions involving q-differentiation see [2, 9, 11–14].

Definition 1.1 *Let $q \in (0,1)$, $b \in \mathbb{C} \setminus \{0\}$, $\gamma \neq 0, -1, -2, \cdots$, $c \in \mathbb{C}$ and let $\mathcal{K}(z)$ be as defined in* (3)*. The function $\mathcal{G}$ is said to belong to class $\chi\mathcal{S}_q(b, \mathcal{K})$, if*

$$1 + \frac{1}{b}\left(\frac{z\mathcal{D}_q\mathcal{G}(z)}{\mathcal{G}(z)} - 1\right) \prec \mathcal{K}(z) \tag{14}$$

and it is said to belong to the class $\chi\mathcal{T}_q(b, \mathcal{K})$, if

$$1 + \frac{1}{b}\left(\frac{z\mathcal{D}_q(\mathcal{D}_q\mathcal{G}(z))}{\mathcal{D}_q\mathcal{G}(z)}\right) \prec \mathcal{K}(z). \tag{15}$$

In this work we gave the estimates on the initial coefficients and on the Fekete-Szegö functionals for two classes of analytic functions.

2 Applicable Lemmas

Let $w \in \nabla$ in (2), then the following lemmas hold true.

Lemma 2.1 ([26]) *Let $w(z) \in \nabla$, then $|w_n| \leqslant 1$ $\forall n \in \mathbb{N}$. Equality occurs for functions $w(z) = e^{i\vartheta}z^n$ $(\vartheta \in [0, 2\pi))$.*

Lemma 2.2 ([10]) *Let $w \in \nabla$, then for $\phi \in \mathbb{C}$,*

$$|w_2 + \phi w_1^2| \leqslant \max\{1; |\phi|\}.$$

Equality holds for functions $w(z) = z$ or $w(z) = z^2$.

3 Main Results

Theorem 3.1 *Let $q \in (0,1)$, $b \in \mathbb{C} \setminus \{0\}$, $\gamma \neq 0, -1, -2, \cdots$, $c \in \mathbb{C}$ and let $\mathcal{K}(z)$ be as defined in* (3)*. If $\mathcal{G}$ belongs to the class $\chi\mathcal{S}_q(b, \mathcal{K})$, then*

$$|a_2| \leqslant \frac{12(\mu)_\lambda(\gamma)_\alpha|b|}{cq},$$

$$|a_3| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2q(1+q)} \max\left\{1, \left|\frac{q+b}{q}\right|\right\},$$

$$|a_4| \leqslant \frac{168(\mu)_{3\lambda}(\gamma)_{3\alpha}|b|}{c^3q(1+q+q^2)} \max\left\{1, \left|\sigma\left[\frac{t}{\sigma} + \left(\frac{q+b}{q}\right) + \left(1 + \frac{2}{\sigma}\right)\right]\right|\right\},$$

and for $\phi \in \mathbb{C}$,

$$|a_3 - \phi a_2^2| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2 q(1+q)} \max\left\{1, \left|\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}q^2 + 5(\mu)_{2\lambda}(\gamma)_{2\alpha}qb - 18\xi(\mu)_{2\lambda}^2(\gamma)_{2\alpha}^2 q(1+q)b}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}q^2}\right|\right\}$$

where

$$\sigma = \frac{([2]_q - 1) + ([3]_q - 1)}{([2]_q - 1)([3]_q - 1)} b \quad \text{and} \quad t = \frac{5}{6} - \frac{b^2}{([2]_q - 1)^2}. \tag{16}$$

Proof. Suppose $\mathcal{G} \in \chi\mathcal{S}_q(b, \mathcal{K})$, then there exists a Schwarz function $w \in \nabla$ of the form (2) such that

$$1 + \frac{1}{b}\left(\frac{z\mathcal{D}_q\mathcal{G}(z)}{\mathcal{G}(z)} - 1\right) = \mathcal{K}(w(z)),$$

so that

$$[z\mathcal{D}_q\mathcal{G}(z) - \mathcal{G}(z)]\mathcal{G}^{-1}(z) = b[\mathcal{K}(w(z)) - 1]. \tag{17}$$

A careful expansion of (17) shows that

$$\begin{aligned}
[z\mathcal{D}_q\mathcal{G}(z) - \mathcal{G}(z)]\mathcal{G}^{-1}(z) &= ([2]_q - 1)\frac{c}{12(\mu)_\lambda(\gamma)_\alpha} a_2 z \\
&+ \left\{([3]_q - 1)\frac{c^2}{40(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_3 - ([2]_q - 1)\frac{c^2}{144(\mu)_\lambda^2(\gamma)_\alpha^2} a_2^2\right\} z^2 \\
&- \left\{\Big(([2]_q - 1) + ([3]_q - 1)\Big)\frac{c^3}{480(\mu)_\lambda(\gamma)_\alpha(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_2 a_3 \right. \\
&\quad \left. - ([2]_q - 1)\frac{c^3}{1728(\mu)_\lambda^3(\gamma)_\alpha^3} a_2^3 - ([4]_q - 1)\frac{c^3}{168(\mu)_{3\lambda}(\gamma)_{3\alpha}} a_4\right\} z^3 \\
&+ \cdots
\end{aligned} \tag{18}$$

and

$$b[\mathcal{K}(w(z)) - 1] = bw_1 z + b(w_2 + w_1^2)z^2 + b(w_3 + 2w_1w_2 + \frac{5}{6}w_1^3)z^3 + \cdots . \tag{19}$$

Equating (18) and (19) gives

$$([2]_q - 1)\frac{c}{12(\mu)_\lambda(\gamma)_\alpha} a_2 = bw_1 \tag{20}$$

$$([3]_q - 1)\frac{c^2}{40(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_3 - ([2]_q - 1)\frac{c^2}{144(\mu)_\lambda^2(\gamma)_\alpha^2} a_2^2 = b(w_2 + w_1^2). \tag{21}$$

$$\begin{aligned}
-\Big[([2]_q - 1) + ([3]_q - 1)\Big]\frac{c^3}{480(\mu)_\lambda(\gamma)_\alpha(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_2 a_3 + ([2]_q - 1)\frac{c^3}{1728(\mu)_\lambda^3(\gamma)_\alpha^3} a_2^3 \\
+ ([4]_q - 1)\frac{c^3}{168(\mu)_{3\lambda}(\gamma)_{3\alpha}} a_4 = b(w_3 + 2w_1w_2 + \frac{5}{6}.w_1^3)
\end{aligned} \tag{22}$$

Now, from (20) we have

$$a_2 = \frac{12(\mu)_\lambda(\gamma)_\alpha b w_1}{c([2]_q - 1)} \tag{23}$$

so that by using (13) we get

$$|a_2| \leqslant \frac{12(\mu)_\lambda(\gamma)_\alpha |b||w_1|}{cq}$$

and the application of Lemma 2.1 achieves the result.

From (21) we have

$$a_3 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2([3]_q - 1)} \left[w_2 + \left(\frac{[2]_q - 1 + b}{[2]_q - 1} \right) w_1^2 \right] \tag{24}$$

so that using (13) gives

$$|a_3| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2 q(1+q)} \left| w_2 + \left(\frac{[2]_q - 1 + b}{[2]_q - 1} \right) w_1^2 \right|$$

and the application of Lemma 2.2 achieves the result.

From (22) we have

$$a_4 = \frac{168(\mu)_{3\lambda}(\gamma)_{3\alpha} b}{c^3([4]_q - 1)} \left[w_3 + \left(\frac{5}{6} - \frac{b^2}{([2]_q - 1))^2} + \frac{\left[([2]_q - 1) + ([3]_q - 1)\right]}{([2]_q - 1)([3]_q - 1)} b \left(\frac{[2]_q - 1 + b}{[2]_q - 1} \right) \right) w_1^3 \right.$$
$$\left. + \left(2 + \frac{\left[([2]_q - 1) + ([3]_q - 1)\right]}{([2]_q - 1)([3]_q - 1)} \right) w_1 w_2 \right]$$

by using (16) we get

$$a_4 = \frac{168(\mu)_{3\lambda}(\gamma)_{3\alpha} b}{c^3([4]_q - 1)} \left[w_3 + \sigma \left(\frac{t}{\sigma} + \left(\frac{[2]_q - 1 + b}{[2]_q - 1} \right) \right) w_1^3 + \left(\frac{2}{\sigma} + 1 \right) w_1 w_2 \right]$$

by using (13) w get

$$|a_4| \leq \frac{168(\mu)_{3\lambda}(\gamma)_{3\alpha}|b|}{c^3 q(1+q+q^2)} \left| w_3 + \sigma \left(\frac{t}{\sigma} + \left(\frac{[2]_q - 1 + b}{[2]_q - 1} \right) \right) w_1^3 + \left(\frac{2}{\sigma} + 1 \right) w_1 w_2 \right|$$

and the application of Lemmas 2.1 and 2.2 achieves the result.

Lastly, from (23) and (24) and for $\phi \in \mathbb{C}$ we have

$$a_3 - \phi a_2^2 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2([3]_q - 1)} \left[w_2 + \left(\frac{[2]_q - 1 + b}{([2]_q - 1)} \right) w_1^2 \right] - \phi \frac{144(\mu)_\lambda^2(\gamma)_\alpha^2 b^2 w_1^2}{c^2([2]_q - 1)^2}$$
$$= \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2([3]_q - 1)} \left[w_2 + \left(1 + \frac{b}{[2]_q - 1} - \frac{18\phi(\mu)_\lambda^2(\gamma)_\alpha^2 b([3]_q - 1)}{5([2]_q - 1)^2(\mu)_{2\lambda}(\gamma)_{2\alpha}} \right) w_1^2 \right]$$

$$a_3 - \phi a_2^2 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}b}{c^2([3]_q - 1}\Bigg[w_2 + \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2 + 5b(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1) - 18\xi(\mu)_\lambda^2(\gamma)_\alpha^2 b([3]_q-1)}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2}\right)w_1^2\Bigg]$$

by using (13) we get

$$a_3 - \phi a_2^2 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}b}{c^2 q(1+q)}\Bigg[w_2 + \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2 + 5b(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1) - 18\xi(\mu)_\lambda^2(\gamma)_\alpha^2 b([3]_q-1)}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2}\right)w_1^2\Bigg]$$

and

$$|a_3 - \phi a_2^2| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2 q(1+q)}\Bigg|w_2 + \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2 + 5b(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1) - 18\xi(\mu)_\lambda^2(\gamma)_\alpha^2 b([3]_q-1)}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}([2]_q-1)^2}\right)w_1^2\Bigg|$$

and the application of Lemma 2.2 achieves the result.

Theorem 3.2 *Let $q \in (0,1)$, $b \in \mathbb{C} \setminus \{0\}$, $\gamma \neq 0, -1, -2, \cdots$, $c \in \mathbb{C}$ and let $\mathcal{K}(z)$ be as defined in* (3). *If $\mathcal{G}$ belongs to the class $\chi\mathcal{T}_q(b,\mathcal{K})$, then*

$$|a_2| \leqslant \frac{12(\mu)_\lambda(\gamma)_\alpha|b|}{c(1+q)},$$
$$|a_3| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2 q(1+q)}\max\{1; |1+b|\},$$

and

$$|a_3 - \xi a_2^2| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha}|b|}{c^2 q(1+q)}\max\left\{1, \left|\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}(1+q)^2(1+b) - 18\xi(\mu)_\lambda^2(\gamma)_\alpha^2 q(1+q)b}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}(1+q)^2}\right|\right\}.$$

Proof. Suppose $\mathcal{G} \in \chi\mathcal{T}_q(b,\mathcal{K})$, then there exists a Schwarz function $w \in \nabla$ of the form (2) such that

$$1 + \frac{1}{b}\left(\frac{z\mathcal{D}_q\mathcal{G}(z)}{\mathcal{D}_q\mathcal{G}(z)}\right) = \mathcal{K}(w(z))$$

so that

$$(z\mathcal{D}_q\mathcal{G}(z))(\mathcal{D}_q\mathcal{G}(z))^{-1} = [\mathcal{K}(w(z)) - 1]b \tag{25}$$

from where we get

$$(z\mathcal{D}_q\mathcal{G}(z))(\mathcal{D}_q\mathcal{G}(z))^{-1} = z - [2]_q\frac{c}{12(\mu)_\lambda(\gamma)_\alpha}a_2 z \tag{26}$$

$$+ \left([3]_q\frac{c^2}{40(\mu)_{2\lambda}(\gamma)_{2\alpha}}a_3 - [2]_q^2\frac{c^2}{144(\mu)_\lambda^2(\gamma)_\alpha^2}a_2^2\right)z^2 + \cdots. \tag{27}$$

Equating (3) and (26) implies that

$$-[2]_q \frac{c}{12(\mu)_\lambda(\gamma)_\alpha} a_2 = bw_1 \tag{28}$$

and

$$[3]_q \frac{c^2}{40(\mu)_{2\lambda}(\gamma)_{2\alpha}} a_3 - [2]_q^2 \frac{c^2}{144(\mu)_\lambda^2(\gamma)_\alpha^2} a_2^2 = b(w_2 + w_1^2). \tag{29}$$

Now from (28) we get

$$a_2 = \frac{-12(\mu)_\lambda(\gamma)_\alpha bw_1}{c[2]_q}. \tag{30}$$

so that by using (13), taking modulus and applying Lemma 2.1 achieves the result.

From (29) we get

$$a_3 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2[3]_q} \left[w_2 + (1+b)w_1^2 \right], \tag{31}$$

and using (13) we get

$$a_3 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2 q(1+q)} \left[w_2 + (1+b)w_1^2 \right]$$

and

$$|a_3| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} |b|}{c^2 q(1+q)} \left| w_2 + (1+b)w_1^2 \right|$$

so that applying Lemma 2.2 achieves the result.

Now from (30) and (31) we get

$$\begin{aligned} a_3 - \varphi a_2^2 &= \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2[3]_q} \left[w_2 + (1+b)w_1^2 \right] - \xi \frac{144(\mu)_\lambda^2(\gamma)_\alpha^2 b^2 w_1^2}{c^2[2]_q^2} \\ &= \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2[3]_q} \left[w_2 + \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2(1+b) - 18\xi k_1^2 b[3]_q}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2} \right) w_1^2 \right] \end{aligned}$$

and using (13) we get

$$a_3 - \varphi a_2^2 = \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} b}{c^2 q(1+q)} \left[w_2 + \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2(1+b) - 18\xi k_1^2 b[3]_q}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2} \right) w_1^2 \right]$$

so that

$$|a_3 - \varphi a_2^2| \leqslant \frac{40(\mu)_{2\lambda}(\gamma)_{2\alpha} |b|}{c^2 q(1+q)} \left| w_2 + \left(\frac{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2(1+b) - 18\xi k_1^2 b[3]_q}{5(\mu)_{2\lambda}(\gamma)_{2\alpha}[2]_q^2} \right) w_1^2 \right|.$$

so that applying Lemma 2.2 achieves the result.

4 Acknowledgement

The authors sincerely appreciate the referees' critical contributions which improved this research.

Список литературы

[1] Abramowitz M., Stegun I.A. (eds.). *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables,* Dover Publications Inc., New York, (1965).

[2] Aral A., Gupta V., Agarwal R.P. *Applications of q-Calculus in Operator Theory,* Springer Science+Business Media, New York, (2013).

[3] Bell E.T. *Exponential polynomials, Ann. Math.,* 35, (1934): 258–277.

[4] Bell E.T. *The iterated exponential integers, Ann. Math.,* 39, (1938): 539–557.

[5] Branges, D.L. *A proof of the Bieberbach conjecture, Acta Math.,* 154(1-2), (1985): 137–152.

[6] Coman D. *The radius of starlikeness for error function, Stud. Univ. Babes Bolyal Math.,* 36, (1991): 13–16.

[7] Duren, P.L., *Univalent Functions*, Springer-Verlag, New York Inc, (1983).

[8] Elbert A., Laforgia A. *The zeros of the complementary error function, Numer. Algorithms,* 49 (1-4), (2008): 153–157.

[9] Jackson F.H. *On q-functions and a certain difference operator, Trans. Roy. Soc. Edinb.,* 46 (2), (1908): 64–72.

[10] Jahangiri J.M., Ramachandran C., Annamalai S. *Fekete-Szegö problem for certain analytic functions defined by hypergeometric functions and Jacobi polynomial. J. Fract. Calc. Appl.,* 9, (2018): 1–7.

[11] Kac V., Cheung P., *Quantum Calculus,* Springer-Verlag Inc., New York, (2002).

[12] Lasode A.O., Opoola T.O. *On a generalized class of bi-univalent functions defined by subordination and q-derivative operator, Open J. Math. Anal.,* 5 (2), (2021): 46–52.

[13] Lasode A.O., Opoola T.O. *Fekete-Szegö estimates and second Hankel determinant for a generalized subfamily of analytic functions defined by q-differential operator, Gulf J. Math.,* 11 (2), (2021): 36–43.

[14] Lasode A.O., Opoola T.O. *Some investigations on a class of analytic and univalent functions involving q-differentiation, Eur. J. Math. Anal.,* 2 (12), (2022): 1–9.

[15] Kumar V., Cho N.E., Ravichandran V., Srivastava H.M. *Sharp coefficient bounds for starlike functions associated with the Bell numbers, Math. Slovaca.,* 69, 2019: 1053–1064.

[16] Nisar K.S., Baleanu D., Qurashi M.A. *Fractional calculus and application of generalized Struve function, SpringerPlus J.,* 5 (910), (2016): 13 pages.

[17] Orhan H., Yagmur N. *Geometric properties of generalized struve functions.* In: The International Congress in honour of Professor H.M. Srivastava, 23–26, Bursa, Turkey, (2012).

[18] Oyekan E.A. *Coefficient estimates and subordination results for certain classes of analytic functions, J. Math. Sci.,* 24 (2), (2013): 75–86.

[19] Oyekan E.A. *Certain geometric properties of functions involving Galuê type Struve function, Ann. Math. Comput. Sci.,* 8, (2022): 43–53.

[20] Oyekan E.A., Awolere I.T. *A new subclass of univalent functions connected with convolution defined via employing a linear combination of two generalized differential operators involving sigmoid function, Maltepe J. Math.,* 2 (2), (2020): 11–21.

[21] Oyekan E.A., Lasode A.O. *Estimates for some classes of analytic functions associated with Pascal distribution series, error function, Bell numbers and q-differential operator, Nigerian J. Math. Appl.,* 32, (2022): 163–173.

[22] Oyekan E.A., Swamy S.R., Opoola T.O. *Ruscheweyh derivative and a new generalized operator involving convolution, Internal. J. Math. Trends Technol.,* 67 (1) (2021): 88–100.

[23] Oyekan E.A., Ojo O.O. *Some properties of a class of analytic functions defined by convolution of two generalized differential operators, Intern. Confer. Contemp. Dev. Math. Sci.,* 23, (2021): 724–742.

[24] Oyekan E.A., Terwase A.J. *Certain characterizations for a class of p-valent functions defined by Sălăgean differential operator, Gen. Math. Notes,* 24 (2), (2014): 1–9.

[25] Ramachandran K., Dhanalakshmi C., Vanitha L. *Hankel determinant for a subclass of analytic functions associated with error functions bounded by conical regions, Internat. J. Math. Anal.,* 11 (2), (2017): 571–581.

[26] Thomas D.K., Tuneski N., Vasudevarao A. *Univalent Functions: A Primer,* Walter de Gruyter Inc., Berlin, (2018).

IRSTI 27.31.17 DOI: https://doi.org/10.26577/JMMCS2023v120i4a6

Zh. Sartabanov[1*], **B. Omarova**[1], **G. Aitenova**[2], **A. Zhumagaziyev**[1]
[1]K. Zhubanov Aktobe Regional University, Kazakhstan, Aktobe
[2]M. Utemisov West Kazakhstan University, Kazakhstan, Uralsk
*e-mail: sartabanov42@mail.ru

INTEGRATING MULTIPERIODIC FUNCTIONS ALONG THE PERIODIC CHARACTERISTICS OF THE DIAGONAL DIFFERENTIATION OPERATOR

In this paper, trajectory of time changing along a helical line is represented by parametric equations in Cartesian coordinates of Euclidean space. On the basis of a cycloidal sweep of a cylindrical surface onto a plane, analytical form of a helix is determined. On its basis, integral surface is determined, which is called the periodic characteristic of the diagonal differentiation operator and its connection with its linear characteristic is established. a) elements of new approach related to the periodic characteristic of diagonal differentiation operator are proposed, b) method for reducing integral along the periodic characteristic to an integral with linear characteristic, c) conditions establishing structure of the integral as sum of linear and multiperiodic functions. Some consequences of these results and recommendations of an algorithmic nature for further expansion of research in this direction are given.
Key words: differentiation operator, periodic characteristic, vector field, infinite cylindrical surface, multiperiodicity, autonomous systems.

Ж. Сартабанов[1*], Б. Омарова[1], Г. Айтенова[2], Ә. Жұмағазиев[1]
[1]Қ. Жұбанов атындағы Ақтөбе өңірлік университеті, Қазақстан, Ақтөбе қ.
[2]М. Өтемісов атындағы Батыс Қазақстан университеті, Қазақстан, Орал қ.
*e-mail: sartabanov42@mail.ru

Көппериодты функцияларды диагонал бойынша дифференциалдау операторының периодты характеристикалары бойында интегралдау

Жұмыста бұрандалы сызық бойымен өзгеретін уақыт траекториясы Евклид кеңістігінің декарттық координаттарындағы параметрлік теңдеулермен ұсынылған. Әрі қарай, цилиндрлік беттің жазықтықтағы циклоидты жазбасы негізінде бұрандалы сызықтың аналитикалық түрі анықталған. Оның негізінде диагонал бойынша дифференциалдау операторының периодты характеристикасы деп аталатын интегралдық бет анықталды және оның сызықтық характеристикасымен байланысы орнатылды. а) диагонал бойынша дифференциалдау операторының периодты характеристикамен байланысты жаңа тәсілдің элементтері, б) периодты характеристика бойымен интегралды сызықтық характеристикалы интегралға дейін келтіру әдісі және в) сызықты және көппериодты функцияның қосындысы түріндегі интегралдың құрылымын орнататын шарттар ұсынылған. Әрі қарай, осы нәтижелердің кейбір салдарлары және осы бағыттағы зерттеулерді одан әрі кеңейту бойынша алгоритмдік сипаттағы ұсыныстар келтірілген.
Түйін сөздер: дифференциалдау операторы, периодты характеристика, векторлық өріс, шексіз цилиндрлік бет, көппериодтылық, автономды жүйелер.

