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AL-FARABI KAZAKH NATIONAL UNIVERSITY

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ANALYTICAL SOLUTION OF INITIAL VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATION WITH SINGULAR PERTURBATION AND PIECEWISE CONSTANT ARGUMENT

The study of differential equations with piecewise constant arguments has been treated widely in the literature. This type of equation, in which techniques of differential and difference equations are combined, models, among others, some biological phenomena , the stabilization of hybrid control systems with feedback discrete controller or damped oscillators.The first studies in this field have been given in 1984, after this, some papers related with stability, oscillation properties and existence of periodic outcomes have been treated by several authors.The manuscript is crafted as follows: Section 2 outlines the primary methodologies adopted throughout the inquiry.Section 3 is dedicated to obtaining the exclusive outcome to the issue. We formulate a series of difference equations overseeing the vector $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i = \overline{1,p}$ which portray the constituents of the outcome. This generalized approach allows for a broader understanding of how to tackle such differential equations across various scenarios. These equations now form a recognized branch of the field of differential equations, and they are frequently used in biological and economic models. Undoubtedly, their applications will continue to increase in the future.
Key words: harmonic oscillator, initial functions, initial value problem.

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Сингулярлы ауытқуы және бөлікті-тұрақты аргументті бар қарапайым дифференциалдық теңдеу үшін бастапқы мән есебінің аналитикалық шешімі

Мақалада жалпыланған түрдегі бөлікті тұрақты аргументті кіші параметрлі жай дифференциалдық теңдеу үшін бастапқы мән есебі қарастырылған. Бөлікті тұрақты аргументті кіші параметрлі біртекті емес дифференциалдық теңдеуге сәйкес біртекті бөлікті тұрақты аргументті сингулярлы ауытқыған дифференциалдық теңдеудің іргелі шешімдер жүйесі құрылды. Шешімнің құрамындағы $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i = \overline{1,p}$ векторын анықтайтын айырымдық теңдеулер жүйесі алынды. Осы құрылған айырымдық теңдеулер жүйесінің шешімі анықталды. Редукция тәсілін қолданып, қойылған бөлікті тұрақты аргументті кіші параметрлі бастапқы мән есебінің шешімінің аналитикалық формуласы алынды. Шешімнің аналитикалық формуласы туралы теорема қорытылып шығарылды. Қарастырылған жалпыланған түрдегі бөлікті тұрақты сингулярлы ауытқыған бастапқы мән есебінің дербес жағдайларда сызықты гармоникалық осциллятор болып табылады.
Түйін сөздер: бөлікті-константалы аргумент, гармоникалық осциллятор, бастапқы функциялар, бастапқы мән есебі.

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Аналитическое решение задачи начального значения для обыкновенного дифференциального уравнения с сингулярным возмущением и кусочно-постоянным аргументом

В статье исследуется задача начального значения для обыкновенного дифференциального уравнения с возмущением малого параметра и кусочно-постоянным аргументом в обобщенном виде. В соответствии с этим уравнением мы разрабатываем систему фундаментальных решений для однородного сингулярно возмущенного дифференциального уравнения, которое зависит от кусочно-постоянного аргумента. Выведем систему разностных уравнений, описывающую вектор $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i = \overline{1,p}$ компонентов решения. Установлено решение полученной системы разностных уравнений. Используя редукционный подход, мы получили аналитическую формулу для решения задачи начального значения для обыкновенного дифференциального уравнения с кусочно- постоянным аргументом в обобщенном виде, включающую малый параметр. Была выведена и доказана теорема, устанавливающая аналитическую формулу для решения. Конкретный пример задачи начального значения в рамках сингулярно возмущенного обыкновенного дифференциального уравнения, зависящего от кусочно- постоянного аргумента в обобщенной форме с малым параметром, соответствует задаче линейного гармонического осциллятора.

Ключевые слова: гармонический осциллятор, начальные функций, задача начального значения.

1 Introduction

We consider the Cauchy problem for a linear differential equation with a singular perturbed piecewise constant argument:

$$\varepsilon y''(t) + A(t)y'(t) + B(t)y(t) + C(t)y\left(\beta(t)\right) = F(t) \tag{1}$$

$$y(0,\varepsilon) = d_0,\ y'(0,\varepsilon) = d_1, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, d_0, d_1 are given constants.

If $t \in [\theta_i, \theta_{i+1})$, $i = \overline{1,p}$, $0 < \theta_1 < \theta_2 < \ldots < \theta_p < T$ a piecewise constant function is defined as $\beta(t) = \theta_i$.

Let the following conditions hold true:

(C1) The roles $A(t), B(t), C(t)$ and $F(t)$ are continuously differentiable in the span $0 \leq t \leq T$.

(C2) $A(t) > 0$, $0 \leq t \leq T$.

Differential equations with piecewise constant argument (EPCA) were proposed for investigations in [1, 2] by founders of the theory, K. Cook, S. Busenberg, J. Wiener, and S. Shah. They are named as differential EPCA. In the last three decades, many interesting results have been obtained, and applications have been realized in this theory. Existence and uniqueness of solutions, oscillations and stability, integral manifolds and periodic solutions, and many other questions of the theory have been intensively discussed. The founders proposed that the method of investigation of these equations is based on a reduction to discrete systems. That is, only values of solutions at moments, which are integers or multiples of integers, were discussed. Moreover, systems must be linear with respect to the values of solutions, if the argument is not deviated. It reduces the theoretical depth of the investigations as well as the number of real-world problems, which can be modeled by using these equations.Through the application of reduction techniques, we derive an analytical formula for solving the IVP for this ODE with a piecewise constant argument and small parameter.

2 Supplementary Resources

This section presents the fundamental system of solutions and discusses the initial functions, along with demonstrating their crucial properties relevant to our research.

2.1 A fundamental system of solutions

According to equation (1), the homogeneous differential equation has the form

$$\varepsilon y''(t) + A(t)y'(t) + B(t)y(t) = 0 \tag{3}$$

Lemma 1 *Under that stipulations (C1) and (C2) hold, the system of fundamental solutions for the homogeneous equation* (3) *can be explicitly expressed in the form denoted by*

$$\begin{gathered} y_1^{(j)}(t,\varepsilon) = y_{10}^{(j)}(t) + O(\varepsilon),\ j = 0,1 \\ y_2^{(j)}(t,\varepsilon) = \frac{1}{\varepsilon^j} e^{-\frac{1}{\varepsilon}\int_{\theta_i}^t A(x)dx} \left((-A(t))^j \cdot y_{20}(t) + O(\varepsilon)\right),\ j = 0,1 \end{gathered} \tag{4}$$

in the interval $\theta_i \le t < \theta_{i+1},\ i = \overline{0,p}$, *where* $y_{10}(t) = \exp\left(-\int_{\theta_i}^t \frac{B(x)}{A(x)}dx\right)$ *and* $y_{20}(t) = \frac{A(\theta_i)}{A(t)},\ \ i = \overline{0,p},\ \ \theta_0 = 0,\ \ \theta_{p+1} = T$.

Proof. First of all, we will consider the $[0,\theta_1)$ interval. The system of fundamental solutions of the homogeneous equation (3) is searched as follows:

$$\begin{gathered} y_1(t,\varepsilon) = y_{10}(t) + \varepsilon y_{11}(t) + \varepsilon^2 y_{12}(t) + \dots \\ y_2(t,\varepsilon) = e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\left(y_{20}(t) + \varepsilon y_{21}(t) + \varepsilon^2 y_{22}(t) + \dots\right), \end{gathered} \tag{5}$$

where $y_{ik}(t),\ i = 1,2,\ \ k = 0,1,2,\dots$ are unknown coefficients.

To determine these coefficients, we take the first and second derivatives of the functions (5):

$$\begin{aligned}
&y_1'(t,\varepsilon) = y_{10}'(t) + \varepsilon y_{11}'(t) + \varepsilon^2 y_{12}'(t) + \dots\\
&y_2'(t,\varepsilon) = \tfrac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\Big(- A(t)y_{20}(t) + \varepsilon\big(y_{20}'(t)\\
&\qquad -A(t)y_{21}(t)\big) + O\left(\varepsilon^2\right)\Big),\\
&y_1''(t,\varepsilon) = y_{10}''(t) + \varepsilon y_{11}''(t) + \varepsilon^2 y_{12}''(t) + \dots\\
&y_2''(t,\varepsilon) = \tfrac{1}{\varepsilon^2} e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\Big(A^2(t)y_{20}(t) + \varepsilon\big(A^2(t)y_{21}(t)\\
&\qquad -2A(t)y_{20}'(t) - A'(t)y_{20}(t)\big) + O\left(\varepsilon^2\right)\Big).
\end{aligned} \tag{6}$$

Substituting formulas (5), (6) into equation (3), equating the coefficients in front of the small parameter ε to the same degree, we get problems defining unknown coefficients $y_{10}(t)$, $y_{20}(t)$:

$$A(t)y_{10}'(t) + B(t)y_{10}(t) = 0,\ y_{10}(0) = 1. \tag{7}$$

The solution to the IVP (7) is determined as $y_{10}(t) = e^{-\int_0^t \frac{B(x)}{A(x)}dx}$.

$$\begin{cases} A(t)y_{20}'(t) + A'(t)y_{20}(t) = 0,\\ y_{20}(0) = 1. \end{cases} \tag{8}$$

The solution to the IVP (8) is determined as $y_{20}(t) = \frac{A(0)}{A(t)}$.

Continuing this process, we will prove in the interval $\theta_i \le t < \theta_{i+1}$, $i = \overline{1,p}$. Lemma 1 is proved.

2.2 The initial functions

Definition 1

$$\varepsilon K_l''(t,s,\varepsilon) + A(t)K_l'(t,s,\varepsilon) + B(t)K_l(t,s,\varepsilon) = 0,\ l = 1,2, \tag{9}$$

$$K_l^{(j)}(s,s,\varepsilon) = \delta_{l-1,j},\ l = 1,2,\ j = 0,1 \tag{10}$$

the functions $K_l(t,s,\varepsilon),;l = 1,2,$ defined for $\theta_i \le s \le t \le \theta_{i+1}$, $i = 0,\dots,p$, represent solutions to the problem described by equations (9) and (10), and are referred to as initial functions. And

$$K_l^{(q)}(t,s,\varepsilon) = \frac{W_{lt}^{(q)}(t,s,\varepsilon)}{W(s,\varepsilon)},\quad l = 1,2,\ q = 0,1. \tag{11}$$

Here, $\delta_{l-1,j}$ represents the Kronecker delta symbol. $W_l(t,s,\varepsilon)$ is a second-order determinant in which the $l-th$ row of the Wronskian $W(s,\varepsilon)$ is replaced by the set of fundamental solutions represented by (4).

Lemma 2 *Assuming stipulations (C1) and (C2) hold. Then the initial functions* $K_l^{(q)}(t,s,\varepsilon)$, $l=1,2$, $q=0,1$ *have the following asymptotic behavior when* $\varepsilon \to 0$ *on* $\theta_i \le t < \theta_{i+1}$, $i=\overline{0,p}$

$$K_1^{(q)}(t,s,\varepsilon) = \frac{y_{10}^{(q)}(t)}{y_{10}(s)} + \varepsilon^{1-q}\frac{(-A(t))^q y_{20}(t) y_{10}'(s)}{A(s)y_{10}(s)y_{20}(s)} e^{-\frac{1}{\varepsilon}\int_s^t A(x)dx}$$
$$+O\left(\varepsilon + \varepsilon^{2-q} e^{-\frac{1}{\varepsilon}\int_s^t A(x)dx}\right),$$
$$K_2^{(q)}(t,s,\varepsilon) = \varepsilon\Big[\frac{y_{10}^{(q)}(t)}{A(s)y_{10}(s)} - \frac{(-A(t))^q y_{20}(t)}{\varepsilon^q \cdot A(s) y_{20}(s)} e^{-\frac{1}{\varepsilon}\int_s^t A(x)dx}$$
$$+O\left(\varepsilon + \frac{1}{\varepsilon^{q-1}} e^{-\frac{1}{\varepsilon}\int_s^t A(x)dx}\right)\Big], q=0,1. \tag{12}$$

Proof. First of all,we consider the interval $[0,\theta_1)$. By leveraging the fundamental solutions (4), we derive the asymptotic behavior of $W(s,\varepsilon)$**:**

$$W(s,\varepsilon) = \begin{vmatrix} y_{10}(s)+O(\varepsilon) & e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(y_{20}(s)+O(\varepsilon)\right) \\ y_{10}'(s)+O(\varepsilon) & \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{20}(s)+O(\varepsilon)\right) \end{vmatrix}$$
$$= \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{10}(s)y_{20}(s)+O(\varepsilon)\right)$$
$$-e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(y_{10}'(s)y_{20}(s)+O(\varepsilon)\right)$$
$$= \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{10}(s)y_{20}(s)+O(\varepsilon)\right) \neq 0. \tag{13}$$

We calculate the asymptotics of the determinant $W_1^{(q)}(t,s,\varepsilon)$, $q=0,1$

$$W_1^{(q)}(t,s,\varepsilon) = \begin{vmatrix} y_{10}^{(q)}(t)+O(\varepsilon) & \frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\left((-A(t))^q y_{20}(t)+O(\varepsilon)\right) \\ y_{10}'(s)+O(\varepsilon) & \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{20}(s)+O(\varepsilon)\right) \end{vmatrix}$$
$$= \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(-A(s)y_{10}^{(q)}(t)y_{20}(s)+O(\varepsilon)\right)$$
$$-\frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\left((-A(t))^q y_{10}'(s)y_{20}(t)+O(\varepsilon)\right). \tag{14}$$

We calculate the asymptotics of the determinant $W_2^{(q)}(t,s,\varepsilon)$, $q=0,1$

$$W_2^{(q)}(t,s,\varepsilon) = \begin{vmatrix} y_{10}(s)+O(\varepsilon) & e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(y_{20}(s)+O(\varepsilon)\right) \\ y_{10}^{(q)}(t)+O(\varepsilon) & \frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\left((-A(t))^q y_{20}(t)+O(\varepsilon)\right) \end{vmatrix}$$
$$= \frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_0^t A(x)dx}\left((-A(t))^q y_{10}(s)y_{20}(t)+O(\varepsilon)\right)$$
$$-e^{-\frac{1}{\varepsilon}\int_0^s A(x)dx}\left(y_{10}^{(q)}(t)y_{20}(s)+O(\varepsilon)\right). \tag{15}$$

Substituting the formulas (13)-(15) into (11) we obtain (12).

Similarly, we also prove for the interval $\theta_i \le t < \theta_{i+1}, \quad i=\overline{1,p}$. By (4) we show the asymptotic behavior of $W(s,\varepsilon)$:

$$W(s,\varepsilon)=\begin{vmatrix} y_{10}(s)+O(\varepsilon)e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s}A(x)dx}\left(y_{20}(s)+O(\varepsilon)\right) \\ y_{10}'(s)+O(\varepsilon)\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s}A(x)dx}\left(-A(s)y_{20}(s)+O(\varepsilon)\right) \end{vmatrix}$$

$$=\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s}A(x)dx}\left(-A(s)y_{10}(s)y_{20}(s)+O(\varepsilon)\right) \tag{16}$$

$$-e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s}A(x)dx}\left(y_{10}'(s)y_{20}(s)+O(\varepsilon)\right)$$

$$=\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s}A(x)dx}\left(-A(s)y_{10}(s)y_{20}(s)+O(\varepsilon)\right)\neq 0.$$

We calculate the asymptotics of the determinant $W_1^{(q)}(t,s,\varepsilon)$,$q=0,1$:

$$W_1^{(q)}(t,s,\varepsilon)=\begin{vmatrix} y_{10}^{(q)}(t)+O(\varepsilon)\frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{t}A(x)dx}\left((-A(t))^q y_{20}(t)+O(\varepsilon)\right) \\ y_{10}'(s)+O(\varepsilon)\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s}A(x)dx}\left(-A(s)y_{20}(s)+O(\varepsilon)\right) \end{vmatrix}$$

$$=\frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s}A(x)dx}\left(-A(s)y_{10}^{(q)}(t)y_{20}(s)+O(\varepsilon)\right) \tag{17}$$

$$-\frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{t}A(x)dx}\left((-A(t))^q y_{10}'(s)y_{20}(t)+O(\varepsilon)\right).$$

We calculate the asymptotics of the determinant $W_2^{(q)}(t,s,\varepsilon)$,$q=0,1$:

$$W_2^{(q)}(t,s,\varepsilon)=\begin{vmatrix} y_{10}(s)+O(\varepsilon)e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s}A(x)dx}\left(y_{20}(s)+O(\varepsilon)\right) \\ y_{10}^{(q)}(t)+O(\varepsilon)\frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{t}A(x)dx}\left((-A(t))^q y_{20}(t)+O(\varepsilon)\right) \end{vmatrix}$$

$$=\frac{1}{\varepsilon^q}e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{t}A(x)dx}\left((-A(t))^q y_{10}(s)y_{20}(t)+O(\varepsilon)\right) \tag{18}$$

$$-e^{-\frac{1}{\varepsilon}\int_{\theta_i}^{s}A(x)dx}\left(y_{10}^{(q)}(t)y_{20}(s)+O(\varepsilon)\right).$$

Putting the formulas(16)-(18) into (11) we obtain (12).Lemma 2 is proven.

3 Main results and methods

3.1 Derivation of analytical solutions

Problem (1)–(2) has the following form for $t \in [0, \theta_1)$:

$$\begin{cases} \varepsilon y''(t) + A(t)y'(t) + B(t)y(t) = F(t) - C(t)d_0, \\ y(0,\varepsilon) = d_0, \\ y'(0,\varepsilon) = d_1 \end{cases} \tag{19}$$

The solution to problem (19) is provided by the following analytical expression:

$$\begin{aligned} y(t,\varepsilon) &= d_0 K_1(t,0,\varepsilon) + d_1 K_2(t,0,\varepsilon) + \tfrac{1}{\varepsilon}\int_0^t K_2(t,s,\varepsilon)\left(F(s) - C(s)d_0\right) ds, \\ y'(t,\varepsilon) &= d_0 K_1'(t,0,\varepsilon) + d_1 K_2'(t,0,\varepsilon) + \tfrac{1}{\varepsilon}\int_0^t K_2'(t,s,\varepsilon)\left(F(s) - C(s)d_0\right) ds, \end{aligned} \tag{20}$$

where $K_l^{(q)}(t,s,\varepsilon), \quad l = 1,2, \quad q = 0,1$ are initial functions.

The Cauchy problem (1)-(2) has the following form for $t \in [\theta_i, \theta_{i+1}), \quad i = \overline{1,p}$:

$$\varepsilon y''(t) + A(t)y'(t) + B(t)y(t) = F(t) - C(t)y(\theta_i), \tag{21}$$

$$\begin{aligned} y(t,\varepsilon)\big|_{t=\theta_i} &= y(\theta_i), \\ y'(t,\varepsilon)\big|_{t=\theta_i} &= y'(\theta_i). \end{aligned} \tag{22}$$

Theorem 1 *Assuming stipulations (C1) and (C2) are gratified, the Cauchy problem defined by equations* (21) *and* (22) *possesses a unique solution over the interval* $t \in [\theta_i, \theta_{i+1}), \quad i = \overline{1,p}$ *, which is able to be formulated in the shape:*

$$\hat{y}(t,\varepsilon) = Q(t,\theta_i,\varepsilon)\hat{y}(\theta_i) + \hat{U}(t,\theta_i,\varepsilon), \ t \in [\theta_i, \ \theta_{i+1}), \ i = \overline{0,p}, \tag{23}$$

where

$$\hat{y}(t,\varepsilon) = \begin{pmatrix} y(t,\varepsilon) \\ y'(t,\varepsilon) \end{pmatrix},$$

$$Q(t,\theta_i,\varepsilon) = \begin{pmatrix} Q_1(t,\theta_i,\varepsilon) & Q_2(t,\theta_i,\varepsilon) \\ Q_1'(t,\theta_i,\varepsilon) & Q_2'(t,\theta_i,\varepsilon) \end{pmatrix}, \tag{24}$$

$$\hat{U}(t,\theta_i,\varepsilon) = \begin{pmatrix} U(t,\theta_i,\varepsilon) \\ U'(t,\theta_i,\varepsilon) \end{pmatrix},$$

where the roles $Q_1^{(q)}(t,\theta_i,\varepsilon), \ \ Q_2^{(q)}(t,\theta_i,\varepsilon), \ \ U^{(q)}(t,\theta_i,\varepsilon), \ q = 0,1$ *and* $\hat{y}(\theta_i)$ *are defined by*

$$Q_1^{(q)}(t,\theta_i,\varepsilon)=K_1^{(q)}(t,\theta_i,\varepsilon)-\tfrac{1}{\varepsilon}\int_{\theta_i}^{t}K_2^{(q)}(t,s,\varepsilon)C(s)ds,\quad q=0,1,$$

$$Q_2^{(q)}(t,\theta_i,\varepsilon)=K_2^{(q)}(t,\theta_i,\varepsilon),\quad q=0,1, \tag{25}$$

$$U^{(q)}(t,\theta_i,\varepsilon)=\tfrac{1}{\varepsilon}\int_{\theta_i}^{t}K_2^{(q)}(t,s,\varepsilon)F(s)ds,\quad q=0,1,$$

$$\begin{aligned}\hat{y}(\theta_i)=&\prod_{j=0}^{i-1}Q\left(\theta_{j+1},\theta_j,\varepsilon\right)\hat{y}(0,\varepsilon)+\sum_{l=1}^{i-1}\prod_{j=l}^{i-1}Q\left(\theta_{j+1},\theta_j,\varepsilon\right)\hat{U}\left(\theta_l,\theta_{l-1},\varepsilon\right)\\&+\hat{U}\left(\theta_i,\theta_{i-1},\varepsilon\right).\end{aligned} \tag{26}$$

Proof. To determine outcome of the issue (1)–(2) for $t\in[\theta_i,\theta_{i+1})$, $i=\overline{1,p}$, we change the variables as $s=t-\theta_i$, $t=\theta_i \quad\Rightarrow\quad s=0$ and as an result we obtain the problem

$$\left[\begin{array}{l}\varepsilon\frac{d^2y}{ds^2}+A(s)\frac{dy}{ds}+B(s)y(s)=F(s)-C(s)y(\theta_i),\\ \qquad y(s,\varepsilon)\big|_{s=0}=y(\theta_i),\quad \frac{dy}{ds}\Big|_{s=0}=y'\ (\theta_i).\end{array}\right. \tag{27}$$

As the problem type in (27) closely resembles the IVP described in (19), the solution to IVP (27) for all $s\in[0,\theta_1)$ is structured as follows:

$$\begin{aligned}y(s,\varepsilon)&=y(\theta_i)K_1(s,\theta_i,\varepsilon)+y'(\theta_i)K_2(s,\theta_i,\varepsilon)\\&\quad+\frac{1}{\varepsilon}\int_0^s K_2(s,p,\varepsilon)\left(F(p)-C(p)y(\theta_i)\right)dp,\\ y'(s,\varepsilon)&=y(\theta_i)K_1'(s,\theta_i,\varepsilon)+y'(\theta_i)K_2'(s,\theta_i,\varepsilon)\\&\quad+\frac{1}{\varepsilon}\int_0^s K_2'(s,p,\varepsilon)\left(F(p)-C(p)y(\theta_i)\right)dp.\end{aligned} \tag{28}$$

By changing variable from s to t in (28) we derive outcome to the issue (21)–(22) for $t\in[\theta_i,\theta_{i+1})$, $i=\overline{1,p}$ in the following form

$$\begin{aligned}y(t,\varepsilon)=&\left(K_1(t,\theta_i,\varepsilon)-\frac{1}{\varepsilon}\int_{\theta_i}^{t}K_2(t,s,\varepsilon)C(s)ds\right)y(\theta_i)\\&+K_2(t,\theta_i,\varepsilon)y'(\theta_i)+\frac{1}{\varepsilon}\int_{\theta_i}^{t}K_2(t,s,\varepsilon)F(s)ds,\end{aligned} \tag{29}$$

$$\begin{aligned}y'(t,\varepsilon)=&\left(K_1'(t,\theta_i,\varepsilon)-\frac{1}{\varepsilon}\int_{\theta_i}^{t}K_2'(t,s,\varepsilon)C(s)ds\right)y(\theta_i)\\&+K_2'(t,\theta_i,\varepsilon)y'(\theta_i)+\frac{1}{\varepsilon}\int_{\theta_i}^{t}K_2'(t,s,\varepsilon)F(s)ds.\end{aligned} \tag{30}$$

In light of (24),(25) and using (29),(30) we obtain (23).

To find the unknown vector $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i=\overline{1,p}$ we put $t=\theta_{i+1}$ into (29) and (30), then we obtain the following difference system of equations for $\begin{pmatrix} y(\theta_i) \\ y'(\theta_i) \end{pmatrix}$, $i=\overline{1,p}$:

$$\begin{aligned} y(\theta_{i+1},\varepsilon) &= \left(K_1(\theta_{i+1},\theta_i,\varepsilon) - \frac{1}{\varepsilon}\int_{\theta_i}^{\theta_{i+1}} K_2(\theta_{i+1},s,\varepsilon)C(s)ds\right) y(\theta_i) \\ &\quad + K_2(\theta_{i+1},\theta_i,\varepsilon)y'(\theta_i) + \frac{1}{\varepsilon}\int_{\theta_i}^{\theta_{i+1}} K_2(\theta_{i+1},s,\varepsilon)F(s)ds, \\ y'(\theta_{i+1},\varepsilon) &= \left(K_1'(\theta_{i+1},\theta_i,\varepsilon) - \frac{1}{\varepsilon}\int_{\theta_i}^{\theta_{i+1}} K_2'(\theta_{i+1},s,\varepsilon)C(s)ds\right) y(\theta_i) \\ &\quad + K_2'(\theta_{i+1},\theta_i,\varepsilon)y'(\theta_i) + \frac{1}{\varepsilon}\int_{\theta_i}^{\theta_{i+1}} K_2'(\theta_{i+1},s,\varepsilon)F(s)ds. \end{aligned} \tag{31}$$

By using the formulas (24),(25) we reduce the system (31) into the following vector form:

$$\hat{y}(\theta_{i+1},\varepsilon) = N\left(\theta_{i+1},\theta_i,\varepsilon\right)\hat{y}(\theta_i) + \hat{P}\left(\theta_{i+1},\theta_i,\varepsilon\right). \tag{32}$$

In view of (32) we obtain

$$\hat{y}(\theta_i,\varepsilon) = N\left(\theta_i,\theta_{i-1},\varepsilon\right)\hat{y}(\theta_{i-1}) + \hat{P}\left(\theta_i,\theta_{i-1},\varepsilon\right). \tag{33}$$

Substituting (33) into (32), we get

$$\begin{aligned} \hat{y}(\theta_{i+1},\varepsilon) &= N\left(\theta_{i+1},\theta_i,\varepsilon\right)\cdot N\left(\theta_i,\theta_{i-1},\varepsilon\right)\hat{y}(\theta_{i-1}) \\ &\quad + N\left(\theta_{i+1},\theta_i,\varepsilon\right)\hat{P}\left(\theta_i,\theta_{i-1},\varepsilon\right) + \hat{P}\left(\theta_{i+1},\theta_i,\varepsilon\right). \end{aligned} \tag{34}$$

Using (32) we have

$$\hat{y}(\theta_{i-1},\varepsilon) = N\left(\theta_{i-1},\theta_{i-2},\varepsilon\right)\hat{y}(\theta_{i-2}) + \hat{P}\left(\theta_{i-1},\theta_{i-2},\varepsilon\right). \tag{35}$$

Putting (35) into (34)we obtain

$$\begin{aligned} \hat{y}(\theta_{i+1},\varepsilon) &= N\left(\theta_{i+1},\theta_i,\varepsilon\right)N\left(\theta_i,\theta_{i-1},\varepsilon\right)N\left(\theta_{i-1},\theta_{i-2},\varepsilon\right)\hat{y}(\theta_{i-2}) \\ &\quad + N\left(\theta_{i+1},\theta_i,\varepsilon\right)N\left(\theta_i,\theta_{i-1},\varepsilon\right)\hat{P}\left(\theta_{i-1},\theta_{i-2},\varepsilon\right) \\ &\quad + N\left(\theta_{i+1},\theta_i,\varepsilon\right)\hat{P}\left(\theta_i,\theta_{i-1},\varepsilon\right) + \hat{P}\left(\theta_{i+1},\theta_i,\varepsilon\right). \end{aligned} \tag{36}$$

Repeating this process till $i+1$ we get

$$\begin{aligned} \hat{y}(\theta_{i+1},\varepsilon) &= \prod_{j=0}^{i} N\left(\theta_{j+1},\theta_j,\varepsilon\right)\hat{y}(0,\varepsilon) \\ &\quad + \sum_{l=1}^{i}\prod_{j=l}^{i} N\left(\theta_{j+1},\theta_j,\varepsilon\right)\hat{P}\left(\theta_l,\theta_{l-1},\varepsilon\right) + \hat{P}\left(\theta_{i+1},\theta_i,\varepsilon\right). \end{aligned} \tag{37}$$

By (37) we obtain (26). Theorem 1 is proven.

Theorem 2 *Assuming conditions (C1) and (C2) hold true, the Cauchy problem defined by equations* (1) *and* (2) *possesses a unique solution over the interval* $[0, T]$*, which can be expressed as follows:*

$$\hat{y}(t,\varepsilon) = \begin{cases} Q(t,0,\varepsilon)\hat{y}(0) + \hat{U}(t,0,\varepsilon), & t \in [0,\theta_1)\,, \\ Q(t,\theta_1,\varepsilon)\hat{y}(\theta_1) + \hat{U}(t,\theta_1,\varepsilon), & t \in [\theta_1,\theta_2)\,, \\ \dots \\ Q(t,\theta_p,\varepsilon)\hat{y}(\theta_p) + \hat{U}(t,\theta_p,\varepsilon), & t \in [\theta_p, T]\,. \end{cases} \tag{38}$$

Here, $\hat{y}(t,\varepsilon)$, $\hat{U}(t,\theta_i,\varepsilon)$, $i = 0,\dots,p$ *are vector functions, and* $Q(t,\theta_i,\varepsilon),;i = 0,\dots,p$ *is a* 2×2 *matrix with elements defined by equations* (24) *and* (25), $\hat{y}(0) = \begin{pmatrix} d_0 \\ d_1 \end{pmatrix}$ *and* $\hat{y}(\theta_i)$, $i = \overline{1,p}$ *is vector functions with the elements which defined by* (26)*.*

4 Conclusion

In this study, we explored the IVP associated with a singularly perturbed ODE that depends on a piecewise constant argument in a generalized form with a small parameter. Employing a reduction approach, we derived an analytical solution formula for this IVP.

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ASYMPTOTIC EXPANSION OF THE SOLUTION FOR SINGULAR PERTURBED LINEAR IMPULSIVE SYSTEMS

In this study, a singularly perturbed linear impulsive system with singularly perturbed impulses is considered. Many books discuss different types of singular perturbation problems. In the present work, an impulse system is considered in which a small parameter is introduced into the impulse equation. This is the main novelty of our study, since other works [25] have only considered a small parameter in the differential equation. A necessary condition is also established to prevent the impulse function from bloating as the parameter approaches zero. As a result, the notion of singularity for discontinuous dynamics is greatly extended. An asymptotic expansion of the solution of a singularly perturbed initial problem with an arbitrary degree of accuracy for a small parameter is constructed. A theorem for estimating the residual term of the asymptotic expansion is formulated, which estimates the difference between the exact solution and its approximation. The results extend those of [32], which formulates an analogue of Tikhonov's limit transition theorem. The theoretical results are confirmed by a modelling example.
Key words: singular perturbation, differential equations with singular impulses, small parameter.

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Сингулярлы ауытқыған сызықты импульсті жүйе шешімінің асимптотикалық жіктелуі

Бұл мақалада импульсті бөлігіде сингулярлы ауытқыған сызықты импульстік жүйе қарастырылады. Көптеген жұмыстарда сингулярлы ауытқыған әртүрлі типтегі есептер қаралды. [25] кітапта және кейбір басқа мақалаларда тек дифференциалдық теңдеуінде кіші параметрі бар импульстік жүйелер қарастырылды. Бұл жұмыста импульсті теңдеуіне кіші параметр енгізілді. Бұл осы зерттеудің басты жаңалығы. Сондай ақ, кіші параметр нөлге ұмтылған кезде импульстік функцияның шексіздікке кетуін болдырмау үшін қажетті қосымша шарт қойылды. Нәтижесінде үзіліссіз динамика теориясы үшін сингулярлық тұжырымдамасы айтарлықтай кеңейтілді. Бұл жұмыста сингулярлы ауытқыған бастапқы есеп шешімінің кез келген дәлдіктегі асимптотикалық жіктелуі құрылды. Асимптотикалық жіктелудің қалдық мүшесін бағалау теоремасы тұжырымдалды және ол нақты шешім мен оның жуықталған шешімінің айырымын бағалайды. Алынған нәтижелер Тихоновтың шектік көшу теоремасы аналогын тұжырымдайтын [32] жұмыстың нәтижелерін кеңейтеді. Теориялық нәтижені растайтын нақты мысал графикалық көрнекілікпен келтірілді.
Түйін сөздер: сингулярлы ауытқу, сингулярлы импульсті дифференциалдық теңдеулер, кіші параметр.

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Асимптотическое разложение решения задачи для сингулярно возмущенных линейных импульсных систем

В статье рассматривается сингулярно возмущенная линейная импульсная система, в которой импульсы также сингулярно возмущены. Во многих книгах обсуждались различные типы задач с сингулярными возмущениями. В [25] и в некоторых других статьях были изучены импульсные системы с малым параметром, присутствующим только в дифференциальных уравнениях. В настоящей работе был введен малый параметр также в уравнение импульса. Это представляет собой главную новизну данного исследования. Более того, было установлено необходимое условие, предотвращающее коллапс импульсной функции при уменьшении параметра до нуля. Этот результат значительно расширяет понятие сингулярности в разрывной динамике. В настоящей работе построено асимптотическое разложение решения сингулярно возмущенной начальной задачи с произвольной степенью точности по малому параметру. Сформулирована теорема об оценке остаточного члена асимптотического разложения, что показывает оценку разности между точным решением и его приближенным решением. Эти результаты расширяют результаты работы [32], в которой сформулирован аналог теоремы Тихонова о предельном переходе. Приведен пример с моделированием, подтверждающий теоретический результат.

Ключевые слова: сингулярное возмущение, дифференциальные уравнения с сингулярными импульсами, малый параметр.

1 Introduction

Singularly perturbed equations have found extensive application as mathematical representations for various phenomena in physics, chemical kinetics [1], mathematical biology [2], hydrodynamics [3], among others. Moreover, these equations are commonly encountered in the exploration of practical engineering and technological challenges [2–5]. A singular wave arises in case when a sudden force is added, for example, an earthquake might lead to a catastrophic tsunami wave [6], a sudden temperature shock might also lead to a thermal tsunami for a porous material [7], and a singular nonlinear oscillator behaves extremely miraculously [8]. Now, the singular wave travelling becomes a hot topic in mathematics [9], especially the quasi-periodic bifurcations [10], singular dissipations [11]. Due to the parameter dependence, the solutions to these problems exhibit non-uniform behavior over time as the parameters approach zero. Many authors are now actively studying singularly perturbed differential equations. There are effective asymptotic methods for singular perturbation problems that allow the construction of uniform approximations with any desired accuracy. The boundary function method is one of them [12]. This method can be used to solve a singularly perturbed problem when the Tikhonov theorem holds in part of the domain.

In this study, we will use the boundary function method to perform an analysis of an impulsive system. The suggested model with singular impulsive can be investigated by developing homotopy perturbation method [13]. Impulsive differential equations play a significant role in multiple scientific fields, including physics, biology, medicine [16], engineering [14] and chemistry [15]. They provide a more accurate representation of certain natural phenomena than ordinary differential equations. Impulsive equations are particularly important for controlling chaos and bifurcation in engineering systems, modelling epidemic scenarios with impulsive births [17], and managing complex dynamic systems [18]. Some of these systems are impulsive in nature and can be affected by small parameters, especially singular ones [19, 20]. However, solving an impulsive differential equation with a singular perturbation is a very complicated task, leading to a lack of research in this particular field. Impulse effects occur in various evolutionary processes that are characterized by abrupt changes of states [21]. They are also present in many systems along with singular perturbations [22–24].

Many papers have discussed different types of singular perturbation problems [12, 25–27]. Consider the following singularly perturbed differential equation

$$\varepsilon z' = f(z, y, t),$$
$$y' = g(z, y, t),$$

where ε represents a small positive real number. In the literature, the result that follows from this equation is known as the Tikhonov's theorem [26, 28, 29]. Bainov and Kovachev [25] were the first to extend the impulsive analogue of Tikhonov's theorem for the system in the form of

$$\varepsilon z' = f(z, y, t), \quad \Delta z|_{t=t_i} = I_i(y(t_i)),$$
$$y' = g(z, y, t), \quad \Delta y|_{t=t_i} = J_i(y(t_i)),$$

where $0 < t_1 < t_2 < ... < t_p < T$ and $i = 1, 2, ..., p$. It is important to note that only the differential equation in their problem has a perturbation singularity. Akhmet and Çağ [30–32] were the first to consider differential equations with singular impulses in addition to differential equations. They presented the following problem

$$\varepsilon z' = f(z, y, t), \qquad y' = g(z, y, t), \tag{1a}$$
$$\varepsilon \Delta z|_{t=\theta_i} = I(z, y, \varepsilon), \quad \Delta y|_{t=\eta_j} = J(z, y), \tag{1b}$$

where z, f and I are m -dimensional vector valued functions, y, g and J are n - dimensional vector valued functions. The impulse system consists of differential equations (1a) and impulse equations (1b). In addition, for the impulse function, the following condition

$$\lim_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} \frac{I(z, y, \varepsilon)}{\varepsilon} = I_0 \neq 0, \tag{$*$}$$

was used, which prevents the blow-up of the impulse function when the parameter approaches zero, where $\overline{y} = \overline{y}(\theta_i)$ representing the values for each impulse moment at $t = \theta_i, i = 1, 2, ..., p$. The main novelty of [32] is to extend Tikhonov's theorem in such a way that in the system (1) the impulse function has small parameter. The singularity of the impulse part of the system is analysed using perturbation theory methods. In [32], the behaviour of solutions in a singularly perturbed system is investigated, differentiating between single-layer and multi-layer dependence on condition ($*$). The results show that the transition to the limit for $y(t, \varepsilon)$ is uniform over the entire interval $0 \leq t \leq T$. However, the transition to the limit for $z(t, \varepsilon)$ is not uniform over the entire interval $0 \leq t \leq T$, but only within the subintervals $\delta \leq t \leq \theta_i, i = 1, 2, ..., p$ for $\delta > 0$, excluding the boundary layers.

The theorems presented in the paper [32] do not provide the precise order of accuracy for the asymptotic approximation $\overline{y}(t)$ for the solution $y(t, \varepsilon)$ in the interval $0 \leq t \leq T$ and $\overline{z}(t)$ for $z(t, \varepsilon)$ outside the boundary layer. Our goal is to construct complete asymptotic expansions with higher degree of accuracy for solutions of systems with singularly perturbed impulses [33], [34]. In this study, we focus on singularly perturbed differential equations with singular impulses and construct a uniform asymptotic approximation of the solution that is valid over the entire interval $0 \leq t \leq T$, using the method of boundary functions.

2 Main Result

In this part of the paper, we consider the inhomogeneous linear differential system where impulses are singularly perturbed. Let us consider the following system

$$\begin{aligned} \varepsilon\frac{dz}{dt} &= A_1(t)z + B_1(t)y + \varepsilon f_1(t), \\ \frac{dy}{dt} &= A_2(t)z + B_2(t)y + f_2(t), \end{aligned} \tag{2}$$

and

$$\begin{aligned} \varepsilon\Delta z|_{t=\theta_i} &= C_1(\theta_i)z + C_2(\theta_i)y + \varepsilon I_1(\theta_i), \\ \Delta y|_{t=\theta_i} &= C_3(\theta_i)y + I_2(\theta_i) \end{aligned} \tag{3}$$

with initial condition

$$z(0,\varepsilon) = z^0, \quad y(0,\varepsilon) = y^0, \tag{4}$$

where $\varepsilon > 0$ is a small positive real number, z^0 and y^0 are assumed to be independent of ε, $0 < \theta_1 < \theta_2 < ... < \theta_p < T, \theta_i, i = 1,2,...,p$, are distinct discontinuity moments in $(0,T)$.

The solution of the problem (2)-(4) as $\varepsilon \to 0$ tends to solve the degenerate system

$$\begin{aligned} 0 &= A_1(t)\overline{z} + B_1(t)\overline{y}, & 0 &= C_1(\theta_i)\overline{z} + C_2(\theta_i)\overline{y}, \\ \frac{d\overline{y}}{dt} &= A_2(t)\overline{z} + B_2(t)\overline{y} + f_2(t), & \Delta\overline{y}|_{t=\theta_i} &= C_3(\theta_i)\overline{y} + I_2(\theta_i), \end{aligned}$$

with initial condition

$$\overline{y}(0) = y^0.$$

We need the following conditions:

$(C1)$ The functions $A_i(t), B_i(t), f_i(t), I_i(t), i = 1,2$, and $C_i(t), i = 1,2,3$, are differentiable infinitely many times on the segment $0 \le t \le T$.

$(C2)$ $A_1(t) < 0, 0 \le t \le T$.

$(C3)$ $1 + \dfrac{C_1(\theta_i)}{\varepsilon} \ne 0, 1 + C_3(\theta_i) \ne 0$.

$(C4)$ $\lim\limits_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} \dfrac{C_1(\theta_i)z + C_2(\theta_i)y + \varepsilon I_1(\theta_i)}{\varepsilon} = I_0 \ne 0$

where $\overline{y} = \overline{y}(\theta_i)-$ are the values for each impulse moment at the points $t = \theta_i, i = 1,2,...,p$.

We will look for the formal asymptotic expansion of the solution of (2)-(4) in the form

$$\begin{aligned} z(t,\varepsilon) &= \overline{z}(t,\varepsilon) + \omega^{(i)}(\tau_i,\varepsilon), \tau_i = \frac{t-\theta_i}{\varepsilon}, \theta_i < t \le \theta_{i+1}, \\ y(t,\varepsilon) &= \overline{y}(t,\varepsilon) + \varepsilon\nu^{(i)}(\tau_i,\varepsilon), \theta_0 = 0, \theta_{p+1} = T, i = \overline{0,p}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \overline{z}(t,\varepsilon) &= \sum_{k=0}^{\infty}\varepsilon^k\overline{z}_k(t), & \overline{y}(t,\varepsilon) &= \sum_{k=0}^{\infty}\varepsilon^k\overline{y}_k(t), \\ \omega^{(i)}(\tau_i,\varepsilon) &= \sum_{k=0}^{\infty}\varepsilon^k\omega_k^{(i)}(\tau_i), & \nu^{(i)}(\tau_i,\varepsilon) &= \sum_{k=0}^{\infty}\varepsilon^k\nu_k^{(i)}(\tau_i). \end{aligned} \tag{6}$$

The coefficients $\omega_k^{(i)}(\tau_i)$ and $\nu_k^{(i)}(\tau_i)$ in the expansions (6) are called boundary functions. The additional conditions are imposed on them:

$$\omega_k^{(i)}(\infty)=0,\ \nu_k^{(i)}(\infty)=0,\quad (i=1,2,...,p.) \tag{7}$$

Substituting the series (5) into the system (2), we obtain

$$\varepsilon(\overline{z}'(t,\varepsilon)+\frac{1}{\varepsilon}\dot{\omega}^{(i)}(\tau_i,\varepsilon))=A_1(t)(\overline{z}(t,\varepsilon)+\omega^{(i)}(\tau_i,\varepsilon))+B_1(t)(\overline{y}(t,\varepsilon)+\varepsilon\nu^{(i)}(\tau_i,\varepsilon))+\varepsilon f_1(t),$$
$$\overline{y}'(t,\varepsilon)+\dot{\nu}^{(i)}(\tau_i,\varepsilon)=A_2(t)(\overline{z}(t,\varepsilon)+\omega^{(i)}(\tau_i,\varepsilon))+B_2(t)(\overline{y}(t,\varepsilon)+\varepsilon\nu^{(i)}(\tau_i,\varepsilon))+f_2(t).$$

The following two equations (8), (9) are obtained to determine the coefficients of the regular and boundary layer parts of the series (6) from the last equations.

$$\begin{aligned}\varepsilon\overline{z}'(t,\varepsilon)&=A_1(t)\overline{z}(t,\varepsilon)+B_1(t)\overline{y}(t,\varepsilon)+\varepsilon f_1(t),\\ \overline{y}'(t,\varepsilon)&=A_2(t)\overline{z}(t,\varepsilon)+B_2(t)\overline{y}(t,\varepsilon)+f_2(t),\end{aligned} \tag{8}$$

and

$$\begin{aligned}\dot{\omega}^{(i)}(\tau_i,\varepsilon)&=A_1(\varepsilon\tau_i+\theta_i)\omega^{(i)}(\tau_i,\varepsilon)+\varepsilon B_1(\varepsilon\tau_i+\theta_i)\nu^{(i)}(\tau_i,\varepsilon),\\ \dot{\nu}^{(i)}(\tau_i,\varepsilon)&=A_2(\varepsilon\tau_i+\theta_i)\omega^{(i)}(\tau_i,\varepsilon)+\varepsilon B_2(\varepsilon\tau_i+\theta_i)\nu^{(i)}(\tau_i,\varepsilon).\end{aligned} \tag{9}$$

Now, represent $A_i(\varepsilon\tau_i+\theta_i), B_i(\varepsilon\tau_i+\theta_i), i=1,2$, in the form of power series in ε,

$$A_i(\varepsilon\tau_i+\theta_i)=A_i(\theta_i)+A_i'(\theta_i)\varepsilon\tau_i+A_i''(\theta_i)\frac{(\varepsilon\tau_i)^2}{2!}+...,$$
$$B_i(\varepsilon\tau_i+\theta_i)=B_i(\theta_i)+B_i'(\theta_i)\varepsilon\tau_i+B_i''(\theta_i)\frac{(\varepsilon\tau_i)^2}{2!}+....$$

In both parts of equations (8), (9), the coefficients are equated according to the powers of ε, we obtain a sequence of ordinary differential equations for coefficients of the expansions in (6).

$$\begin{aligned}\varepsilon^0:0&=A_1(t)\overline{z}_0(t)+B_1(t)\overline{y}_0(t),\\ \overline{y}_0'(t)&=A_2(t)\overline{z}_0(t)+B_2(t)\overline{y}_0(t)+f_2(t),\end{aligned} \tag{10}$$

$$\begin{aligned}\varepsilon^1:\overline{z}_0'(t)&=A_1(t)\overline{z}_1(t)+B_1(t)\overline{y}_1(t)+f_1(t),\\ \overline{y}_1'(t)&=A_2(t)\overline{z}_1(t)+B_2(t)\overline{y}_1(t),\end{aligned} \tag{11}$$

$$\begin{aligned}\varepsilon^k:\overline{z}_{k-1}'(t)&=A_1(t)\overline{z}_k(t)+B_1(t)\overline{y}_k(t),k\geq 2,\\ \overline{y}_k'(t)&=A_2(t)\overline{z}_k(t)+B_2(t)\overline{y}_k(t),\end{aligned} \tag{12}$$

and

$$\begin{aligned}\varepsilon^0:\dot{\omega}_0^{(i)}(\tau_i)&=A_1(\theta_i)\omega_0^{(i)}(\tau_i),\\ \dot{\nu}_0^{(i)}(\tau_i)&=A_2(\theta_i)\omega_0^{(i)}(\tau_i),\end{aligned} \tag{13}$$

$$\begin{aligned}&\varepsilon^k : \dot{\omega}_k^{(i)}(\tau_i) - A_1(\theta_i)\omega_k^{(i)}(\tau_i) = \Gamma_k(\tau_i),\\ &\dot{\nu}_k^{(i)}(\tau_i) - A_2(\theta_i)\omega_k^{(i)}(\tau_i) = \Theta_k(\tau_i),\end{aligned} \tag{14}$$

where functions $\Gamma_k(\tau_i)$ and $\Theta_k(\tau_i)$ are expressed recursively by $\omega_j^{(i)}(\tau_i)$ and $\nu_j^{(i)}(\tau_i)$ with $j < k$.

Consider the interval $t \in [0, \theta_1]$. For the determination of the expansion terms in (6) from equations (8), (12), it is necessary to have the initial conditions.

In order to determine the expansion terms in (6) from equations (8), (12), the initial conditions must be set. In the initial conditions (4) substitute the series (5).

$$\begin{aligned}&\overline{z}_0(0) + \varepsilon\overline{z}_1(0) + ... + \omega_0^{(0)}(0) + \varepsilon\omega_1^{(0)}(0) = z^0,\\ &\overline{y}_0(0) + \varepsilon\overline{y}_1(0) + ... + \varepsilon\nu_0^{(0)}(0) + \varepsilon^2\nu_1^{(0)}(0) = y^0.\end{aligned} \tag{15}$$

In both parts of the equations, equate the coefficients according to powers of ε,

$$\begin{aligned}&\varepsilon^0 : \overline{z}_0(0) + \omega_0^{(0)}(0) = z^0,\\ &\overline{y}_0(0) = y^0,\end{aligned} \tag{16}$$

$$\begin{aligned}&\varepsilon^k : \overline{z}_k(0) + \omega_k^{(0)}(0) = 0,\\ &\overline{y}_k(0) + \nu_{k-1}^{(0)}(0) = 0.\end{aligned} \tag{17}$$

For the leading term $\overline{z}_0(t), \overline{y}_0(t)$ of the regular part of the approximation obtain the systems

$$\begin{aligned}&0 = A_1(t)\overline{z}_0(t) + B_1(t)\overline{y}_0(t),\\ &\overline{y}_0'(t) = A_2(t)\overline{z}_0(t) + B_2(t)\overline{y}_0(t) + f_2(t), \quad \overline{y}_0(0) = y^0,\end{aligned}$$

which obviously coincide with the degenerate system. To find $\omega_0^{(0)}(\tau_0)$, solve the equation

$$\dot{\omega}_0^{(0)}(\tau_0) = A_1(0)\omega_0^{(0)}(\tau_0)$$

with initial condition

$$\omega_0^{(0)}(0) = z^0 - \overline{z}_0(0).$$

Using the second equation in (13) and formula (7), obtain

$$\nu_0^{(0)}(0) = \frac{A_2(0)}{A_1(0)}(z^0 - \overline{z}_0(0)). \tag{18}$$

It remains now to solve equation

$$\dot{\nu}_0^{(0)}(\tau_0) = A_2(0)\omega_0^{(0)}(\tau_0)$$

with initial condition (18).

In this way, all the terms of the approximation of order zero can be defined. Suppose we have already defined all terms up to order $k-1$. To determine the coefficients of ε^k for the approximation $\overline{z}_k(t)$ and $\overline{y}_k(t)$, apply the systems

$$\overline{z}'_{k-1}(t) = A_1(t)\overline{z}_k(t) + B_1(t)\overline{y}_k(t),$$
$$\overline{y}'_k(t) = A_2(t)\overline{z}_k(t) + B_2(t)\overline{y}_k(t), \;\; \overline{y}_k(0) = -\nu^{(0)}_{k-1}(0).$$

To find $\omega^{(0)}_k(\tau_0)$ it is needed to solve the following system

$$\dot{\omega}^{(0)}_k(\tau_0) - A_1(0)\omega^{(0)}_k(\tau_0) = \Gamma_k(\tau_0),$$
$$\omega^{(0)}_k(0) = -\overline{z}_k(0).$$

Using the equation (14) and the condition (7), obtain the following initial condition

$$\nu^{(0)}_k(0) = \frac{A_2(0)}{A_1(0)}\omega^{(0)}_k(0) + \int_0^\infty (\frac{A_2(0)}{A_1(0)}\Gamma_k(s) - \Theta_k(s))ds. \tag{19}$$

Solving the second equation of (14)

$$\dot{\nu}^{(0)}_k(\tau_0) - A_2(0)\omega^{(0)}_k(\tau_0) = \Theta_k(\tau_0)$$

with the initial condition (19), find $\nu^{(0)}_k(\tau_0)$.

Now consider the following interval $t \in (\theta_i, \theta_{i+1}], i = 1, 2, ..., p$. $\overline{z}(\theta_i, \varepsilon)$ and $\overline{y}(\theta_i, \varepsilon)$ are the initial values for this interval. Substituting the given series (5) into the impulsive equation (3), obtain the equalities

$$\varepsilon(\overline{z}(\theta_i+, \varepsilon) + \omega^{(i)}(0, \varepsilon) - \overline{z}(\theta_i, \varepsilon) - \omega^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon)) = C_1(\theta_i)(\overline{z}(\theta_i, \varepsilon) - \omega^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon)) +$$
$$+ C_2(\theta_i)(\overline{y}(\theta_i, \varepsilon) + \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon)) + \varepsilon I_1(\theta_i),$$
$$\overline{y}(\theta_i+, \varepsilon) + \varepsilon\nu^{(i)}(0, \varepsilon) - \overline{y}(\theta_i, \varepsilon) - \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon) = C_3(\theta_i)(\overline{y}(\theta_i, \varepsilon) + \varepsilon\nu^{(i-1)}(\frac{\theta_i - \theta_{i-1}}{\varepsilon}, \varepsilon))$$
$$+ I_2(\theta_i).$$

Taking into account (6), equate the coefficients according to the powers of ε

$$\begin{aligned} &\varepsilon^0 : 0 = C_1(\theta_i)\overline{z}_0(\theta_i) + C_2(\theta_i)\overline{y}_0(\theta_i), \\ &\Delta\overline{y}_0|_{t=\theta_i} = C_3(\theta_i)\overline{y}_0(\theta_i) + I_2(\theta_i), \end{aligned} \tag{20}$$

$$\varepsilon^1 : \omega^{(i)}_0(0) = C_1(\theta_i)\overline{z}_1(\theta_i) + C_2(\theta_i)\overline{y}_1(\theta_i) + I_1(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i},$$
$$\Delta\overline{y}_1|_{t=\theta_i} = C_3(\theta_i)\overline{y}_1(\theta_i) - \nu^{(i)}_0(0),$$
$$\varepsilon^k : \omega^{(i)}_k(0) = C_1(\theta_i)\overline{z}_{k+1}(\theta_i) + C_2(\theta_i)\overline{y}_{k+1}(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i},$$
$$\Delta\overline{y}_k|_{t=\theta_i} = C_3(\theta_i)\overline{y}_k(\theta_i) - \nu^{(i)}_{k-1}(0).$$

In order to determine the approximation of the zero order $\overline{z}_0(t)$ and $\overline{y}_0(t)$, consider the systems

$$\begin{aligned} &0 = A_1(t)\overline{z}_0(t) + B_1(t)\overline{y}_0(t), && 0 = C_1(\theta_i)\overline{z}_0(\theta_i) + C_2(\theta_i)\overline{y}_0(\theta_i), \\ &\overline{y}'_0(t) = A_2(t)\overline{z}_0(t) + B_2(t)\overline{y}_0(t) + f_2(t), && \Delta\overline{y}_0|_{t=\theta_i} = C_3(\theta_i)\overline{y}_0(\theta_i) + I_2(\theta_i). \end{aligned}$$

To find $\omega_0^{(i)}(\tau_i)$, we need to solve the equation

$$\dot{\omega}_0^{(i)}(\tau_i) = A_1(\theta_i)\omega_0^{(i)}(\tau_i), i = 1, 2, ..., p$$

with initial condition

$$\omega_0^{(i)}(0) = C_1(\theta_i)\overline{z}_1(\theta_i) + C_2(\theta_i)\overline{y}_1(\theta_i) + I_1(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i},$$

where $\omega_0^{(i)}(0)$ may be modified as below. From the first equation (10) and (20), obtain

$$\overline{z}_0(t) = -\frac{B_1(t)}{A_1(t)}\overline{y}_0(t), \ \ \overline{z}_0(\theta_i) = -\frac{C_2(\theta_i)}{C_1(\theta_i)}\overline{y}_0(\theta_i),$$

Hence,

$$\frac{B_1(\theta_i)}{A_1(\theta_i)} = \frac{C_2(\theta_i)}{C_1(\theta_i)}. \tag{21}$$

The first equation (11) can be written in the form

$$\overline{z}_1(t) + \frac{B_1(t)}{A_1(t)}\overline{y}_1(t) = \frac{1}{A_1(t)}(\overline{z}_0'(t) - f_1(t)).$$

As a result, using the last equation and equality (21), the initial condition $\omega_0^{(i)}(0)$ is transformed into the form

$$\omega_0^{(i)}(0) = \frac{C_1(\theta_i)}{A_1(\theta_i)}(\overline{z}_0'(\theta_i) - f_1(\theta_i)) + I_1(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}.$$

From the second equation (13) and (7), we find initial condition

$$\nu_0^{(i)}(0) = \frac{A_2(\theta_i)}{A_1(\theta_i)}\omega_0^{(i)}(0), i = 1, 2, ..., p. \tag{22}$$

It is left to solve equation

$$\nu_0^{(i)}(\tau_i) = A_2(\theta_i)\omega_0^{(i)}(\tau_i), i = 1, 2, ..., p.$$

with initial conditions (22).