Ж. Сартабанов[1*], Б. Омарова[1], Г. Айтенова[2], Ә. Жұмағазиев[1]
[1]Актюбинский региональный университет имени К. Жубанова, Казахстан, г. Актобе
[2]Западно-Казахстанский университет имени М. Утемисова, Казахстан, г. Уральск
*e-mail: sartabanov42@mail.ru

Интегрирования многопериодических функций вдоль периодических характеристик операторы дифференцирования по диагонали

© 2023 Al-Farabi Kazakh National University

В работе траектория времени, изменяющиеся по винтовой линии представлена параметрическими уравнениями в декартовых координатах евклидового пространства. Далее, на основе циклоидальной развертки цилиндрической поверхности на плоскость определен аналитический вид винтовой линии. На ее основе определена интегральная поверхность, которая названа периодической характеристикой оператора дифференцирования по диагонали и установлена её связь с его линейной характеристикой. Предложены а) элементы нового подхода, связанного с периодической характеристикой оператора дифференцирования по диагонали, б) метод сведения интеграла вдоль периодической характеристики к интегралу с линейной характеристикой и в) условия, устанавливающие структуру интеграла в виде суммы линейной и многопериодической функции. Далее, приведены некоторые следствия этих результатов и рекомендации алгоритмического характера по дальнейшему расширению исследований такого направления.

Ключевые слова: оператор дифференцирования, периодическая характеристика, векторное поле, бесконечная цилиндрическая поверхность, многопериодичность, автономные системы.

1 Introduction

The main objective of this study was to reduce the integration of multiperiodic functions along the non-periodic characteristics of the operator D to the integration of their along δ-characteristics, that is, along the diagonal of the space of independent variables and to establish the structure of the integral of multiperiodic functions. To solve these issues, the necessary information was provided about the periodic β-characteristics of the operator D and about their scans on the plane. The connection between β and δ characteristics has been established. On this basis, the solution of the main problem, which is important in the theory of multi-frequency oscillations, is given. These questions are studied in detail in the two-dimensional case of time variables, and then their ideas are extended to the multidimensional case. Next, the related a) differentiation operators in the directions of a constant vector are given, b) the vector form of such operators, the components of which are operators with constant vector fields, and c) operators that are compositions of two operators of the form a). Methods for constructing the β-characteristics of these operators are indicated and algorithms for integrating multiperiodic functions along the β-characteristics are given.

Along with the concept of the periodic characteristic of the differentiation operator D, the only novelty of the study is the result on establishing the structure of the integral of a multiperiodic function, which has an application to solving the problem of integrals of quasi-periodic functions. These innovations have become a reality thanks to some methods of work [1–20].

2 Information about periodic characteristics and their scans

2.1. Obviously, if the variable τ changes on the numeric axis R, then the values $\tau + j\theta$ for $j \neq 0$, $\theta = const > 0$ and τ are two different points of the numeric axis (Figure 1):

$$\tau + j\theta \neq \tau,\ \tau \in R,\ j \neq 0,\ \theta = const > 0.$$

Now consider the variable t, which changes on the circle S centered at the origin of the plane xOy with radius r and length $2\pi r = \theta$ (Figure 2).

The peculiarity of the points t of the circle S is that two analytical representations: t and $t + k\theta$, $k \in Z$, the same point corresponds to the circle. Therefore, the geometric identity of two points t and t_0 on a circle (Figure 3) is represented as

$$t = t^0 + k\theta \Leftrightarrow t = t_0,\ t \in S,\ t^0 \in S. \tag{1}$$

2.2. Next, consider the characteristic equation

$$\frac{dt}{d\tau} = 1 \tag{2}$$

of the diagonal differentiation operator of the form D:

$$D = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t}, \tag{3}$$

acting on a plane R^2 with Cartesian coordinates (τ, t).

The characteristic of operator (3) originating from the origin defined by equation (2) is the main diagonal

$$t = \tau$$

coordinate systems, and other characteristics parallel to it are straight:

$$t = t^0 + \tau - \tau^0 \equiv \delta(\tau, \tau^0, t^0),\ (\tau^0, t^0) \in R \times R = R^2. \tag{4}$$

As can be seen from (4), equation (2) has no periodic period θ solutions, and therefore operator (3) on R^2 does not have periodic characteristics.

2.3. Now we associate equation (2) with the circle S, assuming that τ is a time variable, that is, it remains as a parameter, and the variable t, respectively, with τ changes along the circle S, as the solution of the equation under consideration.

This reasoning is justified by the fact that the vector field $v(t) \equiv 1$, given by equation (2), has a periodicity of t with an arbitrarily selected period $\theta = const > 0$. So we have every right to consider equation (2) given at $(\tau, t) \in R \times S$.

Then, according to (1), the solution $t = \beta(\tau, \tau^0, t^0)$ of equation (2) with the starting point $(\tau^0, t^0) \in R \times S$ is periodic:

$$\beta(\tau + \theta, \tau^0, t^0) = \beta(\tau, \tau^0, t^0) = t,\ \tau \in R,\ (\tau^0, t^0) \in R \times S \tag{5}$$

and $t = \beta(\tau, \tau^0, t^0) \in S$.

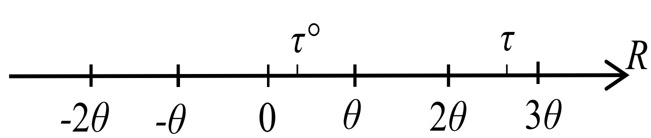


Figure 1: Time of straightline flow

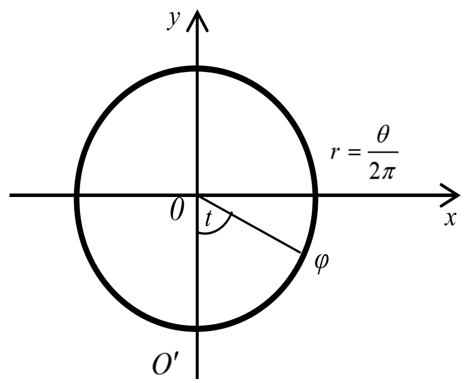


Figure 2: Time of the periodic change

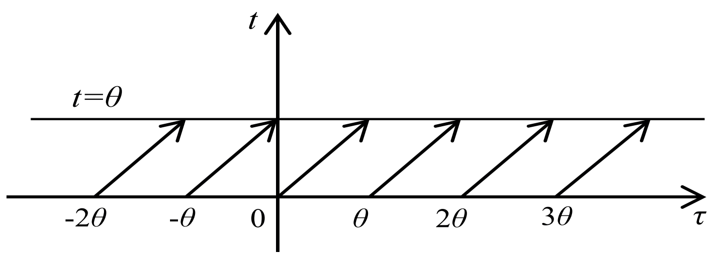


Figure 3: A cycloidal sweep of helical lines

Lemma 1 *Characteristic* (5) *of operator* (3) *defined at* $\tau \in R$, $\tau^0 \in R$ *and* $t^0 \in S$ *with the domain of change* S *has the properties of* θ *periodicity in* τ *and* τ^0, *linearity in* t^0 *and groups as a one-parameter family of motions with parameter* $\sigma = \tau - \tau^0$ *defined by equation* (2).

2.4. To get information about the solution $t = \beta(\tau, 0, 0) \equiv \beta^*(\tau)$ by breaking the circle S at the point O', we position its sweep on a straight line $t = -\tau$ and subject it to a vertical τ-shift.

Then all the arcs on the circle S on the sweep turn into segments, and when combining the plane xOy with the plane τOt from the graph θ-periodic function $t = \beta^*(\tau)$ is represented by the formula

$$t = \theta \left\{\theta^{-1}\tau\right\} \equiv s^*(\tau), \ \tau \in R, \tag{6}$$

where $\{\tau\}$ is the fractional part of the number τ. The graph of this function is shown in figure 3. Note that the sweep function (6) is the projection of the circular helix (Figure 4 on the plane τOt, where $O(0,0,0) = O$, $O'(0,0,r) = O'$, $P'(0,t,\sigma) = P'$, $P(\tau,t,\sigma) = P$, $\psi = \tan\varphi, [0,\psi] \subset Ot = R$, $\varphi = \angle OO'P'$.

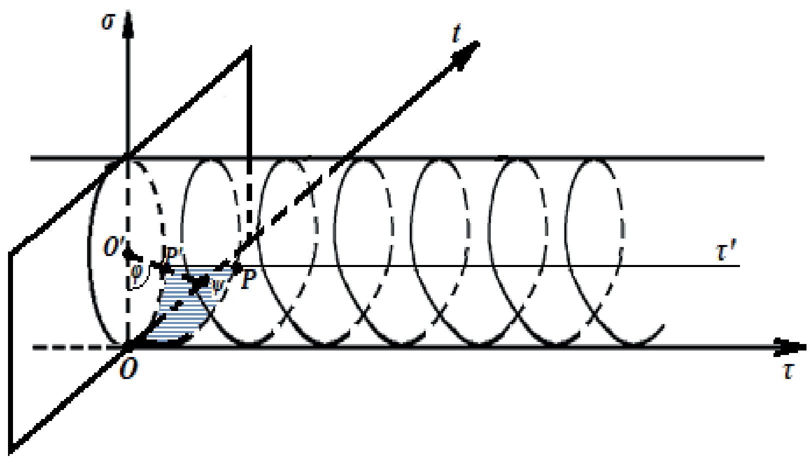


Figure 4: The projection of the circular helix on the plane τOt

Obviously, the function $t = s^*(\tau)$ tolerates discontinuities at points $\tau = k\theta$, $k \in Z$ and in addition to the continuity of the function $t = \beta(\tau)$ on the circle, it represents all other properties concerning the lengths of arcs, periodicity and smoothness between neighboring discontinuity points, and

$$\frac{ds^*(\tau)}{d\tau} = 1 - \sum_{k\in Z} \delta(\tau - k\theta), \quad \frac{ds^*(k\theta - 0)}{d\tau} = \frac{ds^*(k\theta + 0)}{d\tau} = 1, \ k \in Z, \tag{7}$$

where $\delta(\tau)$ is the Dirac delta function. Therefore, from the property (7), additions of the form are suggested

$$\frac{ds^*(k\theta)}{d\tau} = 1, \ k \in Z. \tag{8}$$

Thus, combining (7) and (8), we have a continuous derivative

$$\frac{ds^*(\tau)}{d\tau} = 1, \ \tau \in R \tag{9}$$

of the sweep $t = s^*(\tau)$ of θ-periodic characteristic $t = \beta^*(\tau)$ of the operator D with a range of values S and a definition area R.

Obviously,

$$t = t^0 + s^*(\tau - \tau^0) \equiv s(\tau, \tau^0, t^0) \tag{10}$$

there is a sweep of the solution (5) of equation (2) on the plane τOt.

By virtue of (9), the solution of the general form (10) satisfies equation (2). Due to the autonomy of (2), the solution (10) as a mapping R into itself has the properties of a one-parameter group a) identity, b) inversion and c) compositionality:

$$a)\ s(\tau^0, \tau^0, t^0) = t^0,$$

b) $s(\tau^0, \tau, t) = t^0,$ (11)

c) $s(\sigma, \tau^0, s(\tau^0, \tau, t)) = s(\sigma, \tau, t).$

Note that functions (4) and (5) also have properties *a)*-*b)*. Thus, the following lemma is proved.

Lemma 2 *The sweep* (10) *of the θ-periodic characteristic* (5) *satisfies equation* (2) *and has the properties of θ-periodicity in τ, τ^0, linearity in t^0 and the properties of group a)-b).*

In accordance with lemma 2, it can be shown that a helix originating from a point $(\widetilde{\tau} = 0, t = 0, \sigma = r)$ in Euclidean space $(\widetilde{\tau}, x, y)$ has a parametric equation $\widetilde{\tau} = r\tau$, $x = r\beta^*(\tau)$, $y = r + r\beta^*\left(\tau - \dfrac{\theta}{4}\right)$ where $\varphi = \tau$ is taken into account.

2.5. Thus, along with the natural rectilinear time $t = \tau$, time $t = \beta^*(\tau)$ was introduced, where $\beta^*(\tau)$ changes θ-rotationally and has the property of continuity on the circle S, but it is discontinuous on a flat sweep in the form of a function $t = s^*(\tau)$, where $\beta^*(\tau) = \beta(\tau, 0, 0)$. The transition from $t = \beta^*(\tau)$ to $t = s^*(\tau)$ makes it possible to measure quantities on a manifold $M = S$, for example, associated with the integral of a function $f(\tau, t)$ along β-characteristics a given on R^2 or on a part of $G \subset R^2$. In this regard, it is necessary to consider the difference between $\sigma(\tau)$ rectilinear time τ and rotational time $s^*(\tau)$:

$$\sigma^*(\tau) = \tau - s^*(\tau),\ \tau \in R. \qquad (12)$$

Obviously, for $\tau \in [k\theta, (k+1)\theta]$, $k \in Z$ we have

$$\sigma(\tau) = \begin{cases} k\theta, k\theta \le \tau < (k+1)\theta,\ k \in Z, \\ (k+1)\theta, \tau = (k+1)\theta,\ k \in Z. \end{cases}$$

Hence, $\sigma(\tau)$ is a discontinuous θ-step function with derivative

$$\frac{d\sigma(\tau)}{d\tau} = \theta \sum_{k \in Z} \delta(\tau - k\theta), \quad \frac{d\sigma(k\theta+)}{d\tau} = \frac{d\sigma(k\theta - 0)}{d\tau} = 0.$$

Adding the values $\dfrac{\sigma(k\theta)}{d\tau} = 0$, $k \in Z$ by virtue of the last derivative we have

$$\frac{d\sigma^*(\tau)}{d\tau} = 0, \tau \in R.$$

By analogy with the relations (10), we introduce the function

$$\sigma(\tau, \tau^0, t^0) = t^0 + \sigma^*(\tau - \tau^0) = t, \qquad (13)$$

moreover, by virtue of (12) it has the properties of group a)-b) and the difference of functions (4) and (13) is determined by

$$\delta(\tau, \tau^0, t^0) - \beta(\tau, \tau^0, t^0) = \delta(\tau, \tau^0, t^0) - s(\tau, \tau^0, t^0) = \sigma(\tau - \tau^0) = -\sigma(\tau^0 - \tau). \qquad (14)$$

Since the function $\delta(\xi, \tau, t)$ with respect to the argument t has the property of linearity then from (14) it follows

$$\beta(\xi, \tau, t) = \delta(\xi, \tau, t) + \sigma(\tau - \xi) \equiv \delta(\xi, \tau, t + \sigma(\tau - \xi)),\ \xi \in R. \qquad (15)$$

Lemma 3 *Linear δ-characteristics* (4) *and θ-periodic β-characteristics* (5) *of operator* (3) *are related by relation* (15).

3 Integrals of multiperiodic functions along periodic characteristics

3.1. Further, we note that along with θ-rotational time, we consider ω-rotational time t at $\omega < \theta$. It can be viewed along a circle S_0 by matching the points of a small circle S_0 to the points of an arc of a circle S of length $2\pi\theta > 2\pi\omega$. In the case $\theta k = \omega k_0$, then t along S makes a periodic movement of the period $p = k_0\theta = k\omega$, $(k_0, k) \in Z \times Z$. If there are no such integers k_0, k, then t performs a rotational movement, but non-periodic. Such a circumstance does not violate the (θ, ω)-periodicity of the composition of functions $f(\xi, \eta)$ and $\eta = \beta(\xi, \tau, t)$ by (τ, t), since $f(\xi, \eta)$ is ω-periodic by η, and $\beta(\xi, \tau, t)$ has the property $\beta(\xi, \tau, t + \omega) = \beta(\xi, \tau, t) + \omega$ according to lemma 1.

Thus, we have

$$f(\xi, \beta(\xi, \tau + \theta, t + \omega)) = f(\xi, \beta(\xi, \tau, t) + \omega) = f(\xi, \beta(\xi, \tau, t)), \tag{16}$$

where it is taken into account that $\beta(\xi, \tau + \theta, t) = \beta(\xi, \tau, t)$.

Also note that such a composition is θ-periodic and by ξ:

$$f(\xi + \theta, \beta(\xi + \theta, \tau, t)) = f(\xi, \beta(\xi, \tau, t)),\ \xi \in R,\ (\tau, t) \in R \times S. \tag{17}$$

3.2. According to the Cauchy characteristic method, the equation

$$Dx = f(\tau, t), \quad f(\tau + \theta, t + \omega) = f(\tau, t) \in C^{(0,1)}_{\tau,t}(R \times R) \tag{18}$$

with operator (3) is equivalent to a system of characteristic equations

$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = f(\tau, t), \end{cases} \tag{19}$$

where the first equation (2) does not depend on the second equation of the system (19). Therefore, based on expediency, the first equation can be considered on any variety. In this case, we consider it on the direct product of a straight line (Figure 1) and a circle (Figure 2), which in three-dimensional Euclidean space represents an infinite cylindrical surface with generators parallel to the axis $O\tau$. On this manifold, as we have shown above, equation (2)has the first integral: $\beta(\xi, \tau, t)$-characteristic. Then the second equation along this first integral has the form of the equation

$$\frac{dx}{d\xi} = f(\xi, \beta(\xi, \tau, t)), \tag{20}$$

which is defined when $\xi \in I \subset R$, $(\tau, t) \in R \times S =$ Ц is a cylindrical surface, I is a gap enclosed by the points τ^0 and τ of the axis R, and the right part, according to (16) and (17), has the properties of θ-periodicity by ξ and (θ, ω)-periodicity by (τ, t).

Thus, the problem of (θ, ω)-periodic solutions for equation (18) were reduced to the definition of θ–periodic solutions of the integral equation

$$x(\tau, t) = u(\beta(\tau^0, \tau, t)) + \int_{\tau^0}^{\tau} f(\xi, \beta(\xi, \tau, t))d\xi,\ (\tau, t) \in R \times S, \tag{21}$$

which is derived from equation (20) taking into account the initial condition

$$x|_{\tau=\tau^0} = u(t), \quad u(t+\omega) = u(t) \in C_t^{(1)}(R) \tag{22}$$

for equation (18).

As noted above, the calculation of the integral can be carried out according to lemma 3, using the transition from β–characteristics to their sweeps, and under integrals to δ–characteristics.

Then the solution of the problem of the integral of a multiperiodic continuously differentiable functions along periodic characteristics is represented as

$$x(\tau,t) = u(s(\tau^0,\tau,t)) + \int_{\tau^0}^{\tau} f(\xi,\delta(\xi,\tau,t+\sigma))d\xi, \tag{23}$$

where $\tau^0 \in R$, $(\tau,t) \in R \times R$, $\sigma = \sigma(\tau-\xi)$ and in the process of integration σ behaves as a parameter.

Thus, the following theorem is proved.

Theorem 1 *Multiperiodicity of the solution of the problem* (18), (22) *on integrals smooth* (θ,ω)*–periodic functions are uniquely solved by the integral equation* (21) *along* β*-periodic characteristics, and the representation of its solution in Euclidean space is determined by the ratio* (23) *along* δ*-characteristics.*

Regarding the proof of theorem 1, we note that all its statements are justified by lemmas 1, 2, as well as by the relations (19)-(23). The uniqueness of the solution follows from the equivalence of the problem with the system (19) with the corresponding initial conditions.

3.3. It is possible to determine the structure of the integral of a multiperiodic function along a periodic characteristic.

Theorem 2 *If* $f(\tau,t)$ *is a continuously differentiable multiperiodic function of periods* (θ,ω)*:*

$$f(\tau+\theta,t+\omega) = f(\tau,t) \in C_{\tau,t}^{(0,1)}(R \times R), \tag{24}$$

then its integral along θ*–periodic* β*–characteristics has the following structure:*

$$\int_0^{\tau} f(\xi,\beta(\xi,\tau,t))d\xi = \tau \cdot c(\tau,t) + \varphi(\tau,t), \tag{25}$$

where $c(\tau,t)$ *is a constant along* β*–characteristics, a smooth function:*

$$c(\tau+\theta,t+\omega) = c(\tau,t) \in C_{\tau,t}^{(1,1)}(R \times R), \quad Dc(\tau,t) = 0. \tag{26}$$

and $\varphi(\tau,t)$ *is a multiperiodic of periods* (θ,ω) *a function with the smoothness property:*

$$\varphi(\tau+\theta,t+\omega) = \varphi(\tau,t) \in C_{\tau,t}^{(1,1)}(R \times R). \tag{27}$$

Proof. Let's put

$$c(\tau,t)=\frac{1}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi. \tag{28}$$

By virtue of the condition (24) and the smoothness of $\beta(\xi,\tau,t)$ we have continuity over ξ and smoothness by (τ,t) of the composition of the function $f(\xi,\eta)$ and $\eta=\beta(\xi,\tau,t)$ of the form of a subintegral function $f=(\xi,\beta(\xi,\tau,t))$. This implies the smoothness of $c(\tau,t)$ by (τ,t). From the linearity of $\beta(\xi,\tau,t)$ with respect to t, the multiperiodicity of f by (τ,t) and θ–periodicity $\beta(\tau,t)$ by τ we have (θ,ω) –periodicity of the function $c(\tau,t)$.

Since $D\beta(\xi,\tau,t)=0$, we have

$$Dc(\tau,t)=\frac{1}{\theta}\int_0^\theta \frac{\partial f(\xi,\beta)}{\partial\beta}\cdot D\beta(\xi,\tau,t)d\xi=0.$$

Thus, it is proved that the function (28) satisfies the condition (26).

Next, let's put

$$\varphi(\tau,t)=\int_0^\tau f(\xi,\beta(\xi,\tau,t))d\xi-\frac{\tau}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi. \tag{29}$$

The periodicity of $\varphi(\tau,t)$ by t with the period ω follows from the property $\beta(\xi,\tau,t+\omega)=\beta(\xi,\tau,t)+\omega$ and conditions (24). Now let's check θ-periodicity by τ directly:

$$\varphi(\tau+\theta,t)=\int_0^{\tau+\theta} f(\xi,\beta(\xi,\tau+\theta,t))d\xi-\frac{\tau+\theta}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau+\theta,t))d\xi=$$

$$=\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi+\int_\theta^{\tau+\theta} f(\xi,\beta(\xi,\tau,t))d\xi-\frac{\tau}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi-\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi=$$

$$=\int_0^\theta f(\xi+\theta,\beta(\xi+\theta,\tau,t))d\xi-\frac{\tau}{\theta}\int_0^\theta f(\xi,\beta(\xi,\tau,t))d\xi=\varphi(\tau,t).$$

Since $\beta(\xi+\theta,\tau,t)=\beta(\xi,\tau+\theta,t)=\beta(\xi,\tau,t)$ and $f(\xi,\eta)$ satisfies the condition (24).

Thus, the function (29) satisfies the requirement (27). Smoothness follows from the fact that τ is the upper limit of the integral and integral functions are continuously differentiable due to the smoothness of f and β.

From (28) and (29) follows the identity (25). Theorem 2 is proved.

By putting $t=\tau$, from (25) one can obtain the structure of the integral of the quasi-periodic function $f(t,\tau)=\varphi(\tau)$, generated by the multiperiodic function (24).

4 Periodic characteristics and integration of a function along them in the multidimensional case

Now consider the differentiation operator D in the multidimensional case, which has the form

$$D=\frac{\partial}{\partial\tau}+\sum_{j=1}^m\frac{\partial}{\partial t_j}=\frac{\partial}{\partial\tau}+\left\langle e,\frac{\partial}{\partial t}\right\rangle, \tag{30}$$

where $e = (1, \ldots, 1)$ is m–vector, $\dfrac{\partial}{\partial t} = \left(\dfrac{\partial}{\partial t_1}, \ldots, \dfrac{\partial}{\partial t_m}\right)$ is vector operator, $\tau \in R$, $t = (t_1, \ldots, t_m) \in R \times \ldots \times R = R^m$.

The operator (30) corresponds to the characteristic equation

$$\frac{dt}{d\tau} = e, \tag{31}$$

which is a direct product of m equations

$$\frac{dt_j}{d\tau} = 1, \; j = \overline{1, m}, \tag{32}$$

which, with a fixed j, were studied in points 1.1-1.5.

Consequently, the periodic characteristics of equation (31) are determined by the vector function

$$\beta(\tau, \tau^0, t^0) = \left(\beta(\tau, \tau^0, t_1^0), ..., \beta(\tau, \tau^0, t_m^0)\right), \tag{33}$$

which is a direct product of the characteristics of the systems of equations (32), where $t^0 = (t_1^0, \ldots, t_m^0)$ and the vector notation β is left unchanged, as in the scalar case.

According to (5) the characteristic (33) θ-periodic with respect to τ and τ^0, linear with respect to t^0, and $\tau \in R$, $\tau^0 \in R$, $t^0 \in S^m$.

The system (32), therefore, the equation (30) is autonomous, and $\eta = \beta(\tau, \tau^0, t^0)$, as a family of transformations with the parameter $\sigma = \tau - \tau^0$ has the properties of a) identity, b) reversibility, and c) group. These properties are represented as

$$a) \beta(\tau^0, \tau^0, t^0) = t, \quad b) t^0 = \beta(\tau^0, \tau, t), \quad c) \beta(\sigma, \tau^0, \beta(\tau^0, \tau, t)) = \beta(\sigma, \tau, t), \tag{34}$$

$$\beta(\tau + \theta, \tau^0, t^0) = \beta(\tau, \tau^0 + \theta, t^0) = \beta(\tau, \tau^0, t^0), \quad \beta(\tau, \tau^0, t^0 + \omega) = \beta(\tau, \tau^0, t^0) + \omega, \tag{35}$$

where $\omega = (\omega_1, \ldots, \omega_m)$ is const, $\tau \in R, \tau^0 \in R, \sigma \in R, t_0 \in S^m, \omega \in S^m$;

$$\frac{d\beta(\tau, \tau^0, t^0 n)}{d\tau} = e, \quad D\beta(\tau^0, \tau, t) = 0. \tag{36}$$

Similarly, based on (6)-(11) we have a net of $s(\tau, \tau^0, t^0)$ periodic characteristics $\beta(\tau, \tau^0, t^0)$ in Euclidean space with m-dimensional Cartesian coordinates, having properties similar to (34)-(36):

$$a) s(\tau^0, \tau^0, t^0) = t^0, \quad b) s(\tau^0, \tau, t) = t^0, \quad c) s(\sigma, \tau^0, s(\tau^0, \tau, t)) = s(\sigma, \tau, t), \tag{37}$$

$$s(\tau^0 + \theta, \tau, t) = s(\tau^0, \tau + \theta, t) = s(\tau^0, \tau, t), \quad s(\tau^0, \tau, t + \omega) = s(\tau^0, \tau, t) + \omega, \tag{38}$$

$$\frac{ds(\tau, \tau^0, t^0)}{d\tau} = e, \quad Ds(\tau^0, \tau, t) = 0, \; \tau^0 \in R, \; \tau \in R, \; t \in R^m. \tag{39}$$

About the properties of the characteristics of $\delta(\tau, \tau^0, t^0) = t^0 + \tau - \tau^0$ of the operator (30), defined in Euclidean space can be found from the work [1–3]. They also have properties similar to (34) and (36). The relationship between these characteristics is established by the relations

$$\beta(\xi, \tau, t) = \delta(\xi, \tau, t + \sigma(\tau - \xi)), \tag{40}$$

where the vector function $\sigma(\tau)$ is determined by the relation

$$\sigma(\tau) = e(\tau - s^*(\tau)), \quad es^*(\tau) = s(\tau, 0, 0). \tag{41}$$

Thus, we come to a theorem generalizing lemmas 1, 2, 3.

Theorem 3 *The characteristic $\eta = \beta(\xi, \tau, t)$ with the parameter $\xi \in R$, defined at $(\tau, t) \in R \times S^m$ has the properties of the group* (34)*, periodicity* (35) *and the characteristics of* (36) *for the operator* (30)*, and its sweep in Euclidean space $\zeta = s(\xi, \tau, t)$, $\xi \in R$, $(\tau, t) \in R \times R^m$, related to it by the relations* (40) *and* (41) *also has properties* (37)-(39)*, similar to the property of the characteristic.*

Further, in order to generalize theorems 1, 2 to the multidimensional case, we consider a vector equation of the form

$$\begin{aligned} &Dx = f(\tau, t), \\ &f(\tau + \theta, t + \omega) = f(\tau, t) \in C^{(0,e)}_{\tau,t}(R \times R^m), \\ &x = (x_1, ..., x_n), f = (f_1, ..., f_n), \tau \in R, t = (t_1, ..., t_m) \in R \times ... \times R = R^m \end{aligned} \tag{42}$$

with the operator (30). It is obvious that the vector equation (42) consists of a direct product of scalar equations (18). Therefore, it is possible to formulate the following two theorems without proof.

For certainty, we put to be specific, we set

$$\begin{aligned} &x|_{\tau=\tau^0} = u(t), \\ &u(t + \omega) = u(t) \in C^e_t(R^m). \end{aligned} \tag{43}$$

Theorem 4 *Solution of the problem* (42)-(43) *in the space of multiperiodic functions of periods (θ, ω) is equivalent to the vector integral equation*

$$x(\tau, t) = u(\beta(\tau^0, \tau, t)) + \int_{\tau^0}^{\tau} f(\xi, \beta(\xi, \tau, t))d\xi, (\tau, t) \in R \times S^m,$$

which on the Euclidean net $R \times R^m = R^{m+1}$ of the cylindrical space $R \times S^m$ is represented by equation

$$x(\tau, t) = u(s(\tau^0, \tau, t)) + \int_{\tau^0}^{t} f(\xi, \delta(\xi, \tau, t + \sigma))d\xi,$$

where $\tau^0 \in R, \sigma = \sigma(\tau - \xi)$.

Since the vector function $f(\tau, t)$, the expressions (42), the expressions (24)-(29) acquire a vector form. Therefore, in the following formulation, for brevity, we use the same notation.

Theorem 5 *Under the condition* (24) *for the vector function $f(\tau, t)$, its integral* (25) *is represented using the vector functions* (28) *and* (29)*, which respectively have the properties* (26) *and* (27)*.*

Theorem 5, which defines the structure of the integral of a multiperiodic vector function, is essential in the theories of multiparticle oscillations.

5 Integrals of multiperiodic functions along periodic characteristics of some other differentiation linear operators over vector fields

5.1. The linear operator of differentiation D n the directions of the vector field $v(\tau,t)$ is an operator of the form

$$D=\frac{\partial}{\partial\tau}+\left\langle v(\tau,t),\frac{\partial}{\partial t}\right\rangle, \tag{44}$$

where $\tau\in R$, $t=(t_1,\ldots,t_m)\in R\times\ldots\times R=R^m$, $\dfrac{\partial}{\partial t}=\left(\dfrac{\partial}{\partial t_1},\ldots,\dfrac{\partial}{\partial t_m}\right)$ is vector operator, $v(\tau,t)=(v_1(\tau,t),\ldots,v_m(\tau,t))$ is vector-function of variables $(\tau,t)\in G\subset R^{m+1}$, $\langle\,,\rangle$ is the sign of the scalar product.

When a system defined with the operator D does not depend on (τ,t), then it is called autonomous. Therefore, in this case v is a constant vector: $v=c=(c_1,\ldots,c_m)$, where $c_j\neq 0, j=\overline{1,m}$ are constant coordinates. Then the differentiation operator D has the form

$$D=\frac{\partial}{\partial t}+\left\langle c,\frac{\partial}{\partial t}\right\rangle. \tag{45}$$

If each equation of the system is given by the same differentiation operator (44), then it is said that the system has the same main part or a system with one differentiation operator by (τ,t) in the directions of the vector field

$$\frac{dt}{d\tau}=v(\tau,t). \tag{46}$$

In particular, the vector field (46) can be given by the constant vector $\nu=c$. Then we have a vector field

$$\frac{dt}{d\tau}=c,$$

corresponding to the operator (46).

Each equation of a system consisting of two or more equations can have its differentiation operator of the form (44) or (46). For example, if we restrict ourselves to constant vector fields and the cases $m=1,n=2$, then we have two operators D_1 and D_2 with two variables $\tau\in R$ and $t\in R$ of the form

$$\begin{aligned}D_1&=\frac{\partial}{\partial t}+\nu_1\frac{\partial}{\partial t},\\ D_2&=\frac{\partial}{\partial\tau}+\nu_2\frac{\partial}{\partial\tau}\end{aligned} \tag{47}$$

with constants ν_1 and ν_2.

Of particular interest is the second-order operator $D^2_{1,2}$, which is a composition of two linear operators (47) of the form

$$D^2_{1,2}=D_1D_2.$$

Now let's define the periodic characteristics of these operators.

5.2. The case of systems with a one differentiation operator in the direction of a constant vector. In this case, we are dealing with the operator (45). The operator of autonomous systems also applies to this case. In particular, the operator (45) can have the form (30), where $c = e$. Then θ-periodic by τ characteristic of the structure of the form

$$t = \beta(\tau, \tau^0, t^0) \equiv t^0 + \beta^*(\tau - \tau^0), \quad \beta^*(\tau + \theta) = \beta^*(\tau) \tag{48}$$

corresponds to this operator in accordance with 1, where $(\tau^0, t^0) \in R \times S$, $(\tau, t) \in R \times S$.

Now, in order to build a θ-periodic by τ characteristic, we use the relation (48). To do this, the operator (45) by linear replacement

$$\widetilde{t}_j = c_j^{-1} t_j; \quad t_j = c_j \widetilde{t}_j, \ j = \overline{1, m} \tag{49}$$

is reduced to the form of the operator (30):

$$D_c = \frac{\partial}{\partial \tau} + \sum_{j=1}^m c_j \frac{\partial}{\partial t_j} = \frac{\partial}{\partial \tau} + \sum_{j=1}^m c_j \frac{\partial}{\partial \widetilde{t}_j} \cdot \frac{\partial \widetilde{t}_j}{\partial t_j} = \frac{\partial}{\partial \tau} + \sum_{j=1}^m c_j \frac{\partial}{\partial \widetilde{t}_j} \cdot c_j^{-1} = \frac{\partial}{\partial \tau} + \sum_{j=1}^m c_j \frac{\partial}{\partial \widetilde{t}_j} = D_e.$$

According to (48), the operator D_e has a θ-periodic characteristic

$$\widetilde{t}_j = \widetilde{t}_j^0 + \beta^*(\tau - \tau^0), \ j = \overline{1, m}$$

which, according to (49), we represent as

$$c_j^{-1} = c_j^{-1} t_j^0 + \beta^*(\tau - \tau^0), \ j = \overline{1, m}$$

or as

$$t_j = t_j^0 + c_j \beta^*(\tau - \tau^0), \ j = \overline{1, m}, \tag{50}$$

we have a characteristic system of the operator D_c.

Therefore, in the vector form (50) we write in the form

$$t = t^0 + c\beta^*(\tau - \tau^0), \quad \beta^*(\tau + \theta) = \beta^*(\tau). \tag{51}$$

Thus, (51) is a characteristic of the operator (45), periodic with respect to τ and τ^0 of the period θ, and with respect to t^0 is linear.

And so, next statement is proved.

Statement 1 *The operator* (45) *has a characteristic of the form* $t = t^0 + c\beta^*(\tau - \tau^0) \equiv \widetilde{\beta}(\tau, \tau^0, t^0)$, *which has the properties*

$$\beta(\tau + \theta, \tau^0, t) = \beta(\tau, \tau^0 + \theta, t) = \beta(\tau, \tau^0, t^0), \ \ \beta(\tau, \tau^0, t^0 + \omega) = \beta(\tau, \tau^0, t^0) + \omega, \tag{52}$$

as well as the properties of the group, as a one-parameter family of transformations of a) identity, b) reversibility and c) compositionality.

Note that the properties (52) are a consequence of the property (51), and the properties a)-b) are known from lemma 1.

5.3. The case of systems with two differentiation operators in the directions of constant vectors. Operators of the form (47) belong to this case. According to statement 1, these operators have θ-periodic characteristics of the form

$$t = t^0 + \nu_1\beta^*(\tau, \tau^0) \equiv \beta_1(\tau, \tau^0, t^0), \quad t = t^0 + \nu_2\beta^*(\tau, \tau^0) \equiv \beta_2(\tau, \tau^0, t^0). \tag{53}$$

From statement 1 and the relations (53) we get statement 2:

Statement 2 *Operator $D = (D_1, D_2)$ has θ- periodic by τ and τ^0 characteristics of the form $\bar{t} = (\beta_1(\tau, \tau^0, t^0), \beta_2(\tau, \tau^0, t^0))$, defined by the relations* (53)*, having the properties of linearity in t^0 and groups a)-b).*

5.4. The case of the canonical second-order differentiation operator. First, we give an explanation regarding the name of this case It is well known that the canonical hyperbolic equation

$$\frac{\partial^2 z}{\partial \tau^2} = a^2 \frac{\partial^2 z}{\partial t^2}, \ a = const > 0$$

with the time variable τ and the spatial variable t Euler by linear substitution

$$\beta_1 = \tau + at, \beta_2 = \tau - at$$

led to another canonical form of the equation of this type:

$$\frac{\partial^2 z}{\partial \beta_1 \partial \beta_2} = 0.$$

And immediately obtained the general integral

$$z = f(\beta_1) + \varphi(\beta_2) \equiv f(\tau + at) + \varphi(\tau - at).$$

It is obvious that there is a certain interest of applied scientists about multiperiodic solutions of equation

$$\frac{\partial^2 z}{\partial \beta_1 \partial \beta_2} + c_1 \frac{\partial z}{\partial \beta_1} + c_2 \frac{\partial z}{\partial \beta_2} + c_3 z = f(\tau, t), \ f(\tau + \theta, t + \omega) = f(\tau, t)$$

with constants or (θ, ω)-periodic by (τ, t) by the coefficients c_1, c_2 and c_3. The integration of this equation in the homogeneous case was handled by Laplace and we propose the cascade method.

The study of such a problem, combining a classical question with a modern problem of the theory of oscillations, on the basis of periodic characteristics is important in the development of the theory of multiperiodic systems. The existence of periodic characteristics of this operator is proved by statement 2.

5.5. Integration of multiperiodic functions along the periodic characteristics of the corresponding differentiation operators. It is carried out in accordance with the methodology used in the proofs of lemma 3 and theorem 1. To implement this technique, 1) it is necessary to determine the δ-characteristic of the operator in Euclidean space, 2) it is necessary to establish a connection between the δ-characteristic and the β-characteristic in accordance with 3, and then 3) prove analogs of theorems 1, 2. The described algorithm for integrating many periodic functions is used to create the foundations of the theory of multiperiodic systems with corresponding differentiation operators in the directions of vector fields.

Each differentiation operator along the characteristics passes to an ordinary differentiation operator (finding the full derivative), and the corresponding system becomes a system of ordinary differential equations. Consequently, multiperiodic solutions based on characteristics are converted into quasi-periodic solutions of a system of ordinary differential equations. Thus, we have applications of the developed theory to the study of many particular oscillations in physical and technical systems.

Acknowledgment

This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP19676629).

References

[1] Харасахал В.Х. Почти периодические решения обыкновенных дифференциальных уравнений. – Алма-Ата: Наука, 1970.

[2] Умбетжанов Д.У. Почти многопериодические решения дифференциальных уравнений в частных производных. – Алма-Ата: Наука, 1979.

[3] Умбетжанов Д.У. Почти периодические решения эволюционных уравнений. – Алма-Ата: Наука, 1990.

[4] Умбетжанов Д.У., Сартабанов Ж.А. О необходимом и достаточном условии многопериодичности решения одной системы уравнений в частных производных с одинаковой главной частью // В книге: Математика и механика. – 1972. – Т. 7, Ч. 2. – С. 22-27.

[5] Сартабанов Ж.А. Периодты функциялар және кейбір қарапайым дифференциалдық теңдеулердің периодты шешімдері. – Алматы: РБК, 2001.

[6] Сартабанов Ж.А. Об одном способе изучения периодических решений уравнений в частных производных специального вида // Известия АН КазССР. серия физ.-мат. – 1989. – №1. – С. 42-48.

[7] Мухамбетова А.А., Сартабанов Ж.А. Устойчивость решений систем дифференциальных уравнений. – Актобе: ПринтА, 2007.

[8] Кульжумиева А.А., Сартабанов Ж.А. Периодические решения системы дифференциальных уравнений с многомерным временем. – Уральск: РИЦ ЗКГУ, 2013.

[9] Малкин И.Г. Методы Ляпунова и Пуанкаре в теории нелинейных колебаний. – М: Едиториал УРСС, 2004.

[10] Андронов А.А., Вит А.А., Хайкин С.Э. Теория колебаний. – М: Наука, 1981.

[11] Понтрягин Л.С. Обыкновенные дифференциальные уравнения. – М: Наука, 1965.

[12] Арнольд В.И. Дополнительные главы теории обыкновенных дифференциальных уравнений. – М: Наука, 1978.

[13] Sartabanov Zh.A., Omarova B.Zh., Kerimbekov A. Research of multiperiodic solutions of perturbed linear autonomous systems with differentiation operator on the vector field // News of the National Academy of Sciences of the Republic of Kazakhstan-series Physico-mathematical. – 2020. – 329, No. 1. – Pp. 5-13.

[14] Omarova B.Zh., Sartabanov Zh.A. On multiperiodic solutions of perturbed nonlinear autonomous systems with the differentiation operator on a vector field // Eurasian Math. J. – 2021. – 12:1. – Pp. 68-81.

[15] Zhumagaziyev A.Kh., Sartabanov Zh.A., Sultanaev Ya.T. On a new method for investigation of multiperiodic solutions of quasilinear strictly hyperbolic system // Azerbaijan Journal of Mathematics. – 2022. – 12:1. – Pp. 32-48.

[16] Sartabanov Z.A., Zhumagaziyev A.K., Abdikalikova G.A. Multiperiodic solution of linear hyperbolic in the narrow sense system with constant coefficients // Bulletin of the Karaganda University-Mathematics. – 2020. – 98, No. 2. – Pp. 125-140.

[17] Sartabanov Z.A., Zhumagaziyev A.K., Abdikalikova G.A. On one method of research of multiperiodic solution of block-matrix type system with various differentiation operators // News of the National Academy of Sciences of the Republic of Kazakhstan-Series Physico-Mathematical. – 2020. – 330, No. 2. – Pp. 149-158.

[18] Sartabanov Z.A., Aitenova G.M., Abdikalikova G.A. Multiperiodic solutions of quasilinear systems of integro-differential equations with D_c-operator and ϵ-period of hereditarity // Eurasian Math. J. – 2022. – 13:1. – Pp. 86–100.

[19] Sartabanov Z.A., Aitenova G.M., Abdikalikova G.A. Multiperiodic solution of the initial-boundary value problem for an integro-differential equation of the parabolic type // Russian Mathematics. – 2022. – 66:8. – Pp. 46–55.

[20] Aitenova G.M., Sartabanov Zh.A., Abdikalikova G.A. Multiperiodic bounded oscillations in quasilinear finite-hereditary integro-differential systems convection-diffusion type // Lobachevskii Journal of Mathematics. – 2022. – 43:8. – Pp. 2046-2056.

References

[1] Kharasakhal V.Kh., *Pochti periodicheskie reshenija obyknovennyh differencialnyh uravnenij* [Almost-periodic solutions of ordinary differential equations] (Alma-Ata: Nauka, 1970).

[2] Umbetzhanov D.U., *Pochti mnogoperiodicheskie reshenija differencialnyh uravnenij v chastnyh proizvodnyh* [Almost multiperiodic solutions of partial differential equations] (Alma-Ata: Nauka, 1979).

[3] Umbetzhanov D.U., *Pochti periodicheskie reshenija ievoljucionnyh uravnenij* [Almost periodic solutions of evolution equations] (Alma-Ata: Nauka, 1990).

[4] Umbetzhanov D.U., Sartabanov Zh.A., "O neobhodimom i dostatochnom uslovii mnogoperiodichnosti reshenija odnoj sistemy uravnenij v chastnyh proizvodnyh s odinakovoj glavnoj chastju. [On the necessary and sufficient condition for the multiperiodicity solutions of one system of partial differential equations with the same main part]", *Mathematics and Mechanics* V. 7, No 2 (1972): 22-27.