In order to determine the coefficients of ε^k for the approximation $\overline{z}_k(t)$ and $\overline{y}_k(t)$, have the systems

$$\begin{aligned} &\overline{z}_{k-1}'(t) = A_1(t)\overline{z}_k(t) + B_1(t)\overline{y}_k(t), \\ &\overline{y}_k'(t) = A_2(t)\overline{z}_k(t) + B_2(t)\overline{y}_k(t), \quad \Delta\overline{y}_k|_{t=\theta_i} = C_3(\theta_i)\overline{y}_k(\theta_i) - \nu_{k-1}^{(i)}(0). \end{aligned}$$

To find $\omega_k^{(i)}(\tau_i)$ it is needed to solve the system

$$\begin{aligned} &\dot{\omega}_k^{(i)}(\tau_i) - A_1(\theta_i)\omega_k^{(i)}(\tau_i) = \Gamma_k(\tau_i), \\ &\omega_k^{(i)}(0) = C_1(\theta_i)\overline{z}_{k+1}(\theta_i) + C_2(\theta_i)\overline{y}_{k+1}(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i}, \end{aligned}$$

where $\omega_k^{(i)}(0)$ can be changed as follows. The first equation (12) can be written in the form

$$\overline{z}_{k+1}(t)+\frac{B_1(t)}{A_1(t)}\overline{y}_{k+1}(t)=\frac{\overline{z}'_k(t)}{A_1(t)}.$$

Using the last equation and equality (21), the initial condition $\omega_k^{(i)}(0)$ is transformed as follows

$$\omega_k^{(i)}(0)=\frac{C_1(\theta_i)}{A_1(\theta_i)}\overline{z}'_k(\theta_i)-\Delta\overline{z}_k|_{t=\theta_i}.$$

By using the equation (14) and the condition (7), we get the following initial condition

$$\nu_k^{(i)}(0)=\frac{A_2(\theta_i)}{A_1(\theta_i)}\omega_k^{(i)}(0)+\int_0^{\infty}(\frac{A_2(\theta_i)}{A_1(\theta_i)}\Gamma_k(s)-\Theta_k(s))ds. \tag{23}$$

Then the boundary functions $\nu_k^{(i)}(\tau_i), i=1,2,...,p$ can be determined by the second equation of (14) and the initial condition (23).

Functions $\Gamma_k(\tau_i)$ and $\Theta_k(\tau_i)$ possess the exponential estimate. Therefore, it can be proved that the following inequalities hold,

$$|\omega_k^{(i)}(\tau_i)|\le K\exp(-\gamma\tau_i), i=1,2,...,p,$$
$$|\nu_k^{(i)}(\tau_i)|\le K\exp(-\gamma\tau_i), i=1,2,...,p,$$

where K and γ are positive numbers.

Thus, the coefficients of the expansions (6) are obtained at least up to order $k=n$. On the basis of above discussion one can conclude that the following assertion is correct.

Theorem 1 *Under conditions $(C1)-(C4)$, the series (5) is the asymptotic expansion as $\varepsilon\to 0$ for the solution $z(t,\varepsilon), y(t,\varepsilon)$ of the problem (2)-(4) in the interval $0\le t\le T$, i.e. the following estimate holds*

$$|z(t,\varepsilon)-Z_n(t,\varepsilon)|=O(\varepsilon^{n+1}),\ \ 0\le t\le T,$$
$$|y(t,\varepsilon)-Y_n(t,\varepsilon)|=O(\varepsilon^{n+1}),\ \ 0\le t\le T,$$

where

$$Z_n(t,\varepsilon)=Z_n^{(i)}(t,\varepsilon), Y_n(t,\varepsilon)=Y_n^{(i)}(t,\varepsilon), \theta_i<t\le\theta_{i+1},$$
$$Z_n^{(i)}(t,\varepsilon)=\sum_{k=0}^{n}\varepsilon^k\overline{z}_k(t)+\sum_{k=0}^{n}\varepsilon^k\omega_k^{(i)}(\tau_i), \tau_i=\frac{t-\theta_i}{\varepsilon},$$
$$Y_n^{(i)}(t,\varepsilon)=\sum_{k=0}^{n}\varepsilon^k\overline{y}_k(t)+\varepsilon\sum_{k=0}^{n}\varepsilon^k\nu_k^{(i)}(\tau_i), i=1,2,...,p.$$

3 Example

Consider the following system with impulsive singularity

$$\begin{aligned} &\varepsilon\frac{dz}{dt} = -(t+2)z-(t+1)y-\varepsilon t, \quad \varepsilon\Delta z|_{t=\theta_i} = (\theta_i+2)z+(\theta_i+1)y+4\varepsilon\theta_i, \\ &\frac{dy}{dt} = -(t+2)z-12y, \qquad\qquad \Delta y|_{t=\theta_i} = 12y-2z \end{aligned} \tag{24}$$

with initial conditions

$$z(0,\varepsilon)=3, \quad y(0,\varepsilon)=2. \tag{25}$$

where $\theta_i = \dfrac{i}{3}, i=1,2,3,4.$

The solution of problem (24)-(25) as $\varepsilon \to 0$ tends to solve the degenerate system

$$\begin{aligned} &0=-(t+2)\overline{z}-(t+1)\overline{y}, \quad 0=(\theta_i+2)\overline{z}+(\theta_i+1)\overline{y}, \\ &\frac{d\overline{y}}{dt} = -(t+2)\overline{z}-12\overline{y}, \qquad \Delta\overline{y}|_{t=\theta_i} = 12\overline{y}-2\overline{z} \end{aligned}$$

with initial condition

$$\overline{y}(0)=2.$$

From the first line, we find the root $\overline{z}=\varphi=-\dfrac{(t+1)}{(t+2)}\overline{y}$. One can verify that condition $(C4)$ is valid

$$\lim_{(z,y,\varepsilon)\to(\varphi,\overline{y},0)} \frac{(\theta_i+2)z+(\theta_i+1)y+4\varepsilon\theta_i}{\varepsilon} = 4\theta_i \neq 0.$$

The solution $z(t,\varepsilon)$ of system (24) with initial value (25) has multi-layers near $t=0$ and $t=\theta_i+, i=1,2,3,4$. It is clear from Figure 1 that there are multi-layers.

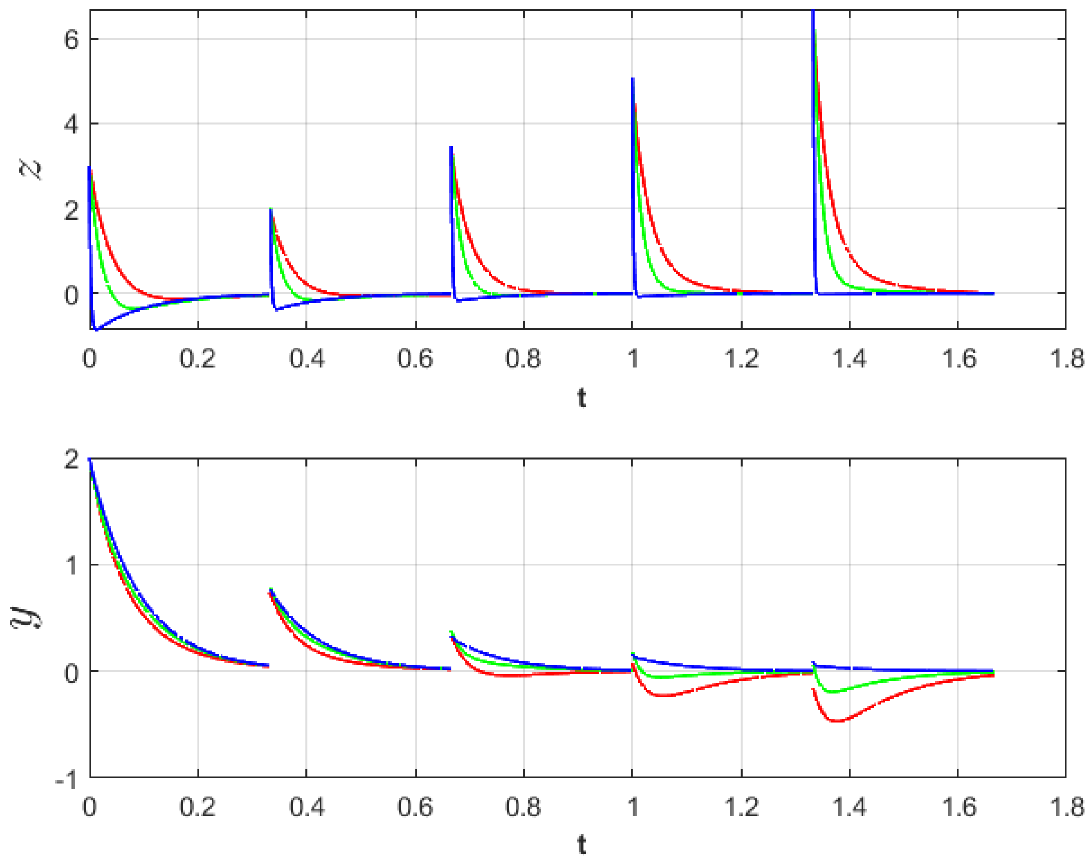

Figure 1: Red, green and blue represent the solution $z(t,\varepsilon), y(t,\varepsilon)$ of (24) with initial conditions $z(0,\varepsilon) = 3$ and $y(0,\varepsilon) = 2$, for $\varepsilon : 0.1, 0.05, 0.005$, respectively.

4 Conclusion

In this article, the singular linear impulsive system is considered. The boundary function method is used to construct the required asymptotic solutions. The asymptotic expansion of solutions with arbitrary degree of accuracy on a small parameter is constructed. To verify the theoretical results, an illustrative example is provided through simulation.

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THE NONLOCAL SOLVABILITY CONDITIONS FOR A SYSTEM WITH CONSTANT TERMS AND COEFFICIENTS OF THE VARIABLE t

We consider the Cauchy problem for a system of quasilinear differential equations with constant terms and coefficients of the variable t. We investigate the solvability of the Cauchy problem for a system of quasilinear differential equations with constant terms and coefficients of the variable t using the additional argument method. A theorem on the existence and uniqueness of the local solution of the Cauchy problem for a system of quasilinear differential equations with constant terms and coefficients of the variable t is formulated. We obtain sufficient conditions for the existence and uniqueness of a nonlocal solution of the Cauchy problem in original coordinates for a system of quasilinear differential equations with constant terms and coefficients of the variable t. A theorem on the existence and uniqueness of the nonlocal solution of the Cauchy problem for a system of quasilinear differential equations with constant terms and coefficients of the variable t is formulated. A theorem on the existence and uniqueness of the nonlocal solution of the Cauchy problem for a system of quasilinear differential equations with constant terms and coefficients of the variable t is proved. The proof of the nonlocal solvability of the Cauchy problem for a system of quasilinear differential equations with constant terms and coefficients of the variable t relies on global estimates.

Key words: Cauchy problem, quasilinear system, functions, global estimates.

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t айнымалы еркін мүшелері мен коэффициенттері бар жүйе үшін локалді емес шешімділік шарттары

t айнымалы еркін мүшелері мен коэффициенттері бар квазисызықты дифференциалдық теңдеулер жүйесі үшін Коши есебін қарастырамыз. Қосымша аргумент әдісі арқылы t айнымалы еркін мүшелері мен коэффициенттері бар квазисызықты дифференциалдық теңдеулер жүйесі үшін Коши есебін шешімділікке зерттейміз. t айнымалы еркін мүшелері мен коэффициенттері бар квазисызықты дифференциалдық теңдеулер жүйесі үшін Коши есебінің локалді шешімі бар және жалғыздығы туралы теорема тұжырымдалған. t айнымалы еркін мүшелері мен коэффициенттері бар квазисызықты дифференциалдық теңдеулер жүйесі үшін бастапқы координаталардағы Коши есебінің локальды емес шешімінің бар болуы мен жалғыздығының жеткілікті шарттарын аламыз. Еркін мүшелері және t айнымалы коэффициенттері бар квазисызықты дифференциалдық теңдеулер жүйесі үшін Коши есебінің локалді емес шешімі бар және жалғыздығы туралы теорема тұжырымдалған. t айнымалы еркін мүшелері мен коэффициенттері бар квазисызықты дифференциалдық теңдеулер жүйесі үшін Коши есебінің локальды емес шешімінің бар болуы және жалғыздығы туралы теорема дәлелденді. t айнымалы еркін мүшелері мен коэффициенттері бар квазисызықты дифференциалдық теңдеулер жүйесі үшін Коши есебінің локальды емес шешімділігінің дәлелі глобалді априорлық бағалауларға негізделген.

Түйін сөздер: Коши есебі, квазисызықты жүйе, функциялар, глобалді бағалаулар.

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Условия нелокальной разрешимости для системы со свободными членами и коэффициентами переменного t

Мы рассматриваем задачу Коши для системы квазилинейных дифференциальных уравнений со свободными членами и коэффициентами переменного t. Мы исследуем разрешимость задачи Коши для системы квазилинейных дифференциальных уравнений со свободными членами и коэффициентами переменного t с помощью метода дополнительного аргумента. Сформулирована теорема о существовании и единственности локального решения задачи Коши для системы квазилинейных дифференциальных уравнений со свободными членами и коэффициентами переменного t. Мы получаем достаточные условия существования и единственности нелокального решения задачи Коши в исходных координатах для системы квазилинейных дифференциальных уравнений со свободными членами и коэффициентами переменного t. Сформулирована теорема о существовании и единственности нелокального решения задачи Коши для системы квазилинейных дифференциальных уравнений со свободными членами и коэффициентами переменного t. Доказана теорема о существовании и единственности нелокального решения задачи Коши для системы квазилинейных дифференциальных уравнений со свободными членами и коэффициентами переменного t. Доказательство нелокальной разрешимости задачи Коши для системы квазилинейных дифференциальных уравнений со свободными членами и коэффициентами переменного t основано на глобальных оценках.
Ключевые слова: задача Коши, квазилинейная система, функции, глобальные оценки.

1 Introduction

We consider the system:

$$\begin{cases} \partial_t u\,(t,x) + (a(t)u(t,x) + b(t)v(t,x) + a_1(t))\partial_x u\,(t,x) = f_1(t,x), \\ \partial_t v(t,x) + (c(t)u(t,x) + g(t)v(t,x) + a_2(t))\partial_x v(t,x) = f_2(t,x), \end{cases} \tag{1}$$

where $u(t,x),\ v(t,x)$ are unknown functions, $f_1,\ f_2,\ \ a(t),\ b(t),\ c(t),\ g(t),\ \ a_1(t),\ a_2(t)$ are given functions,

$$a(t) > 0,\ b(t) > 0,\ c(t) > 0,\ g(t) > 0, \qquad t \in [0,T],$$

subject to the initial conditions:

$$u(0,x) = \varphi_1(x),\ v(0,x) = \varphi_2(x), \tag{2}$$

where $\varphi_1(x), \varphi_2(x)$ are given functions.

The problem (1), (2) is considered on

$$\Omega_T = \{(t,x)\,|0 \le t \le T, x \in (-\infty,+\infty), T > 0\}.$$

The system (1) appear in various problems in natural sciences, for instance, in describing the spreading of finite intensity perturbation under non-stationary one-dimensional ow of ideal gas [1,2].

In the present work, we determine sufficient conditions for the existence and uniqueness of a nonlocal solution of the Cauchy problem (1), (2), where $f_1,\ f_2,\ \ a(t),\ b(t),\ c(t),\ g(t),$ $a_1(t),\ a_2(t)$ are given functions,

$$a(t) > 0,\ b(t) > 0,\ c(t) > 0,\ g(t) > 0, \qquad t \in [0,T].$$

We investigate the solvability of the Cauchy problem (1), (2) using the additional argument method. The method of an additional argument allows us to obtain sufficient conditions for the existence and uniqueness of a nonlocal solution of the Cauchy problem (1), (2) in original coordinates.

2 Material and Methods

We use the additional argument method. For the problem (1), (2) we write the extended characteristic system [3–9]:

$$\frac{d\eta_1(s,t,x)}{ds} = a(s)w_1(s,t,x) + b(s)w_3(s,t,x) + a_1(s), \tag{3}$$

$$\frac{d\eta_2(s,t,x)}{ds} = c(s)w_4(s,t,x) + g(s)w_2(s,t,x) + a_2(s), \tag{4}$$

$$\frac{dw_1(s,t,x)}{ds} = f_1(s,\eta_1), \tag{5}$$

$$\frac{dw_2(s,t,x)}{ds} = f_2(s,\eta_2), \tag{6}$$

$$w_3(s,t,x) = w_2(s,s,\eta_1),\ \ w_4(s,t,x) = w_1(s,s,\eta_2), \tag{7}$$

$$w_1(0,t,x) = \varphi_1(\eta_1(0,t,x)),\ \ w_2(0,t,x) = \varphi_2(\eta_2(0,t,x)),\ \ \eta_i(t,t,x) = x,\ \ i = 1,2. \tag{8}$$

Unknown functions $\eta_i,\ w_j,\ i = 1,2,\ j = \overline{1,4}$, depend not only on t and x, but also on additional argument s. Integrating equations (3)–(6) with respect to the argument s and taking into considerations conditions (7), (8), we obtain an equivalent system of integral equations:

$$\eta_1(s,t,x) = x - \int_s^t (a(\nu)w_1 + b(\nu)w_3 + a_1(\nu))d\nu, \tag{9}$$

$$\eta_2(s,t,x) = x - \int_s^t (c(\nu)w_4 + g(\nu)w_2 + a_2(\nu))d\nu, \tag{10}$$

$$w_1(s,t,x) = \varphi_1(\eta_1(0,t,x)) + \int_0^s f_1(\nu,\eta_1)d\nu, \tag{11}$$

$$w_2(s,t,x) = \varphi_2(\eta_2(0,t,x)) + \int_0^s f_2(\nu,\eta_2)d\nu, \tag{12}$$

$$w_3(s,t,x) = w_2(s,s,\eta_1), \;\; w_4(s,t,x) = w_1(s,s,\eta_2). \tag{13}$$

Substituting (9), (10) into (11)–(13), we get

$$\begin{aligned} w_1(s,t,x) &= \varphi_1(x - \int_0^t (a(\nu)w_1 + b(\nu)w_3 + a_1(\nu))d\nu) + \\ &+ \int_0^s f_1(\nu, x - \int_\nu^t (a(\tau)w_1 + b(\tau)w_3 + a_1(\tau))d\tau)d\nu, \end{aligned} \tag{14}$$

$$\begin{aligned} w_2(s,t,x) &= \varphi_2(x - \int_0^t (c(\nu)w_4 + g(\nu)w_2 + a_2(\nu))d\nu) + \\ &+ \int_0^s f_2(\nu, x - \int_\nu^t (c(\tau)w_4 + g(\tau)w_2 + a_2(\tau))d\tau)d\nu, \end{aligned} \tag{15}$$

$$w_3(s,t,x) = w_2(s,s,x - \int_s^t (a(\nu)w_1 + b(\nu)w_3 + a_1(\nu))d\nu), \tag{16}$$

$$w_4(s,t,x) = w_1(s,s,x - \int_s^t (c(\nu)w_4 + g(\nu)w_2 + a_2(\nu))d\nu). \tag{17}$$

Lemma 1 *Let $w_1(s,t,x)$ and $w_2(s,t,x)$ satisfy the system of integral equations (14)–(17). Assume that $w_1(s,t,x)$, $w_2(s,t,x)$ together with their first order derivatives are continuously differentiable and bounded. Then the pair of functions*

$$u(t,x) = w_1(t,t,x), \;\; v(t,x) = w_2(t,t,x)$$

is a solution to the problem (1), (2) on Ω_{T_0}, where T_0 is a constant.

The Lemma 1 can be proven in the same way as in [9].

The proof of the nonlocal solvability of the Cauchy problem (1), (2) relies on global estimates.

3 Existence of a local solution

We denote $\Gamma_T = \{(s,t,x) | \, 0 \le s \le t \le T, \; x \in (-\infty, +\infty), \; T > 0\}$,

$$C_\varphi = \max\{\sup_R \left|\varphi_i^{(l)}\right| |i = 1,2, \; l = \overline{0,2}\},$$

$$l=\max\{\sup_{[0,T]}a(t),\ \sup_{[0,T]}b(t),\ \sup_{[0,T]}c(t),\ \sup_{[0,T]}g(t)\},$$

$$C_f=\max\{\sup_{\Omega_T}\left|f_1(t,x)\right|,\sup_{\Omega_T}\left|f_2(t,x)\right|,\sup_{\Omega_T}\left|\partial_x f_1(t,x)\right|,\sup_{\Omega_T}\left|\partial_x f_2(t,x)\right|\},$$

$$\|G\|=\sup_{\Gamma_T}\left|G(s,t,x)\right|,\ \ \|f\|=\sup_{\Omega_T}|f(t,x)|,$$

$\bar{C}^{\alpha_1,\alpha_2,\ldots\alpha_n}(\Omega_*)$ is the space of functions continuous and bounded, together with its derivatives up to order α_m w.r.t. mth argument, $m=\overline{1,n}$ on unbounded subset $\Omega_*\subset R^n, n=1,2....,$

$C([0,T])$ is the space of continuous functions on $[0,T]$.

Theorem 1 *Suppose that*

$$\varphi_1,\varphi_2\in\bar{C}^2(R),\ \ f_1,f_2\in\bar{C}^{2,2}(\Omega_T),\ \ a,b,c,g,a_1,a_2\in C([0,T]),$$

$$T\leqslant\min(\frac{C_\varphi}{4C_f},\frac{3}{40C_\varphi l}),$$

$$a(t)>0,\ \ b(t)>0,\ \ c(t)>0,\ \ g(t)>0\ \ \textit{on}\ [0,T],$$

$$\varphi_1'(x)\geq0,\ \ \varphi_2'(x)\geq0\ \ \textit{on}\ R,$$

$$\partial_x f_1(t,x)\geq0,\ \ \partial_x f_2(t,x)\geq0\ \ \textit{on}\ \Omega_T.$$

Then for each

$$T\leq\min(\frac{C_\varphi}{4C_f},\frac{3}{40C_\varphi l}),$$

the Cauchy problem (1), (2) has a unique solution

$$u(t,x),v(t,x)\in\bar{C}^{1,2}(\Omega_T)$$

which can be found from the system of integral equations (14)–(17).

The Theorem 1 can be proven in the same way as in [4–8].

4 Existence of a nonlocal solution

Theorem 2 *Suppose that*

$$\varphi_1,\varphi_2\in\bar{C}^2(R),\ \ f_1,f_2\in\bar{C}^{2,2}(\Omega_T),\ \ a,b,c,g,a_1,a_2\in C([0,T]),$$

$$a(t)>0,\ \ b(t)>0,\ \ c(t)>0,\ \ g(t)>0\ \ \textit{on}\ [0,T],$$

$$\varphi_1'(x)\geq0,\ \ \varphi_2'(x)\geq0\ \ \textit{on}\ R,$$

$$\partial_x f_1(t,x)\geq0,\ \ \partial_x f_2(t,x)\geq0\ \ \textit{on}\ \Omega_T.$$

Then for any $T>0$ the Cauchy problem (1), (2) has a unique solution

$$u(t,x),v(t,x)\in\bar{C}^{1,2}(\Omega_T)$$

which can be found from the system of integral equations (14)–(17).