[5] Sartabanov Zh.A., *Periodty funkcijalar zhane kejbir karapajym differencialdyk tendeulerdi? periodty sheshimderi* [Periodic functions and periodic solutions of some ordinary differential equations] (Almaty: RBK, 2001).

[6] Sartabanov Zh.A., "Ob odnom sposobe izuchenija periodicheskih reshenij uravnenij v chastnyh proizvodnyh special'nogo vida. [On one way to study periodic solutions of partial differential equations of a special form]", *Izvestiya RAN. ser. matem.* No 1 (1989): 42-48.

[7] Mukhambetova A.A., Sartabanov Zh.A., *Ustoichivost reshenji sistem differentsialnykh uravnenij s mnogomernym vremenem* [Stability of solutions of the systems of differential equations with multidimensional time] (Aktobe: Print A, 2007).

[8] Kulzhumieva A.A., Sartabanov Zh.A., *Periodicheskie reshenija sistem differentsialnykh uravnenij s mnogomernym vremenem* [Periodic solutions of the systems of differential equations with multidimensional time] (Uralsk: RIC WKSU, 2013).

[9] Malkin I.G., *Metody Ljapunova i Puankare v teorii nelinejnyh kolebanij* [The methods of Lyapunov and Poincare in the theory of nonlinear oscillations] (M.: Editorial URSS, 2004).

[10] Andronov A.A., Vitt A.A., Khaikin S.E., *Teorija kolebanij* [Theory of oscillators] (M.: Nauka, 1981).

[11] Pontryagin L.S., *Obyknovennye differencial'nye uravnenija* [Ordinary differential equations] (M.: Nauka, 1965).

[12] Arnold V.I., *Dopolnitelnye glavy teorii obyknovennyh differencial'nyh uravnenij* [Additional chapters in the theory of ordinary differential equations] (M.: Nauka, 1978).

[13] Sartabanov Zh.A., Omarova B.Zh., Kerimbekov A., "Research of multiperiodic solutions of perturbed linear autonomous systems with differentiation operator on the vector field", *News of the National Academy of Sciences of the Republic of Kazakhstan-series Physico-mathematical.* 329, No. 1 (2020): 5-13.

[14] Omarova B.Zh., Sartabanov Zh.A., "Lagrange formula for differential operators self-adjoint restrictions of the maximal operator on a tree" , *Eurasian Math. J.,* 12:1 (2021): 68-81.

[15] Zhumagaziyev A.Kh., Sartabanov Zh.A., Sultanaev Ya.T, "On a new method for investigation of multiperiodic solutions of quasilinear strictly hyperbolic system" , *Azerbaijan Journal of Mathematics,* 12:1 (2022): 32-48.

[16] Sartabanov Z.A., Zhumagaziyev A.K., Abdikalikova G.A., "Multiperiodic solution of linear hyperbolic in the narrow sense system with constant coefficients" , *Bulletin of the Karaganda University-Mathematics.* 98, No. 2 (2020): 125-140.

[17] Sartabanov Z.A., Zhumagaziyev A.K., Abdikalikova G.A., "On one method of research of multiperiodic solution of block-matrix type system with various differentiation operators" , *News of the National Academy of Sciences of the Republic of Kazakhstan-series Physico-mathematical.* 330, No. 2 (2020): 149-158.

[18] Sartabanov Z.A., Aitenova G.M., Abdikalikova G.A., "Multiperiodic solutions of quasilinear systems of integro-differential equations with D_c-operator and ϵ-period of hereditarity" , *Eurasian Math. J.,* 13:1 (2022): 86-100.

[19] Sartabanov Z.A., Aitenova G.M., Abdikalikova G.A., "Multiperiodic solution of the initial-boundary value problem for an integro-differential equation of the parabolic type" , *Russian Mathematics,* 66:8 (2022): 46–55.

[20] Aitenova G.M., Sartabanov Zh.A., Abdikalikova G.A., "Multiperiodic bounded oscillations in quasilinear finite-hereditary integro-differential systems convection-diffusion type" , *Lobachevskii Journal of Mathematics,* 43:8 (2022): 2046-2056.

2-бөлім **Раздел 2** **Section 2**

Механика **Механика** **Mechanics**

IRSTI 30.03.19 DOI: https://doi.org/10.26577/JMMCS2023v120i4a7

B.Z. Kenzhegulov[1], N.I. Vatin[2], C.B. Kenzhegulova[3], D.B. Alibiyev [4],
A.Sh. Kazhikenova[4], O. Khabidolda[4*]
[1]Dosmukhamedov Atyrau University, Kazakhstan, Atyrau
[2]Peter the Great St.Petersburg Polytechnic University, Russia, St.Petersburg
[3]K. Sagadiyev University of International Business, Kazakhstan, Almaty
[4]Karaganda University named after Academician E.A. Buketov, Kazakhstan, Karaganda
[*]e-mail: oka-kargtu@mail.ru

NUMERICAL MODELING OF THE TEMPERATURE DISTRIBUTION FIELD IN A COMPLEX SHAPE STRUCTURAL ELEMENT

As you know, many parts of internal combustion engines, gas turbine power plants, steam generators of nuclear power plants and manufacturing industries experience thermal effects of various forms. At the same time, a process of thermal expansion occurs on these parts and, as a result, a thermal stress-strain state arises on them with a value that in some cases can exceed the limit value. Therefore, knowledge of the stationary field of temperature distribution in the volume of partially thermally insulated parts of a complex configuration while there is a heat flux and heat exchange in parts of its surface is an urgent task. At the same time, it is very difficult to take into account all inhomogeneous boundary conditions when solving the problem of stationary heat conduction. Therefore, a new numerical method is proposed, based on the law of conservation of total thermal energy alongside with the finite element method. In this case, the procedure for minimizing the total thermal energy involves quadrilateral bilinear finite elements. Partial thermal insulation, the heat flux supplied to the local surface, and the process of heat exchange through the local surface area and ambient temperature are taken into account. Nodal temperature values are determined.
Key words: mathematical model, channel-shaped body (beam), heat flow, cross-section, functional, heat exchange, thermal insulation, temperature distribution field, form functions.

Б.З. Кенжегулов [1], Н.И. Ватин[2], С.Б. Кенжегулова[3], Д.Б. Әлібиев[4],
А.Ш. Кажикенова[4], О. Хабидолда.[4*]
[1]Х. Досмұхамедов атындағы Атырау университеті, Қазақстан, Атырау қ.
[2]Ұлы Петр Санкт-Петербург политехникалық университеті, Ресей, Санкт-Петербург қ.
[3]К. Сағадиев атындағы Халықаралық бизнес университеті, Қазақстан, Алматы қ.
[4]Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қазақстан, Қарағанды қ.
[*]e-mail: oka-kargtu@mail.ru

Күрделі пішінді дененің түйін нүктелеріндегі температураның таралуын сандық пішіндеу

Өздеріңіз білетіндей, іштен жанатын қозғалтқыштардың, газ турбиналық электр станцияларының, атом электр станцияларының, бу генераторларының және өңдеу өнеркәсібінің көптеген бөліктері әртүрлі формада болады және олар әртүрлі жылу әсерлерін сезінеді. Сонымен бірге бұл бөліктерде термиялық кеңею процесі жүреді де нәтижесінде оларда кейбір жағдайларда шекті мәннен асатын мәнмен термиялық кернеу-деформациялық күй пайда болады.

© 2023 Al-Farabi Kazakh National University

Сондықтан оның бетінің бөліктерінде жылу ағыны және жылу алмасу болған кезде күрделі конфигурацияның ішінара жылу оқшауланған бөліктерінің көлеміндегі тұрақты күйдегі температураның таралу өрісін білу өзекті мәселе болып табылады. Сонымен қатар тұрақты күйдегі жылу өткізгіштік мәселесін шешуде барлық біртекті емес шекаралық шарттарды есепке алу өте қиын. Сондықтан шекті элементтер әдісімен үйлесімде жалпы жылу энергиясының сақталу заңына бағытталған жаңа сандық әдіс ұсынылады. Бұл жағдайда жалпы жылу энергиясын азайту процедурасы төртбұрышты екі сызықты ақырлы элементтерді пайдалана отырып қолданылады. Ішінара жылу оқшаулау, жергілікті жерге берілетін жылу ағыны мен жылу алмасу процесі және қоршаған орта температурасы ескеріледі. Түйінді нүктелерінің температура мәндері анықталады.
Түйін сөздер: математикалық модель (пішін), арна тәрізді дене (арқалық), жылу ағыны, қима, функционал, жылу алмасу, жылу оқшаулау, температураның таралу өрісі, пішін функциялары.

Б.З. Кенжегулов[1], Н.И. Ватин[2], С.Б. Кенжегулова[3], Д.Б. Алибиев[4], А.Ш. Кажикенова[4], О. Хабидолда[4]*
[1] Атырауский университет имени Х. Досмухамедова, Казахстан, г. Атырау
[2] Санкт-Петербургский политехнический университет Петра Великого, Россия, г. Санкт-Петербург
[3] Университет Международного Бизнеса имени К. Сагадиева, Казахстан, г. Алматы
[4] Карагандинский университет имени академика Е.А. Букетова, Казахстан, г. Караганда
*e-mail: oka-kargtu@mail.ru

Численное моделирование поля распределения температуры в конструкционном элементе сложной формы

Как известно, многие детали двигателей внутреннего сгорания, газотурбинных электростанций, парогенераторов атомных электростанций и предприятий обрабатывающей промышленности испытывают тепловые воздействия различной формы. При этом на этих деталях происходит процесс теплового расширения и, как следствие, на них возникает термическое напряженно-деформированное состояние величиной, которая в ряде случаев может превышать предельное значение. В данной статье мы показываем, что знание стационарного поля распределения температуры в объеме частично теплоизолированных деталей сложной конфигурации при наличии теплового потока и теплообмена на участках ее поверхности является актуальной задачей. В то же время учесть все неоднородные граничные условия при решении задачи стационарной теплопроводности очень сложно. Поэтому предлагается новый численный метод, ориентированный на закон сохранения полной тепловой энергии в сочетании с методом конечных элементов. При этом используется процедура минимизации полной тепловой энергии с использованием билинейных конечных элементов четырехугольной формы. Учитываются частичная тепловая изоляция, тепловой поток, подведенный к локальной поверхности, и процесс теплообмена через площадь локальной поверхности и температуру окружающей среды. Определены узловые значения температуры.

Ключевые слова: математическое модель, швеллеро подобное тело (балка), тепловой поток, поперечное сечение, функционал, теплообмен, теплоизоляция, поле распределения температуры, функции формы.

1 Introduction

In the thermomechanical process, the main characteristic that has a significant impact on the strength of the load-bearing structural elements is an intensive temperature increase. Temperature is one of the most important characteristics of the growth process and affects the morphology and crystal structure of heat-resistant alloys. Depending on the parameters of the structure body, the distribution of the temperature field in its different parts is uneven. It should be noted that the simultaneous influence on the distribution of temperature over the volume of the body and such external factors as various forms of local thermal insulation, the property of heat transfer, and the temperature of the heat source. Consequently, during the thermomechanical process, in some parts of the structural elements, the temperature will be acceptable, and in some – critical, which leads to rapid wear of structural elements and to the loss of their physical qualities. In this regard, the exact calculation of the distribution of the temperature field at each nodal point of multidimensional bodies of complex shape is relevant [1- 4].

This article discusses a technique for constructing a mathematical model and the accompanying computational algorithm that allow solving problems of studying the patterns of distributing the temperature field in a complicated-shape structural element where there is a heat flux, heat transfer and partial thermal insulation on their local surfaces.

At present, in our country and abroad, there are many works devoted to the problem of the influence of a thermomechanical process on a change in the structure and composition of the material of any technical installation or design. This article takes into account the simultaneous influence of the heat flow on the body, partial thermal insulation and local heat transfer. A computational algorithm is presented for solving a problem obtained by discretizing bodies of complex shape made of heat-resistant alloys using quadratic finite elements [2, 5].

The purpose of this article is to show the regularity of the distribution of the temperature field using a numerical study of heat transfer in the presence of heat flow, thermal insulation and heat transfer. The objectives of the study are to determine the temperature value at each nodal point of a multidimensional body to develop a computational algorithm based on minimizing the total thermal energy functional.

2 Research methodology

To illustrate the proposed numerical method and the corresponding computational algorithm, consider the following problem. Given a "channel-like"body of unlimited length $-\infty \leq z \leq \infty$ (Figure 1). The outer side and inner surface of which are thermally insulated along the entire length. Through the areas of the upper surface $y = h, 0 \leq x \leq (r+2l)$, $y = h, 0 \leq x \leq (r+2l)$, $-\infty \leq z \leq \infty$ heat exchange with its environment takes place. In this case, the heat transfer coefficient is h_{oc}, and the ambient temperature is T_{oc}. On the surface $y = 0, [(0 \leq x \leq l$ and $(r+l) \leq x \leq (r+2l)]$,$-\infty \leq z \leq \infty$ a heat flux of q - constant intensity is supplied. It is required to determine the steady temperature distribution field in the volume of the structural element under consideration. To do this, first, the initial cross section, which is shown in Figure 1 is discretized by quadrangular finite elements.

Within each finite element, we represent the temperature distribution field as [1, 2, 6]

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 y = \phi_1(x,y)\cdot T_1 + \phi_2(x,y)\cdot T_2 + \phi_3(x,y)\cdot T_3 + \phi_4(x,y)\cdot T_4 \quad (1)$$

where $\phi_i(x,y)$ are the shape function for a quadrangular finite element with four nodes [1]:

$$\begin{aligned} &\phi_1(x,y) = \tfrac{(b-x)(a-y)}{4ab}; \qquad \phi_2(x,y) = \tfrac{(b+x)(a+y)}{4ab};\\ &\phi_3(x,y) = \tfrac{(b+x)(a+y)}{4ab}; \qquad \phi_4(x,y) = \tfrac{(b-x)(a+y)}{4ab} \end{aligned} \quad (2)$$

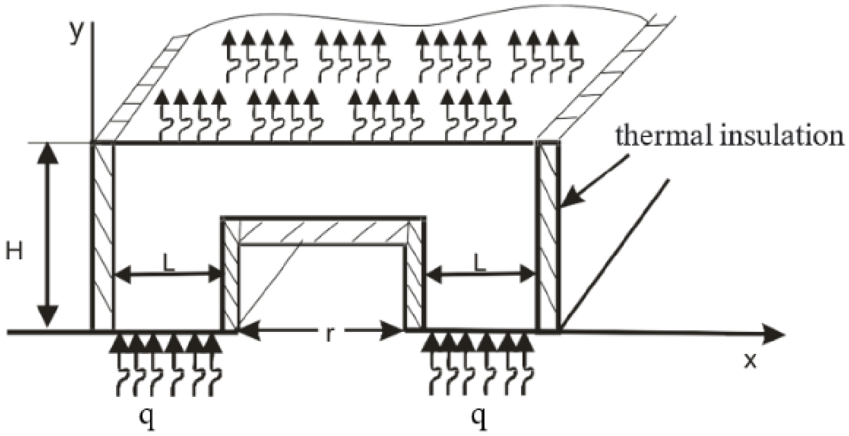


Figure 1: The design scheme of the problem under consideration in the cross section of a structural element

where the size of the finite element along the direction of the coordinate axes x and y is ($2b$, $2a$) (Figure 2)

$$J=\int_V \frac{1}{2}\left[K_{xx}\left(\frac{\partial T}{\partial x}\right)^2+K_{yy}\left(\frac{\partial T}{\partial y}\right)^2\right]dv+\int_{S(x=0)} qTdS+\int_{S(x=A)}\frac{h}{2}\left(T-T_{oc}\right)^2 dS \quad (3)$$

where V is the volume of the timber in question; *S(x=0)* - the surface area of the beam (x=0), where the heat flow ; $S(x$=A) - the surface area (x=A) of the beam through which heat is exchanged with the environment h; K_{xx}; K_{yy}; $\left(\frac{W}{cm\cdot{}^{\circ}C}\right)$ - the coefficient of thermal conductivity of the timber under consideration, respectively, in the directions of the coordinate axes ox and oy.

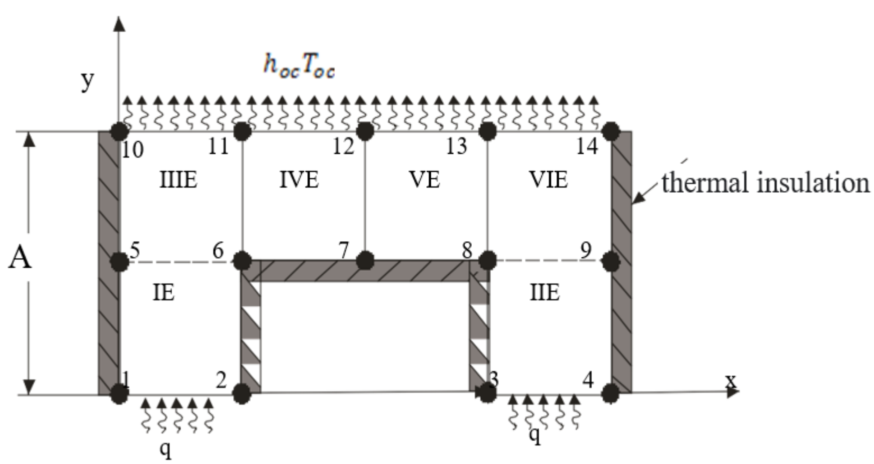


Figure 2: Discretization of the computational domain in the context of a structural element.

3 Results and Discussion

The cross-sectional area of the timber in question (which has the shape of a rectangular quadrangle) is discretized using coordinate lines into quadrangular finite elements. The number of discrete finite elements will be $m\text{x}n$ (respectively, on the axes ox and oy). For each element, there is constructed a local coordinate system oxy, so that the origin coincides with the geometric center of the element, as shown in Figure 3. The numbering of the element nodes is shown in this Figure. The coordinates of the element nodes in the local coordinate system will be as follows $1\,(-a;-b)\,;2\,(a;-b)\,;3\,(a;b)\,;4\,(-a;b)$:

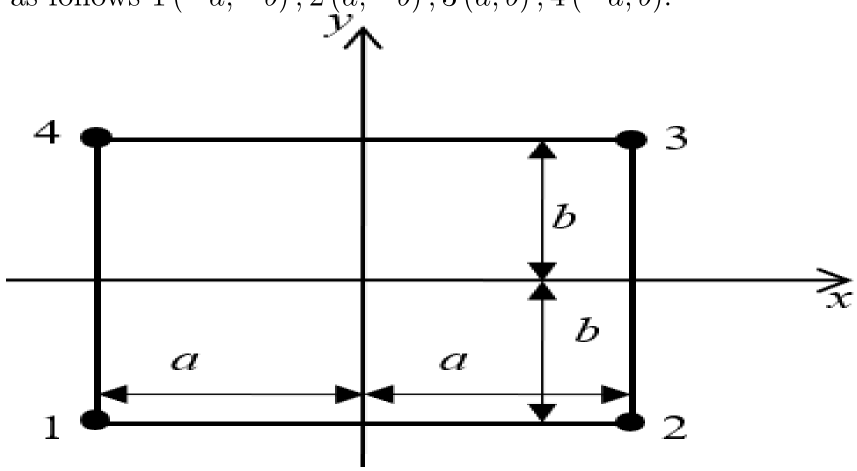


Figure 3: The diagram of constructing a local coordinate system.

These form functions will possess the following properties:

1) in the first node, i.e. when $x = -a;\ y = -b \phi_1(-a;-b) = 1;\ \phi_2(-a;-b) = \varphi_3(-a;-b) = \phi_4(-a;-b) = 0;$

2) in the second node, i.e. when $x = a;\ y = -b\ \ \phi_1(a;-b) = 0;\quad \phi_2(a;-b) = 1;\quad \varphi_3(a;-b) = \phi_4(a;-b) = 0;$

3) in the third node, i.e. when $x = a; y = b\ \ \phi_1(a;b) = 0;\ \ \phi_2(a;b) = 0;\ \ \varphi_3(a;b) = 1;\ \ \phi_4(a;b) = 0;$

4) in the fourh node, i.e. when $x = -a;\quad y = b\ \ \phi_1(-a;b) = \phi_2(-a;b) = \varphi_3(-a;b) = 0;\quad \phi_4(-a;b) = 1;$

5) 1) in the first node, i.e. when $x = -a;\ y = -b$
$\varphi_1(-a;-b) = 1;\ \ \varphi_2(-a;-b) = \varphi_3(-a;-b) = \varphi_4(-a;-b) = 0;$

2) in the second node, i.e. when $x = a;\ \ y = -b$
$\varphi_1(a;-b) = 0;\quad \varphi_2(a;-b) = 1;\quad \varphi_3(a;-b) = \varphi_4(a;-b) = 0;$

3) in the third node, i.e. when $x = a; y = b$
$\varphi_1(a;b) = 0;\ \ \varphi_2(a;b) = 0;\ \ \varphi_3(a;b) = 1;\ \ \varphi_4(a;b) = 0;$

4) in the fourh node, i.e. when $x = -a;\quad y = b$
$\varphi_1(-a;b) = \varphi_2(-a;b) = \varphi_3(-a;b) = 0;\quad \varphi_4(-a;b) = 1;$

6) $\sum_{i=1}^{4} \frac{\partial \phi_i}{\partial x} = 0$- at any point in a discrete finite element.

In addition, from (1), (2) the values of temperature gradients at any point of a discrete element are easily determined:

$$\frac{\partial T}{\partial x} = \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial x} T_i; \frac{\partial T}{\partial y} = \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial y} T_i$$

The expression is also defined:

$$\begin{aligned}&\left(\tfrac{\partial T}{\partial x}\right)^2 = \left(\sum_{i=1}^{4} \tfrac{\partial \phi_i}{\partial x} T_i\right)^2 = \left[-\tfrac{b-y}{4ab}T_1 + \tfrac{b-y}{4ab}T_2 + \tfrac{b+y}{4ab}T_3 - \tfrac{b-y}{4ab}T_4\right]^2 = \\ &= \tfrac{b^2-2by+y^2}{16a^2b^2} \cdot T_1^2 - 2 \cdot \tfrac{b^2-2by+y^2}{16a^2b^2} \cdot T_1T_2 - 2 \cdot \tfrac{b^2-y^2}{16a^2b^2} \cdot T_1T_3 + 2 \cdot \tfrac{b^2-y^2}{16a^2b^2} \cdot T_1T_4 + \\ &+ \tfrac{b^2-2by+y^2}{16a^2b^2} \cdot T_2^2 + 2 \cdot \tfrac{b^2-y^2}{16a^2b^2} \cdot T_2T_3 - 2 \cdot \tfrac{b^2-y^2}{16a^2b^2} \cdot T_2T_4 + \\ &+ \tfrac{b^2+2by+y^2}{16a^2b^2} \cdot T_3^2 - 2 \cdot \tfrac{b^2+2by+y^2}{16a^2b^2} \cdot T_3T_4 + \tfrac{b^2+2by+y^2}{16a^2b^2} \cdot T_4^2;\end{aligned} \tag{4}$$

$$\begin{aligned}&\left(\tfrac{\partial T}{\partial y}\right)^2 = \left(\sum_{i=1}^{4} \tfrac{\partial \phi_i}{\partial y} T_i\right)^2 = \left[-\tfrac{a-x}{4ab}T_1 - \tfrac{a+x}{4ab}T_2 + \tfrac{a+x}{4ab}T_3 + \tfrac{a-x}{4ab}T_4\right]^2 = \\ &= \tfrac{a^2-2ax+x^2}{16a^2b^2} \cdot T_1^2 - 2 \cdot \tfrac{a^2-x^2}{16a^2b^2} \cdot T_1T_2 - 2 \cdot \tfrac{a^2-x^2}{16a^2b^2} \cdot T_1T_3 + 2 \cdot \tfrac{a^2-2ax+x^2}{16a^2b^2} \cdot T_1T_4 + \\ &+ \tfrac{a^2+2ax+x^2}{16a^2b^2} \cdot T_2^2 - 2 \cdot \tfrac{a^2+2ax+x^2}{16a^2b^2} \cdot T_2T_3 - 2 \cdot \tfrac{a^2-x^2}{16a^2b^2} \cdot T_2T_4 + \\ &+ \tfrac{a^2+2ax+x^2}{16a^2b^2} \cdot T_3^2 + 2 \cdot \tfrac{a^2-x^2}{16a^2b^2} \cdot T_3T_4 + \tfrac{a^2-2ax+x^2}{16a^2b^2} \cdot T_4^2;\end{aligned} \tag{5}$$

For clarity of the proposed computational algorithm, we consider the cross section of the timber under consideration as one discrete quadrangular element, as shown in Figure 4.

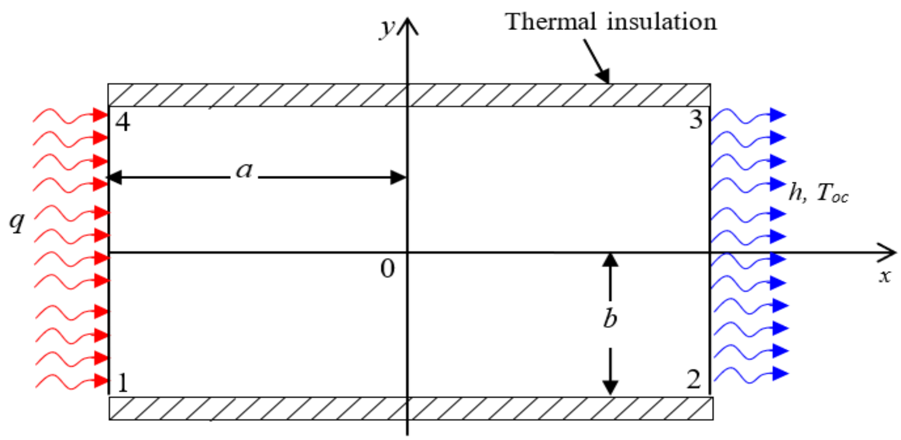


Figure 4: The calculation diagram of the problem.

Now for one discrete element, we calculate the integral over the volume. Here we use the following formula:

$$\int_V f(x;y)\, dV = L \int_{-a}^{a} \int_{-b}^{b} f(x;y)\, dxdy \tag{6}$$

Using (6) we calculate the integral:

$$J_{11} = \int_V \frac{1}{2} \left[K_{xx} \left(\frac{\partial T}{\partial x} \right)^2 \right] dV \tag{7}$$

In calculating this integral, we use the expression (4). As a result, we have:

$$1) \int_{-a}^{a}\int_{-b}^{b} \frac{b^2-2by+y^2}{16a^2b^2} T_1^2 dxdy = \frac{2aT_1^2}{16a^2b^2}\int_{-b}^{b}\left(b^2-2by+y^2\right)dy =$$

$$= \frac{2aT_1^2}{16a^2b^2}\left[2b^3-0+\frac{2b^3}{3}\right] = \frac{T_1^2}{ab^2}\cdot\frac{8b^3}{3} = \frac{b}{3a}T_1^2; \tag{8}$$

$$2) \int_{-a}^{a}\int_{-b}^{b} \left(-2\cdot\frac{b^2-2by+y^2}{16a^2b^2}\cdot T_1T_2\right)dxdy = -\frac{T_1T_2}{8a^2b^2}\cdot 2a\int_{-b}^{b}\left(b^2-2by+y^2\right)dy =$$

$$= -\frac{T_1T_2}{4ab^2}\left[2b^3+\frac{2b^3}{3}\right] = -\frac{2b}{3a}T_1T_2; \tag{9}$$

$$3) \int_{-a}^{a}\int_{-b}^{b} \left(-2\cdot\frac{b^2-y^2}{16a^2b^2}\cdot T_1T_3\right)dxdy = -\frac{T_1T_3}{8a^2b^2}\cdot 2a\left[2b^3-\frac{2b^3}{3}\right] = -\frac{b}{3a}T_1T_3; \tag{10}$$

$$4) \int_{-a}^{a}\int_{-b}^{b} \left(-2\cdot\frac{b^2-y^2}{16a^2b^2}\cdot T_1T_4\right)dxdy = \frac{b}{3a}T_1T_4; \tag{11}$$

$$5) \int_{-a}^{a}\int_{-b}^{b} \left(\frac{b^2-2by+y^2}{16a^2b^2}\cdot T_2^2\right)dxdy = \frac{2aT_2^2}{16a^2b^2}\left[2b^3+\frac{2b^3}{3}\right] = \frac{b}{3a}T_2^2; \tag{12}$$

$$6) \int_{-a}^{a}\int_{-b}^{b} \left(2\cdot\frac{b^2-y^2}{16a^2b^2}\cdot T_2T_3\right)dxdy = \frac{T_2T_3}{8a^2b^2}\cdot 2a\cdot\frac{4b^3}{3} = \frac{b}{3a}T_2T_3; \tag{13}$$

$$7) \int_{-a}^{a}\int_{-b}^{b} \left(-2\cdot\frac{b^2-y^2}{16a^2b^2}\cdot T_2T_4\right)dxdy = -\frac{b}{3a}T_2T_4; \tag{14}$$

$$8) \int_{-a}^{a}\int_{-b}^{b} \left(\frac{b^2+2by+y^2}{16a^2b^2}\cdot T_3^2\right)dxdy = \frac{T_3^2}{16a^2b^2}\cdot 2a\cdot\frac{8b^3}{3} = \frac{b}{3a}T_3^2; \tag{15}$$

$$9) \int_{-a}^{a}\int_{-b}^{b} \left(-2\cdot\frac{b^2+2by+y^2}{16a^2b^2}\cdot T_3T_4\right)dxdy = -\frac{2b}{3a}T_3T_4; \tag{16}$$

$$10) \int_{-a}^{a}\int_{-b}^{b}\left(\frac{b^2+2by+y^2}{16a^2b^2}\cdot T_4^2\right)dxdy=\frac{b}{3a}T_4^2. \tag{17}$$

Substituting (8) - (17) into (7), we find the integrated form J_{11}:

$$\begin{aligned}J_{11}&=\int_V \tfrac{1}{2}\left[K_{xx}\left(\tfrac{\partial T}{\partial x}\right)^2\right]dV=\tfrac{LK_{xx}}{2}\int_{-a}^{a}\int_{-b}^{b}\left(\tfrac{\partial T}{\partial x}\right)^2 dxdy=\\&=\tfrac{LK_{xx}}{2}\int_{-a}^{a}\int_{-b}^{b}\left[\tfrac{b^2-2by+y^2}{16a^2b^2}\cdot T_1^2-2\cdot\tfrac{b^2-2by+y^2}{16a^2b^2}\cdot T_1T_2-2\cdot\tfrac{b^2-y^2}{16a^2b^2}\cdot T_1T_3+\right.\\&+2\cdot\tfrac{b^2-y^2}{16a^2b^2}\cdot T_1T_4+\tfrac{b^2-2by+y^2}{16a^2b^2}\cdot T_2^2+2\cdot\tfrac{b^2-y^2}{16a^2b^2}\cdot T_2T_3-2\cdot\tfrac{b^2-y^2}{16a^2b^2}\cdot T_2T_4+\\&\left.+\tfrac{b^2+2by+y^2}{16a^2b^2}\cdot T_3^2-2\cdot\tfrac{b^2+2by+y^2}{16a^2b^2}\cdot T_3T_4+\tfrac{b^2+2by+y^2}{16a^2b^2}\cdot T_4^2\right]dxdy=\\&=\tfrac{bLK_{xx}}{6a}\left[T_1^2-2T_1T_2-T_1T_3+T_1T_4+T_2^2+T_2T_3-T_2T_4+T_3^2-2T_3T_4+T_4^2\right].\end{aligned} \tag{18}$$

Examining the last expression, we find that the sum of the coefficients in front of the nodal temperature values will be zero. Indeed, from (18) we find that (1-2-1 + 1 + 1 + 1-1 + 1-2 + 1) = 0.

Next, similarly, we find the integrated form expression

$$J_{22}=\int_V \frac{1}{2}\left[K_{yy}\left(\frac{\partial T}{\partial y}\right)^2\right]dV. \tag{19}$$

Using (5) we find that

$$1) \int_{-a}^{a}\int_{-b}^{b}\frac{a^2-2ax+x^2}{16a^2b^2}T_1^2dxdy=\frac{T_1^2}{16a^2b^2}\cdot 2b\cdot\frac{8a^3}{3}=\frac{a}{3b}T_1^2; \tag{20}$$

$$2) \int_{-a}^{a}\int_{-b}^{b}\left(2\cdot\frac{a^2-x^2}{16a^2b^2}\cdot T_1T_2\right)dxdy=\frac{2T_1T_2}{16a^2b^2}\cdot 2b\cdot\frac{4a^3}{3}=\frac{a}{3b}\cdot T_1T_2; \tag{21}$$

$$3) \int_{-a}^{a}\int_{-b}^{b}\left(-2\cdot\frac{a^2-x^2}{16a^2b^2}\cdot T_1T_3\right)dxdy=-\frac{2aT_1T_3}{6b}=-\frac{a}{3b}\cdot T_1T_3; \tag{22}$$

$$4) \int_{-a}^{a}\int_{-b}^{b}\left(-2\cdot\frac{a^2-2ax+x^2}{16a^2b^2}T_1T_4\right)dxdy=-\frac{2a}{3b}T_1T_4; \tag{23}$$

$$5) \int_{-a}^{a}\int_{-b}^{b}\left(\frac{a^2+2ax+x^2}{16a^2b^2}T_2^2\right)dxdy=\frac{a}{3b}T_2^2; \tag{24}$$

$$6) \int_{-a}^{a}\int_{-b}^{b}\left(-2\cdot\frac{a^2+2ax+x^2}{16a^2b^2}T_2T_3\right)dxdy=-\frac{2a}{3b}T_2T_3; \tag{25}$$

$$7)\int_{-a}^{a}\int_{-b}^{b}\left(-2\cdot\frac{a^2-x^2}{16a^2b^2}\cdot T_2T_4\right)dxdy=-\frac{a}{3b}\cdot T_2T_4; \tag{26}$$

$$8)\int_{-a}^{a}\int_{-b}^{b}\left(\frac{a^2+2ax+x^2}{16a^2b^2}T_3^2\right)dxdy=\frac{a}{3b}T_3^2; \tag{27}$$

$$9)\int_{-a}^{a}\int_{-b}^{b}\left(2\cdot\frac{a^2-x^2}{16a^2b^2}\cdot T_3T_4\right)dxdy=\frac{a}{3b}\cdot T_3T_4; \tag{28}$$

$$10)\int_{-a}^{a}\int_{-b}^{b}\left(\frac{a^2-2ax+x^2}{16a^2b^2}T_4^2\right)dxdy=\frac{a}{3b}T_4^2. \tag{29}$$

Substituting (20) - (29) into (19) we define the integrated form J_{22}:

$$J_{22}=\int_V\frac{1}{2}\left[K_{yy}\left(\frac{\partial T}{\partial y}\right)^2\right]dV=\frac{LK_{yy}}{2}\int_{-a}^{a}\int_{-b}^{b}\left(\frac{\partial T}{\partial y}\right)^2dxdy=$$
$$=\frac{aLK_{yy}}{6a}\left[T_1^2+T_1T_2-T_1T_3-2T_1T_4+T_2^2-2T_2T_3-T_2T_4+T_3^2+T_3T_4+T_4^2\right].$$

In this expression, the sum of the coefficients in front of the nodal temperature values will be zero [2, 7, 8]. Then there is found the expression for J_1:

$$J_1=J_{11}+J_{22}=\frac{bLK_{xx}}{6a}\left[T_1^2-2T_1T_2-T_1T_3+T_1T_4+T_2^2+T_2T_3-T_2T_4+T_3^2-2T_3T_4+T_4^2\right]+$$
$$+\frac{aLK_{yy}}{6a}\left[T_1^2+T_1T_2-T_1T_3-2T_1T_4+T_2^2-2T_2T_3-T_2T_4+T_3^2+T_3T_4+T_4^2\right].$$

Now from (3) we find:

$$J_2=\int_{S(x=-a)}qTdS=Lq\int_{-b}^{b}\left[\phi_1(x;y)T_1+\phi_2(x;y)T_2+\phi_3(x;y)T_3+\phi_4(x;y)T_4\right]|_{x=-a}\,dy=$$
$$=\frac{Lq}{2b}\int_{-b}^{b}\left[(b-y)T_1+(b+y)T_4\right]dy=\frac{Lq}{2b}\left[2b^2T_1+2b^2T_4\right]=Lqb\left[T_1+T_4\right].$$

From (3) we calculate:

$$\begin{aligned}J_3&=\int_{S(x=a)}\frac{h}{2}(T-T_{oc})^2dS=\frac{hL}{2}\int_{-b}^{b}\left[\sum_{i=1}^{4}\phi_i(x;y)T_i-T_{oc}\right]_{x=a}^2dy=\\&=\frac{hL}{2}\int_{-b}^{b}\left[\phi_1(x;y)T_1+\phi_2(x;y)T_2+\phi_3(x;y)T_3+\phi_4(x;y)T_4-T_{oc}\right]_{x=a}^2dy=\\&=\frac{hL}{2}\int_{-b}^{b}\left[\frac{b^2+2by+y^2}{4b^2}T_2^2+2\frac{b^2-y^2}{4b^2}T_2T_3-2\frac{b-y}{2b}T_2T_{oc}+\right.\\&\left.+\frac{b^2+2by+y^2}{4b^2}T_3^2-2\frac{b+y}{2b}T_3T_{oc}+T_{oc}^2\right]dy.\end{aligned} \tag{30}$$

Now in (30) we calculate each integral separately:

$$1)\int_{-b}^{b}\left[\frac{b^2-2by+y^2}{4b^2}T_2^2\right]dy=\frac{1}{4b^2}\left(2b^3+\frac{2b^3}{3}\right)T_2^2;$$

$$2)\int_{-b}^{b}\left[2\cdot\frac{b^2-y^2}{4b^2}T_2T_3\right]dy=\frac{1}{2b^2}\left(2b^3-\frac{2b^3}{3}\right)T_2T_3;$$

$$3)\int_{-b}^{b}\left[2\cdot\frac{b-y}{4b^2}T_2T_{oc}\right]dy=\frac{1}{b}\left(2b^2-0\right)T_2T_{oc};$$

$$4)\int_{-b}^{b}\left[\frac{b^2+2by+y^2}{4b^2}T_3^2\right]dy=\frac{1}{4b^2}\left(2b^3+\frac{2b^3}{3}\right)T_3^2;$$

$$5)\int_{-b}^{b}\left[2\cdot\frac{b+y}{2b}T_3T_{oc}\right]dy=2bT_3T_{oc};$$

$$6)\int_{-b}^{b}T_{oc}^2dy=2bT_{oc}^2. \tag{31}$$

Substituting (31) into (30) we find the integrated form J_3:

$$J_3=\int_{S(x=a)}\frac{h}{2}\left(T-T_{oc}\right)^2dS=\frac{hL}{2}\left[\frac{1}{4b^2}\left(2b^3+\frac{2b^3}{3}\right)T_2^2+\frac{1}{2b^2}\left(2b^3-\frac{2b^3}{3}\right)T_2T_3-\right.$$
$$\left.\frac{1}{b}\left(2b^2-0\right)T_2T_{oc}+\frac{1}{4b^2}\left(2b^3+\frac{2b^3}{3}\right)T_3^2-2bT_3T_{oc}+2bT_{oc}^2\right]=$$
$$=\frac{hL}{2}\left[\frac{2b}{3}T_2^2+\frac{2b}{3}T_2T_3-2bT_2T_{oc}+\frac{2b}{3}T_3^2-2bT_3T_{oc}\right]=$$
$$=\frac{hLb}{3}\left[T_2^2+T_2T_3-3T_2T_{oc}+T_3^2-3T_3T_{oc}+3T_{oc}^2\right].$$

It should also be noted here that in the expression in the bracket the sum of the coefficients will be equal to zero.

Given the expressions J_1; J_2 and J_3, from (3) there is determined the final integrated form of the J functional that characterizes the total thermal energy of the timber under study, taking into account the simultaneous presence of a heat flux, thermal insulation and heat transfer:

$$J=J_1+J_2+J_3=\frac{bLK_{xx}}{6a}\left[T_1^2-2T_1T_2-T_1T_3+T_1T_4+T_2^2+T_2T_3-T_2T_4+T_3^2-2T_3T_4+T_4^2\right]+$$
$$+\frac{aLK_{yy}}{6a}\left[T_1^2+T_1T_2-T_1T_3-2T_1T_4+T_2^2-2T_2T_3-T_2T_4+T_3^2+T_3T_4+T_4^2\right]+bLq\left[T_1+T_4\right]+$$
$$+\frac{bLh}{3}\left[T_2^2+T_2T_3-3T_2T_{oc}+T_3^2-3T_3T_{oc}+3T_{oc}^2\right].$$

Further, minimizing the functional J with respect to the nodal values of the temperatureT_1, T_2, T_3 and T_4 we find the resolving system of linear algebraic equations

$$1)\frac{\partial J}{\partial T_1}=0\Rightarrow\frac{bLK_{xx}}{6a}\left[2T_1-2T_2-T_3+T_4\right]+\frac{aLK_{yy}}{6a}\left[2T_1+T_2-T_3-2T_4\right]+bLq=0;$$

$$2)\frac{\partial J}{\partial T_2}=0\Rightarrow\frac{bLK_{xx}}{6a}\left[-2T_1+2T_2+T_3-T_4\right]+\frac{aLK_{yy}}{6a}\left[T_1+2T_2-2T_3-T_4\right]++\frac{bLq}{3}\left[2T_2+T_3-3T_{oc}\right]=0;$$

$$3)\frac{\partial J}{\partial T_3}=0\Rightarrow$$

$$\frac{bLK_{xx}}{6a}\left[-T_1+T_2+2T_3-2T_4\right]+\frac{aLK_{yy}}{6a}\left[-T_1-2T_2+2T_3+T_4\right]++\frac{bLq}{3}\left[T_2+2T_3-3T_{oc}\right]=0;$$

$$4)\frac{\partial J}{\partial T_4}=0\Rightarrow\frac{bLK_{xx}}{6a}\left[T_1-T_2-2T_3+2T_4\right]+\frac{aLK_{yy}}{6a}\left[-2T_1-T_2+T_3+2T_4\right]+bLq=0.$$

For convenience, we discretize with 6 elements. The global numbering of elements and nodes is shown in (Figure 2). Now, for all the finite elements, there is an expression for the J functional that characterizes its total thermal energy, taking into account the existing boundary conditions [2, 6-14].