Proof. Differentiating (1) with respect to x and denoting

$$p(t,x)=\partial_x u(t,x),\quad r(t,x)=\partial_x v(t,x),$$

we obtain the system of equations:

$$\begin{cases} \partial_t p+(a(t)u+b(t)v+a_1(t))\partial_x p=-a(t)p^2-b(t)pr+\partial_x f_1, \\ \partial_t r+(c(t)u+g(t)v+a_2(t))\partial_x r=-g(t)r^2-c(t)pr+\partial_x f_2, \\ p(0,x)=\varphi_1'(x),\quad r(0,x)=\varphi_2'(x). \end{cases} \tag{18}$$

We add two equations to the system of equations (9)–(13):

$$\begin{cases} \frac{d\gamma_1(s,t,x)}{ds}=-a(s)\gamma_1^2(s,t,x)-b(s)\gamma_1(s,t,x)\gamma_2(s,s,\eta_1)+\partial_x f_1(s,\eta_1), \\ \frac{d\gamma_2(s,t,x)}{ds}=-g(s)\gamma_2^2(s,t,x)-c(s)\gamma_1(s,s,\eta_2)\gamma_2(s,t,x)+\partial_x f_2(s,\eta_2), \end{cases} \tag{19}$$

subject to the conditions:

$$\gamma_1(0,t,x)=\varphi_1'(\eta_1),\quad \gamma_2(0,t,x)=\varphi_2'(\eta_2). \tag{20}$$

We rewrite (19), (20) as follows:

$$\begin{cases} \gamma_1(s,t,x)=\varphi_1'(\eta_1)+\int\limits_0^s[-a(\nu)\gamma_1^2-b(\nu)\gamma_1\gamma_2(\nu,\nu,\eta_1)+\partial_x f_1]d\nu, \\ \gamma_2(s,t,x)=\varphi_2'(\eta_2)+\int\limits_0^s[-g(\nu)\gamma_2^2-c(\nu)\gamma_2\gamma_1(\nu,\nu,\eta_2)+\partial_x f_2]d\nu. \end{cases} \tag{21}$$

As in [4–8], we can prove the existence of a continuously differentiable solution to the problem (21). Therefore,

$$\gamma_1(t,t,x)=p(t,x)=\frac{\partial u}{\partial x},\gamma_2(t,t,x)=r(t,x)=\frac{\partial v}{\partial x}.$$

As in [4,5], we can prove that for all t and x on Ω_T

$$\|u\|\leqslant C_\varphi+TC_f,\ \|v\|\leqslant C_\varphi+TC_f. \tag{22}$$

From (19),(20), we obtain

$$\begin{cases} \gamma_1(s,t,x)=\varphi_1'(\eta_1)\exp(-\int\limits_0^s(a(\nu)\gamma_1+b(\nu)\gamma_2)\,d\nu)+ \\ +\int\limits_0^s\partial_x f_1\exp(-\int\limits_\tau^s(a(\nu)\gamma_1+b(\nu)\gamma_2)\,d\nu)d\tau, \\ \gamma_2(s,t,x)=\varphi_2'(\eta_2)\exp(-\int\limits_0^s(c(\nu)\gamma_1+g(\nu)\gamma_2)d\nu)+ \\ +\int\limits_0^s\partial_x f_2\exp(-\int\limits_\tau^s(c(\nu)\gamma_1+g(\nu)\gamma_2)\,d\nu)d\tau. \end{cases} \tag{23}$$

Since

$$a(t)>0,\ b(t)>0,\ c(t)>0,\ g(t)>0 \text{ on } [0,T],$$

$$\varphi_1'(x)\geq 0,\ \varphi_2'(x)\geq 0 \text{ on } R,$$

$$\partial_x f_1(t,x) \geq 0,\ \partial_x f_2(t,x) \geq 0 \text{ on } \Omega_T,$$

it follows from (23) that $\gamma_1 \geq 0,\ \gamma_2 \geq 0$ on Γ_T. Therefore,

$$\|\gamma_i\| \leqslant C_\varphi + TC_f,\ i = 1, 2.$$

Since $\gamma_1(t,t,x) = \partial_x u,\ \gamma_2(t,t,x) = \partial_x v$, then for all t and x on Ω_T we obtain the estimates

$$\|\partial_x u\| \leqslant C_\varphi + TC_f,\ \|\partial_x v\| \leqslant C_\varphi + TC_f. \tag{24}$$

As in [4–8], for all t and x we obtain the estimates

$$|\partial^2_{x^2} u| \leq E_1 ch\left(T\sqrt{C_1C_2}\right) + \frac{E_2C_1 + C_3}{\sqrt{C_1C_2}} sh\left(T\sqrt{C_1C_2}\right) + C_1C_4T^2, \tag{25}$$

$$|\partial^2_{x^2} v| \leq E_2 ch\left(T\sqrt{C_1C_2}\right) + \frac{E_1C_2 + C_4}{\sqrt{C_1C_2}} sh\left(T\sqrt{C_1C_2}\right) + C_2C_3T^2, \tag{26}$$

where $E_1,\ E_2,\ C_1,\ C_2,\ C_3,\ C_4$ are constants.

Owing to the global estimates (22), (24)–(26), we can extend the solution to any given segment $[0,T]$. We take $u(T_0,x),\ v(T_0,x)$ for the initial values, using Theorem 1, we extend the solution to the segment $[T_0,\ T_1]$. Then for the initial values we take $u(T_1,x),\ v(T_1,x)$, using Theorem 1, we extend the solution to the segment $[T_1,\ T_2]$. As a result, we can extend the solution to any given segment $[0,T]$ in finitely many steps.

The uniqueness of a solution to the Cauchy problem (1), (2) is proved with the help of estimates similar to those used in the proof of the convergence of successive approximations.

Example 1 *We consider the system:*

$$\begin{cases} \partial_t u\,(t,x) + ((10t+1)u(t,x) + (9t^6+2)v(t,x) + 150t)\partial_x u\,(t,x) = t + 19arctg2x, \\ \partial_t v(t,x) + ((2t^2+5)u(t,x) + (5t^3+7)v(t,x) - 71t)\partial_x v(t,x) = -\frac{t+17}{e^{5x}+11}, \end{cases} \tag{27}$$

where $u(t,x),\ v(t,x)$ are unknown functions, subject to the initial conditions:

$$u(0,x) = \varphi_1(x) = -\frac{1}{e^{19x}+3},\ v(0,x) = \varphi_2(x) = 12 + arctg6x. \tag{28}$$

The problem (27), (28) is considered on $\Omega_T = \{(t,x)\,|0 \leq t \leq T, x \in (-\infty,+\infty), T > 0\}$.

We have

$$a(t) = 10t+1,\ b(t) = 9t^6+2,\ c(t) = 2t^2+5,\ g(t) = 5t^3+7,$$

$$a_1(t) = 150t,\ a_2(t) = -71t,\ f_1(t,x) = t + 19arctg2x,\ f_2(t,x) = -\frac{t+17}{e^{5x}+11},$$

$$\varphi_1'(x) = \frac{19e^{19x}}{(e^{19x}+3)^2},\ \varphi_2'(x) = \frac{6}{1+36x^2},$$

$$\partial_x f_1(t,x) = \frac{38}{1+4x^2},\ \partial_x f_2(t,x) = \frac{5e^{5x}(t+17)}{(e^{5x}+11)^2}.$$

Since

$$\varphi_1, \varphi_2 \in \bar{C}^2(R),\ f_1, f_2 \in \bar{C}^{2,2}(\Omega_T),\ a, b, c, g, a_1, a_2 \in C([0,T]),$$

$$a(t) = 10t + 1 > 0,\ b(t) = 9t^6 + 2 > 0,$$

$$c(t) = 2t^2 + 5 > 0,\ g(t) = 5t^3 + 7 > 0 \ on\ [0,T],$$

$$\varphi_1'(x) = \frac{19e^{19x}}{(e^{19x}+3)^2} > 0,\ \varphi_2'(x) = \frac{6}{1+36x^2} > 0\ on\ R,$$

$$\partial_x f_1(t,x) = \frac{38}{1+4x^2} > 0,\ \partial_x f_2(t,x) = \frac{5e^{5x}(t+17)}{(e^{5x}+11)^2} > 0\ on\ \Omega_T,$$

then by Theorem 2, the Cauchy problem (27), (28) has a unique solution

$$u(t,x), v(t,x) \in \bar{C}^{1,2}(\Omega_T).$$

We consider the system (27) subject to the initial conditions:

$$u(0,x) = \varphi_1(x) = -\frac{1}{e^{23x}+13},\ v(0,x) = \varphi_2(x) = 24 + arctgx. \tag{29}$$

The problem (27), (29) is considered on $\Omega_T = \{(t,x)\,|0 \le t \le T, x \in (-\infty,+\infty), T > 0\}$.

Since

$$\varphi_1, \varphi_2 \in \bar{C}^2(R),\ f_1, f_2 \in \bar{C}^{2,2}(\Omega_T),\ a, b, c, g, a_1, a_2 \in C([0,T]),$$

$$a(t) = 10t + 1 > 0,\ b(t) = 9t^6 + 2 > 0,$$

$$c(t) = 2t^2 + 5 > 0,\ g(t) = 5t^3 + 7 > 0 \ on\ [0,T],$$

$$\varphi_1'(x) = \frac{23e^{23x}}{(e^{23x}+13)^2} > 0,\ \varphi_2'(x) = \frac{1}{1+x^2} > 0\ on\ R,$$

$$\partial_x f_1(t,x) = \frac{38}{1+4x^2} > 0,\ \partial_x f_2(t,x) = \frac{5e^{5x}(t+17)}{(e^{5x}+11)^2} > 0\ on\ \Omega_T,$$

then by Theorem 2, the Cauchy problem (27), (29) has a unique solution

$$u(t,x), v(t,x) \in \bar{C}^{1,2}(\Omega_T).$$

5 Conclusion

We have obtained sufficient conditions for the existence and uniqueness of a nonlocal solution of the Cauchy problem (1), (2), where f_1, f_2, $a(t)$, $b(t)$, $c(t)$, $g(t)$, $a_1(t)$, $a_2(t)$ are given functions, $a(t) > 0,\ b(t) > 0,\ c(t) > 0,\ g(t) > 0, \quad t \in [0,T]$.

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A ROBUST NUMERICAL METHOD FOR SINGULARLY PERTURBED SOBOLEV PERIODIC PROBLEMS ON B-MESH

This article examines periodic Sobolev reports with a singular deviation, which causes significant difficulties in numerical approximation due to the presence of sharp or boundary layers. A stable quantitative method for the effective solution of such problems in the Bakhvalov lattice, a special grid for the deviant action of the solution, is proposed. Singularly perturbed periodic Sobolev problems create significant difficulties in numerical approximation due to the presence of sharp layers or boundary layers. Our proposed reliable numerical method for efficiently solving such problems on the Bakhvalov grid, a specialized grid, is designed to account for the singular behavior of the solution. First, an asymptotic analysis of the exact solution is performed. Then a finite difference scheme is created by applying quadrature interpolation rules to an adaptive network. The stability and convergence of the presented algorithm in a discrete maximum norm is analyzed. The results show that the proposed approach provides an accurate approximation of the solution for singular problems while maintaining computational efficiency.
Key words: Difference scheme, error estimate, periodic boundary value problem, singular perturbation, Sobolev differential equation.

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В-тордағы сингулярлы ауытқыған Соболев периодты проблемалары үшін тұрақты сандық әдісі

Бұл мақалада Соболевтің сингулярлық ауытқуы бар мерзімді есептері қарастырылады, бұл өткір немесе шекаралық қабаттардың болуына байланысты сандық жуықтауда айтарлықтай қиындықтар тудырады. Бахвалов торында мұндай мәселелерді тиімді шешудің тұрақты сандық әдісі, шешімнің ауытқу әрекеті үшін арнайы тор ұсынылған. Соболевтің ерекше ашуланған мерзімді міндеттері өткір қабаттардың немесе шекаралық қабаттардың болуына байланысты сандық жуықтауда айтарлықтай қиындықтар туғызады. Бахвал торында, мамандандырылған торда осындай мәселелерді тиімді шешу үшін біз ұсынатын сенімді сандық әдіс шешімнің сингулярлық мінез-құлқын есепке алуға арналған. Алдымен нақты шешімге асимптотикалық талдау жасалады. Содан кейін адаптивті желіге квадратуралық интерполяция ережелерін қолдану арқылы ақырлы айырмашылық схемасы жасалады. ұсынылған алгоритмнің тұрақтылығы мен конвергенциясы дискретті максималды нормада талданады. Нәтижелер ұсынылған тәсіл есептеу тиімділігін сақтай отырып, сингулярлық есептер үшін шешімнің дәл жуықтауын қамтамасыз ететінін көрсетеді.
Түйін сөздер: Айырмашылық схемасы, қатені бағалау, периодты шекаралық есеп, сингулярлық бұзылыс, Соболевтің дифференциалдық теңдеуі.

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Устойчивый численный метод для сингулярных возмущенных периодических проблем Соболева на В-сетке

В данной статье рассматриваются периодические отчеты Соболева с сингулярным отклонением, что вызывает значительные трудности в численном приближении из-за наличия острых или пограничных слоев. Предложен устойчивый количественный метод эффективного решения таких задач в решетке Бахвалова, специальная сетка для отклоняющегося действия решения. Сингулярно возмущенные периодические задачи Соболева создают значительные трудности при численной аппроксимации из-за наличия резких слоев или пограничных слоев. Предлагаемый нами надежный численный метод для эффективного решения таких задач на Бахваловской сетке, специализированной сетке, предназначен для учета сингулярного поведения решения. Сначала проводится асимптотический анализ точного решения. Затем создается конечно-разностная схема путем применения квадратурных правил интерполяции к адаптивной сети. Анализируется устойчивость и сходимость представленного алгоритма в дискретной максимальной норме. Результаты показывают, что предложенный подход обеспечивает точное приближение решения для сингулярных задач при сохранении вычислительной эффективности.
Ключевые слова: Разностная схема, оценка погрешности, периодическая краевая задача, сингулярное возмущение, дифференциальное уравнение Соболева.

1 Introduction

In this study, we consider the following singularly perturbed initial-periodic boundary value problem in the domain $\overline{D} = \overline{\Omega} \times [0, T]$; $\overline{\Omega} = [0, l]$, $\Omega = (0, l)$, $D = \Omega \times (0, T]$:

$$Lu \equiv L_1[u_{tt}] + L_2 u = f(x, t), \quad (x, t) \in D, \tag{1}$$

$$u(x, 0) = \varphi(x), \quad x \in \overline{\Omega}, \tag{2}$$

$$u_t(x, 0) = \psi(x), \quad x \in \overline{\Omega} \tag{3}$$

$$u(0, t) = u(l, t), \quad t \in (0, T], \tag{4}$$

$$u_t(0, t) = u_t(l, t), \quad t \in (0, T], \tag{5}$$

where

$$L_1[u_{tt}] \equiv -\varepsilon u_{xxtt} + a(x) u_{tt},$$
$$L_2[u(x, t)] \equiv -\varepsilon u_{xx} + b(x, t) u(x, t),$$

and $0 < \varepsilon \ll 1$ perturbation parameter; the functions a, b, f and φ are sufficiently smooth, l-periodic, and $a(x) \geq \alpha > 0$, $b^{*} \geq b(x, t) > 0$.

This study presents numerical solutions for partial differential equations with a second derivative with respect to time in the highest order term and small parameters in that term. These equations are commonly found in mathematical physics and fluid mechanics and are used in various fields such as transmission lines, electron plasma waves, and ion-acoustic waves in plasmas [14].

Previous research has looked at problems similar to the one we are investigating in regular classical difference schemes. In this study, we are focusing on the singular-perturbed version of the problem, where small parameters affect the coefficient of the higher-order derivative. One unique aspect of this issue is that when ? is small, the solution changes quickly around boundary points in x and the derivatives of the solution become unbounded. As a result, the traditional difference scheme is not effective with a uniform mesh, as the approximate solution deviates from the exact solution as the steps in the schema decrease [12]. In this study, a three-level difference scheme is introduced for the problem under investigation. This scheme was developed utilizing linear basis functions, interpolation quadrature formulas with integral terms, and the weight function as outlined by Duru and Gunes (2022). The stability and convergence criteria of the proposed difference scheme were analyzed, and the convergence speed was assessed for each scenario.The research also delves into the existence, uniqueness, and smoothness of the exact solutions to similar problems. Moreover, numerous mathematicians have investigated the presence, distinctiveness, and regularity of the precise solution to such issues [17]. Our main objective is to develop a reliable and stable finite difference scheme for addressing problems (1)–(5), incorporating interpolating quadrature rules and linear basis functions in the construction process.

2 Asymptotic Estimates

Lemma 1 *The following estimation is true for the solution $u(x,t)$ of the problems* (1)–(5)

$$\left\|\frac{\partial^{k+s}u}{\partial t^k \partial x^s}\right\| \leq C\{\varepsilon^{-\frac{s}{2}}\left[\|f\|_{L_2(D)} + \|\varphi\| + \varepsilon\|\varphi'\| + \|\psi\| + \varepsilon\|\psi'\|\right] +$$

$$+s(s-1)\left[\|\varphi''\| + \|\psi''\|\right]\}, \quad k,s = 0,1,2. \tag{6}$$

Proof. Multiplying both sides of the differential equation (1) as a scaler with $\dfrac{\partial u}{\partial t}$, we have

$$\left\|\frac{\partial u}{\partial t}\right\|^2 \leq \Psi_* e^{c_* t} + \int_0^t \left\{\alpha^{-1}(b^*\|u\|^2 + \|f\|^2)e^{c_*(t-s)}\right\} ds, \tag{7}$$

where $\Psi_* = \alpha^{-1}\left[\varepsilon\|\psi'\|^2 + (a(x)\psi,\psi) + \varepsilon\|\varphi'\|^2\right]$, $c_* = \alpha^{-1}(1+b^*)$.

In (7), using the following inequality for the arbitrary function $\vartheta(t) \in C^1$

$$\frac{1}{2T}\vartheta^2(t) - \frac{1}{T}\vartheta^2(0) \leq \int_0^t |\vartheta'(s)|^2 ds \tag{8}$$

and by taking $\xi \in (0,t)$ instead of t by integrating we get

$$\|u\|^2 \leq C_1\left[\|\varphi\|^2 + \varepsilon\|\varphi'\|^2 + \|\psi\|^2 + \varepsilon\|\psi'\|^2\right] + C_2\int_0^t \left[\|u(s)\|^2 + \|f(s)\|^2\right] ds.$$

From integral inequality

$$\|u\|^2 \le C\left(\|f\|^2_{L_2(D)} + \|\varphi\|^2 + \varepsilon\|\varphi'\|^2 + \|\psi\|^2 + \varepsilon\|\psi'\|^2\right).$$

This is proof of Lemma 1 for the case $k = s = 0$. Lemma 1 is true for the case $k = 1,\ s = 0$. From this inequality and (7), the estimate (6) is

$$\varepsilon\left\|\frac{\partial^2 u}{\partial t\partial x}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \varepsilon\left\|\frac{\partial u}{\partial x}\right\|^2 \le$$

$$\le C\left\{\|f\|^2_{L_2(D)} + \|\phi\|^2 + \varepsilon\|\phi'\|^2 + \|\Psi\|^2 + \varepsilon\|\Psi'\|^2\right\}.$$

Thus, the lemma is proved for the cases $k = s = 1$ and $k = 0,\ s = 1$.

Now, if both sides of equation (1) are multiplied by $\dfrac{\partial^2 u}{\partial t^2}$ as the scalar, the following inequality is obtained

$$\varepsilon\left\|\frac{\partial^3 u}{\partial t^2\partial x}\right\|^2 + \alpha\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 \le C\left(\varepsilon\left\|\frac{\partial u}{\partial x}\right\|^2 + \|u\|^2 + \|f\|^2\right).$$

From the results for $\|u\|$ and $\left\|\dfrac{\partial u}{\partial x}\right\|$, Lemma 1 is proved for the cases $k = 2,\ s = 0$, and $k = 2,\ s = 1$.

Similarly, the other cases are proved by multiplying the differential equation (1) by $\dfrac{\partial^4 u}{\partial x^2\partial t^2}$ as a scalar.

3 The Finite Difference Scheme

In this section the finite difference scheme is constructed by using interpolating quadrature rules We use the interpolation quadrature rules when constructing the difference scheme [2].Now we give the node points of Bakhvalov mesh.

3.1 Bakhvalov Mesh

In this subsection, adaptive mesh points are presented. For these points, the mesh generation function that Bakhvalov 1969, mentioned in his paper is used.

Let ω denote the mesh on D, where $\omega = \omega_N \times \omega_\tau$

$$\omega_N = \{x_i = ih,\ i = 1, 2, \ldots, N-1;\ h_i = x_i - x_{i-1}\},$$

$$\omega_\tau = \left\{t_j = j\tau,\ j = 1, 2, \ldots, M;\ \tau = \frac{T}{M}\right\}, \omega_N^+ = \omega_N \cup \{x = 0, l\},\ \overline{\omega}_\tau = \omega_\tau \cup \{t = 0\}.$$

Let the mesh function v be defined on ω_N. The notations are as the following

$$v_x = \frac{v_{i+1} - v_i}{h_{i+1}},\ v_{\overline{x}} = \frac{v_i - v_{i-1}}{h_i},\ v_{\hat{x}} = \frac{v_{i+1} - v_i}{\hbar_i},\ v_{\overline{x}\hat{x}} = \frac{v_x - v_{\overline{x}}}{\hbar_i}.$$

Let the function $g(t)$ be defined on mesh ω_τ. Then the formulas are the following:

$$g_t = \frac{g_{j+1} - g_i}{\tau}, \; g_t = \frac{g_j - g_{j-1}}{\tau}, \; g_{\bar{t}t} = \frac{g_{j+1} - 2g_j + g_{j-1}}{\tau^2} \quad \text{(Samarskii, 2001).}$$

Bakhvalov mesh points (Boglaev 1984; Boglaev, 2006) are as the following

$$x_i = \begin{cases} -\alpha^{-1}\varepsilon \ln\Big(1 - \Big(1-\varepsilon\Big)\frac{4i}{N}\Big), \quad I = 0, 1, \ldots, \frac{N}{4}, \; x_i \in [0, \sigma_1], \quad \text{if} \quad \sigma_1 < \frac{l}{4}; \\ -\alpha^{-1}\varepsilon \ln\Big(1 - \Big(1 - e^{-\frac{\alpha l}{4\varepsilon}}\Big)\frac{4i}{N}\Big), \quad i = 0, 1, \ldots, \frac{N}{4}, \; x_i \in [0, \sigma_1], \quad \text{if} \quad \sigma_1 = \frac{l}{4}; \\ \sigma_1 + \Big(i - \Big(\frac{N}{4}\Big)\Big)h^{(1)}, \quad i = \frac{N}{4} + 1, \ldots, \frac{3N}{4}, \quad x_i \in [\sigma_1, \sigma_2], \; h^{(1)} = \frac{2(\sigma_2 - \sigma_1)}{N}; \\ \sigma_2 - \alpha^{-1}\varepsilon \ln\Big(1 - \Big(1-\varepsilon\Big)\Big(\frac{4\Big(i - \frac{3N}{4}\Big)}{N}\Big)\Big), \quad i = \frac{3N}{4} + 1, \ldots, N, \quad x_i \in [\sigma_2, l]; \\ \sigma_2 - \alpha^{-1}\varepsilon \ln\Big(1 - \Big(1 - e^{-\frac{\alpha l}{4\varepsilon}}\Big)\Big(\frac{4\Big(i - \frac{3N}{4}\Big)}{N}\Big)\Big), \quad i = \frac{3N}{4} + 1, \ldots, N, \quad x_i \in [\sigma_2, l], \quad \sigma_2 = \frac{3l}{4}, \end{cases}$$

where

$$\sigma_1 = \min\left\{\frac{l}{4}, -\alpha^{-1}\varepsilon \ln \varepsilon\right\}, \sigma_2 = l - \sigma_1.$$

The approach of generating difference method is through the integral identity:

$$\tau^{-1} \int_{t_{j-1}}^{t_{j+1}} \left[\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} L_1\left[\frac{\partial^2 u}{\partial t^2}\right] \varphi_i(x)dx + \hbar_i^{-1} \int_{x_{j-1}}^{x_{j+1}} L_2[u]\varphi_i(x)dx\right] \chi_j(t)dt =$$

$$= \tau^{-1} \int_{t_{j-1}}^{t_{j+1}} \left[\hbar_i^{-1} \left[\int_{x_{j-1}}^{x_{j+1}} f(x,t)\varphi_i(x)dx\right]\right] \chi_j(t)dt. \tag{9}$$

where the basis functions

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) \equiv \frac{(x - x_{i-1})}{h_i}, & x \in [x_{i-1}, x_i], \\ \varphi_i^{(2)}(x) \equiv \frac{(x_{i+1} - x)}{h_{i+1}}, & x \in [x_i, x_{i+1}], \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

and

$$\chi_j(t) = \begin{cases} \chi_j^{(1)}(t) \equiv \frac{(t - t_{i-1})}{\tau}, & t \in [t_{j-1}, t_j], \\ \chi_j^{(2)}(t) \equiv \frac{(t_{i+1} - t)}{\tau}, & t \in [t_j, t_{j+1}], \\ 0, & t \notin (t_{j-1}, t_{j+1}). \end{cases}$$

Applying interpolating quadrature rules in [1], we find:

$$\ell u_i^j = \ell_1(u_{\bar{t}t,i}^j) + \ell_2(u_i^j) + R_j^i = f_i^j,$$

where

$$\ell_1(u_{\bar{t}t,i}^j) = -\varepsilon u_{\bar{t}t\bar{x}\hat{x},i}^j + a_i u_{\bar{t}t}^j, \quad \ell_2(u_i^j) = -\varepsilon u_{\bar{x}\hat{x},i} + b_i^j u_i^j.$$

Here the remainder terms are denoted by

$$R_j^i = \varepsilon(R^{(0)})_x + R^{(1)},$$

where

$$R^{(0)} = R_1^{(0)}, \quad R^{(1)} = R^* + R_1^{(1)} - R_2,$$

$$R^* = \tau^{-1} \int_{t_{j-1}}^{t_{j+1}} R_i^* \chi_j(t)dt, \quad R_i^* = R_1^*(t) + R_2^*(t) - R_3^*(t),$$

and

$$R_1^{(0)} = \tau^{-1} \int_{t_{j-1}}^{t_{j+1}} d\eta \left[\frac{\partial^2 u(x_i,\eta)}{\partial t^2}\right]_x \left[\int_{\eta}^{t_{j+1}} T_1(t-\eta)\chi_j(t)dt - T_1(t_j-\eta)\right],$$

$$R_1^{(1)} = \tau^{-1} \int_{t_{j-1}}^{t_{j+1}} d\eta b(x_i,\eta) \left[\frac{\partial^2 u(x_i,\eta)}{\partial t^2}\right]_x \left[\int_{\eta}^{t_{j+1}} T_1(t-\eta)\chi_j(t)dt - T_1(t_j-\eta)\right],$$

$$R_2 = \tau^{-1} \int_{t_{j-1}}^{t_{j+1}} d\eta \left[\frac{\partial^2 f(x_i,\eta)}{\partial t^2}\right]_x \left[\int_{\eta}^{t_{j+1}} T_1(t-\eta)\chi_j(t)dt - T_1(t_j-\eta)\right],$$

and in R_i^* we have

$$R_{1,i}^*(t) = \hbar_i^{-1} \int_{x_{i-1}}^{x_{j+1}} [a(x) - a(x_i)] \frac{\partial u}{\partial t} \varphi_i(x)dx$$

$$R_{2,i}^*(t) = \left[h^{-1} \int_{x_{i-1}}^{x_{i+1}} L_2[u]\varphi_i(x)dx - (-\varepsilon u_{\bar{x}\hat{x},i} + b(x_i,t)u(x_i,t))\right],$$

$$R_{3,i}^*(t) = \hbar_i^{-1} \int_{x_{j-1}}^{x_{j+1}} [f(x,t) - f(x_i,t)]\varphi_i(x)dx.$$

Then, it follows that

$$\ell u \equiv -\varepsilon u_{\bar{t}t\bar{x}\hat{x},i}^j + a_i u_{\bar{t}t}^j - \varepsilon u_{\bar{x}\hat{x},i} + b_i^j u_i^j + R = f_i^j, \quad (x,t) \in \omega. \tag{10}$$

For the initial condition (3), the following relation is written:

$$\hbar_i^{-1}\tau^{-1}\int\limits_{t_0}^{t_1}\int\limits_{x_{i-1}}^{x_{i+1}}(Lu-f)\varphi_i(x)\chi_0^{(2)}(t)dxdt=0,\quad x_i\in\omega.$$

From here, we get

$$-\varepsilon u^0_{t\overline{x}\hat{x}}+a_iu^0_t+r=\phi,\quad \phi=-\varepsilon\psi_{\overline{x}\hat{x}}+a_i\psi_i+\frac{\tau}{2}\varepsilon\varphi_{\overline{x}\hat{x}}-\frac{\tau}{2}b^0_i\varphi_i+\frac{\tau}{2}f^0_i \tag{11}$$

with the remainder term

$$r=-\varepsilon(r^{(0)})_x+r^{(1)},$$

that

$$r^{(0)}=\int\limits_{t_0}^{t_1}d\eta\left(\frac{\partial^2 f(x_i,\eta)}{\partial t^2}\right)_x\left[\int\limits_{\eta}^{t_1}T_1(t-\eta)\chi_0^{(2)}(t)dt-T_1(t_j-\eta)\right],$$

$$r^{(1)}=\int\limits_{t_0}^{t_1}R_i^*(t)\chi_0^{(2)}(t)dt$$

$$+\int\limits_{t_0}^{t_1}d\eta b(x_i,\eta)\frac{\partial^2 u(x_i,\eta)}{\partial t^2}\left[\int\limits_{\eta}^{t_1}T_1(t-\eta)\chi_0^{(2)}(t)dt-T_1(t_j-\eta)\right]$$

$$-\int\limits_{t_0}^{t_1}d\eta\frac{\partial^2 f(x_i,\eta)}{\partial t^2}\left[\int\limits_{\eta}^{t_1}T_1(t-\eta)\chi_0^{(2)}(t)dt-T_1(t_j-\eta)\right], \tag{12}$$

where the basis function $\chi_0(t)$ is given by

$$\chi_0(t)=\begin{cases}\chi_0^{(2)}(t)\equiv\dfrac{t_1-t}{\tau}, & t\in(t_0,t_1),\\ 0, & t\notin(t_0,t_1).\end{cases}$$