The integrated form of this functional for all discrete elements is as follows:

$$\begin{aligned}J=&\left(\tfrac{aLK_{xx}}{6b}\right)_{IE}\left[T_1^2-2T_1T_2-T_1T_6+T_1T_5+T_2^2+T_2T_6-T_2T_5+T_6^2-2T_6T_5+T_5^2\right]+\\+&\left(\tfrac{bLK_{yy}}{6a}\right)_{IE}\left[T_1^2+T_1T_2-T_1T_6-2T_1T_5+T_2^2-2T_2T_6-T_2T_5+T_6^2+T_6T_5+T_5^2\right]+\\+&(aLq)_{IE}\left[T_1+T_2\right]+\left(\tfrac{aLK_{xx}}{6b}\right)_{IIE}\left[T_3^2-T_3T_4-T_3T_9+T_3T_8+T_4^2+T_4T_9-T_2T_8+\right.\\+&\left.T_9^2-2T_9T_8+T_8\right]+\left(\tfrac{bLK_{yy}}{6a}\right)_{IIE}\left[T_3^2+T_3T_4-T_3T_9-2T_3T_8+T_4^2-2T_4T_9-T_4T_8+\right.\end{aligned}$$

$$\begin{aligned}&\left.+T_9^2+T_9T_8+T_8^2\right]+(aLq)_{IIE}\left[T_3+T_4\right]+\left(\tfrac{aLK_{xx}}{6b}\right)_{IIIE}\left[T_5^2-2T_5T_6-T_5T_{11}+T_5T_{10}+\right.\\&\left.+T_6^2+T_6T_{11}-T_6T_{10}+T_{11}^2-2T_{11}T_{10}+T_{10}^2\right]+\left(\tfrac{bLK_{yy}}{6a}\right)_{IIIE}\left[T_5^2+T_5T_6-T_5T_{11}-2T_5T_{10}+\right.\\&\left.+T_6^2-2T_6T_{11}-T_6T_{10}+T_{11}^2+T_{11}T_{10}+T_{10}^2\right]+\left(\tfrac{bLh}{3}\right)_{IIIE}\left[T_{11}^2+T_{11}T_{10}-3T_{11}T_e+T_{10}^2-\right.\\&\left.-3T_{10}T_e+3T_e^2\right]+\left(\tfrac{aLK_{xx}}{6b}\right)_{IVE}\left[T_6^2-2T_6T_7-T_6T_{12}+T_6T_{11}+T_7^2-T_7T_{12}-T_7T_{11}+T_{12}^2-\right.\\&\left.-2T_{12}T_{11}+T_{11}^2\right]+\left(\tfrac{bLK_{yy}}{6a}\right)_{IVE}\left[T_6^2+T_6T_7-T_6T_{12}-2T_6T_{11}+T_7^2-2T_7T_{12}-T_7T_{11}+T_{12}^2+\right.\\&\left.+T_{12}T_{11}+T_{11}^2\right]+\left(\tfrac{bLh}{3}\right)_{IVE}\left[T_{12}^2+T_{12}T_{11}-3T_{12}T_e+T_{11}^2-3T_{11}T_e+3T_e^2\right]+\end{aligned}$$

$$\begin{aligned}&+\left(\tfrac{aLK_{xx}}{6b}\right)_{VE}\left[T_7^2-2T_7T_8-T_7T_{13}+T_7T_{12}+T_8^2+T_8T_{13}-T_8T_{12}+T_{13}^2-2T_{13}T_{12}+T_{12}^2\right]+\\&+\left(\tfrac{bLK_{yy}}{6a}\right)_{VE}\left[T_7^2+T_7T_8-T_7T_{13}-2T_7T_{12}+T_8^2-2T_8T_{13}-T_8T_{12}+T_{13}^2+T_{13}T_{12}+T_{12}^2\right]+\\&+\left(\tfrac{bLh}{3}\right)_{VE}\left[T_{13}^2+T_{13}T_{12}-3T_{13}T_e+T_{12}^2-3T_{12}T_e+3T_e^2\right]+\\&+\left(\tfrac{aLK_{xx}}{6b}\right)_{VIE}\left[T_8^2-2T_8T_9-T_8T_{14}+T_8T_{13}+T_9^2+T_9T_{14}-T_9T_{13}+T_{14}^2-2T_{14}T_{13}+T_{13}^2\right]+\\&+\left(\tfrac{bLK_{yy}}{6a}\right)_{VIE}\left[T_8^2+T_8T_9-T_8T_{14}-2T_8T_{13}+T_9^2-2T_9T_{14}-T_9T_{13}+T_{14}^2+T_{14}T_{13}+T_{13}^2\right]+\\&+\left(\tfrac{aLh}{3}\right)_{VIE}\left[T_{14}^2+T_{14}T_{13}-3T_{14}T_e+T_{13}^2-3T_{13}T_e+3T_e^2\right]\end{aligned}$$

Further, minimizing the last functional over nodal values, we obtain the following system of linear algebraic equations with respect to T_i:

$$\frac{\partial J}{\partial T_i} = 0, (i = 1 \div 14).$$

Solving the last system by the Gaussian method, we determine the nodal values of temperatures, and according to them, according to (1), the temperature value at any point of each finite element. Inparticular, withthefollowinginitial [1, 2]:

$K_{xx} = K_{yy} = 72 \left[\frac{W}{cm \cdot {}^\circ C}\right]$; a=b=1cm; $q = -100 \left[\frac{W}{cm^2}\right]$; $h_e = 6 \left[\frac{W}{cm^2 \cdot {}^\circ C}\right]$;

$T_e = 40^\circ C$; r=2 cm; l=1cm.

Wefindthat

$$T_1 = T_4 = 52.895^\circ C; T_2 = T_3 = 53.017^\circ C; T_5 = T_9 = 50.482^\circ C; T_6 = T_8 = 49.874^\circ C;$$

$$T_7 = 48.658^\circ C; T_{10} = T_{14} = 48.573^\circ C; T_{11} = T_{13} = 48.304^\circ C; T_{12} = 48.152^\circ C.$$

It can be seen from the obtained results that due to the symmetrical formulation of the problem under consideration, the process of the steady distribution of the temperature field in the section of the beam will also be symmetrical.

4 Conclusions

The proposed mathematical model, based on the law of conservation and change of thermal energy, allows us to solve a class of multidimensional problems of steady thermal conductivity for structural elements of any configuration, where there is a heat flux and a temperature, partial thermal insulation, and heat transfer.

Because of the symmetry of the nodal points of the problem under consideration, in this work the results of the numerical solution are symmetrical, i.e., there are the same temperature values.

The exact calculation of distributing the temperature field at each nodal point is determined by formula (1). Based on the energy principle combined with the finite element method, the steady-state temperature distribution field in the volume of a partially thermally insulated beam in the presence of a heat flux and heat exchange is studied numerically. A numerical solution is given for specific initial data. A numerical study of the convergence and accuracy of the obtained numerical solutions is carried out.

References

[1] Segerlind, L. (1979). Primenenie metoda konechnyh elementov [Application of the finite element method]. Moscow: Mir [in Russian]

[2] Kenzhegulov, B. (2021). Numerical modeling of multidimensional temperature and one-dimensional nonlinear thermomechanical processes in heat-resistant alloys. Atyrau: ASU Press publishing House.

[3] Kenzhegulov, B., Shazhdekeyeva, N., Myrzhasheva, A. N., Kabylhamitov, G. T., Tuleuova, R. U. (2020). Necessary Optimality Conditions for Determining of The Position of The Boundary of Oil Deposit. International Journal of Engineering Research and Technology, Vol. 13, 1204-1209.

[4] Kenzhegulov, B., Kultan, J., Alibiyev, D.B., Kazhikenova, A.Sh. (2020). Numerical Modelling of Thermomechanical Processes in Heat-Resistant Alloys. Bulletin of the Karaganda University, Physics Series. 2(98),101-108.

[5] Kenzhegulov, B., Shazhdekeyeva, N., Myrzhasheva, A. N., Tuleuova, R. U. (2020). A Numerical Method for Determining the Dependence of The Thermally Stressed State of a Rod on Ambient Temperature with The Simultaneous Presence of Thermal Processes. PeriodicoTcheQuimica, Vol. 17, 765- 780.

[6] Kenzhegulov, B., Tuleuova, R., Myrzasheva, A., Shazhdekeyeva, N., Kabylkhamitov, G. (2021). Mathematical modelling and development of a computational algorithm for the study of thermo-stressed state of a heat-resistant alloy. Site of journal "Periodicals of Engineering and Natural Sciences". Retrieved from http://pen.ius.edu.ba/index.php/pen/issue/view/31.

[7] Kenzhegulov, B., Shazhdekeyeva, N., Myrzasheva, A., Kabylkhamitov, G., Tuleuova, R. (2021). Numerical methods for solving improper problems of filtration theory. Journal of Applied Engineering Science, Vol. 19, br. 1, 98-108.

[8] Kenzhegulov, B., Kenzhegulova, S.B., Alibiyev, D. B., Kazhikenova, A. Sh. (2022). Finite element element modeling of heat propagation of a complete rod of constant cross-section. Bulletin of the Karaganda University, Physics Series. 4(108),94-105.

[9] Wang, J. (2020). Kinetic and Strength Calculation of Age-Hardening Phases in Heat-Resistant Aluminum Alloys with Silver. Materials Science Forum, Vol. 993 (pp. 1051-1056).

[10] Adaskin, A. M., Kirillov, A. K., Kutin, A. A. (2020). Improving the Cutting of Heat-Resistant Chromium Alloy. Russian Engineering Research, Vol. 40, no. 9, 748-750.

[11] Min, P. G., Sidorov, V. V., Vadeev, V. E., Kramer, V. V. (2020). Development of Corrosion and Heat Resistant Nickel Alloys and their Production Technology with the Aim of Import Substitution. Power Technology and Engineering, Vol. 54, no. 2, 225-231.

[12] Facco, A., Couvrat, M., Magne, D., Roussel, M., Guillet, A., Pareige, C. (2020). Microstructure Influence on Creep Properties of Heat-Resistant Austenitic Alloys with High Aluminum Content. Materials Science and Engineering A, Vol. 783, article number 139276.

[13] Kvasnytska, Y. H., Ivaskevych, L. M., Balytskyi, O. I., Maksyuta, I. I., Myalnitsa, H. P. (2020). HighTemperature Salt Corrosion of a Heat-Resistant Nickel Alloy. Materials Science, vol. 56, no. 3, 432-440.

[14] Patrin, P. V., Karpov, B. V., Aleshchenko, A. S., Galkin, S. P. (2020). Capability Process Assessment of Radial-Displacement Rolling of Heat-Resistant Alloy HN73MBTYU. Steel in Translation, Vol. 50, no. 1, 42-45.

IRSTI 55.30.03, 55.30.31, 55.30.13 DOI: https://doi.org/10.26577/JMMCS2023v120i4a8

M.K. Zhumadillayev[1*], **M.K. Baktybayev**[2], **A.B. Baratova**[1], **A.N. Mussulmanbekova**[1], **A. Adilkhan**[1]

[1]Institute of Mechanics and Engineering Science named after academician U.A. Dzholdasbekov, Kazakhstan, Almaty
[2]Satbayev University, Kazakhstan, Almaty
*e-mail: yojellyfish1992@gmail.com

DESIGN OF CONTROL SYSTEMS FOR A ROBOT FOR CLEANING PUBLIC SPACES

The aim of this article is to examine the control mechanism for a cleaning robot employed in public areas, focusing on the development of a controller for the cleaning robot. The motor control system block diagram for the surface cleaning robot is created based on the principle of Pulse-Width Modulation (PWM) for speed control. Each module's functions in the control system are separated and elaborated. The article presents a proposal for software and hardware design, adopting a thinking model based on the AVR microprocessor. By using RS485 and PC communication, following an agreed protocol, the control system facilitates the robot's forward and backward movements, rotation, and operation with a DC or stepper motor. Consequently, it enables the surface cleaning robot to perform its work more effectively.

Key words: a robot designed for cleaning surfaces, The AVR Mega8535 microcontroller, control system, modular design.

М.К. Жумадиллаев[1*], М.К. Бактыбаев[2], А.Б. Баратова[1], А.Н. Мусулманбекова[1], А. Әдилхан[1]

[1]Академик Ө.А. Жолдасбеков атындағы механика және машинатану институты, Қазақстан, Алматы қ.
[2]Satbayev University, Қазақстан, Алматы қ.
*e-mail: yojellyfish1992@gmail.com

Қоғамдық кеңістіктерді тазалауға арналған роботтың басқару жүйелерін жобалау

Бұл мақаланың мақсаты – тазалау роботының контроллерін әзірлеуге назар аудара отырып, қоғамдық орындарда қолданылатын тазалау роботының басқару механизмін зерттеу. Бет тазалау роботының қозғалтқышты басқару жүйесінің құрылымдық схемасы жылдамдықты басқаруға арналған импульстік ен модуляциясы (PWM) принципіне негізделген. Басқару жүйесіндегі әрбір модульдің функциялары бөлініп, өңделеді. Мақалада AVR микропроцессоры негізіндегі ойлау моделін пайдалана отырып бағдарламалық және аппараттық құралдарды жобалау ұсынысы берілген. RS485 пайдалану және келісілген хаттамаға сәйкес ДК-мен байланыс орнату, басқару жүйесі роботтың алға және артқа қозғалысын, айналуын және тұрақты ток қозғалтқышымен немесе қадамдық қозғалтқышпен жұмысын жеңілдетеді. Сондықтан бұл бет тазалау роботына өз жұмысын тиімдірек орындауға мүмкіндік береді.

Түйін сөздер: бет тазалау роботы, AVR Mega8535 микроконтроллері, басқару жүйесі, модульдік дизайн.

М.К. Жумадиллаев[1*], М.К. Бактыбаев[2], А.Б. Баратова[1], А.Н. Мусулманбекова[1], А. Адилхан[1]

[1]Институт механики и машиноведения имени академика У.А. Джолдасбекова, Казахстан, г. Алматы
[2]Satbayev University, Казахстан, г. Алматы
*e-mail: yojellyfish1992@gmail.com

Проект системы управления роботом для уборки общественных помещений

© 2023 Al-Farabi Kazakh National University

Целью данной статьи является изучение механизма управления роботом-уборщиком, используемым в общественных местах, с упором на разработку контроллера для робота-уборщика. Блок-схема системы управления двигателем робота для очистки поверхностей создана на основе принципа широтно-импульсной модуляции (ШИМ) для управления скоростью. Функции каждого модуля в системе управления разделены и проработаны. В статье представлено предложение по проектированию программного и аппаратного обеспечения, использующее модель мышления на основе микропроцессора AVR. Используя RS485 и связь с ПК в соответствии с согласованным протоколом, система управления облегчает движения робота вперед и назад, вращение и работу с двигателем постоянного тока или шаговым двигателем. Следовательно, это позволяет роботу для очистки поверхностей выполнять свою работу более эффективно.

Ключевые слова: робот для уборки поверхностей, микроконтроллер AVR Mega8535, система управления, модульная конструкция.

1 Introduction

Surface cleaning robots are public space protection equipment, mainly used to collect garbage in the restroom. In addition to their primary function of collecting garbage in public spaces, surface cleaning robots offer several additional benefits. For instance, they can be used as tools for evaluating the quality of indoor air, which is critical for ensuring the health and safety of individuals who spend time in such spaces. Furthermore, they provide numerous societal benefits, such as reducing the workload of human cleaners and promoting a cleaner, more hygienic environment. Finally, the potential for promising applications of surface cleaning robots is vast, as they can be further developed and modified to serve other purposes beyond their current functions. These various benefits have been noted and supported by research in the field [1, 2, 3, 4, 5]. At present, the responsibility for cleaning public spaces falls upon human cleaners due to the absence of surface cleaning robot systems that possess flexibility, mobility and high intelligence. However, with recent advancements in robotics technology, the development of a control system for surface cleaning robots that can communicate with the main computer and control system has become increasingly feasible. Such a system would have theoretical significance by furthering our understanding of the capabilities and limitations of robotic systems. In addition, the development of such a control system would have significant applied value, as it would improve the efficiency and effectiveness of surface cleaning in public spaces, reduce the workload of human cleaners, and promote a cleaner, more hygienic environment. This potential has been recognized and supported by research in the field [5, 6, 7, 8, 9, 10].

2 Literature review

Robots are used in many areas, even in everyday life (22). Robots for home use are on the rise. Robot vacuum cleaners are especially known (23). Among the various robots that exist in the world, only some robots can be used specifically for human household chores. Among these robots, there is one particular type of robot that is very useful for everyone, and that is the cleaning robot (24). A simple automatic robot that uses some predefined algorithms and programs to clean a given area is called a cleaning robot. The main application of this robot is to reduce human intervention in the cleaning process, which can be time consuming. These robots can be used anywhere, i.e., in offices, homes, factories, etc. These robots can be activated with the push of a button or can be pre-configured to activate at a specific time (22).

As shown in Fig. 1, the whole system adopts a public space cleaning robot, previously developed in the laboratory, based on the SCARA system, as a carrier. SCARA robots are a new type of industrial robots that combine high precision and movement speed. The design of SCARA robots consists of two arms connected together, attached to the base. Thus, SCARA robots are four-axis robots with freedom of movement along the $X-Y-Z$ axes, rotational movement is also carried out along the Z axis. SCARA robots are ideal for moving loads from point A to point B in a horizontal plane, if little movement is required along the Z axis. Due to the rotational movement of the Z axis, SCARA robots can not only move objects, but also screw them, which makes these robots indispensable on assembly lines. The surface cleaning robot system is composed of three working motors, each serving a specific purpose. The front stepper motor is responsible for collecting debris, while the left and right rear DC motors control the $X-Y$ axes to enable movement and navigation. The control system receives information from the main computer, which it then processes and utilizes to execute commands that control the three motors, ultimately working to accomplish the desired objectives. The cleaning robots are also equipped to organize the information received in real-time, utilizing the user protocol to send data to the monitoring center via the RS485 serial port. Upon arrival at the host computer, the batch data is recovered and displayed on the control interface of the host computer, providing valuable information about the status and performance of the cleaning robot. This sophisticated system serves as a testament to the rapid advancements in robotics technology, making it possible to design and implement intelligent and efficient solutions to complex problems in a wide range of industries.

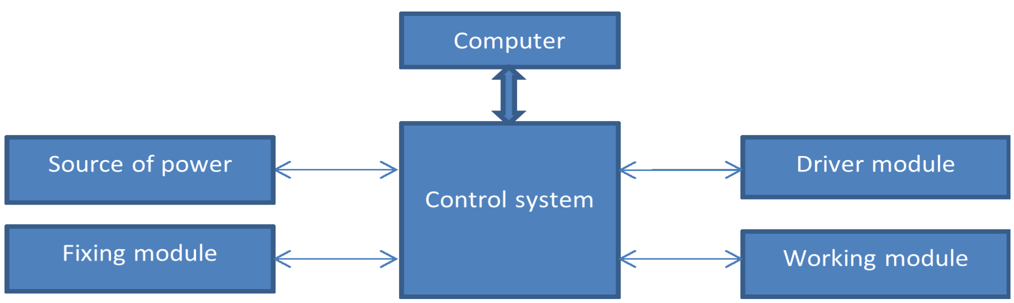
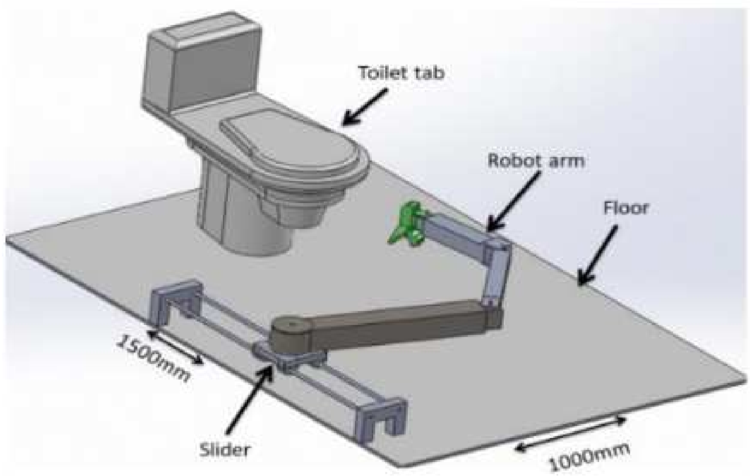


Figure 1: Block diagram of the robot control system

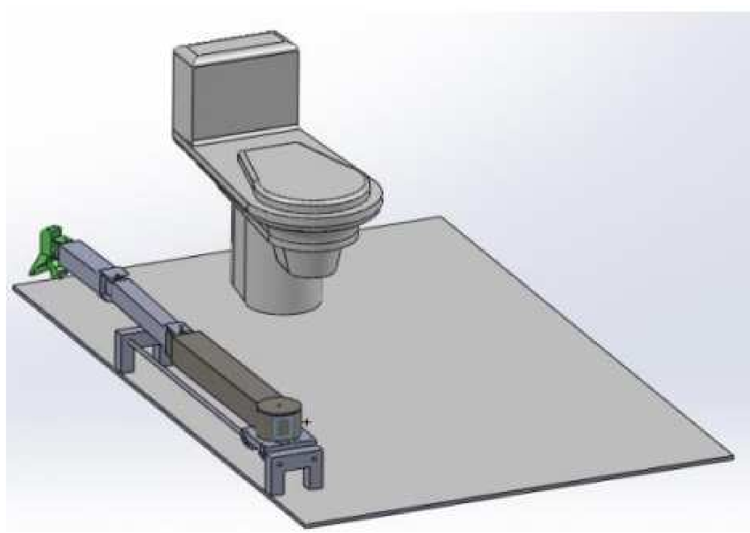
3 Robot design concept

The design of the robot is determined by the working environment and the task. The exploration robot comes with some limitations; the proposed robot can only work in a structured and predefined workspace. In this study, bathroom dimensions should be no more than 1000 mm wide and 1500 mm deep. According to the standard toilet cubicle systems in Western European countries, the dimensions of the public toilet cubicle should be 850 mm wide and 1500 mm deep.

Depending on the robot performing the task, the robot arm must be accessible anywhere in the workspace without blind spots and reasonably compact. Therefore, in this study, we proposed a multi-joint robotic arm with a similar structure to the SCARA robot (see Fig. 2. a, b). As shown in Figure 2b, the robot aligns itself along the slider guide after cleaning the surface. Traditionally, in such a tight workspace, continuum arms have an advantage due to their flexible structure. However, in this task, the Robot Manipulator must have a rigid link to control a heavy tool [21].



a)



b)

Figure 2: a) Robot working mode; b) Robot standby mode

4 Results and discussion

After analyzing the requirements for control systems of public area cleaning robots, it was determined that the control system's hardware circuit should consist of several important components, including the motor control circuit, external watchdog, simulation download module circuit, RS485 interface circuit, and power management circuit. The motor control circuit is responsible for driving both the DC and stepper motors, which are crucial for the robot's movement and cleaning capabilities. A leak detection circuit is also included to ensure the safe and proper operation of the robot. The general structure of the control system can be visualized in Figure 3, which provides an overview of the various components and their interconnectivity. By carefully designing and integrating these circuits, the control system can effectively control the movements and actions of the cleaning robot, ensuring its proper functioning and efficient operation.

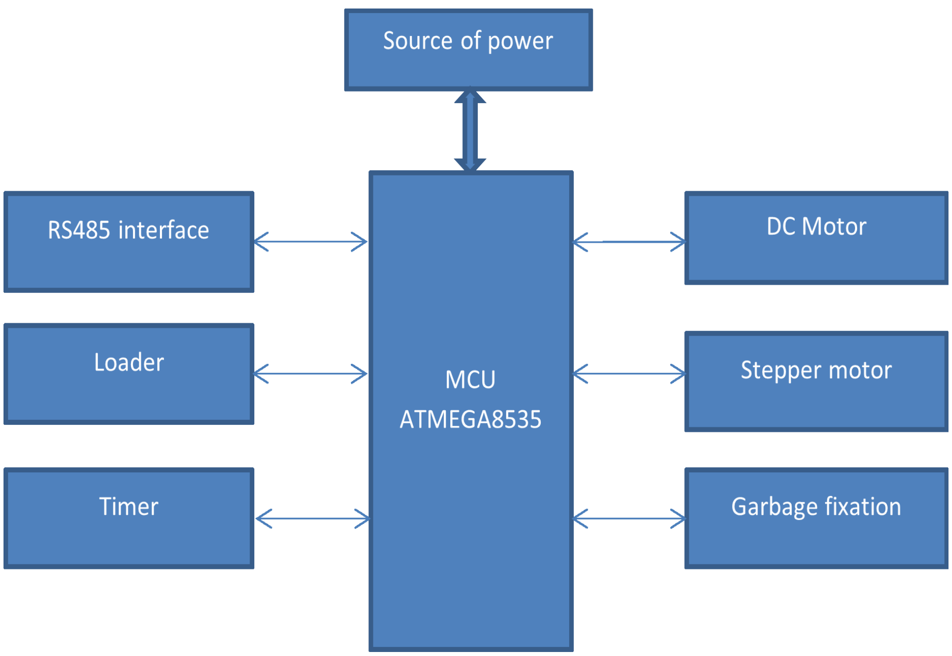


Figure 3: General scheme of building the control system hardware

4.1 Microprocessor

The ATmega8535 microprocessor serves as the fundamental component of the surface cleaning robot control system, providing a high throughput of up to 1 Million Instructions Per Second per Megahertz (MIPS/MHz). This ensures that the system can process data and execute commands quickly, without compromising on power consumption. By utilizing the ATmega8535 microprocessor, the surface cleaning robot control system is equipped with a reliable and efficient core that can effectively manage and execute the necessary tasks required for the proper functioning of the system.

4.2 DC motor drive module

In order to achieve the necessary movements of the surface cleaning robot, such as moving forward, backward, and turning, it is crucial to have precise control over the speed and direction of the motor's rotation. The speed control of a permanent magnet DC motor can be accomplished through two primary methods: armature series resistance and supply voltage reduction. However, using the armature series resistance method can result in instability at low speeds and discontinuous speed changes. Conversely, reducing the supply voltage can provide smoother speed control without altering the motor's mechanical characteristics [13]. During a single cycle of positive and negative changes in the PWM, the armature voltage at both ends of the motor experiences changes twice, which enables determining the average voltage through the following formula [14]:

$$U_0 = \left(\frac{t_1}{T} - \frac{T - t_1}{T}\right) U_s = \left(2\frac{t_1}{T-1}\right) U_s = (2\alpha - 1)U_s$$

α – duty cycle, $\alpha = t1/T$.

The duty cycle is a crucial parameter in the pulse-width modulation (PWM) technique that controls the speed of a motor by varying the conduction time of a switch in a periodic signal. In the given equation, the duty cycle represents the ratio of the switch's on-time to the period T, while the conversion range α lies between 0 and 1. In a bipolar reversing PWM drive, the average armature voltage of the motor is determined by the value of α. For instance, when $\alpha = 0$ and $U0 = -US$, the motor rotates in the reverse direction and achieves maximum speed; when $\alpha = 1$ and $U0 = US$, the motor rotates forward and attains the maximum speed; and when $\alpha = 1/2$ and $U0 = 0$, the motor comes to a halt. The duty cycle plays a crucial role in controlling the speed and direction of the motor, making it a vital parameter in motor control systems.

4.3 Stepper Motor Driver Module

A stepper motor is a type of actuator that can convert an electrical signal into precise angular movement. Unlike other types of motors, the stepper motor can be precisely controlled by varying the number of pulses it receives, allowing for highly accurate positioning. In addition, the speed and acceleration of the stepper motor can also be controlled by changing the frequency of the pulses, making it a versatile option for a wide range of applications. To drive a stepper motor, a square wave current is typically used, which is achieved through the use of a stepper motor drive. A breakdown driver is a type of stepper motor drive that employs electronic damping technology to reduce or eliminate low frequency vibrations that may affect the motor's performance. By minimizing vibrations, breakdown drivers can improve the motor's positioning accuracy, reduce operating noise, and ensure smooth operation [15].

Figure 4 illustrates that stepper drives are comprised of three signal ends: pulse signal (PUL), direction signal (DIR), and enable signal (ENA). These signal ends are connected to the MCU's PB3, PB1, and PB0, as well as the negative electrodes which are grounded. Through programming, the driver can be controlled to regulate the stepper motor's movements to achieve the desired outcome. With this method, precise control over the motor's speed, direction, and positioning can be obtained.

4.4 An overview of the program's control system structure

Based on the analysis of the system's functional requirements and the results of the hardware module design, the software for the control system of the surface cleaning robot

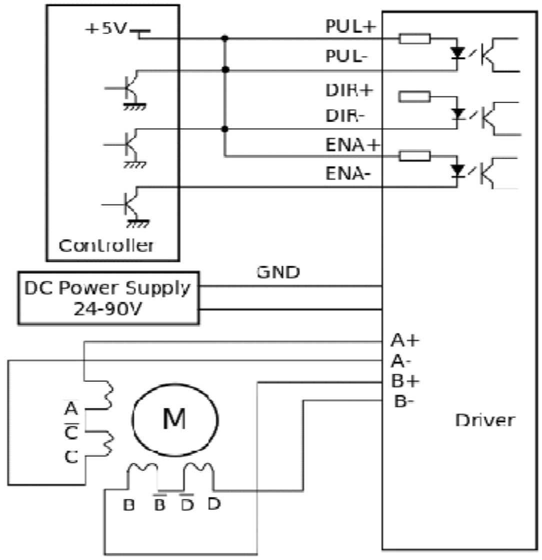


Figure 4: Stepper Motor Drive Interface Block

is composed of several subroutines in addition to the main module program. The general structure of this software is depicted in Figure 5. The software consists of several important components, including the main program, the internal initialization routine, the UART serial communication program, the DC motor control program, and the leakage detection interrupt program. These components have been developed based on a careful review of existing literature and established practices in the field of robotic control systems [16, 17, 18, 19, 20].

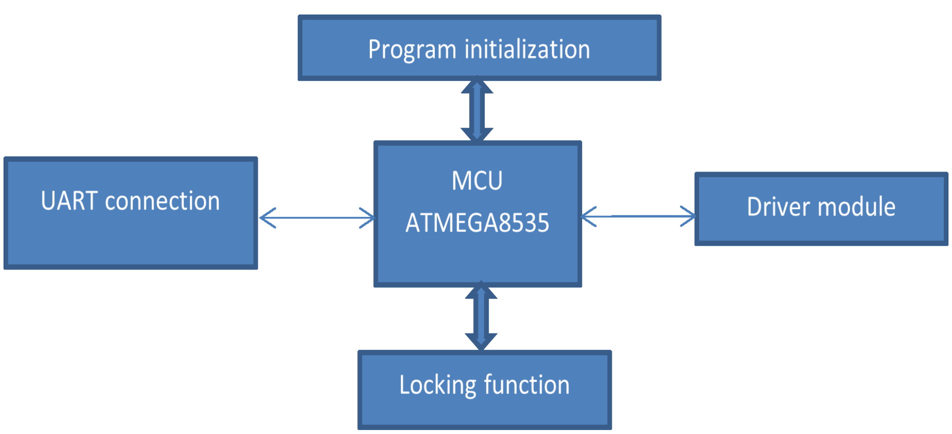


Figure 5: General structure of the system software

4.5 The process of creating software for the main controller module

Once the master device applies power to the system, the parameters are initialized and the port addresses are assigned. Additionally, two leakage detection signals are read, and the motor driver module routine is executed based on the instruction sent by the master to modify the speed and direction of the motor's rotation. This entire process results in the surface cleaning robot transitioning into the working state. If we take a closer look at the robotic surface cleaning system, we can see that the main program, as illustrated in figure 6, plays a crucial role in managing these tasks and ensuring the smooth functioning of the entire system.

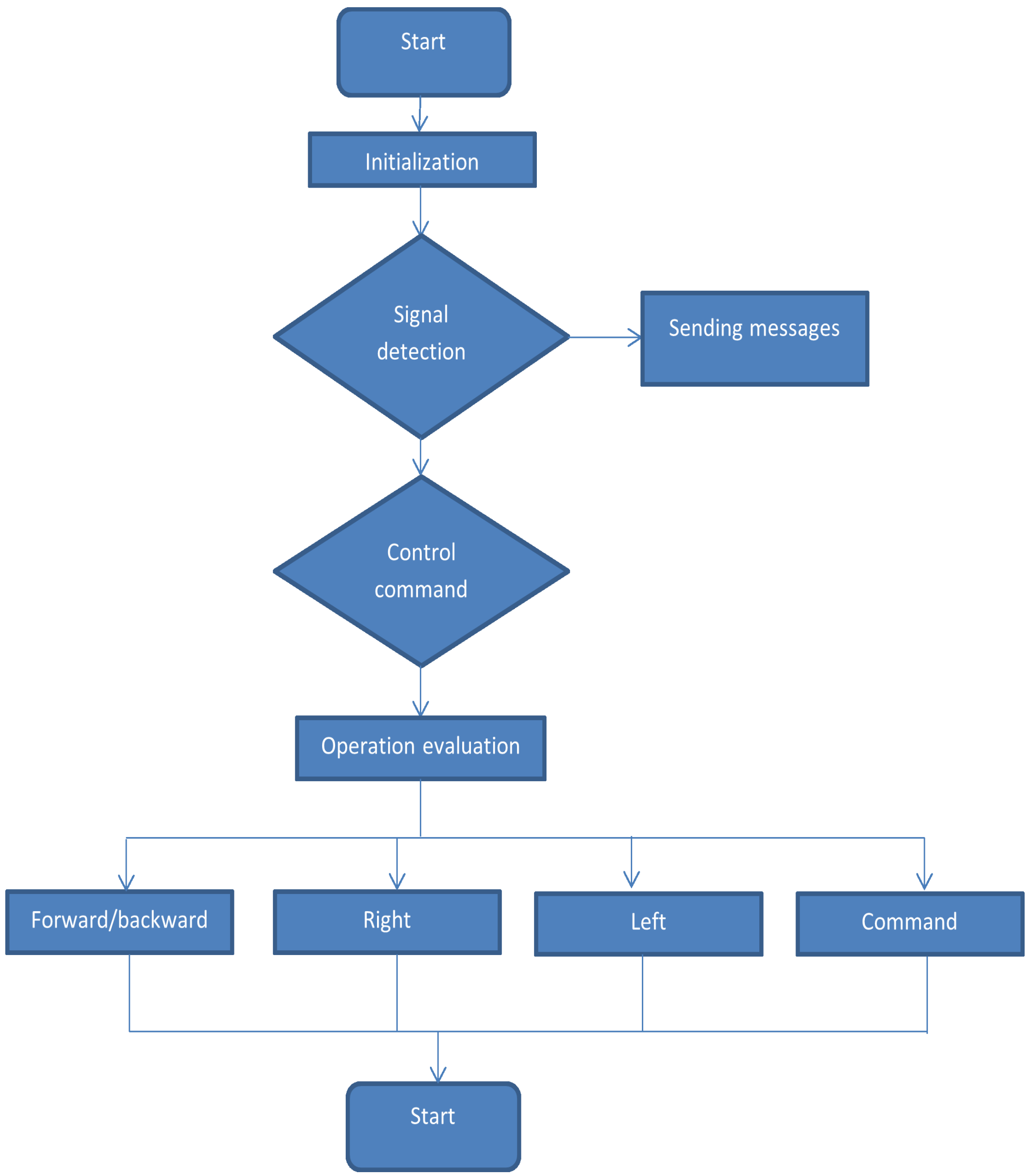


Figure 6: Scheme of the main operating mode of the system

5 Conclusion

The current study provides a comprehensive analysis of the public area cleaning robot control system design in both hardware and software aspects. It introduces an innovative control scheme that integrates a main computer and a microcontroller, which brings a series of benefits to the entire cleaning robot control system, such as fast response time, decreased power consumption, durability and real-time control capability. As a result, the overall system stability and reliability are significantly improved. With the aid of the main computer and the microcontroller, surface cleaning robots can be accurately controlled to perform their cleaning tasks with high precision and efficiency.

References

[1] Maske C.R., Ashrafiwan H., Haas G., McCloskey R. Flynn T., "Identification of a Control Oriented Nonlinear Dynamic USV Model" , *American Conference on Management,* (2008): 562–567.

[2] Fill Youb Lee, Bong Huan Jun, Pan Mook Lee, Kihun Kim, "Implementation and testing of the ISiMI100 unmanned aerial vehicle for a member of the unmanned aerial vehicle fleet" , *Proc. Oceans,* (2008): 1–6.

[3] Bellingham J.G., "New Oceanographic Applications of Autonomous Submersibles" , *Cambridge. MA.MTS Journal,* 31(3).

[4] Jianhua Wang, Wei Gu, Jianxin Zhu, Jubiao Zhang, "Unmanned surface vehicle for multi-purpose use" , *Proc. International Conference on Electronic Computer Technology,* Shanghai, P. R. China (2009): 358–361.

[5] Caccia M., Bibuli M., Bono R., Bruzzone G., "Aluminum-hulled USV for coastal and seabed monitoring" , *Proc. OCEANS 2009-EUROPE,* Bremen Germany (2009): 1–5.

[6] Jianhua Wang, Wei Gu, Jianxin Zhu, Jubiao Zhang, "Unmanned surface vehicle for multi-purpose use" , *Proc. International Conference on Electronic Computer Technology,* Shanghai, P. R. China (2009): 358–361.

[7] Manley J., "Unmanned surface vehicles, 15 years of development" , *Proc. Oceans 2008 MTS/IEEE Quebec Conference and Exhibition,* (2008).

[8] Caccia M., Bono R. Bruzzone G., Veruggio G., "Unmanned underwater vehicles for scientific applications and robotics research: the ROMEO project" , *Journal of the Society for Marine Technology,* 24(2) (2000): 3–17.

[9] Almeida C. et al., "Radar Based Collision Detection Developments on the USV ROAZ II" , *Proc. OCEANS 2009-EUROPE,* Bremen, Germany (2009): 1–6.

[10] Zhai Yu-yi, Ma Jin-ming, Yao Zhi-liang, Gong Zhen-bang, "Design of a system for lifting and lowering a subminiature adlittoral robot" , *Optics and Precision Engineering,* 12(3) (2004): 299–302.

[11] Zhai Yuyi, Tang Haibin, Ma Jinming, Chen Weihua, "Structure Improvement Design for Sub-Small Submersibles" , *Electromechanical integration,* 12(2) (2006): 22–24.

[12] Lee, Wonseok, Bang, Young-Bong, Lee, Kyung-Min, Shin, Bu-Hyun, Paik, Jamie Kyujin, Kim, In-Su, "Motion learning method for complex robot links using motor current" , *International Journal of Control, Automation and Systems,* 8(5) (2010): 1072–1081.

[13] Description of vehicle motion for scenario 2. ATmega8535(L), User's Manual[Z]. Rev.2502B-09/02.

[14] Li Lin, Wu Jian Xin, "Designing a high-precision angular position tracking system for a stepper motor" , *Electric drive,* 4 (2003): 25–27.

[15] Zhang, Lixiao, Luo, Delin, Su, Longjiang, Cao, Da, Luo, Zhifeng, "Development of DC motor soft start based on AVR microcontroller" , *Applied Mechanics and Materials,* 55-57 (2011): 382–387.

[16] Wu Gongxing, Sun Hanbing, Zou Jin, Wang Lei, "Basic traffic control strategy for a jet-powered USV" , *Proc. ICMA International Conference on Mechatronics and Automation,* Changchun, China (2009): 611–616.

[17] Singh P. et al., "Fuzzy Logic Modeling of an Unmanned Surface Vehicle (USV) Hybrid Power System" , *Proc. the 13th International Conference on Intelligent Systems Application to Power Systems,* (2005): 1–7.

[18] Blank J., Bishop B.E., "In-Situ Simulation of a High-Speed Autonomous Surface Vessel" , *Proc. 40th Southeast Symposium on Systems Theory,* (2008).

[19] Qiao Wei Yuan, Qiang Chen, Sawaya K., "MUSIC- based DOA finding and polarization estimation using USV with polarization sensitive array antenna" , *Radio and Wireless Symposium, IEEE2006,* (2006): 339–342.

[20] Azamat Yeshmukhametov, Ainur Baratova, Ainur Salemkhan, Zholdas Buribayev, Kassymbek Ozhikenov, Yedilkhan Amirgaliyev, "Design and modeling of self-sustainable bathroom floor cleaning robot system" , *The 21st International Conference on Control, Automation and Systems (ICCAS 2021)* Jeju, Korea (2021).

[21] Youngkak Ma, Seungwoo Kim, Dongik Oh and Youngwan Cho, "A study on the development of home mess- cleanup robot McBot" , *IEEE/ASME international conference on advanced mechatronics,* Xian, China (2008).

[22] Yunbo Hong, Rongchuan Sun, Rui Lin, Shumei Yu and Lining Sun, "Mopping module design and experiments of a multifunction floor cleaning robot" , *Proceeding of the 11th World Congress on Intelligent Control and Automation,* Shenyang, China (2014).

[23] Zheng Zhao, Weihai Chen, Chen C.Y. Peter and Xingming Wu, "A novel navigation system for indoor cleaning robot" , *IEEE International Conference on Robotics and Biomimetics (ROBIO),* Qingdao, China (2016).

3-бөлім | Раздел 3 | Section 3

Информатика | Информатика | Computer Science

IRSTI 82.13.13 DOI: https://doi.org/10.26577/JMMCS2023v120i4a9

N.T. Amanov[1], B.A. Dzhaparov[2], R.E. Nurbatyrova[2*]
[1]Institute of information and administrative technologies, Kazakhstan, Almaty
[2]Al-Farabi Kazakh National University, Kazakhstan, Almaty
*e-mail: raushan.nurbatyrova@mail.ru

THE TASK OF FORMING ELECTRONIC FILES IN THE ELECTRONIC DOCUMENT FLOW SYSTEM

In connection with the transition of organizations and enterprises of the country from paper to electronic document management and the adoption of regulatory requirements for electronic document management systems and archival storage, the task of creating electronic files becomes relevant. The paper proposes an approach to solving the problem of generating electronic files, which is characterized by sufficient generality and simplicity in the creation of algorithmic and (or) software. The task of forming electronic files is solved by implementing a sequence of operations for binary (paired) comparison of the metadata of an electronic document within the existing (ordered in some way) feature space. Within the framework of the proposed approach, it is possible to obtain in an explicit form the conditions for the belonging of an electronic document to the corresponding electronic file. Fulfillment (non-fulfillment) of these conditions is equivalent to the truth (falsehood) of some very specific predicate, and the structure of this predicate makes it possible to implement the mentioned sequence of comparison operations. In practice, the approach allows you to effectively solve the problem of automated distribution of an electronic document in electronic cases, when some of the details of the electronic document are known.
Key words: management documentation support, electronic document, details of an electronic file, metadata, electronic document management system, nomenclature of files, formation of an electronic file, theory of predicates, binary comparison.

Н.Т. Аманов[1], Б.А. Джапаров[2], Р.Е. Нурбатырова[2*]
[1]Ақпараттық және әкімшілік технологиялары институты, Қазақстан, Алматы қ.
[2]Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан, Алматы қ.
*e-mail: raushan.nurbatyrova@mail.ru

Электрондық құжат айналымы жүйесінде электрондық істерді қалыптастыру мәселесі

Еліміздің ұйымдары мен кәсіпорындарының қағаз құжат айналымынан электрондық құжат айналымына көшуіне және электрондық құжат айналымы мен архивтік сақтау жүйелеріне қойылатын нормативтік талаптардың қабылдануына байланысты электрондық істерді қалыптастыру мәселесі өзекті бола бастады. Жұмыста электрондық істерді қалыптастырудың алгоритмдік және (немесе) бағдарламалық қамтамасыз етудің жеткілікті дәрежеде жалпылығымен және қарапайымдылығымен сипатталатын тәсілі ұсынылған. Электрондық істерді қалыптастыру электрондық құжаттар метадеректерінің (қандай да бір жолмен реттелген) белгілер кеңістігінде екілік (жұптық) салыстыру операциялары тізбегін жүзеге асыру арқылы шешіледі. Ұсынылған тәсілдің шеңберінде электрондық құжаттың қандай электрондық іске тиесілігі шарттарын айқын түрде анықтауға болады. Бұл шарттарды орындау (орындамау) қандай да бір нақты предикаттың ақиқаттығына (жалғандығына) тең және бұл предикаттың құрылымы жоғарыда аталған салыстыру операцияларының тізбегін жүзеге асыруға мүмкіндік береді. Тәжірибеде бұл тәсіл электрондық құжат деректемелерінің белгілі бір бөлігі анықталған жағдайда, электрондық құжаттарды автоматтандырылған түрде электрондық істерге тарату мәселесін тиімді түрде шешеді.

© 2023 Al-Farabi Kazakh National University

Түйін сөздер: басқаруды құжаттамамен қамтамасыз ету, электрондық құжат, электрондық іс реквизиттері, метадеректер, электрондық құжат айналымы жүйесі, істер номенклатурасы, электрондық істерді қалыптастыру, предикаттар теориясы, бинарлық салыстыру.

Н.Т. Аманов[1], Б.А. Джапаров[2], Р.Е. Нурбатырова[2*]
[1]Институт информационных и административных технологий, Казахстан, г. Алматы
[2]Казахский Национальный университет им. аль-Фараби, Казахстан, г. Алматы
*e-mail: raushan.nurbatyrova@mail.ru

Задача формирования электронных дел в системе электронного документооборота

В связи с переходом организаций и предприятий страны от бумажного к электронному документообороту и принятием нормативных требований к системам электронного документооборота и архивного хранения становится актуальной задача формирования электронных дел. В работе предложен подход к решению задачи формирования электронных дел, отличающийся достаточной общностью и простотой при создании алгоритмического и (или) программного обеспечения. Задача формирования электронных дел решается путем реализации последовательности операций бинарного (парного) сравнения метаданных электронного документа в рамках имеющегося (упорядоченного некоторым образом) пространства признаков. В рамках предложенного подхода удается получить в явном виде условия принадлежности электронного документа к соответствующему электронному делу. Выполнение (невыполнение) этих условий эквивалентно истинности (ложности) некоторого вполне конкретного предиката, а структура этого предиката и позволяет реализовать упомянутую последовательность операций сравнения. На практике подход позволяет эффективно решать задачу автоматизированного распределения электронного документа по электронным делам, когда часть реквизитов электронного документа заведомо известна.