For the periodical boundary conditions, we use the integral identity as the form

$$\hbar_i^{-1}\tau^{-1}\int\limits_{t_{j-1}}^{t_{j+1}}\int\limits_{x_0}^{x_1}(Lu-f)\varphi_0(x)\chi_j(t)dxdt=0,\quad t\in\omega_\tau.$$

From here, we analogously find:

$$-\varepsilon u^j_{\overline{t}t\overline{x}\hat{x},N}+a_0u^j_{\overline{t}t,N}-\varepsilon u^j_{\overline{x}\hat{x},N}+b^j_0u^j_i+r^*=f^j_0,\quad t\in\omega_\tau. \tag{13}$$

Here, the remainder term is shown that

$$r^* = -\varepsilon r^{*(0)} + r^{*(1)}$$

where

$$r^{*(0)} = \tau^{-1} \int\limits_{t_{j-1}}^{t_{j+1}} d\eta \left(\frac{\partial^2 f(x_i, \eta)}{\partial t^2} \right)_{x,0}$$

$$\left[\int\limits_{\eta}^{t_1} T_1(t-\eta)\chi_j(t)dt - T_1(t_j - \eta) \right],$$

and

$$r^{*(1)} = \tau^{-1} \int\limits_{t_{j-1}}^{t_{j+1}} r_i^*(t)\chi_j(t)dt + {r_1}^{*(1)} - {r_2}^*.$$

Here, we can write the ${r_1}^{*(1)}$ and ${r_2}^*$ terms in the last the remainder term as follows

$${r_1}^{*(1)} = \tau^{-1}\hbar_i^{-1} \int\limits_{t_{j-1}}^{t_{j+1}} d\eta b(x_0, \eta) \frac{\partial^2 u(0,\eta)}{\partial t^2}$$

$$\left[\int\limits_{\eta}^{t_1} T_1(t-\eta)\chi_j(t)dt - T_1(t_j - \eta) \right],$$

$${r_2}^* = \tau^{-1}\hbar_i^{-1} \int\limits_{t_{j-1}}^{t_{j+1}} d\eta \frac{\partial^2}{\partial t^2} f_0(\eta)$$

$$\left[\int\limits_{\eta}^{t_1} T_1(t-\eta)\chi_j(t)dt - T_1(t_j - \eta) \right],$$

$$\tau^{-1} \int\limits_{t_{j-1}}^{t_j} r_i^*(t)\chi_j(t)dt = \tau^{-1} \int\limits_{t_{j-1}}^{t_{j+1}} [r_{1,i}^*(t) + r_{2,i}^*(t) - r_{3,i}^*(t)]\chi_j(t)dt,$$

where

$$r_{1,i}^*(t) = \int\limits_{x_0}^{x_1} [a(x) - a_0] \frac{\partial^2 u}{\partial t^2} {\varphi_0}^{(2)}(x)dx,$$

$$r_{2,i}^{*}(t)=\int\limits_{x_0}^{x_1} L_2u\varphi_0^{(2)}(x)dx - l_2^{(1)}[u],$$

$$r_{3,i}^{*}(t)=\int\limits_{x_0}^{x_1} [f(x,t)-f_0]\varphi_0{}^{(2)}(x)dx$$

and the basis function $\varphi_0(x)$ is defined by

$$\varphi_0(x)=\begin{cases}\varphi_0^{(2)}(x)\equiv\dfrac{(x_0-x)}{h_1}, & x_0<x<x_1,\\ \varphi_N^{(1)}(x)\equiv\dfrac{(x-x_{N-1})}{h_N}, & x_{N-1}<x<x_N,\\ 0, & x\notin(x_0,x_1)\cup(x_{N-1},x_N).\end{cases}$$

Based on the relations (10), (11) and (13), we propose the following difference scheme for the approximating (1)–(4):

$$\ell y\equiv-\varepsilon y^j_{\bar{t}t\overline{x}\hat{x},i}+a_iy^j_{\overline{t}t}-\varepsilon y_{\overline{x}\hat{x},i}+b_i^jy_i^j=f_i^j,\quad (x,t)\in\omega \tag{14}$$

$$y(x,0)=\varphi(x),\quad x\in\omega_N, \tag{15}$$

$$\ell^{(0)}y\equiv-\varepsilon y^0_{\bar{t}\overline{x}\hat{x}}+a_iy^0_t=\phi \tag{16}$$

$$y(0,t)=y(l,t),\quad y(h_1,t)=y(l+h_N,t),\quad t\in\omega_\tau, \tag{17}$$

$$\ell^{(1)}y\equiv-\varepsilon y^j_{\bar{t}t\overline{x}\hat{x},N}+a_0y^j_{\overline{t}t,N}-\varepsilon y^j_{\overline{x}\hat{x},N}+b_0^jy_i^j=f_0^j,\quad t\in\omega_\tau, \tag{18}$$

where

$$\phi=-\varepsilon\psi_{\overline{x}\hat{x}}+a_i\psi_i+\frac{\tau}{2}\varepsilon\varphi_{\overline{x}\hat{x}}-\frac{\tau}{2}b_i^0\varphi_i+\frac{\tau}{2}f_i^0,\quad i=1,2,\ldots,N;\quad j=1,2,\ldots,M.$$

4 Error Analysis

Let u be the solution of (1)–(5) and y be the solution of (14)–(18). The error function $z=y-u$ is a solution to the following discrete problem:

$$\ell_1(z^j_{\bar{t},i})+\ell_2(z_i^j)=R_j^i. \tag{19}$$

$$z(0,t_j)=0,\quad 0\le i\le N, \tag{20}$$

$$z(0,t)=z(l,t),\quad z(h_1,t)=z(l+h_N,t),\quad t\in\omega_\tau, \tag{21}$$

$$\ell^{(0)}z\equiv-\varepsilon z^0_{\bar{t}t\bar{x}\hat{x}}+a_iyz^0_t=r \tag{22}$$

$$\ell^{(1)}z\equiv-\varepsilon z^j_{\bar{t}t\bar{x}\hat{x},N}+a_0z^j_{\bar{t}t,N}-\varepsilon z^j_{\bar{x}\hat{x},N}+b^j_0yz^j_i=r^*,\quad t\in\omega_\tau, \tag{23}$$

Lemma 2 *Under the conditions:* $C_0\tau<1\left(C_0=\max\left(\frac{b^*+1}{2\alpha},\gamma_*^{-1}\right)\right)$, $1-\frac{\tau^2}{2}\geq\gamma_*>0$, *the following estimates are satisfied for the solution of the problem* (19)–(23)

$$\|z_t\|+\varepsilon\|z_{t\hat{x}}\|\leq C(N^{-2}+\tau^2). \tag{24}$$

See Duru, 2004 for proof.

5 Numerical Results

Let's write the problem (14)–(18) explicitly:

$$A_1y^{j+1}_N-C_1y^{j+1}_1+B_1y^{j+1}_2=-F_1,\quad i=1,\ldots,N-1$$

$$A_iy^{j+1}_{i+1}-C_iy^{j+1}_i+B_iy^{j+1}_{i-1}=-F_i,\quad i=1,\ldots,N-1;\quad j=2,\ldots,M-1 \tag{25}$$

$$A_Ny^{j+1}_{N-1}-C_Ny^{j+1}_N+B_Ny^{j+1}_1=-F_N,$$

$$A^*_1y^1_{i-1}-C^*_1y^1_i+B^*_1y^1_{i+1}=-F^*_i,\quad i=1,\ldots,N-1,$$

where

$$A_i=-\varepsilon\tau^{-2}\hbar_i^{-1}h_i^{-1},$$

$$B_i=-\varepsilon\tau^{-2}\hbar_i^{-1}h_{i+1}^{-1},$$

$$C_i=-\varepsilon\hbar_i^{-1}\tau^{-2}\left(h_{i+1}^{-1}+h_i^{-1}\right)-a_i\tau^{-2},$$

$$F_i=-f^j_i+\left(b^j_i-2a_i\tau^{-2}\right)y^j_i+a_i\tau^{-2}y^{j-1}_i+\varepsilon\left(2\tau^{-2}-1\right)y^j_{\bar{x}\hat{x},i}-\varepsilon\tau^{-2}y^{j-1}_{\bar{x}\hat{x},i}$$

$$A^*_1=-\varepsilon\tau^{-1}\hbar_i^{-1}h_i^{-1},$$

$$B^*_1=-\varepsilon\tau^{-1}\hbar_i^{-1}h_{i+1}^{-1},$$

$$C^*_1=-\varepsilon\hbar_i^{-1}\tau^{-1}\left(h_{i+1}^{-1}+h_i^{-1}\right)-a_i\tau^{-1},$$

$$F^*_1=\varepsilon\psi_{\bar{x}\hat{x}}-a_i\psi_i-\varepsilon\left(\frac{\tau}{2}-\tau^{-1}\right)\varphi_{\bar{x}\hat{x}}+\left(\frac{\tau}{2}-a_i\tau^{-1}\right)b^0_i\varphi_i-\frac{\tau}{2}f^0_i,$$

$$A_N = -\varepsilon\tau^{-2}\hbar_0^{-1}h_0^{-1},$$
$$B_N = -\varepsilon\tau^{-2}\hbar_0^{-1}h_1^{-1},$$
$$C_N = -\varepsilon\hbar_0^{-1}\tau^{-2}\left(h_1^{-1}+h_0^{-1}\right) - a_N\tau^{-2},$$
$$F_N = -f_0^j + (b_0^j - 2a_0\tau^{-2})y_N^j + a_i\tau^{-2}y_N^{j-1} + \varepsilon(2\tau^{-2}-1)y_{\overline{x}\hat{x},N}^j - \varepsilon\tau^{-2}y_{\overline{x}\hat{x},N}^{j-1}.$$

The linear equation system (25) will be solved by the elimination algorithm given below [16]. For these coefficients, the elimination algorithm is defined as follows

$$\alpha_2 = \frac{B_1}{C_1},\quad \gamma_2 = \frac{A_1}{C_1},\quad \beta_2 = \frac{F_1}{C_1},\quad C_i - \alpha_i A_i \neq 0,$$
$$\alpha_{i+1} = \frac{B_i}{C_i - \alpha_i A_i},\quad \beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - \alpha_i A_i},\quad \gamma_{i+1} = \frac{A_i\gamma_i}{C_i - \alpha_i A_i}\quad i = 1,\ldots,N-1,$$
$$p_{N-1} = \beta_N,\quad q_{N-1} = \beta_N + \gamma_N,$$
$$p_i = \alpha_{i+1}p_{i+1} + \beta_{i+1},\quad q_i = \alpha_{i+1}q_{i+1} + \gamma_{i+1},\quad i = N-1, dots, 1,$$
$$y_N = \frac{\beta_{N+1} + \alpha_{N+1}p_1}{1 - \alpha_{N+1}q_1 - \gamma_{N+1}},$$
$$y_i = p_i + y_N q_i,\ i = N-1,\ldots,0.$$

To test the order of uniform convergence of the samples, we define the absolute errors and the convergence rate as follows

$$e^N = \max_{0\le i<N}\left|y_i^N - y_{2i}^{2N}\right|$$

and

$$p = \frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

Example 1. Consider the following periodic problem on $D = (0,1)\times(0,1]$

$$-\varepsilon\frac{\partial^4 u}{\partial t^2\partial x^2} + (1+e^{\sin(2\pi x)})\frac{\partial^2 u}{\partial t^2} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (t+\sin(2\pi x))u = e^{-t}\sin(t)(e^{\sin(2\pi x)} + \sin(2\pi x) + t + 1)$$
$$u(x,0) = \sin(2\pi x),$$
$$\frac{\partial u}{\partial t}(x,0) = -\sin(2\pi x),$$
$$u(0,t) = u(1,t),$$
$$\frac{\partial u}{\partial x}(x,0) - \frac{\partial u}{\partial x}(x,1) = 0.$$

The obtained results are given in Table 1.

Table 1: *Maximum point-wise errors and the order of convergence rate for $N = 8$ and $M = 10$.*

ε	r_0	p_1	r_1	p_2	r_2	$\overline{p}$
2^{-3}	0.09348526	1.7747	0.02732148	1.9396	0.00712245	1.8571
2^{-4}	0.09675640	1.7142	0.02948816	1.8696	0.00806897	1.7919
2^{-5}	0.09331897	1.7085	0.02855243	1.8846	0.00773220	1.7965
2^{-6}	0.09402811	1.7444	0.0280626	1.8611	0.00772448	1.8027

Example 2. Consider the problem on $D = (0, 1) \times (0, 1]$

$$-\varepsilon\frac{\partial^4 u}{\partial t^2 \partial x^2} + (1 + e^{\cos(2\pi x)})\frac{\partial^2 u}{\partial t^2} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (t + cos(2\pi x)t)u =$$

$$= e^{-t}\sin(t)(e^{\cos(2\pi x)} + \cos(2\pi x) + t + 1)$$

$$u(x, 0) = -sin(\pi x),$$

$$\frac{\partial u}{\partial t}(x, 0) = -sin(\pi x),$$

$$u(0, t) = u(1, t),$$

$$\frac{\partial u}{\partial x}(x, 0) - \frac{\partial u}{\partial x}(x, 1) = 0.$$

The computed results are summarized in Table 2.

Table 2: *Maximum point-wise errors and the order of convergence rate for N=8 and M=10.*

ε	r_0	p_1	r_1	p_2	r_2	$\overline{p}$
2^{-3}	0.10025268	1.8574	0.02766639	1.9126	0.00734822	1.8850
2^{-4}	0.09669872	1.8467	0.02688412	1.9344	0.00703359	1.8905
2^{-5}	0.09337445	1.8413	0.02605667	1.9343	0.00681726	1.8878
2^{-6}	0.10048995	1.8736	0.02742225	1.9239	0.00722681	1.8987

6 Discussion and Conclusion

We suggested a new difference scheme to solve singularly perturbed equations. By using the energy inequalities, the stability of the solution to the continuous problem was shown. Difference schemes were constructed using interpolation quadrature rules with integral terms and weight functions as linear basis functions. The remainder term with integral form relieves the conditions on the solution, such as continuity. The linear basis functions are chosen in such a way that the method error of the terms with the highest order derivative is zero. Error analysis is performed in discrete norm and the convergence rate is $O(N^{-2} + \tau^2)$.

Different numerical methods can be used for the problem we are considering. Exponential fitted difference schemes and similarly established difference schemes on a piecewise regular network can be used. Each of these has advantages and disadvantages. The method we used in this study is advantageous in terms of memory savings and computational cost in the computer.

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EXPANSIONS OF KAMPÉ DE FÉRIET HYPERGEOMETRIC FUNCTIONS

When studying the properties of the hypergeometric functions in two variables, expansion formulas are very important, allowing one to represent a function of two variables in the form of an infinite sum of products of several hypergeometric Gaussian functions, and this in turn facilitates the process of studying the properties of functions in two variables. Burchnall and Chaundy, in 1940–41, using the symbolic method, obtained more than 15 pairs of expansions for the second-order double hypergeometric Appell and Humbert functions. In order to find expansion formulas for functions depending on three or more variables, Hasanov and Srivastava introduced symbolic operators, with the help of which they were able to expand a whole class of hypergeometric functions of several variables. Hasanov, Turaev and Choi defined so-called H-operators that make it possible to find expansions for generalized hypergeometric functions of one variable. In addition, applications of these H-operators to the expansion of the hypergeometric functions of two and three variables of second order are known. On the other hand, thanks to the Kampé de Fériet functions, solutions of the boundary value problems for some degenerate and singular partial differential equations can be written in explicit forms. In this paper, expansion formulae are obtained for the hypergeometric Kampé de Fériet functions of the superior order. Some Kampé de Fériet functions are expanded in terms of the Appell and Humbert functions as illustrative examples.

Key words: Hypergeometric function in two variables, Kampé de Fériet function, generalized hypergeometric function, expansion formula, Burchnall-Chaundy method, symbolic H-operator, Appell and Humbert hypergeometric functions.

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Кампе-де-Ферье гипергеометриялық функцияларын жіктеу формулалары

Екі айнымалы гипергеометриялық функциясын зерттеу үшін екі айнымалы функциясын бірнеше Гаусс гипергеометриялық функциялар көбейтідісінің шексіз қосындысы ретінде көрсетуге мүмкіндік беретін жіктеу формулалары өте маңызды, бұл өз кезегінде екі айнымалы функцияларының қасиеттерін зерттеу процесін жеңілдетеді. Берчнелл мен Ченди 1940-1941 жылдары символдық әдіспен Аппель мен Гумберттің екінші ретті гипергеометриялық функциялары үшін 15-тен астам жіктеу жұбын алды. Үш немесе одан да көп айнымалыларға тәуелді функциялардың жіктеу формулаларын табу үшін Хасанов және Сривастава 2006-2007 жылдары символдық операторларды енгізді, олардың көмегімен бірнеше айнымалы гипергеометриялық функцияларының бүкіл класын жіктей алды.

Алайда, бұл символдық операторлар екінші ретті гипергеометриялық функциялармен шектелді, сондықтан 2010 жылы Хасанов, Тураев және Чой жоғары ретті бір айнымалы жалпыланған гипергеометриялық функцияларын жіктеуге мүмкіндік беретін Н-операторларын енгізді. Сонымен қатар, осы Н-операторларының екі және үш айнымалы екінші ретті гипергеометриялық функцияларын жіктеуге арналған қосымшалары белгілі. Екінші жағынан, Кампе-де-Ферье функцияларының көмегімен кейбір өзгешеленген және сингулярлық дербес туындылы дифференциалдық теңдеулер үшін шеттік есептердің шешімдері айқын түрде жазылуы мүмкін. Бұл жұмыста жоғары ретті Кампе-де-Ферьенің гипергеометриялық функциялары үшін жіктеу формулалары алынды. Көрнекі мысалдар ретінде Кампе-де-Ферьенің кейбір функциялары Аппель мен Гумберттің екінші ретті гипергеометриялық функциялары бойынша жіктеледі.
Түйін сөздер: екі айнымалы гипергеометриялық функциясы, Кампе-де-Ферье функциясы, жалпыланған гипергеометриялық функция, жіктеу формуласы, Берчнелл-Чанди әдісі, символдық H-операторы, Аппель мен Гумберттің гипергеометриялық функциялары

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Разложения гипергеометрических функций Кампе-де-Ферье

Для исследования гипергеометрической функции двух переменных очень важны формулы разложения, которые позволяют представить функцию двух переменных в виде бесконечной суммы произведений нескольких гипергеометрических функций Гаусса, а это, в свою очередь, облегчает процесс изучения свойств функций двух переменных. Берчнелл и Ченди в 1940–41 гг. символическим методом получили более 15 пар разложений для гипергеометрических функций Аппеля и Гумберта второго порядка. Чтобы найти формулы разложения функций, зависящих от трех и более переменных, Хасанов и Сривастава в 2006–07 гг. ввели символические операторы, с помощью которых они смогли разложить целый класс гипергеометрических функций нескольких переменных. Однако, эти символические операторы ограничились гипергеометрическими функциями второго порядка, поэтому в 2010 г. Хасанов, Тураев и Чой ввели в рассмотрение так называемые Н-операторы, позволяющие разложить обобщенные гипергеометрические функции одной переменной высокого порядка. Кроме того, известны приложения этих Н-операторов к разложению гипергеометрических функций двух и трех переменных второго порядка. С другой стороны, благодаря функциям Кампе-де-Ферье решения краевых задач для некоторых вырождающихся и сингулярных уравнений в частных производных могут быть записаны в явном виде. В данной работе получены формулы разложения для гипергеометрических функций Кампе-де-Ферье высшего порядка. В качестве наглядных примеров, некоторые функции Кампе-де- Ферье разложены по гипергеометрическим функциям Аппеля и Гумберта второго порядка.
Ключевые слова: гипергеометрическая функция двух переменных, функция Кампе-де-Ферье, обобщенная гипергеометрическая функция, формула разложения, метод Берчнелла-Ченди, символический H-оператор, гипергеометрические функции Аппеля и Гумберта.

1 Introduction

A great interest in the theory of hypergeometric functions (that is, hypergeometric functions of one, two and more variables) is motivated essentially by the fact that solutions of many applied problems involving thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, and quantum mechanics and potential theory are obtainable with the help of hypergeometric (higher and special or transcendent) functions. Such kinds of functions are often referred to as special functions of mathematical physics.

A sum of the following power series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\qquad |z|<1, \tag{1}$$

is known as the Gaussian hypergeometric function, where a, b, c are independent of z. We call a, b, c the parameters of the hypergeometric function; they are arbitrary complex numbers with $c\neq 0,-1,-2,\ldots$. Here $(\nu)_n$ is a Pochhammer symbol:

$$(\nu)_0:=1,\quad (\nu)_n:=\nu(\nu+1)...(\nu+n-1)=\frac{\Gamma(\nu+n)}{\Gamma(\nu)};$$

$\Gamma(z)$ is a well-known gamma function.

The great success of the theory of hypergeometric series in one variable has stimulated the development of a corresponding theory in two and more variables. Appell has defined, in 1880, four series, F_1 to F_4 (cf. equations (3) to (6) *infra*) which are all analogous to Gauss' $F(a,b;c;z)$. Picard has pointed out that one of these series is intimately related to a function studied by Pochhammer in 1870, and Picard and Goursat also constructed a theory of Appell's series which is analogous to Riemann's theory of Gauss' hypergeometric series. P.Humbert has studied confluent hypergeometric series in two variables (cf. equations (7) to (13) *infra*). An exposition of the results of the French school together with references to the original literature are to be found in the monograph by Appell and Kampé de Fériet [1], which is the standart work on the subject until the middle of the last century. This work also contains an extensive bibliography of all relevant papers up to 1926. In 1953, a five-volume book on special functions appeared, the first book [2] of which, dedicated to hypergeometric functions, contains brief but very clearly written conclusions of the main properties of the functions under study, from which a person who does not know the theory can study it. But it also includes numerous lists of formulas relating to the most important special functions. Currently, the monograph by Srivastava and Karlsson [3], published in 1985, is highly respected among researchers.

Gauss' hypergeometric series (1) has been generalized [4] by the introduction of p parameters of the nature of a, b, and of q parameters of the nature of c. The ensuing series

$${}_pF_q\left[\begin{array}{l}a_1,...,a_p;\\ b_1,...,b_q;\end{array} z\right]:={}_pF_q\left(a_1,...,a_p;b_1,...,b_q;z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{z^n}{n!}, \tag{2}$$

is known as the generalized hypergeometric series of the order r, where $r=\max(p,\,q+1)$.

Just as the Gaussian series $F(a,b;c;z)$ was generalized to ${}_pF_q$ by increasing the numbers of the numerator and denominator parameters, the four Appell series were unified and generalized by Kampé de Fériet [5] who defined a general hypergeometric series in two variables [1, p. 150, equation (29)]. The notation introduced by Kampé de Fériet for his double hypergeometric series of superior order was subsequently abbreviated by Burchnall and Chaundy [6, p. 112]. We recall here (see Section 2 *infra*) the definition of a more general double hypergeometric series (than the one defined by Kampé de Fériet) in a slightly modified notation (see, for example, [7]). Although the double hypergeometric series defined in [7] reduces to the Kampé de Fériet series in the special case, yet it is usually referred to in the literature as the Kampé de Fériet series.

There are many works devoted to the Kampé de Fériet hypergeometric function, but here we note only some works in which the issues of convergence [8] and reducibility [9, 10] are studied, summation formulas [11] and transformations [12], integral representations [13] are obtained. .

The hypergeometric function in one variable has been sufficiently fully studied in all respects, therefore, for the study of the hypergeometric function of two variables, expansion formulas are very important, which allow us to represent the hypergeometric function of two variables in the form of an infinite sum of products of either two hypergeometric functions in one variable, or relatively well-studied functions in two variables , and this, in turn, facilitates the process of studying the properties of the functions in two variables under consideration.

In 1940–41, Burchnall and Chaundy [6,14] systematically presented a number of expansion and decomposition formulas for double hypergeometric functions (of second order only) in series of simpler hypergeometric functions. In 2006–07, Hasanov and Srivastava [15, 16] introduced operators generalizing the Burcnall-Chaundy operators and found expansion formulas for many triple hypergeometric functions, and they proved recurrent formulas when the dimension of hypergeometric function exceeds three. Hasanov, Turaev and Choi [17] defined the so-called H-operators that make it possible to find expansions for generalized hypergeometric functions of one variable. In addition, applications of these operators to the expansion of second-order hypergeometric functions in two and three variables are known [18, 19]. When the order of the hypergeometric function exceeds two, an applications of H-operators to some third- and fourth-order Kampé de Fériet functions can be found in the works [20, 21]. Namely, thanks to the Kampé de Fériet functions, solutions to boundary value problems for some degenerate partial differential equations can be written in explicit forms [22, 23].

This work is devoted to the application of one-dimensional and two-dimensional H-operators to hypergeometric Kampé de Fériet functions of superior order. To give an examples, some Kampé de Fériet functions are expanded in terms of the Appell and Humbert hypergeometric functions.

2 Hypergeometric functions in two variables and its generalizations

Hypergeometric Appell functions are usually defined as the sums of the following series [24]:

$$F_1\left(a, b, b'; c; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| < 1, \quad |y| < 1, \tag{3}$$

$$F_2\left(a, b, b'; c, c'; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| + |y| < 1, \tag{4}$$

$$F_3\,(a,a',b,b';c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1,\ \ |y|<1, \tag{5}$$

$$F_4\,(a,b;c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\quad \sqrt{x}+\sqrt{y}<1, \tag{6}$$

Seven confluent forms of the four Appell functions were defined by Humbert [25]:

$$\Phi_1\,(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1, \tag{7}$$

$$\Phi_2\,(b,b';c;x,y)=\sum_{m,n=0}^{\infty}\frac{(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{8}$$

$$\Phi_3\,(b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{9}$$

$$\Psi_1\,(a,b;c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1, \tag{10}$$

$$\Psi_2\,(a;c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{11}$$

$$\Xi_1\,(a,a',b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1, \tag{12}$$

$$\Xi_2\,(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<1. \tag{13}$$

In equations (3) – (13) all parameters $a,a',b,b';c,c'$ and variables x,y take complex values and, as usual, the denominator parameters $c,c'\neq 0,-1,-2,...$ are neither zero nor a negative integer.

The Kampé de Fériet hypergeometric function of order (P,Q) is defined by [7](see also [3, p.27]:

$$F^{p:\,k;l}_{q:\,r;s}\left[\begin{array}{l}(a_p):\,(b_k);\,(c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right]=\sum_{m,n=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_{m+n}\prod\limits_{j=1}^{k}(b_j)_m\prod\limits_{j=1}^{l}(c_j)_n}{\prod\limits_{j=1}^{q}(\alpha_j)_{m+n}\prod\limits_{j=1}^{r}(\beta_j)_m\prod\limits_{j=1}^{s}(\gamma_j)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{14}$$

$$= \sum_{m,n=0}^{\infty} \frac{[a,p]_{m+n}\,[b,k]_m\,[c,l]_n}{[\alpha,p]_{m+n}\,[\beta,k]_m\,[\gamma,l]_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

where p, q, k, l, r, s are nonnegative integers with $p + k + l \neq 0$ and $q + r + s \neq 0$; $P = \max(p + k,\ p + l)$ is an order by x, and $Q = \max(q + r + 1,\ q + s + 1)$ is an order by y; $(a_p) := (a_1, ..., a_p)$ is a vector of p components; for convergence,

1. $p + k < q + r + 1,\ \ p + l < q + s + 1,\ \ |x| < \infty,\ \ |y| < \infty;$

2. $p + k = q + r + 1,\ \ p + l = q + s + 1,$

$$\begin{cases} |x|^{\frac{1}{p-q}} + |y|^{\frac{1}{p-q}} < 1, & p > q, \\ \max\{|x|,\ |y|\} < 1, & p \leq q; \end{cases}$$

3. $p + k = q + r + 1,\ \ p + l < q + s + 1,\ \ p \leq q,\ \ |x| < 1,\ \ |y| < \infty;$

4. $p + k < q + r + 1,\ \ p + l = q + s + 1,\ \ p \leq q,\ \ |x| < \infty,\ \ |y| < 1.$

Hypergeometric functions of superior order are usually divided into two types. If $p+k = q+r+1 = p+l = q+s+1 = P = Q$, then the Kampé de Fériet hypergeometric function is called *a complete hypergeometric function*, otherwise, *a confluent hypergeometric function.*

3 Operator identities for the Kampé de Fériet functions of superior order

Burchnall and Chaundy [7, 14] expanded the Appell and Humbert hypergeometric functions into series in terms of simpler hypergeometric functions. Their method is based on a mutually inverse pair of symbolic operators

$$\nabla(h) = \frac{\Gamma(h)\,\Gamma(h+\delta+\sigma)}{\Gamma(h+\delta)\,\Gamma(h+\sigma)}, \Delta(h) = \frac{\Gamma(\delta+h)\,\Gamma(\sigma+h)}{\Gamma(h)\,\Gamma(\delta+\sigma+h)},$$

where $\Gamma(z)$ is a gamma function and

$$\delta \equiv x\frac{\partial}{\partial x}, \quad \sigma \equiv y\frac{\partial}{\partial y}, \tag{15}$$

These symbolic forms are used to obtain a large number of expansions of Appell's functions in terms of each other, of Appell's functions in terms of products of ordinary hypergeometric functions, or vice versa. By these methods Burchnall and Chaundy obtained 15 pairs of expansions involving Appell's functions $F_1 - F_4$, defined in (3) – (6), and ordinary hypergeometric functions, и обычные гипергеометрические функции, as well as a significant number of expansions containing confluent hypergeometric functions Φ, Ψ and Ξ, defined by equations (7) – (13). Burchnall-Chaundy expansions have applications in applied problems. For example, it is precisely thanks to the expansion of the Appel function F_2 [14, equation (26)] solutions to several boundary value problems were written in explicit forms [26, 27].

However, the Burchnall-Chaundy method was limited to second order functions in two variables. In order to find expansion formulas for generalized hypergeometric functions (of superior order) defined in (2), Hasanov, Turaev and Choi [17] first introduced symbolic operators:

$$H_x\left(A,B\right)=\frac{\Gamma\left(B\right)\Gamma\left(A+\delta\right)}{\Gamma\left(A\right)\Gamma\left(B+\delta\right)}, \tag{16}$$

$$H_y\left(A,B\right)=\frac{\Gamma\left(B\right)\Gamma\left(A+\sigma\right)}{\Gamma\left(A\right)\Gamma\left(B+\sigma\right)}, \tag{17}$$

where δ и σ are defined in (15). Hereinafter, A and B can take complex values, and $A\neq B$ and $A,\ B\neq 0,\ -1,-2,....$. Currently, multidimensional analogues of the operators (16) and (17) are known: in the work [18] a symbolic operator of the form

$$H_{x,y}\left(A,B\right)=\frac{\Gamma\left(B\right)\Gamma\left(A+\delta+\sigma\right)}{\Gamma\left(A\right)\Gamma\left(B+\delta+\sigma\right)}, \tag{18}$$

are applied to the expansion of second order confluent hypergeometric functions in two variables included in Horn's list [2, p. 225] . In the work [28], using the one-dimensional operator (16), infinite summation formulas for second order hypergeometric Lauricella functions in three variables are proved. H-operators are applied to the transformation of second order hypergeometric functions in three and four variables [29].

Further study of the properties of symbolic H-operators showed that they can be used in the expansion of hypergeometric functions of several variables when their order exceeds 2.

The main result of this work is

Theorem 1 *The following operator identities are valid:*

$$F^{p:k+1;l}_{q:r+1;s}\left[\begin{array}{l}(a_p):\ (b_k),A;\ (c_l);\\ (\alpha_q):(\beta_r),B;(\gamma_s);\end{array}x,y\right]=H_x\left(A,\ B\right)F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):\ (b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right], \tag{19}$$

$$F^{p:k;l+1}_{q:r;s+1}\left[\begin{array}{l}(a_p):\ (b_k);\ (c_l),C;\\ (\alpha_q):(\beta_r);(\gamma_s),D;\end{array}x,y\right]=H_y\left(C,\ D\right)F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right], \tag{20}$$

$$F^{p:k+1;l+1}_{q:r+1;s+1}\left[\begin{array}{l}(a_p):\ (b_k),A;\ (c_l),\ C;\\ (\alpha_q):\ (\beta_r),B;(\gamma_s),D;\end{array}x,y\right]$$

$$=H_x\left(A,\ B\right)H_y\left(C,\ D\right)F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right], \tag{21}$$

$$F^{p+1:k;l}_{q+1:r;s}\left[\begin{array}{l}(a_p),\ A:(b_k);\ (c_l);\\ (\alpha_q),B:(\beta_r);(\gamma_s);\end{array}x,y\right]=H_{x,y}(A,\ B)F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):\ (b_k);\ (c_l);\\ (\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right]. \tag{22}$$

Proof. Using the Burchnall-Chaundy method [7], we have

$$F^{p:k+1;l}_{q:r+1;s}\left[\begin{array}{l}(a_p):(b_k),\ A;\ (c_l);\\(\alpha_q):(\beta_r),B;(\gamma_s);\end{array}x,y\right]=\frac{\Gamma(A+\delta)\Gamma(B)}{\Gamma(A)\Gamma(B+\delta)}F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\ (c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right],$$

$$F^{p:k;l+1}_{q:r;s+1}\left[\begin{array}{l}(a_p):(b_k);\ (c_l),C;\\(\alpha_q):(\beta_r);(\gamma_s),D;\end{array}x,y\right]=\frac{\Gamma(C+\sigma)\Gamma(D)}{\Gamma(C)\Gamma(D+\sigma)}F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\ (c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right],$$

$$F^{p:k+1;l+1}_{q:r+1;s+1}\left[\begin{array}{l}(a_p):(b_k),A;\ (c_l),\ C;\\(\alpha_q):(\beta_r),B;(\gamma_s),D;\end{array}x,y\right]$$
$$=\frac{\Gamma(A+\delta)\Gamma(B)}{\Gamma(A)\Gamma(B+\delta)}\frac{\Gamma(C+\sigma)\Gamma(D)}{\Gamma(C)\Gamma(D+\sigma)}F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\ (c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right],$$

$$F^{p+1:k;l}_{q+1:r;s}\left[\begin{array}{l}(a_p),\ A:(b_k);(c_l);\\(\alpha_q),B:(\beta_r);\ \ \gamma_s;\end{array}x,y\right]=\frac{\Gamma(A+\delta+\sigma)\Gamma(B)}{\Gamma(A)\Gamma(B+\delta+\sigma)}F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\ (c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array}x,y\right].$$

Next, using the definitions of H-operators, we obtain the required identities. Q.E.D.

Corollary 1 *If $k=l$ and $r=s$, then the Kampé de Fériet function is expressed by the generalized hypergeometric function of $x+y$:*

$$F^{p:1;1}_{q:1;1}\left[\begin{array}{l}(a_p):A;\ C;\\(\alpha_q):B;\ D;\end{array}x,y\right]=H_x\left(A,\ B\right)H_y\left(C,\ D\right){}_pF_q\left[(a_p);(\alpha_q);x+y\right],\tag{23}$$

$$F^{p:1;0}_{q:1;0}\left[\begin{array}{l}(a_p):\ A;\ -;\\(\alpha_q):B;\ -;\end{array}x,y\right]=H_x\left(A,\ B\right){}_pF_q\left[(a_p);(\alpha_q);x+y\right],\tag{24}$$

$$F^{p:0;1}_{q:0;1}\left[\begin{array}{l}(a_p):-;\ C;\\(\alpha_q):-;\ D;\end{array}x,y\right]=H_y\left(C,\ D\right){}_pF_q\left[(a_p);(\alpha_q);x+y\right],\tag{25}$$

where ${}_pF_q$ is a generalized hypergeometric function, defined in (2).

Corollary 2 *If $p=q=1$, then the Kampé de Fériet function is expressed through the product of two generalized hypergeometric functions:*

$$F^{1:k;l}_{1:r;s}\left[\begin{array}{l}A:(b_k);\ (c_l);\\B:(\beta_r);(\gamma_s);\end{array}x,y\right]=H_{x,y}(A,\ B)\ {}_kF_r\left[(b_k);\ (\beta_r);x\right]\ \ {}_lF_s\left[(c_l);\ (\gamma_s);y\right],\tag{26}$$

where ${}_kF_r$ and ${}_lF_s$ are generalized hypergeometric functions, defined in (2).

Symbolic forms (19) – (26) are used to obtain a large number of expansions of Kampé de Fériet functions in terms of each order of lower order, of Kampé de Fériet functions in terms of products of two generalized hypergeometric functions:

$$F^{p:k+1;l}_{q:r+1;s}\left[\begin{array}{l}(a_p):(b_k),A;\ (c_l);\\(\alpha_q):(\beta_r),B;(\gamma_s);\end{array}x,y\right]=\sum_{m=0}^{\infty}(-1)^m\frac{(B-A)_m}{m!\,(B)_m}\times$$

$$\times \frac{[a,p]_m\,[b,k]_m}{[\alpha,q]_m\,[\beta,r]_m} x^m F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+m:(\,b_k)+m;\,(c_l);\\(\alpha_q)+m:(\,\beta_r)+m;(\gamma_s);\end{array} x,y\right], \tag{27}$$

$$F^{p:k;l+1}_{q:r;s+1}\left[\begin{array}{l}(a_p):\,(b_k);\,(c_l),\,C;\\(\alpha_q):(\beta_r);(\gamma_s),\,D;\end{array} x,y\right] = \sum_{n=0}^{\infty}(-1)^n \frac{(D-C)_n}{n!\,(D)_n}\times$$

$$\times \frac{[a,p]_n\,[c,l]_n}{[\alpha,q]_n\,[\gamma,s]_n} y^n F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+n:\,(b_k);\,(c_l)+n;\\(\alpha_q)+n:(\beta_r);(\gamma_s)+n;\end{array} x,y\right], \tag{28}$$

$$F^{p:k+1;l+1}_{q:r+1;s+1}\left[\begin{array}{l}(a_p):(b_k),A;\,(c_l),\,C;\\(\alpha_q):(\beta_r),B;(\gamma_s),D;\end{array} x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n} \frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times \frac{[a,p]_{m+n}\,[b,k]_m\,[c,l]_n}{[\alpha,q]_{m+n}\,[\beta,r]_m\,[\gamma,s]_n} x^m y^n F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+m+n:(b_k)+m;\,(c_l)+n;\\(\alpha_q)+m+n:(\beta_r)+m;(\gamma_s)+n;\end{array} x,y\right], \tag{29}$$

$$F^{p+1:k;l}_{q+1:r;s}\left[\begin{array}{l}(a_p),\,A:(b_k);\,(c_l);\\(\alpha_q),B:(\beta_r);(\gamma_s);\end{array} x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n} \frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times \frac{[a,p]_{m+n}\,[b,k]_m\,[c,l]_n}{[\alpha,q]_{m+n}\,[\beta,r]_m\,[\gamma,s]_n} x^m y^n F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+m+n:(b_k)+m;\,(c_l)+n;\\(\alpha_q)+m+n:(\beta_r)+m;(\gamma_s)+n;\end{array} x,y\right], \tag{30}$$

$$F^{p:1;1}_{q:1;1}\left[\begin{array}{l}(a_p):\,A;\,\,C;\\(\alpha_q):B;\,\,D;\end{array} x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n} \frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times \frac{[a,p]_{m+n}}{[\alpha,q]_{m+n}} x^m y^n\,{}_pF_q\left[\begin{array}{l}(a_p)+m+n;\\(\alpha_q)+m+n;\end{array} x+y\right], \tag{31}$$

$$F^{p:1;0}_{q:1;0}\left[\begin{array}{l}(a_p):A;\,\,-;\\(\alpha_q):B;\,\,-;\end{array} x,y\right] = \sum_{m=0}^{\infty}(-1)^m \frac{(B-A)_m}{m!\,(B)_m}\frac{[a,p]_m}{[\alpha,q]_m} x^m\,{}_pF_q\left[\begin{array}{l}(a_p)+m;\\(\alpha_q)+m;\end{array} x+y\right], \tag{32}$$

$$F^{p:0;1}_{q:0;1}\left[\begin{array}{l}(a_p):-;\,\,C;\\(\alpha_q):-;\,\,D;\end{array} x,y\right] = \sum_{n=0}^{\infty}(-1)^n \frac{(D-C)_n}{n!\,(D)_n}\frac{[a,p]_n}{[\alpha,q]_n} y^n\,{}_pF_q\left[\begin{array}{l}(a_p)+n;\\(\alpha_q)+n;\end{array} x+y\right], \tag{33}$$

$$F^{1:k;l}_{1:r;s}\left[\begin{array}{l}A:(b_k);\,(c_l);\\B:(\beta_r);(\gamma_s);\end{array} x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n} \frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times \frac{[b,k]_m\,[c,l]_n}{[\beta,r]_m\,[\gamma,s]_n} x^m y^n {}_kF_r\left[(b_k)+m;\;(\beta_r)+m;x\right]\;{}_lF_s\left[(c_l)+n;\;(\gamma_s)+n;y\right]. \tag{34}$$

Hereinafter,the notation $(\nu_\lambda)+\mu$ denotes the vector $(\nu_1+\mu,...,\nu_\lambda+\mu)$.

Let us show the validity of the established expansions (27) – (34) . To give an example, by Gauss' formula [2, p. 112, equation (46)]

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, Re(c-a-b)>0, c\neq 0,-1,-2,\dots, \tag{35}$$

we have symbolically for $H_x(A,\ B)$

$$H_x\left(A,B\right)=\sum_{m=0}^{\infty}\frac{(B-A)_m(-\delta)_m}{(B)_m m!}. \tag{36}$$

Similarly, we have

$$H_y\left(C,D\right)=\sum_{n=0}^{\infty}\frac{(D-C)_n(-\delta)_n}{(D)_n n!}. \tag{37}$$

By virtue of Poole's formula [30, p. 26]

$$(-\delta)_m f(x)=(-1)^m x^m\frac{d^m f(x)}{dx^m}, \tag{38}$$

we get

$$(-\delta)_m F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p):(b_k);\,(c_l);\\(\alpha_q):(\beta_r);(\gamma_s);\end{array} x,y\right]$$

$$=(-1)^m\frac{[a,p]_m\,[b,k]_m}{[\alpha,q]_m\,[\beta,r]_m}x^m F^{p:k;l}_{q:r;s}\left[\begin{array}{l}(a_p)+m:(b_k)+m;\,(c_l);\\(\alpha_q)+m:(\beta_r)+m;(\gamma_s);\end{array} x,y\right]$$

and therefore, by virtue of (36), the symbolic form (19) leads to the expansion (27).

Further, using the summation formula for the hypergeometric Appell function F_1 in the form [3, p.34, equation (7)]

$$F_1(a,b,b';c;1,1)=\frac{\Gamma(c)\Gamma(c-a-b-b')}{\Gamma(c-a)\Gamma(c-b-b')},\;\; Re(c-a-b-b')>0,\;\; c\neq 0,-1,-2,...,$$

we have

$$H_{x,y}\left(A,B\right)=\sum_{m,n=0}^{\infty}\frac{(B-A)_{m+n}(-\delta)_m(-\sigma)_n}{(B)_{m+n}m!n!}.$$

Using the Poole's formula (38) по x и y, from the symbolic form (22) we obtain an expansion (30). The symbolic form (37) is used when proving the expansions (28), (29), (31) и (33).

Note that when proving the expansions (31) – (33) we should keep in mind the well-known identity [31, p. 52]:

$$\sum_{m,n=0}^{\infty}f(m+n)\frac{x^m}{m!}\frac{y^n}{n!}=\sum_{k=0}^{\infty}f(k)\frac{(x+y)^k}{k!}.$$

4 Examples

Some Kampé de Fériet functions can be expanded into Appell and Humbert functions (of second order!):

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix} a: A, b; C, b'; \\ c: \ B\ ; \ D\ ; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m (D-C)_n}{m!n! (B)_m (D)_n} \times$$

$$\times \frac{(a)_{m+n} (b)_m (b')_n}{(c)_{m+n}} x^m y^n F_1 (a+m+n, b+m, b'+n; c+m+n; x, y), \tag{39}$$

$$F^{1:2;2}_{0:2;2}\left[\begin{matrix} a: A, b; C, b'; \\ -: B, c; D, c'; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m (D-C)_n}{m!n! (B)_m (D)_n} \times$$

$$\times \frac{(a)_{m+n} (b)_m (b')_n}{(c)_m (c')_n} x^m y^n F_2 (a+m+n, b+m, b'+n; c+m, c'+n; x, y), \tag{40}$$

$$F^{0:3;3}_{1:1;1}\left[\begin{matrix} -: A, a, b; C, a', b'; \\ c: \quad B; \quad D\ ; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m (D-C)_n}{m!n! (B)_m (D)_n} \times$$

$$\times \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}} x^m y^n F_3 (a+m, a'+n, b+m, b'+n; c+m+n; x, y), \tag{41}$$

$$F^{2:1;1}_{0:2;2}\left[\begin{matrix} a, b: \quad A; \quad C; \\ -: B, c; D, c'; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m (D-C)_n}{m!n! (B)_m (D)_n} \times$$

$$\times \frac{(a)_{m+n} (b)_{m+n}}{(b)_m (b')_n} x^m y^n F_4 (a+m+n, b+m+n; c+m, c'+n; x, y), \tag{42}$$

$$F^{1:2;1}_{1:1;1}\left[\begin{matrix} a: A, b; C; \\ c: \ B\ ; D; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m (D-C)_n}{m!n! (B)_m (D)_n} \times$$

$$\times \frac{(a)_{m+n} (b)_m}{(c)_{m+n}} x^m y^n \Phi_1 (a+m+n, b+m; c+m+n; x, y), \tag{43}$$

$$F^{0:2;2}_{1:1;1}\left[\begin{matrix} -: A, b; C, b'; \\ c: \quad B; \quad D; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{(B-A)_m (D-C)_n}{m!n! (B)_m (D)_n} \times$$

$$\times \frac{(b)_m (b')_n}{(c)_{m+n}} x^m y^n \Phi_2 (b+m, b'+n; c+m+n; x, y), \tag{44}$$

$$F^{0:2;1}_{1:1;1}\left[\begin{array}{l} -: A,b;\ C; \\ c:\ \ B\ ;\ D; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n\,(b)_m}{m!n!\,(B)_m\,(D)_n\,(c)_{m+n}}\times$$

$$\times x^m y^n \Phi_3\left(b+m;c+m+n;x,y\right), \tag{45}$$

$$F^{1:\,2;\,1}_{0:\,2;\,2}\left[\begin{array}{l} a: A,\ b;\quad C; \\ -: B,c; D,c'; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m}{(c)_m\,(c')_n}x^m y^n \Psi_1\left(a+m+n,b+m;c+m,c'+n;x,y\right), \tag{46}$$

$$F^{1:\,1;\,1}_{0:\,2;\,2}\left[\begin{array}{l} a:\quad A;\quad C; \\ -: B,c; D,c'; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{(a)_{m+n}}{(c)_m\,(c')_n}x^m y^n \Psi_2\left(a+m+n;c+m,c'+n;x,y\right), \tag{47}$$

$$F^{0:3;2}_{1:1;1}\left[\begin{array}{l} -: A,a,b;\ C,a'; \\ c:\qquad B;\qquad D; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{(a)_m\,(a')_n\,(b)_m}{(c)_{m+n}}x^m y^n \Xi_1\left(a+m,a'+n,b+m;c+m+n;x,y\right), \tag{48}$$

$$F^{0:3;1}_{1:1;1}\left[\begin{array}{l} -: A,a,b;\ C; \\ c:\qquad B; D; \end{array} x,y\right]=\sum_{m,\,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_m\,(D-C)_n}{m!n!\,(B)_m\,(D)_n}\times$$

$$\times\frac{(a)_m\,(b)_m}{(c)_{m+n}}x^m y^n \Xi_2\left(a+m,b+m;c+m+n;x,y\right), \tag{49}$$

$$F^{2:1;1}_{2:0;0}\left[\begin{array}{l} a,A:\ b;\ b'; \\ c,B: -; -; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_{m+n}}x^m y^n F_1\left(a+m+n,b+m,b'+n;c+m+n;x,y\right), \tag{50}$$

$$F^{2:1;1}_{1:1;1}\left[\begin{array}{l} a,A: b; b'; \\ \quad B: c; c'; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_m\,(c')_n}x^m y^n F_2\left(a+m+n,b+m,b'+n;c+m,c'+n;x,y\right), \tag{51}$$

$$F^{1:2;2}_{2:0;0}\left[\begin{matrix} A: a,b;a',b'; \\ B,c: \quad -; \quad -; \end{matrix}\ x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{(c)_{m+n}}x^m y^n F_3\left(a+m,a'+n,b+m,b'+n;c+m+n;x,y\right), \tag{52}$$

$$F^{3:0;0}_{1:1;1}\left[\begin{matrix} a,b,A: -;-; \\ B:\ c;\ c'; \end{matrix}\ x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_{m+n}}{(c)_m\,(c')_n}x^m y^n F_4\left(a+m+n,b+m+n;c+m,c'+n;x,y\right), \tag{53}$$

$$F^{2:1;0}_{2:0;0}\left[\begin{matrix} a,A:\ b;-; \\ c,B: -;-; \end{matrix}\ x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m}{(c)_{m+n}}x^m y^n \Phi_1\left(a+m+n,b+m;c+m+n;x,y\right), \tag{54}$$

$$F^{1:1;1}_{2:0;0}\left[\begin{matrix} A:\ b;\ b'; \\ c,B: -;-; \end{matrix}\ x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(b)_m\,(b')_n}{(c)_{m+n}}x^m y^n \Phi_2\left(b+m,b'+n;c+m+n;x,y\right), \tag{55}$$

$$F^{1:1;0}_{2:0;0}\left[\begin{matrix} A:\ b;-; \\ c,B: -;-; \end{matrix}\ x,y\right] = \sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}\,(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(b)_m}{(c)_{m+n}}x^m y^n \Phi_3\left(b+m;c+m+n;x,y\right), \tag{56}$$

$$F^{2:1;0}_{1:1;1}\left[\begin{matrix} a,A: b;-; \\ B: c;\ c'; \end{matrix}\ x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}\,(b)_m}{(c)_m\,(c')_n}x^m y^n \Psi_1\left(a+m+n,b+m;c+m,c'+n;x,y\right), \tag{57}$$

$$F^{2:0;0}_{1:1;1}\left[\begin{matrix} a,A: -;-; \\ B:\ c;\ c'; \end{matrix}\ x,y\right] = \sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_{m+n}}{(c)_m\,(c')_n}x^m y^n \Psi_2\left(a+m+n;c+m,c'+n;x,y\right), \tag{58}$$

$$F^{1:2;1}_{2:0;0}\left[\begin{array}{c} A:a,b;a';\\ c,B:\quad -;-; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_m\,(a')_n\,(b)_m}{(c)_{m+n}}x^m y^n \Xi_1\left(a+m,a'+n,b+m;c+m+n;x,y\right), \tag{59}$$

$$F^{1:2;0}_{2:0;0}\left[\begin{array}{c} A:a,b;-;\\ c,B:\quad -;-; \end{array} x,y\right]=\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{(B-A)_{m+n}}{m!n!\,(B)_{m+n}}\times$$

$$\times\frac{(a)_m\,(b)_m}{(c)_{m+n}}x^m y^n \Xi_2\left(a+m,b+m;c+m+n;x,y\right). \tag{60}$$

Note, these expansions (27) – (34) and (39) – (60) can be proved without symbolic methods by comparing coefficients of equal powers of x and y on both sides. Indeed, for example, consider the expansion (27), the right-hand side of which we denote by

$$K:=\sum_{j=0}^{\infty}(-1)^j\frac{(B-A)_j}{j!\,(B)_j}\frac{[a,p]_j\,[b,k]_j}{[\alpha,p]_j\,[\beta,k]_j}x^j F^{p:k;l}_{q:r;s}\left[\begin{array}{c} a_p+j:b_k+j;\,c_l;\\ \alpha_q+j:\beta_r+j;\gamma_s; \end{array} x,y\right]. \tag{61}$$

Using the definition of the Kampé de Fériet function (14), the infinite sum (61) we represent in the form

$$K:=\sum_{n=0}^{\infty}\frac{[c,l]_n}{n!\,[\gamma,s]_n}y^n\sum_{j,m=0}^{\infty}\frac{(-1)^j\,(B-A)_j}{j!\,(B)_j}\frac{[a,p]_{m+n+j}\,[b,k]_{m+j}}{(m-j)!\,[\alpha,p]_{m+n+j}\,[\beta,k]_{m+j}}x^{m+j}.$$

Considering the following easily verifiable equality for power series

$$\sum_{j,m=0}^{\infty}A(j,m)x^{j+m}=\sum_{m=0}^{\infty}\sum_{j=0}^{m}A(j,\,m-j)x^m$$

and the definition of the Gaussian hypergeometric function, we obtain

$$K:=\sum_{m=0}^{\infty}\frac{[a,p]_{m+n}\,[b,k]_m}{m!\,[\alpha,p]_{m+n}\,[\beta,k]_m}F\left(-m,B-A;B;1\right)x^m\sum_{n=0}^{\infty}\frac{[c,l]_n}{n!\,[\gamma,s]_n}y^n.$$

Hence, by virtue of the Gauss' formula (35) for the value of the hypergeometric function at unity, we have

$$K:=\sum_{m,\,n=0}^{\infty}\frac{[a,p]_{m+n}\,[b,k]_m\,(A)_m\,[c,l]_n}{m!n!\,[\alpha,p]_{m+n}\,[\beta,k]_m\,(B)_m\,[\gamma,s]_n}x^m y^n.$$

Q.E.D.

5 Applications

Using the properties (expansion formulas, transformations) of the Kampé de Fériet hypergeometric functions, the authors of the works [21] and [22] managed to write an explicit solutions to Cauchy problems for hyperbolic equations of the second kind with single line

$$y^m u_{xx} - u_{yy} - \lambda^2 y^m u = 0, \;\; -1 < m < 0, \;\; \lambda \in R, \;\; y > 0$$

and two lines

$$y^m u_{xx} - x^n u_{yy} = 0, \;\; -1 < m, n < 0, x > 0, \;\; y > 0$$

of the degeneration, respectively.