Ключевые слова: документационное обеспечение управления, электронный документ, реквизиты электронного дела, метаданные, система электронного документооборота, номенклатура дел, формирование электронного дела, теория предикатов, бинарное сравнение.

1 Introduction

The increasing pace of transition of the organization and enterprise of the country from paper to electronic document management causes a rapid increase in the volume of electronic documents (ED), which must be accepted for archival storage and used for public administration purposes. The directive documents [1, 2] define the requirements for systems of documentary support for management, electronic document management (EDMS) and archival storage of ED in the Republic of Kazakhstan.

In accordance with regulatory documents and historical tradition, the unit of archival storage of documents on paper is an archival file, which contains a finite number of archival documents, combined by subject matter, taking into account the importance and timing of their archival storage. In the case of electronic document management, this means that EDs must also be distributed among the relevant electronic files (formation of electronic files), which are subject to subsequent transfer to departmental and then state storage [3, 4]. Thus, the task of forming electronic files is to distribute a huge array of EDs executed over a calendar period (year) in electronic files, which will be stored, first, in the information system of the departmental archive of the organization, and then, possibly, in the information system of the state archive.

Currently, the interaction of the EDMS and the systems of temporary and long-term archival storage of ED in state bodies is carried out according to a "simplified"scheme, due

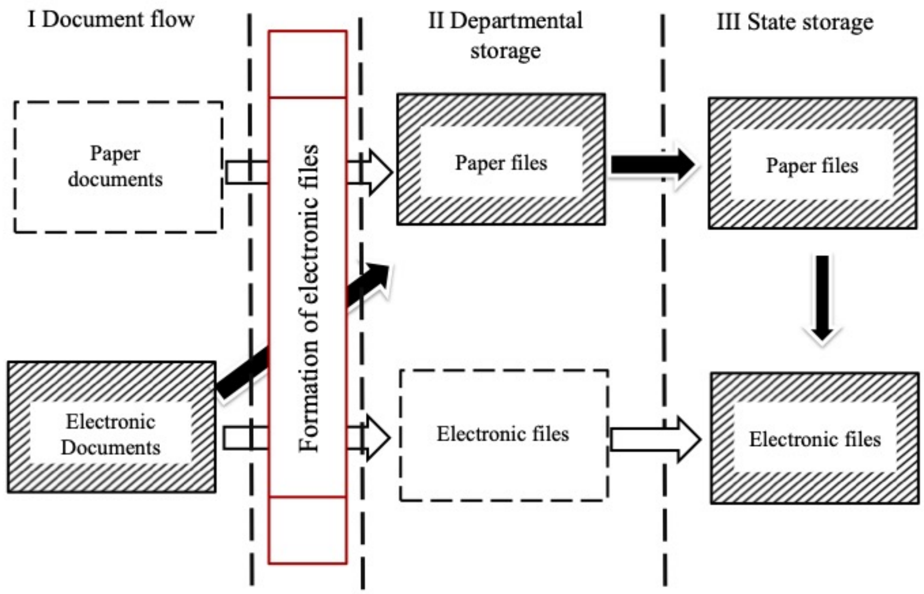


Figure 1: The current state of the formation of cases from traditional documents and electronic documents.

to the unresolved tasks of automating the processes of forming electronic files. In practice, it is performed manually, after making paper copies from the completed ED and in the future operating only with documents on paper. This nullifies the huge efforts to automate the organization's workflow, since at the output of the EDMS we are again dealing with paper documents that are manually distributed among cases according to its nomenclature of files.

Therefore, the automation of the now widely used routine manual actions of clerks and archivists for the distribution of completed documents on paper cases is an urgent task due to the lack of a unified scientific and methodological approach, the lack of effective algorithmic and software development, as well as the presence of a huge array of ED that has accumulated in the EDMS servers of the organization and requiring accelerated transfer to archival storage.

2 Methods

2.1 Statement of the problem of the formation of electronic files from ED

The process of forming electronic files from completed EDs consists in distributing a set of EDs between a set of electronic files in accordance with a finite set of common features (details) characteristic of all EDs included in one electronic file. In accordance with regulatory requirements, each case should include: case index, its name, start and end date, storage period in accordance with the List of typical archival documents with storage periods [5], article number of the departmental list of documents with their storage periods, name structural subdivision to whose information field of activity the electronic file belongs, etc. "tuple which will unambiguously identify a specific electronic file, which will include this ED.

The scheme for recognition of ED attributes proposed in [6] (determining the values of tuple elements) begins with the definition of a set of knowledge areas (activity) in accordance with a given classifier, which include ED stored in the archive and newly arriving in the archive. The initial data are a list of standard documents with their storage periods, a nomenclature of cases for the current year, which corresponds to a certain set A of all electronic files in the archive A_{ij}, $1 \leq i \leq e$, $1 \leq j \leq j(i)$, where i - kind (number) of the classifier, $j(i)$ is the number of templates in the i-th classifier, e is a natural number corresponding to the number of classifiers, $e \geq 1$.

The set of electronic files A_{ij} combines all EDs compiled according to the template $a_i j$, related to the i-th classifier i (for example, "Foreign economic activity") and with the j-th type of electronic document template (for example, "Orders"), $1 \leq i \leq$, $1 \leq j \leq j(i)$, $e \geq 1$. Each case A_{ij} can belong to one of the knowledge areas (classifier) C_i from a certain set of knowledge areas C, which is represented as a union of classifiers of cases in each knowledge area, i.e. each field of knowledge has its own unique classifier: $C = \{C_1, C_2, \ldots, C_e\}$, $e \geq 1$, where each classifier $_i$ can be represented as a set (set) of EDs related to the i-th classifier with the j-th type of models (templates), $C_i = \{a_{i1}, a_{i2}, \ldots, a_{ij(i)}\}$, where $1 \leq i \leq e$, $1 \leq j \leq j(i)$.

To determine for each executed ED a specific electronic file, in [6–9] it is proposed to carry out the following sequence of actions. Initial data are set: nomenclature of files for the current year (set of cases A); criteria for attributing ED to a particular field of knowledge (classifier), as well as to a specific electronic file; requirements for the size of the electronic file (volume). Based on the initial data, a set of areas of knowledge is formed, according to the subject-matter and industry-specific features, and within the framework of the existing feature space, the executor distributes ED in electronic files.

Indeed, when the next ED is submitted for consideration, certain elements of its structure (metadata) are sequentially checked for compliance with one or another predetermined criteria (features) of referring to a specific electronic file, i.e. functions containing logical comparison operations and taking values 1 (true) or 0 (false) are executed, which in mathematical terminology are called "predicates". A predicate in programming is an expression that uses one or more values with a boolean result [10].

Based on the distribution results, appropriate marks are made in the ED accounting forms and details are formed, after which the ED is attached to the electronic file. However, in case of overflow of the memory allocated for a single electronic volume (file), they are automatically closed and the next volume of the file is formed. After the electronic file is closed, an automatic procedure for the formation and execution of its electronic inventory is performed.

We have proposed and substantiated the following approaches to solving the problem mentioned above:

- Transition from the language of predicate calculus to the function of distributing ED in electronic files and its advantage.

- Generalization of the ED distribution function for electronic files in the case of establishing more than 2 details.

- A new approach to solving the problem in the presence of an ED registration and control card.

3 Results

3.1 Mathematical model of function $F : D \to A$ distribution of ED in electronic files

The set of templates mentioned above, constituting the corresponding classifier, can be built as follows:

1. Denote by A the set of all electronic files in the departmental archive of the organization.
2. Denote by A_i the set of all electronic files from A related to C_i, that is, to the area of knowledge (classifier) with the number $1 \le i \le e$.

Assuming that each case in A satisfies the conditions for belonging to one and only one area, it is clear that $A = A_1 \cup A_2 \cup \ldots \cup A_e$.

Let with a fixed number of the i-th classifier (nomenclature), $1 \le i \le e$, the archive has a set A_{ij} files, where $1 \le j \le j(i)$, where $j(i)$ - the number of files in the i-th classifier. Then obviously, $A_i = \{A_{i1}, A_{i2}, \ldots, A_{ij}(i)\}$, $1 \le i \le e$. Thus

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1j(1)} \\ A_{21} & A_{22} & \ldots & A_{2j(2)} \\ \ldots & \ldots & \ldots & \ldots \\ A_{e1} & A_{e2} & \ldots & A_{ej(e)} \end{pmatrix}. \tag{1}$$

For every file $_{ij}$, $1 \le i \le e$, $1 \le j \le j(i)$, which combines all EDs related to the i-th type of the classifier (nomenclature) and the j-th number of the electronic file in the set $_i$, enter a unique template a_{ij} (distracting from the structure of the template, in order to analyze the methodology, it can be considered a_{ij} the name of the template). Then, by definition, the classifier $_i$ consists of a set of predefined templates a_{ij}, т.е. $C_i = \{a_{i1}, a_{i2}, \ldots, a_{ij(i)}\}, 1 \le i \le e, 1 \le j \le j(i)$.

It is assumed that there is a set D of all executed ED, which includes incoming documents d, each incoming document d is placed in a single case from A, and criteria for assigning d to a particular area are set C_i, as well as to a specific electronic case, matching pattern a_{ij}, $1 \le i \le e$, $1 \le j \le j(i)$.

Let's introduce the distribution function of ED in electronic files $F : D \to A$, $F(d) = A_{ij} \in A$ for some i and j (which must be determined).

The presence of appropriate criteria allows us to set predicates P and P_y:

$$P(C_i, d) = \begin{cases} 1, & \text{если } d \text{ satisfies the criterion for classifying ED as } C_i \\ 0, & \text{otherwise} \end{cases}$$

and

$$P_y(a_{ij}, C_i, d) = \begin{cases} 1, & \text{if } d \text{ matches pattern } a_{ij} \\ 0, & \text{otherwise} \end{cases}$$

In [6], it is proposed to determine the value of the function $F(d)$ through the value of an appropriately constructed predicate P_M based on the already introduced predicates P и P_y, that is, based on the following equivalence:

$$F(d) = A_{ij} \iff P_M(i, j, d) - \text{true}$$

(or, which is the same, $P_M(i, j, d) = 1$, if, as usual, we assume that the value is true – это 1, a false – this 0)

In view of the fact that the explicit expression for the predicate proposed in this work P_M incorrect both syntactically and semantically (in a number of subformula members it is necessary to replace the predicates P и P_y on their negations, there is also confusion in the indices), below we have made the appropriate corrections in the work and given a specific type of predicate P_M to calculate the function value $F(d)$ distribution of the document, which, in turn, can undoubtedly simplify the mathematical calculations for the formation of electronic files and the definition of their metadata. At the same time, we immediately pass from the language of predicate calculus to the corresponding interpretation, which allows, without loss of generality, to consider P_M the function of its parameters, which takes values only 0 or 1, and the logical operation "and"(conjunction) corresponds to the usual multiplication.

To do this, note that the function

$$F(d) = A_{ij} \iff \begin{cases} P(C_l, d) = 0, & 1 \le l \le i - 1; P(C_i, d) = 1; \\ P_y(a_{im}, C_i, d) = 0, & 1 \le m \le j - 1; \\ P_y(a_{ij}, C_i, d) = 1 \end{cases}$$

where $1 \le i \le e$, $1 \le j \le j(i)$.

Labeling for convenience

$$Q(C_i, d) = 1 - P(C_i, d),$$
$$Q_y(a_{ij}, C_i, d) = 1 - P_y(a_{ij}, C_i, d)$$

(actually, it's a negation P и P_y respectively) and introducing the function

$$P_M(i, j, d) = \prod_{l=1}^{i-1} Q(C_l, d) \cdot P(C_i, d) \cdot \prod_{m=1}^{j-1} Q_y(a_{im}, C_i, d) \cdot P_y(a_{ij}, C_i, d),$$

defined on the set $= \{(i, j) | 1 \le i \le e, 1 \le j \le j(i)\}$, we see that this function takes the values 0 or 1 and only these values. Wherein

$$F(d) = A_{ij} \iff P_M(i, j, d) = 1.$$

The proposed model for finding the value of the distribution function $F : D \to A$ solves the original problem, however, compared to the original model, it is very simple and convenient for practical use in order to automate the procedure for generating electronic files (there is no need to use the apparatus of mathematical logic and graph theory).

If we denote

$$f = \max_{1 \le i \le e} j(i), N = e \cdot f,$$

then it is obvious that the number of operations of the ED selection algorithm has $O(N)$ time complexity,), that is, of polynomial complexity, and therefore, it is well implemented (in terms of speed).

Thus, by implementing a sequence of operations for binary (paired) comparison of the metadata of an electronic document within the existing feature space of features, we distribute each ED over a maximum of N electronic files.

3.2 Generalization of the distribution function model $F : D \to A$ in case of more than 2 details

Note that the technique considered in [5] makes it possible to match a specific ED with a specific number of an electronic file based on the results of recognizing two features: 1) the number of the classifier, 2) the type of the ED template. However, in practice, the solution of the desired task of distributing ED in electronic cases requires the establishment of a larger number of features, such as the details of the types of formalized models (templates), areas of knowledge (classifiers), articles of the List of standard documents with retention periods, retention periods, structural unit numbers, etc.

We noted above that the symbols a_{ij} serve as template names, where $1 \le i \le e, 1 \le j \le j(i)$. This allows you to determine whether the document belongs to the case number j in the knowledge area C_i, i.e. (due to the one-to-one correspondence a_{ij} and A_{ij}) to the file A_{ij}. In fact, for obtaining more detailed information or for the purpose of further ordering, these symbols alone, of course, are not enough. To be able to do this, we propose (in line with the arguments already given) to slightly improve the methodology.

For simplicity, we will assume that there is one more sign (props) of an electronic document and there is an appropriate criterion for classifying this document in some part of the electronic file A_{ij}. This means that we must "divide"the file A_{ij} subsection according to the attribute values. We enumerate (encode) all possible values (or absence) of this feature (props) by natural numbers from 1 to k. Then it is obvious that every file A_{ij} splits into k subdivisions (moreover, some of them may be empty, i.e., do not contain documents at all).

In accordance with the above, we will make some changes to our model, namely: Every file A_{ij} is a set of subfiles $A_{ij} = \{A_{ij1}, A_{ij2}, \ldots, A_{ijk}\}$, where A_{ijs} consists of documents d with attribute value code equal to s.

Let us designate the criterion for attributing an electronic document d to a specific attribute code r, i.e. set the predicate

$$S(r, d) = \begin{cases} 1, & \text{if attribute code in document } d \text{ is equal to } r, \\ 0, & \text{otherwise,} \end{cases}$$

where $1 \le r \le k$ an denote $T(r, d) = 1 - S(r, d)$.

$$R(i, j, p, d) = P_M(i, j, d) \cdot \prod_{q=1}^{p-1} T(q, d) \cdot S(p, d),$$

defined on the set $= \{(r, i, j) | 1 \le r \le k, 11 \le i \le e, 1 \le j \le j(i)\}$, we see that this function takes the value 0 or 1 and only these values.

Along with the function F, we can introduce its “modernized” analogue, i.e. function $G : D \to A$, s.t. $G : d \to A_{ijs}$. Then

$$G(d) = A_{ijp} \iff R(i, j, p, d) = 1.$$

The generalization of the ED distribution function model to the case of several features is quite obvious. Thus, it is possible to continue expanding the ED distribution function for electronic files on a finite set of cases, according to the nomenclature of the organization's files, which will significantly simplify the solution of the initial problem of ED distribution and, accordingly, can serve as a mathematical tool for automating the formation of electronic files in the EDMS.

3.3 A new approach to automating the formation of electronic files in the EDMS.

The method proposed in [7] for automatically generating electronic files from ED obtained by scanning completed paper documents allows you to automatically set the number of an electronic file by sequentially recognizing (determining) a set of ED details in the form of a metadata tuple. Thus, the task of forming electronic files of the EDMS is reduced to determining the set of ED-in, the details of which will correspond to the values of the elements of the tuple describing the electronic file. This means that the intellectual activity of officials of a state governing body or a state institution to determine "their"electronic file for each executed ED is modeled and automated in the form of a sequence of procedures for recognizing the details (metadata) applied to the ED, determining the type of formalized ED, determining the structural unit (official) to whose zone of informational responsibility the ED belongs, the articles of the standard list of documents with retention periods to which the ED belongs, the definition of its storage period and, finally, the nomenclature of electronic files.

It should be noted that this technique [7] does not take into account the main feature of the functioning of modern EDMS, in which the origin of each ED (accounting, registration) is accompanied by the creation of its registration and control card (RCC), which contains a number of its important details. The presence of the RCM ED, which is filled in from the moment of registration of the ED, greatly facilitates the task of automating the formation of electronic files from the ED, since the RCM from the very beginning indicates the number of the responsible structural unit and the number of the electronic file to which this ED belongs, i.e. case index, type of document, etc. However, in practice, in many cases, the RCC, as a rule, cannot knowingly contain the article number from the List of standard documents with retention periods, the retention period, and some other ED details. In this case, to ensure the completeness of the data in the RCC, one can resort to the method proposed in [7] in order to establish the exact values of the missing details.

4 Discussion

The paper develops and simplifies the approach to automating the formation of electronic files in the EDMS, considered in [6]. For further development of the approach under consideration, it is necessary (by analogy with the technique proposed in [7]) to establish the optimal structure of templates a_{ij} (in terms of the names of the corresponding attributes) and specify the attribute of the document d belonging to the knowledge area (classifier) $_i$, which is obviously equivalent to knowing explicit form of the predicates P_y and P.

5 Conclusion

The paper proposes a mathematical model for the problem of generating electronic files from ED, which boils down to sequential recognition and establishment of a complete set of details and registration data for each completed ED, i.e. "tuple which will unambiguously identify a specific electronic file, which will include this ED.

The proposed model for finding the value of the distribution function $F : D \to A$ solves the original problem, however, compared to the original model (from [6]), it is very simple and convenient for practical use in order to automate the procedure for generating electronic files, while there is no need to use the apparatus of mathematical logic and graph theory. If

$$f = \max_{1 \le i \le e} j(i), N = e \cdot f,$$

then the number of operations of the ED selection algorithm is $O(N)$, that is, polynomial complexity, and therefore, it is well implemented (in terms of speed).

Thus, by implementing a sequence of operations for binary (paired) comparison of the metadata of an electronic document within the existing feature space of features, we distribute each ED over a maximum of N electronic files.

The generalization of the ED distribution function model to the case of several features is quite obvious. Thus, it is possible to continue expanding the ED distribution function for electronic files on a finite set of files, according to the nomenclature of the organization's files, which will significantly simplify the solution of the initial problem of ED distribution and, accordingly, can serve as a mathematical tool for automating the formation of electronic files in the EDMS.

Known methods for the automated generation of electronic files from ED obtained by scanning completed paper archival documents are based on the sequential determination of details, when at the final stage the index of the file is determined, where the executed ED will be distributed. However, in modern EDMS, the index (file number and responsible structural unit) of the electronic file is known, but a number of important details may be missing. In this case, the application of the methodology we propose and its mathematical apparatus gives an adequate and efficient solution to the desired problem.

References

[1] Decree of the Government of the Republic of Kazakhstan dated September 20, 2018 No. 576 “On approval of the Rules for the acquisition, storage, accounting and use of documents of the National Archival Fund and other documents by state and special state archives”

[2] Decree of the Government of the Republic of Kazakhstan dated October 31, 2018 No. 703 “On approval of the Rules for documenting, managing documentation and using electronic document management systems in government and non-government organizations”

[3] Mustafina, A. (2019). Creation of the e-archive in Kazakhstan. Atlanti, 29(2). Retrieved from https://journal.almamater.si/index.php/Atlanti/article/view/325

[4] Estaeva M. Formation process and preservation of a natural electronic archive. June 2009 Proceedings of the American Society for Information Science and Technology 45(1):1 – 9 DOI:10.1002/meet.2008.1450450270

[5] Order acting Minister of Culture and Sports of the Republic of Kazakhstan dated September 29, 2017 No. 263 On approval of the List of standard documents generated in the activities of state and non-state organizations, indicating the storage period

[6] Volkov I.K. Methodology for the formation of electronic cases // Scientific and technical bulletin of information technologies, mechanics and optics. 2019. V. 19. No. 4. S. 650–656. doi: 10.17586/2226-1494-2019-19-4-650-656

[7] I. D. Korolev, V. S. Nazintsev, A. S. Mezentsev, and A. P. Makhnev, "Methods of Formation of Electronic Files in an Electronic Document Management System," Science-Intensive Technologies in Space Research of the Earth. 2020. V. 12. No. 4. S. 65–75. doi: 10.36724/2409-5419-2020-12-4-65-75

[8] Volkov I.K., Korolev I.D., Medvedev A.N. Problematic issues in the development of electronic archives. Science Yesterday, Today, Tomorrow, 2016, no. 11, pp. 13–18. (in Russian)

[9] Volkov I.K., Korolev I.D. The formation of the electronic cases to the electronic document management system. Estestvennye i Matematicheskie Nauki v Sovremennom Mire, 2017, no. 1, pp. 36–41. (in Russian)

[10] Bondarenko M.F., Shabanov-Kushnarenko Yu.P. About Algebra final predicates // Bionics of intelligence. 2011. No. 3(77). pp. 3–13.

ANNIVERSARY

70th anniversary of Doctor of Physical and Mathematical Sciences, Professor Baltabek Kanguzhin

The editor-in-chief of the journal, Professor Kanguzhin Baltabek Esmatovich, turned 70 years old!

Dear Baltabek Esmatovich, Your scientific works are an invaluable contribution to the development of society! We believe that for many years to come You will delight us with new discoveries, bold decisions and interesting works!

The editorial board of the magazine sincerely wishes You great success, goodness and prosperity!

© 2023 Al-Farabi Kazakh National University

МАЗМҰНЫ – СОДЕРЖАНИЕ – CONTENTS