Applications of Kampé de Fériet functions and other hypergeometric functions can be found in more recent works [10–13] and [32].

6 Conclusion

In conclusion, we note that in 1940–41, Burchnall and Chaundy found expansion formulas for a certain class of hypergeometric functions of two variables of order 2. In this paper, expansion formulas were established for hypergeometric Kampé de Fériet functions, the order of which exceeds 2. As examples, some the Kampé de Fériet functions are expanded into the well-known hypergeometric functions of Appel and Humbert. Applications of Kampé de Fériet hypergeometric functions to finding explicit solutions of boundary value problems for degenerate partial differential equations are indicated.

For the future, it would be interesting to solve the problem of decomposing a triple hypergeometric functions $F^{(3)}[x, y, z]$ of superior order (for definition, see [3, p. 44, equation (14)]) into known lower order hypergeometric functions of one, two, and three variables.

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ON THE OPTIMAL DISCRETIZATION OF THE SOLUTION POISSON'S EQUATION

The paper studies the problem of discretizing the solution of the Poisson equation with the right-hand side f belonging to the multidimensional periodic Sobolev class. The research methodology is based on considering the problem of discretizing the solution of the Poisson equation as one of the concretizations of the general problem of optimal recovery of the operator Tf and using well-known statements of approximation theory. Within the framework of this general optimal recovery problem, we first estimate from above the smallest discretization error δ_N of the solution of the Poisson equation in the Hilbert metric using the discretization operator $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ constructed from a finite set of Fourier coefficients of the function f. A lower estimate, coinciding in order with the upper estimate, for the smallest error δ_N was obtained by involving all linear functionals defined on the multidimensional Sobolev class. It should be noted that the optimal discretization operator $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ better approximates the solution under consideration in the Hilbert metric than any discretization operator constructed from values f at given points. Poisson's equation is an elliptic partial differential equation and describes many physical phenomena such as electrostatic field, stationary temperature field, pressure field and velocity potential field in hydrodynamics. Therefore, the relevance of the research conducted here is beyond doubt.

Key words: Poisson's equation, discretization operator, optimal discretization, Fourier coefficients, discretization error, linear functionals, Sobolev class

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Пуассон теңдеуі шешімін оптималды дискреттеу туралы

Жұмыста оң жағындағы f функциясы көпөлшемді периодты Соболев класына тиесілі Пуассон теңдеуінің шешімін дискреттеу есебі зерттелген. Зерттеу әдіснамасы Пуассон теңдеуінің шешімін дискреттеу есебін Tf операторын оптималды қалыптастырудың жалпы есебінің көп нақтылануларының бірі ретінде қарастыруға негізделген және жуықтаулар теориясының белгілі тұжырымдарын пайдалануға бағытталған. Осы оптималды қалыптастырудың жалпы есебі аясында алдымен Пуассон теңдеуінің шешімін f функциясының Фурье коэффициенттерінің ақырлы жиынтығы бойынша құрылған $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ дискреттеу операторымен гильберттік метрикада дискреттеуде пайда болатын ең аз δ_N қателігі жоғарыдан бағаланған. Одан әрі ең аз δ_N қателігінің, реті бойынша жоғарғы бағамен беттесетін, төменгі бағасы көпөлшемді периодты Соболев класында анықталған барлық сызықтық функцияларды қарастыру нәтижесінде алынған. $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ оптималды дискреттеу операторы қарастырылып отырылған шешімді гильберттік метрикада f функциясының берілген нүктелердегі мәндері бойынша құрылған кез келген оператордан жақсы жуықтайтынын атап өткен жөн. Пуассон теңдеуі дербес туындылы эллипстік дифференциалдық теңдеулер қатарына жатады және электростатикалық өріс, температураның стационарлық өрісі, қысым өрісі, сондай – ақ, гидродинамикадағы жылдамдық потенциалының өрісі сияқты біраз физикалық құбылыстарды сипаттайды. Сондықтан, осы жұмыста жүргізілген зерттеудің өзектілігі ешқандай күмән туғызбайтыны сөзсіз.

Түйін сөздер: Пуассон теңдеуі, дискреттеу операторы, оптималды дискреттеу, Фурье коэффициенттері, дискреттеу қателігі, сызықтық функционалдар, Соболев класы

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Об оптимальной дискретизации решения уравнения Пуассона

В работе изучена задача дискретизации решения уравнения Пуассона с правой частью f принадлежащей многомерному периодическому классу Соболева. Методология исследования основана на рассмотрении задачи дискретизации решения уравнения Пуассона как одной из конкретизации общей задачи оптимального восстановления оператора Tf и в использовании известных утверждений теории приближений. В рамках этой общей задачи оптимального восстановления сначала оценена сверху наименьшая погрешность δ_N дискретизации решения уравнения Пуассона в гильбертовой метрике с помощью оператора дискретизации $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$, построенного по конечному набору коэффициентов Фурье функции f. Оценка снизу, совпадающая по порядку с оценкой сверху, наименьшей погрешности δ_N получена в результате привлечения всех линейных функционалов, определенных на многомерном периодическом классе Соболева. Следует отметить, что оптимальный оператор дискретизации $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ лучше приближает в гильбертовой метрике рассматриваемое решение, чем любой оператор дискретизации, построенный по значениям f в заданных точках. Уравнение Пуассона является эллиптическим дифференциальным уравнением в частных производных и описывает многие физические явления такие, как электростатическое поле, стационарное поле температуры, поле давления и поле потенциала скорости в гидродинамике. Поэтому актуальность проведенного здесь исследования не вызывает сомнений.

Ключевые слова: уравнение Пуассона, оператор дискретизации, оптимальная дискретизация, коэффициенты Фурье, погрешность дискретизации,линейные функционалы, класс Соболева

1 Introduction

The paper considers the Poisson equation

$$\triangle u \equiv \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_s^2} = f \tag{1}$$

with the right -hand side f from the multidimensional periodic Sobolev class $W_2^r \equiv W_2^r[0,1]^s$ with parameters $r > 0$ and $s \in \mathbb{N}\backslash\{1\}$, where $\triangle$ is the Laplace operator, $s \in \{2, 3, \ldots\}, x = (x_1, \ldots, x_s), u = u(x), f = f(x)$.

It is easy to check that if $r > s/2$ and $\hat{f}(0) \neq 0$ are true, then for any boundary condition there is a function $\omega = \omega(x) \in C[0,1]^s$ with $\triangle\omega = 1$ on $[0,1]^s$ such that the solution to equation (1) has the form

$$u_\omega(x; f) = \omega(x)\hat{f}(0) - \frac{1}{4\pi^2} \sum_{m \in Z^s}^{*} \frac{\hat{f}(m)}{(m,m)} \exp\{-2\pi i(m,x)\}, \tag{2}$$

here and everywhere below, the $*$ sign above the sum sign means that the vector $m = (0, \ldots, 0) \in Z^s$ does not participate in the summation. Conversely, any function of the form (2) satisfies equation (1). Since the multiple functional series from (2) is an infinite object, the problem arises of discretizing (approximating) the solution with a finite object and establishing the accuracy of the discretization error.

The first result on discretization of the solution $u_\omega(x;f)$ was obtained in [1] under the condition that the right-hand side of (1) is odd and a one-periodic function $f(x) = f(x_1,\ldots,x_s)$ on each variable $x_1,\ldots,x_s$ belonging to the Korobov class. There a discretization operator is proposed, constructed on the value of the function at the points

$$(\{a_1k/N\},\ldots,\{a_sk/N\})\,, k\in\{1,\ldots,N\} \tag{3}$$

and approximating solution

$$u(x;f) = -\frac{1}{4\pi^2}\sum_{m\in Z^s}^{*}\frac{\hat{f}(m)}{(m,m)}\exp\{-2\pi i(m,x)\}$$

with accuracy

$$\mathcal{O}\left(\frac{(\ln N)^{r\beta/2+s}}{N^{(r-1)/2+1/s}}\right), \tag{4}$$

where $\{d\}$−the fractional part of the number d, $a_1,\ldots,a_s$ is the optimal coefficients on modulus N and index β.

In [2], the authors, using nodes of a modified Korobov grid, constructed a discretization operator $\Lambda_N(x;f)$, that approximates solution (2) in the metric of space $L^p(2\le p\le\infty)$ with accuracy

$$\mathcal{O}\left(\frac{(\ln N)^{(r+2/s)(s-1)}}{N^{r-(1-1/p-2/s)}}\right)\quad\text{and}\quad\mathcal{O}\left(\frac{(\ln N)^{r(\beta+s)+s}}{N^r}\right) \tag{5}$$

in case $1-\frac{1}{p}-\frac{2}{s}>0$ and $1-\frac{1}{p}-\frac{2}{s}\le 0$ accordingly. Comparing (4) and (5), we conclude that the estimates from (5), corresponding to the case $p=\infty$, are almost “square times” better than the estimate (4).

Further in [3], using the nodes of the Smolyak grid (see, for example, [4], [5]) and the results of the article [6], a discretization operator $(Jf)_N(x)$ was constructed such that

$$\sup_{f\in E_s^r}\|u_\omega(x;f)-(Jf)_N(x)\|_p \underset{\omega,s,r}{\ll}\begin{cases}\frac{(\ln N)^{(r+2/s)(s-1)}}{N^{r-(1-1/p-2/s)}}, 1-\frac{1}{p}-\frac{2}{s}>0,\\ \frac{(\ln N)^{(r+2/s)(s-1)}}{N^r}, 1-\frac{1}{p}-\frac{2}{s}<0,\\ \frac{(\ln N)^{(r(s-1)+2s-1-s/p}}{N^r}, 1-\frac{1}{p}-\frac{2}{s}=0.\end{cases} \tag{6}$$

Thus, in the case $1-\frac{1}{p}-\frac{2}{s}>0$ of the estimate (5) coincides with the estimate (6), and in the other two cases the differences are only in the exponents of logarithms.

In [7], to discretize solutions $u_\omega(x;f)$ with the right side $f\in E_s^r$ discretization operators were used, constructed from a finite set of Fourier coefficients of functions f, and the following results were obtained: firstly, a specific discretization operator $\Psi_N(x;f)$ was proposed, which is optimal in order in the power scale in the metric of space L^2 ; secondly, the error in calculating the Fourier coefficients was found, preserving the optimality of the discretization operator; thirdly, $\Psi_N(x;f)$ has a simpler form than the discretization operators $\Lambda_N(x;f)$ and $(Jf)_N(x;f)$. It should be noted that the order of estimating the error of the discretization operator $\Psi_N(x;f)$ in one case is better than the estimates of the errors of discretization operators $\Lambda_N(x;f)$ and $(Jf)_N(x;f)$, in the other case it coincides with them. Finally, in [8] the error in calculating the values of the function at points (3) was found, preserving the order of estimating the error of the discretization operator from [9] with the algorithm for finding optimal coefficients. The solution $u_\omega(x;f), f \in E_s^r$ discretization operators proposed in [2], [3] and [7] are optimal in order in the power scale. For the first time, the optimal discretization operator for the solution $u_\omega(x;f)$ was constructed in [9], in the case when f belongs to the multidimensional periodic Nikol'skii – Besov class $B^r_{q,\theta}(0,1)^s$ with parameters $s \in \mathbb{N}\backslash\{1\}, r > s/2, 1 \leq \theta \leq \infty, 1 \leq q \leq 2$. Here we note that to construct the optimal discretization operator from [9], the nodes of the uniform grid of a unit $s-$ dimensional cube were used.

In this work, using a finite set of Fourier coefficients of a function $f \in W_2^r$, an optimal discretization operator for the solution $u_\omega(x;f)$ is constructed.

2 Research methodology

Achieving the aim of the research is carried out as a result of considering the problem of discretization of the solution $u_\omega(x;f)$ as one of the concretizations of the problem of optimal recovery of the operator, formulated in [10] based on work [11]. Now we present from [10] the formulation of the problem of recovering the operator with some clarifications. Let be given normed spaces X and Y, consisting of functions $f : \Omega_X \mapsto R$ and $g : \Omega_Y \mapsto R$, respectively, a functional class $F \subset X$, operator $T : F \mapsto Y$ and also a function

$$\varphi_N \equiv \varphi_N(z_1, \ldots, z_N; y) : \mathbb{C}^N \times \Omega_Y \mapsto \mathbb{C}(N = 1, 2, \ldots),$$

which for every fixed $(z_1, \ldots, z_N)$ as a function of a variable y belongs to the space Y. Next, denoting by the symbol $\{(l^{(N)}, \varphi_N)\}$ the set of all possible pairs $(l^{(N)}, \varphi_N)$ formed from a $N-$ dimensional vector $l^{(N)} = \left(l_N^{(1)}, \ldots, l_N^{(N)}\right)$ with components $l_N^{(1)} : F \mapsto \mathbb{C}, \ldots, l_N^{(N)} : F \mapsto \mathbb{C}$ and function φ_N, for a given class F, space Y, operator $T : F \mapsto Y$, set $D_N \subset \{(l^{(N)}, \varphi_N)\}$, we determine the quantity

$$\delta_N(D_N, T, F)_Y = \inf_{(l^{(N)},\varphi_N)\in D_N} \delta_N\left((l^{(N)}, \varphi_N), T, F\right)_Y, \tag{7}$$

where $\delta_N\left((l^{(N)}, \varphi_N), T, F\right)_Y = \sup\limits_{f\in F} \left\|(Tf)(\cdot) - \varphi_N(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); \cdot)\right\|_Y$.

In the following presentation, we will call a numerical function $\varphi_N(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); y)$ of a variable y a computing unit. For sequences $\{a_n\}_{n\geq1}$ and $\{b_n\}_{n\geq1}$ with positive members we will use a notation $a_n \underset{\alpha,\beta,\ldots}{\ll} b_n$, that means the existence of a some quantity $C_k(\alpha, \beta, \ldots) > 0, k = 1, 2, \ldots$, depending only on the parameters indicated in brackets, such that $a_n \leq C_k(\alpha, \beta, \ldots)b_n$ for all $n \in \mathbb{N}$. And the simultaneous fulfillment of the inequalities $a_n \underset{\alpha,\beta,\ldots}{\ll} b_n$ and $b_n \underset{\alpha,\beta,\ldots}{\ll} a_n$ is written in the form $a_n \underset{\alpha,\beta,\ldots}{\succ\prec} b_n$. According to the above notations and definitions, the problem of optimal recovery of the operator $T : F \mapsto Y$ by computing units $(l^{(N)}, \varphi_N) \equiv \varphi_N(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); \cdot) \in D_N$ in the metric of space Y is to find the exact order of quantity (7) (i.e., to determine a sequence $\{\Psi_N\}_{N\geq1}$ of positive members that satisfies the relation

$$\delta_N(D_N, T, F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \psi_N,$$

here $\alpha, \beta, \ldots$ are parameters of class F and space Y) and in the indication of the computing unit releasing the established exact order, i.e. such that

$$\delta_N((\overline{l}^{(N)}, \overline{\varphi}_N), T, F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \psi_N.$$

A computing unit that releases exact order is called an optimal computing unit. The concretization in (7) of the class F, space Y, operator $T : F \mapsto Y$ and of the set D_N gives rise to various optimal recovery problems (see, for example, [10]- [14]). Further, following the works [2], [3] and [7] in this case $(Tf)(\cdot) = u_\omega(\cdot; f)$ instead of the terms "recovery" and "computing unit" we will use the terms "discretization" and "discretization operator" respectively.

In this paper, the discretization problem $u_\omega(x; f)$ is considered as concretization

$$(Tf)(\cdot) = u_\omega(\cdot; f), F = W_2^r[0,1]^s, Y = L^2[0,1]^s, D_N = L_N,$$

where L_N is the set of all pairs $\{(l^{(N)}, \varphi_N)\}$ with linear functionals

$$l_N^{(1)} : W_2^r \mapsto \mathbb{C}, \ldots, l_N^{(N)} : W_2^r \mapsto \mathbb{C},$$

and a discretization operator $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ with the following properties is constructed:

I. The discretization operator $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ approximates in the metric of space $L^2[0,1]^s$ the solution $u_\omega(x; f)$ of equation (1) with the right-hand side $f \in W_2^r$ with accuracy $\frac{C_1(s,r)}{N^{(r+2)/s}}$, i.e., the inequality

$$\sup_{f \in W_2^r} \left\| u_\omega(\cdot; f) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f); \cdot) \right\|_{L^2} \underset{s,r}{\ll} \frac{1}{N^{(r+2)/s}}; \tag{8}$$

holds for $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$;

II. Estimate (8) cannot be improved in order;

III. The discretization operator $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ is optimal, i.e. an arbitrarily chosen discretization operator $(l^{(N)}, \varphi_N) \in L_N$ does not improve estimate (8) in order.

3 Main Result

First, we give the definition of the class W_2^r. The multidimensional periodic Sobolev class $W_2^r \equiv W_2^r[0,1]^s$ is the set of all one-periodic functions $f(x) = f(x_1, \ldots, x_s)$ by each variable $x_1, \ldots, x_s$ and summable on $[0,1]^s$, whose trigonometric Fourier coefficients satisfy the condition

$$\sum_{m \in Z^s} |\hat{f}(m)|^2 (\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r}) \leq 1, \tag{9}$$

where $(m, x) = m_1 x_1 + \ldots + m_s x_s, \overline{m}_j = \max\{1; |m_j|\} (j = 1, \ldots, s)$.

Now, using properties I – III, we formulate the main result of this work.

Theorem. Let numbers $s \in \mathbb{N}\backslash\{1\}$ and $r > s/2$ be given. Then for each $N = (2n+1)^s, n \in \mathbb{N}$ one has the relations

$$\delta_N(L_N, (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2} \underset{s,r}{\succ\prec}$$

$$\underset{s,r}{\succ\prec} \delta_N\left((\tilde{l}^{(N)}, \widetilde{\varphi}_N), (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r\right)_{L^2} \underset{s,r}{\succ\prec} \frac{1}{N^{(r+2)/s}}, \tag{10}$$

here $l^{(N)}$ consists of components

$$\tilde{l}_N^{(1)}(f) = \hat{f}(\widetilde{m}^{(1)}) = \hat{f}(0), \tilde{l}_N^{(2)}(f) = \hat{f}(\widetilde{m}^{(2)}), \ldots, \tilde{l}_N^{(N)}(f) = \hat{f}(\widetilde{m}^{(N)})$$

and the function $\varphi_N \equiv \varphi_N(z_1, \ldots, z_N; \cdot)$ is defined by the equality

$$\varphi_N(z_1, \ldots, z_N; x) = -\frac{1}{4\pi^2}\sum_{k=1}^{N} z_k d_k(x) \exp\{2\pi i(\widetilde{m}^{(k)}, x)\},$$

where $\left\{\widetilde{m}^{(1)} = 0, \widetilde{m}^{(2)}, \ldots, \widetilde{m}^{(N)}\right\}$ is some ordering of the set

$$A_n = \{m \in Z^s : |m_1| \le n, \ldots, |m_s| \le n\},$$

$$d_k(x) = \begin{cases} \left(\widetilde{m}^{(k)}, \widetilde{m}^{(k)}\right)^{-1}, k \in \{2, \ldots, N\}, \\ -4\pi^2\omega(x), k = 1. \end{cases}$$

4 Proof

Everywhere below, for each integer vector $m = (m_1, \ldots, m_s)$ we put $\|m\| = \max\{|m_1|, \ldots, |m_s|\}$. The symbol $|B|$ will denote the number of elements of a finite set B.

When estimating the quantity $\delta_N(L_N, (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2}$ from below, we use the following lemma from [13].

Lemma. Let $s \ge 1$ be a given integer. Then, for each integer $N \ge 1$, for any set

$$G \equiv \{m^{(1)}, \ldots m^{(N')}\} \subset Z^s$$

such that $N' = |G| \ge 2N$ and $|G| \underset{s}{\succ\prec} N$, and for arbitrary linear functionals $l_1, \ldots, l_N$, defined at lest on the set of all trigonometric polynomials with spectrum in G, there exists complex numbers $\{c_k\}_{k=1}^{N'}$, satisfying conditions $\sum\limits_{k=1}^{N'} |c_k| \ge N, \sum\limits_{k=1}^{N'} |c_k|^2 = N$; further, if $\chi(x) = \sum\limits_{k=1}^{N'} e^{2\pi i(m^{(k)}, x)}$, then $l_1(\chi) = 0, \ldots, l_N(\chi) = 0$ and $\|\chi\|_\infty \ge N, \|\chi\|_2 = \sqrt{N}$.

First, let us estimate from above the quantity $\delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N), (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2}$. For any function $f \in W_2^r$ the equality

$$\widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f); x) = \omega(x)\hat{f}(0) - \frac{1}{4\pi^2}\sum_{m \in A_n}^{*} \frac{\hat{f}(m)}{(m, m)} \exp\{2\pi i(m, x)\}$$

holds. Therefore, according to (2) and Parseval's equality we obtain

$$\left\|u_\omega(x;f)-\widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f),\ldots,\tilde{l}_N^{(N)}(f);\cdot)\right\|_{L^2}=\frac{1}{4\pi^2}\left(\sum_{m\in Z^s\setminus A_n}\frac{|\hat{f}(m)|^2}{(m,m)^2}\right)^{1/2}. \tag{11}$$

Further,

$$\sum_{m\in Z^s\setminus A_n}\frac{|\hat{f}(m)|^2}{(m,m)^2}=\sum_{m\in Z^s\setminus A_n}\frac{|\hat{f}(m)|^2(\overline{m}_1^{2r}+\ldots+\overline{m}_s^{2r})}{(m_1^2+\ldots+m_s^2)^2(\overline{m}_1^{2r}+\ldots+\overline{m}_s^{2r})}\le$$
$$\le\sum_{m\in Z^s\setminus A_n}\frac{|\hat{f}(m)|^2(\overline{m}_1^{2r}+\ldots+\overline{m}_s^{2r}))}{\overline{m}_1^{2r+4}+\ldots+\overline{m}_s^{2r+4}}.$$

For each $m\in Z^s\setminus A_n$ there is an index $\theta\in\{1,\ldots,s\}$ such that $|m_\theta|>n$. Therefore, continuing the expression written above taking into account the inequalities

$$\frac{1}{\overline{m}_1^{2r+4}+\ldots+\overline{m}_s^{2r+4}}\le\frac{1}{\overline{m}_\theta^{2r+4}}\le\frac{1}{n^{2r+4}}\underset{s}{\ll}\frac{1}{N^{(2r+4)/s}}$$

and (9) we arrive at the inequality $\sum\limits_{m\in Z^s\setminus A_n}\frac{|\hat{f}(m)|^2}{(m,m)^2}\underset{s}{\ll}\frac{1}{N^{(2r+4)/s}}$.

Therefore, according to (11), the inequality

$$\left\|u_\omega(x;f)-\widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f),\ldots,\tilde{l}_N^{(N)}(f);\cdot)\right\|_{L^2}\underset{s}{\ll}\frac{1}{N^{(r+2)/s}}. \tag{12}$$

is true, from where, taking into account the arbitrariness of the function $f\in W_2^r$, we arrive at the estimate

$$\delta_N\left((\tilde{l}^{(N)},\widetilde{\varphi}_N),(Tf)(\cdot)=u_\omega(\cdot;f),W_2^r\right)_{L^2}\underset{s}{\ll}\frac{1}{N^{(r+2)/s}}. \tag{13}$$

Let linear functionals

$$l_N^{(1)}:W_2^r\mapsto\mathbb{C},\ldots,l_N^{(N)}:W_2^r\mapsto\mathbb{C} \tag{14}$$

and function $\varphi_N(z_1,\ldots,z_N;y)\,:\,\mathbb{C}^N\times[0,1]^s\mapsto\mathbb{C}$ be given. For some $C_2(s,r)>0$ the conditions $|U_N|>2N$ and $|U_N|\underset{s}{\succ\prec}N$ are satisfied for the set

$$U_N=\{m\in Z^s:1\le\|m\|\le C_2(s,r)N^{1/s}\}.$$

Therefore, due to the above lemma for linear functionals (14), there are complex numbers such that

$$\sum_{m\in U_N}|c_m|^2=N \tag{15}$$

and if $g_N(x) = \sum\limits_{m\in U_N} c_m \exp\{2\pi i(m,x)\}$, then

$$l_N^{(1)}(g_N) = 0, \ldots, l_N^{(N)}(g_N) = 0. \tag{16}$$

The function

$$f_N(x) = \frac{C_3(s,r)}{N^{r/s}\sqrt{N}} g_N(x)$$

belongs to the class W_2^r. Indeed, taking into account (15) and the relation $|U_N| \underset{s}{\succ\prec} N$, we obtain

$$\sum_{m\in U_N} |\hat{f}(m)|^2 \left(\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r}\right) \underset{s,r}{\ll} \sum_{m\in U_N} \frac{|c_m|^2}{N^{2r/s}N} \left(\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r}\right) \underset{s,r}{\ll}$$
$$\underset{s,r}{\ll} \frac{1}{N^{2r/s}N} \sum_{m\in U_N} |c_m|^2 \left(\overline{m}_1^{2r} + \ldots + \overline{m}_s^{2r}\right) \underset{s,r}{\ll} \frac{1}{N} \sum_{m\in U_N} |c_m|^2 \leq 1.$$

Since $0 \notin U_N$, then $\hat{f}_N(0) = 0$. Therefore, from equality (2) follows

$$u_\omega(x; f_N) = -\frac{C_3(s,r)}{4\pi^2 N^{s/r}\sqrt{N}} \sum_{m\in U_N} \frac{c_m \exp\{-2\pi i(m,x)\}}{(m,m)}$$

Because

$$\|u_\omega(\cdot; f_N)\|_{L^2} \underset{s,r}{\gg} \frac{1}{N^{r/s}\sqrt{N}} \left(\sum_{m\in U_N} \frac{|c_m|^2}{(m,m)}\right)^{1/2}, \tag{17}$$

then due to equalities $(m,m) = m_1^2 + \ldots + m_s^2 \underset{s}{\ll} \|m\|^2$ and (15)

$$\|u_\omega(\cdot; f_N)\|_{L^2} \underset{s,r}{\gg} \frac{1}{N^{(r+2)/s}}. \tag{18}$$

According to (16) the equalities $l_N^{(1)}(f_N) = 0, \ldots, l_N^{(N)}(f_N) = 0$ are true. Hence, taking into account the inclusion $f \in W_2^r$ we have

$$\sup_{f\in W_2^r} \left\| u_\omega(\cdot; f_N) - \varphi_N(l_N^{(1)}(f_N), \ldots, l_N^{(N)}(f_N); \cdot) \right\|_{L^2} \geq$$
$$\geq \frac{1}{2}\Big(\|u_\omega(\cdot; f_N) - \varphi_N(l_N^{(1)}(f_N), \ldots, l_N^{(N)}(f_N); \cdot)\|_{L^2} +$$
$$+ \|u_\omega(\cdot; -f_N) - \varphi_N(l_N^{(1)}(-f_N), \ldots, l_N^{(N)}(-f_N); \cdot)\|_{L^2} \Big) =$$
$$= \frac{1}{2}\Big(\|u_\omega(\cdot; f_N) - \varphi_N(0, \ldots, 0; \cdot)\|_{L^2} +$$
$$+ \|u_\omega(\cdot; -f_N) - \varphi_N(0, \ldots, 0; \cdot)\|_{L^2} \Big) \geq \|u_\omega(\cdot; f_N)\|_{L^2},$$

whence, due to (18), we obtain

$$\delta_N(L_N, (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2} \underset{s,r}{\gg} \frac{1}{N^{(r+2)/s}}. \tag{19}$$

Therefore, according to (13) and

$$\delta_N(L_N, (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2} \leq \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N), (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2}$$

there are relations (10). The theorem is proven.

5 Conclusions

1) If we take as a set D_N the set P_N of all pairs $(l^{(N)}, \varphi_N)$ with functionals

$$l_N^{(1)}(f) = f(\xi^{(1)}), \ldots, l_N^{(N)}(f) = f(\xi^{(N)}),$$

where $\xi^{(i)} \in [0,1]^s$ for each $i \in \{1, \ldots, s\}$ then, due to the fact that the Sobolev class W_2^r coincides with the Nikolskii–Besov class $B_{2,2}^r$ from Theorem 3.2, formulated and proven in [9], we obtain the following inequality $\delta_N(P_N, (Tf)(\cdot) = u_\omega(\cdot; f), W_2^r)_{L^2} \underset{s,r}{\gg} \frac{1}{N^{r/s}}$.

This inequality allows us to assert that any discretization operator $(l^{(N)}, \varphi_N)$ constructed from the values of the function at given points , including the optimal discretization operator

$$\overline{\varphi}_N\left(f(\xi^{(1)}), \ldots, f(\xi^{(N)}), x\right) = \frac{1}{N} \sum_{\xi^{(n)} \in S_N} f(\xi^{(n)}) \times \left(\omega(x) - \frac{1}{4\pi^2} \sum_{\|k\| < t/2}^{*} \frac{\exp(2\pi i(k, x - \xi^{(n)}))}{(k,k)}\right)$$

from [9], where

$$S_N = \left\{\xi^{(n)} = \left(\frac{n_1}{t}, \ldots, \frac{n_s}{t}\right), n \in Z^s, 0 \leq n_j < t (j = 1, \ldots, s)\right\}$$

for each $N = t^s (t = 1, 2, \ldots)$, approximates the solution $u_\omega(x; f)$ in the metric of space L^2 worse than the discretization operator $\left(\tilde{l}^{(N)}, \widetilde{\varphi}_N\right)$ from the theorem above.

2) The theorem we formulated is a new result in approximation theory, numerical analysis and computational mathematics. Due to the optimality of the discretization operator we constructed, this research can be continued by considering the problem of finding the limit error of the optimal discretization operator, the formulation of which is presented in [10].

3) Another direction of development of the research carried out here is the consideration of other periodic functional classes ensuring the absolute convergence of the multiple functional series (2) for each $f \in F$, and normed spaces Y.

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2-бөлім **Раздел 2** **Section 2**

Механика **Механика** **Mechanics**

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COMPLEX RESISTANCE OF A COMPRESSED-BENT ROD TAKING INTO ACCOUNT ELASTIC COMPLIANCE OF ITS SUPPORT

This work deals with the urgent problem in the mechanics of a deformable solid: studying factors of the stress-strain state of a single-span statically indeterminate beam with complex boundary conditions that is under conditions of complex resistance (axial compression with plane transverse bending). To solve the problem, both analytical methods (the method of forces in the form of “five” support moments, the method of initial parameters) and the numerical finite difference method with a “linear” grid with density n=8 were used. The necessary resolving equations and matrices are given to take into account changes in the rigidity parameter of the right hinged support and variations in concentrated and uniformly distributed loads, both along the axis of the beam and across it. In the final form, diagrams of deflections, bending moments and shear forces were constructed for specific values of bending rigidity and the degree of elastic compliance of the right hinge-yielding support. Reliability of the theoretical principles and applied results obtained by the authors is confirmed on the basis of the given alternative calculation methods.

Key words:complex resistance, elastic compliance of supports, stress-strain state, statically indeterminate structure, five-moment equation, fictitious reaction of supports, resolution matrix, finite difference method, boundary conditions, initial method parameters.

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Серпімді-иілгіш тіректі ескергендегі сығыла-иілген өзектің күрделі кедергісі

Бұл жұмыс деформацияланатын қатты дене механикасының өзекті мәселесі - күрделі кедергі жағдайында орналасқан күрделі шекаралық шарттары бар бір аралықты статикалық анықталмаған арқалықтың кернеулі-деформациялық күйінің факторларын зерттеуге арналған (жазық көлденең иілумен осьтік сығылу). Мәселені шешу үшін аналитикалық әдістер де ("бес"тірек моменттері түріндегі күштер әдісі, бастапқы параметрлер әдісі) және жиілігі n=8 болатын "сызықтық"торы бар сандық ақырлы айырым әдісі қолданылды. Қажетті шешу теңдеулер мен матрицалар оң жағы топсалы тіректің қатаңдық параметрінің өзгеруін және арқалық өсінің бойымен де өзгеретін шоғырланған және біркелкі таралған жүктемелердің өзгеруіне мүмкіндік береді. Соңында иілу қатаңдығының нақты мәндері үшін майысу, иілу моменттері мен көлденең күштерінің және оң жақ топсалы-икемді тіректерінің серпімді икемді дәрежесінің эпюралары тұрғызылды. Авторлар алған теориялық қағидалар мен қолданбалы нәтижелердің сенімділігі берілген балама есептеу әдістерінің негізінде расталады.

Түйін сөздер: күрделі кедергі, тіректердің серпімді иілгіштігі, кернеулі-деформациялық күй, статикалық анықталмаған конструкция, бес моменттік теңдеу, тіректердің жалған реакциясы, шешу матрицасы, ақырлы айырым әдісі, шекаралық шарттар, бастапқы параметрлер әдісі.

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Сложное сопротивление сжато-изогнутого стержня с учетом упругой податливости ее опоры

Данная работа посвящена актуальной задаче механики деформируемого твердого тела – исследование факторов напряженно-деформируемого состояния (НДС) однопролетной статически неопределимой балки со сложными граничными условиями, находящейся в условиях сложного сопротивления (осевое сжатие с плоским поперечным изгибом). Для решения поставленной задачи применены как аналитические методы (метод сил в виде «пяти» опорных моментов, метод начальных параметров), так и численный метод конечных разностей при «линейной» сетке с густотой $n = 8$. Приведены необходимые разрешающие уравнения и матрицы, позволяющие учитывать изменение параметра жесткости правой шарнирной опоры и варьирования сосредоточенными и равномерно-распределенными нагрузками, как вдоль оси балки, так и поперек её. В конечном виде построены эпюры прогибов, изгибающих моментов и поперечных сил для конкретных значений изгибной жесткости и степени упругой податливости правой шарнирно-податливой опоры. Достоверность полученных авторами теоретических положений и прикладных результатов подтверждена на основе приведенных альтернативных методов расчета.

Ключевые слова: Сложное сопротивление, упругая податливость опор, напряженно-деформированное состояние, статически неопределимая конструкция, уравнение пяти моментов, фиктивная реакция опор, разрешающая матрица, метод конечных разностей, граничные условия, метод начальных параметров.

Purpose and tasks of the study. The purpose of this work is to study the issues of the stress-strain state of a rod structure in the form of a statically indeterminate beam with an elastic compliant support at the right end by analytical and numerical methods based on the equations of five support moments and using the method of initial parameters. The following tasks are solved:

- reviewing scientific literature at the time of the study;
- using the numerical finite difference method with the formation of resolution matrices with the “density” of the linear grid n=8;
- using the five moment equations to obtain the initial moment diagram (taking into account the compliance of the right support);
- checking reliability of the obtained theoretical and applied results based on an alternative method of initial parameters;
- studying the effect of the rigidity of the right support on the parameters of the stress-strain state of the studied single-span beam under the simultaneous action of both axial and transverse loads.

1 Introduction

Structures used in various branches of technology (construction, transport, mechanical engineering, mine and underground construction) are often in the state of complex resistance (a combination of several types of stress: tension-compression, bending, torsion, etc.). The task is complicated in the presence of support points that have a high compliance coefficient, which significantly affects the stress-strain state of the structures under study.

It is known that the level of stress-strain state of structures is significantly affected by the presence of so-called extra connections, the number of which determines the degree of their static indetermination.

The analysis of the stress-strain state of such structures is widely reflected in the scientific works of domestic and foreign scientists. The essence of the problem lies in obtaining final results with complex loading patterns (both in the axial and transverse directions), in the presence, alongside with rigid supports, of supports that have significant compliance in the direction of their settlement. Elastic compliance is characterized by damping (cushioning) of supports, which affects the magnitude of deflections and internal forces in statically indeterminate beams (single-span and multi-span). Among the results obtained in this area of studies, the following can be noted.

In works [1, 2], oblique bending of beams under eccentric compression and torsion is considered, and the calculation of cylindrical helical springs for their axial compression is also given. Work [3] sets out a method of calculating the strength of structural elements subject to variants of complex resistance (eccentric tension-compression. oblique bending); various reference data for carrying out relevant engineering calculations are also provided here.

In work [4], the compliance of compressed rods with elastic support is studied. taking into account their supercritical behavior; in this case, large deflections of rods with hinged supports are studied (the problem of nonlinear elastic resistance). Based on the results of the analytical study, resolving nonlinear integral-differential equations were obtained; the variant of loss of stability in the form of a mechanical "clap" was analyzed.

Study [5] provides an analysis of the stress-strain state of a rod on an elastic foundation with initial deflections and elastic-yielding fastenings at the ends (the following types of deformation are taken into account: bending. linear. shear). In a particular case, for linearly elastic fastenings at the ends of the rod. its supercritical behavior was also studied.

In work [6], the problem of supercritical behavior of an elastically fixed (non-rotating) rod is considered on the basis of the iterative method in determining its displacements.

In works [7-9], the flexibility of supporting steel beams on columns (racks) was studied. In works [10. 11]. a method was proposed to take into account compliance in the nodes of metal structures. Study [12] deals with the issues of regulating the level of supports in metal structures by changing the values of their rigidity (or compliance).

The authors of works [13-15] studied the operation of continuous beams of constant and variable cross sections both when they are supported on an elastic continuous space. and when they are supported on elastic-yielding supports; The study was carried out using the analytical method of five supporting points. The Winkler model was used for the elastic foundation of the structure.

Based on the above review of previously published scientific works, this article continues studying this field of mechanics of a deformable solid for cases of complex resistance in the presence of an elastic-yielding support at one of the ends of a single-span beam.

In this regard, this work sets the purpose of studying structures taking into account the above factors.

2 Theoretical propositions and calculation methods

The object of analysis is a single-span statically indeterminate beam in the compressed-bent state in the presence of a hinged support with the corresponding compliance coefficient "C" (Fig. 1, a).

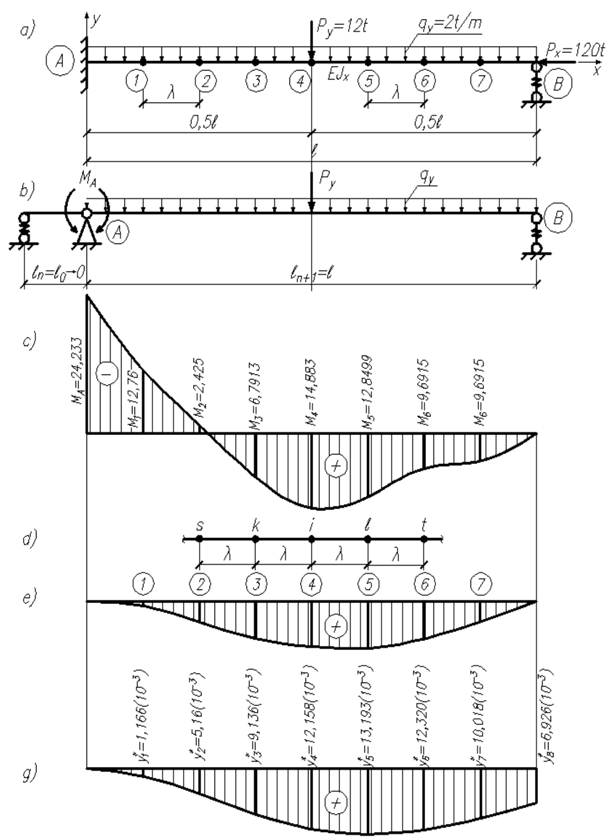

Figure 1: Design diagram of a compressed-bent beam with an elastic-yielding support B: a) design diagram of a beam with an elastic-yielding support B; b) the basic force method system; c) diagram of bending moments (M_0) from shear forces P_y, q_y; d) a fragment of a regular linear grid; f) diagram of deflections (y_{C}, m) (without taking into account the settlement of the support B); g) calculated diagram of deflections (M) taking into account B settlement

To solve the purpose, there should be used the numerical finite difference method [16. 17]. The initial differential equation of compressed-bent rods has the form [18]:

$$\frac{d^2y}{dx^2} + \lambda^2 y = -\frac{M_0}{EJ}. \tag{1}$$

where $y = y(x)$ is the required function of the beam deflection

$$\propto^2 = \frac{N_x}{EJ} \tag{2}$$

is the parameter of the longitudinal axial load P_x effect (Fig. 1,a) ($N_x = -P_x$); M_0 is a bending moment from the initial transverse load $P_y, \;\; q_y$ (Fig.1,a); EJ is the beam flexural rigidity.

Before applying the finite difference method to implement equation (1), it is necessary to construct a diagram of bending moments from a given load $P_y, \;\; q_y$, taking into account the elastic compliance of support "B". To do this, there should be used the method of equation of five support moments [18, 19]. The basic system of the force method is given in Figure 1, b (C is the compliance coefficient of the support B).

For node A (Fig. 1, b) ($M_B = 0; M_O = 0$): ($\ell_n = \ell_0 = 0$; $\ell_{n+1} = \ell$)

$$\delta_{A.A} M_A + \delta_{A.B} M_B + \Delta_{AP} = 0. \text{ or } \delta_{A.A} M_A + \Delta_{AP} = 0. \tag{3}$$

$$\delta_{A.A} = \delta_{n.n} = \frac{\ell}{3EJ} + \frac{C}{\ell^2}; \; \Delta_{AP} = \Delta_{nP} = \frac{A_{\mathrm{A}}^{\Phi}}{EJ} + \frac{C}{\ell} R_B. \tag{4}$$

$$A_{\mathbf{A}}^{\mathbf{\Phi}} = \frac{3}{48} P_y \ell^2 + \frac{q_y \ell^3}{24} = \frac{3}{48} 12(6)^2 + \frac{2(6)^3}{24} = 27 + 18 = 45(tm^2)$$

$$R_B = \frac{P_y}{2} + \frac{q_y \ell}{2} = 6 + 6 = 12(t).$$

$$\delta_{A.A} = 0.560483 \cdot 10^{-3} . \Delta_{A.P} = \frac{45}{3.8 \cdot 10^3} + \frac{0.87 \cdot 10^{-3}}{6} \cdot 12 = 13.58205 \cdot 10^{-3}.$$

According to (3):

$$M_A = -\frac{\Delta_{A.P}}{\delta_{A.A}} = -\frac{13.58205 \cdot 10^{-3}}{0.560483 \cdot 10^{-3}} = 24.233 \;\; (tm) \, . \tag{5}$$

Let's write down equation (1) in finite differences for the i-th node of the regular linear grid (Fig.1, d):

$$\frac{1}{\lambda^2} (y_k - 2y_i + y_\ell) + \alpha_i^2 y_i = -\frac{M_{O.i}}{EJ}.$$

or

$$y_i\left(-2+\alpha_i^2\lambda^2\right)+(y_k+y_\ell)=-\frac{M_O\lambda^2}{EJ} \tag{6}$$

Let's pre-calculate the load parameters $\alpha_i(i=1.2.3.7.8)$ $(N_i=\pm P_x=120\ t)$

$$\alpha_i\lambda^2=(0.125\ell)^2\,\frac{N_i}{EJ}=0.015625\left(\frac{120\cdot 36}{3.8\cdot 10^3}\right)=0.01776316;$$

$$\Delta_{1P}=\ \frac{M_{0.1}}{EJ}=\frac{12.7673}{3.8\cdot 10^3}=3.3598\cdot 10^{-3}\ (0.75)^2=1.8899{\cdot}10^{-3};$$

$$\Delta_{2P}=\ 0.3593\cdot 10^{-3}\ ;\quad \Delta_{3P}=\ -1.0053{\cdot}10^3;\ \Delta_{4P}=-2.2036\cdot 10^{-3};$$

$$\Delta_{5P}=-1.9021\cdot 10-3;\ \Delta_{6P}=-1.436\cdot 10^{-3};\quad \Delta_{7P}=-0.8006\cdot 10^{-3}.$$

The system of resolving finite-difference equations for the beam obtained from expression (5) (Fig. 1, a) is given in Table 1.

Table 1: Matrix for calculating a compressed-bent rod (Fig. 1, a)

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	Right part
1	-1.982237	1						$1.8899\cdot 10^{-3}$
2	1	-1.982237	1					$0.3593\cdot 10^{-3}$
3		1	-1.982237	1				$-1.0053\cdot 10^{3}$
4			1	-1.982237	1			$-2.2031{\cdot}10^{-3}$
5				1	-1.982237	1		$-1.9021{\cdot}10^{-3}$
6					1	-1.982237	1	$-1.436\cdot 10^{-3}$
7						1	-1.982237	$-0.8005{\cdot}10^{-3}$

In the matrix form system of linear algebraic equations (SLAE) (6) presented in Table 1 wull take the form

$$A\cdot\overrightarrow{y}=\overrightarrow{P}. \tag{7}$$

The SLAE (6) solution gives the vector of deflections at the nodes of the linear grid (Fig.1, a), that is:

$$\overrightarrow{y}=A^{-1}\cdot\overrightarrow{P} \tag{8}$$

A^{-1} is the reverse matrix.

To assess reliability of the obtained results of deflections $\overrightarrow{y}$ (9), let's construct a diagram of deflections for the beam (Fig. 2) (without taking into account elastic compliance of the support B (taking $c=0$):

$$\delta_{A.A}=\frac{\ell}{3EJ}=0.526316{\cdot}10^{-3};\Delta_{A.P}=11.842{\cdot}10^{-3}; \tag{9}$$

According to (3): $M_A = -\frac{11.842 \cdot 10^{-3}}{0.52631 \cdot 10^{-3}} = -22.5\ tm.$

Comparing values (5, 10), one establishes that the presence of an elastic-yielding support "B" in the beam (Fig. 2, a) increases the value of the bending moment M_A in relation to the rigid support by (7.7%):

$$\delta = \frac{24.233 - 22.5}{22.500} \cdot 100\% = 7.7\%.$$

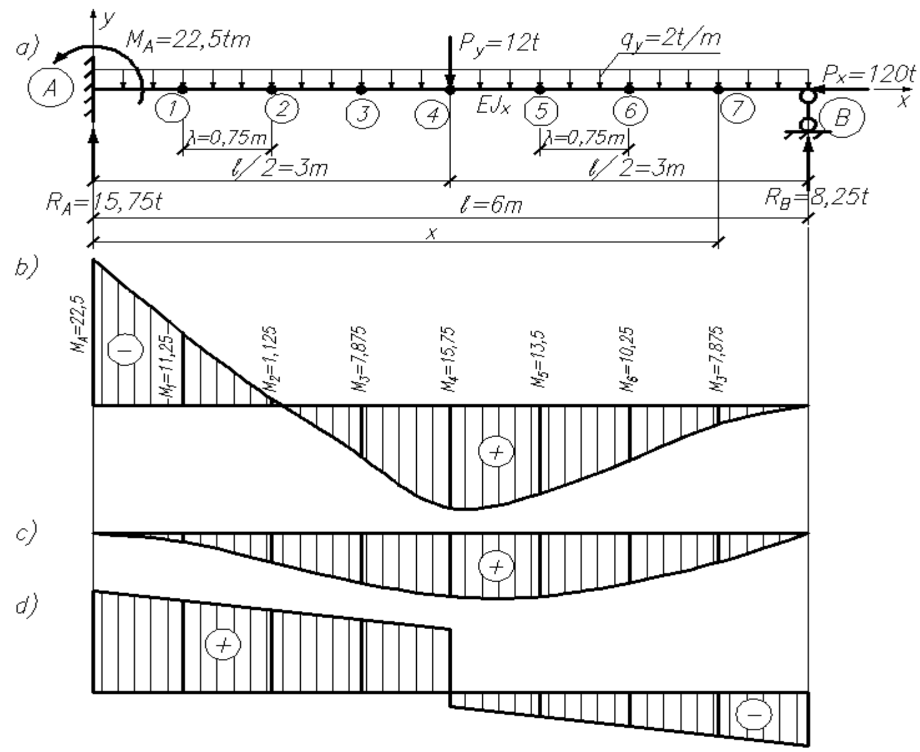

Figure 2: Calculation of a compressed-bent beam with rigid support B: a) design diagram of a beam with rigid supports A. B; b) diagram of bending moments M_x. tm (without taking into account the axial landing forces of the support B); c) diagram of deflections yi (without taking into account the axial force P_x); d) diagram of transverse forces (without taking into account the axial force P_x)

Figure 2, b shows a diagram of the bending moments of a compressed-bent beam with absolutely rigid supports A and B (without taking into account the effect of the axial force P_x, constructed by the initial parameters method [20]). ($M_0 = M_A = -22.5tm;\ Q_0 = Q_A = R_A = 15.571t$) ($y_0 = y_A = 0;\ \theta_0 = \theta_A = 0.0$):

$$EJy = -22.5\ \frac{(x-0)^2}{2} + 15.75\frac{(x-0)^3}{6} - 12\frac{(x-3)^3}{6} - 2\frac{(x-0)^4}{24};$$

or

$$EJy = -11.25\ (x)^2 + 2.625(x)^3 - 2(x-3)^3 - 0.0833(x)^4; \tag{10}$$

a) $(x = 0.75m)$: $y_1 = \frac{1}{3.8 \cdot 10^3}(-5.247069) = -1.38081 \cdot 10^{-3}\,(m)$;
b) $(x = 1.5m$: $y_2 = -4.4408 \cdot 10^{-3}\,(m)$;
c) $(x = 2.25m$: $y_3 = -7.6813 \cdot 10^{-3}\,(m)$;
d) $(x = 3.0m)$: $y_4 = -9.7698 \cdot 10^{-3}\,(m)$;
e) $(x = 3.75m)$: $y_5 = -9.7626 \cdot 10^{-3}\,(m)$;
f) $(x = 4.5\text{мм}m$: $y_6 = -7.771 \cdot 10^{-3} m$;
g) $(x = 5.25m)$: $y_7 = -4.2944 \cdot 10^{-3}\,(m)$.

The diagram of the beam deflections with absolutely rigid supports A and B (without taking into account the effect of the axial force P_x) is shown in Figure 2. c.

Let's calculate the ordinates of the diagram $M_x^{(2)}$ (Fig. 2,b):

$$M_7 = R_B \cdot 0.75 - q\frac{(0.75)^2}{2} = 8.25 \cdot 0.75 - (0.75)^2 = 5.625\,(tm);$$

$$M_6 = 10.225\,(tm); \quad M_5 = 13.50\,(tm); \quad M_4 = 15.75\,(tm); \tag{11}$$

$$M_3 = 7.785\,(tm); \; M_2 = -1.125\,(tm); \; M_1 = -11.25\,(tm); \; M_A = M_{on} = -22tm.$$

Let's build the diagram of the beam transverse forces with absolutely rigid supports A and B (without taking into account the effect of the axial force P_x):

$$Q_A = R_A = 15.75t; \; Q_1 = 14.25; \; Q_2 = 12.75; \; Q_3 = 11.25; \tag{12}$$

$$Q_4^{\text{лев}} = 15.75 - 2.3 = 9.75; Q_4^{right} = 9.75 - P_y = 9.75 - 12 = -2.25.$$

3 Research results

Based on equation (7) using the data in Table 1, the values of deflections at the nodes of the linear grid were obtained (Fig. 1, a) taking into account the elastic compliance of the hinge support "B":

$$\begin{aligned} &y_1^* = 1.013 \cdot 10^{-3}\,(m); \; y_2^* = 3.914 \cdot 10^{-3}\,(m); \quad y_3^* = 7.167 \cdot 10^{-3}\,(m); \\ &y_4^* = 9.402 \cdot 10^{-3}\,(m); y_5^* = 9.417 \cdot 10^{-3}\,(m); \quad y_6^* = 7.514 \cdot 10^{-3}\,(m); \\ &y_7^* = 4.161 \cdot 10^{-3}\,(m). \end{aligned} \tag{13}$$

The diagrams of deflections y_i^* $(i = 1.2.3.4.5.6.7)$ are presented in Figure 1,e.

Let's determine the settlement (displacement. deflection) of the elastic-yielding support "B" [(c=0.87m/t); the value of the support compliance coefficient]: ($R_B = 7.961\ t$ (Fig. 1,b)

$$y_B = R_B = 7.61 \cdot 0.87 \cdot 10^{-3}. \text{ or } y_B = 6.926 \cdot 10^{-3}\,(m). \tag{14}$$

Taking into account expression (14) based on the method of initial parameters [20], let's construct a diagram of deflections y_i^* (Fig. 1.f), calculating the ordinates of this diagram (the origin is at point "B"). Along the cross section *(m-n)* (Fig. 1.b) there is:

$$EJy = -EJ \cdot 6.926 \cdot 10^{-3} + Q_0 EJx + \frac{7.961}{6} \cdot x^3 + \frac{12}{6} \cdot (x-3)^3 + \frac{2}{24} \cdot x^4 \tag{15}$$

With the initial conditions, $x = 6m$; $y_A = 0.0$; from (15) there are determined the θ_0 values:

$$EJ\theta_0 = -16.38 \tag{16}$$

By substituting (16) into (15) (with $EJ = 3.8 \cdot 10^{-3} tm^2$) there is obtained:

$$y = 10^{-3}(-6.926 - 4.31x + 0.3492x^3 - 0.5623\left(x-3\right)^3 - 0.02137 \cdot x^4\right). \tag{17}$$

According to (17) there are calculated deflections at the design nodes $i = 1.2.3.4.5.6.7$. B:

$$\begin{gathered} y_A = 0.0;\ y_1 = 1.6627 \cdot 10^{-3};\ y_2 = 5.1607 \cdot 10^{-3};\ y_3 = 9.1369 \cdot 10^{-3}; \\ y_4 = 12.1585 \cdot 10^{-3};\ y_5 = 13.1936 \cdot 10^{-3};\ y_6 = 12.3206 \cdot 10^{-3}; \\ y_7 = 10.018 \cdot 10^{-3};\ y_B = 6.926 \cdot 10^{-3}. \end{gathered} \tag{18}$$

Based on [18], p. 467-468 and with the use of equations of the initial parameters method, $K = 1.15 \cdot 10^3$ (t/m) is the spring stiffness coefficient in the support "B", the origin of the point (at point "B"). The initial parameters (at point "B") are as follows:

$$M(0) = 0;\ Q(0) = Ky(0); y(0) \neq 0;\ y'(0) \neq 0; \tag{19}$$

$$P_y = 12t; P_x = 120t;\ K = 1.15 \cdot 10^3 \ \left(\frac{\text{T}}{\text{M}}\right);\ EJ = 3.8 \cdot 10^3 \ \left(tm^2\right);\ \ell = 6m;$$

$R_B = 7.961t$ (support reaction); $\alpha^2 = \frac{P_x}{EJ} = \frac{120t}{3.8 \cdot 10^3 \text{T}m^2}$ is the parameter of the axial force P_x; $\alpha = 0.1777$.

The boundary conditions (at the point A):

$$y(\ell) = 0;\quad y'(\ell) = 0. \tag{20}$$

The equation of the method of initial parameters ([18], formulas $2.35 \div 2.38$, p. 461) (taking into account condition (19)):

$$y(z) = y(0) + y'(0) \cdot \frac{\sin\alpha z}{\alpha} - \frac{Ky(0)}{\alpha^3 EJ}(\alpha z - \sin\alpha z) + \frac{R_B}{6EJ}z^3 - \frac{P_y}{6EJ}(z-3)^3 - \frac{q_y}{24EJ}z^4; \tag{21}$$

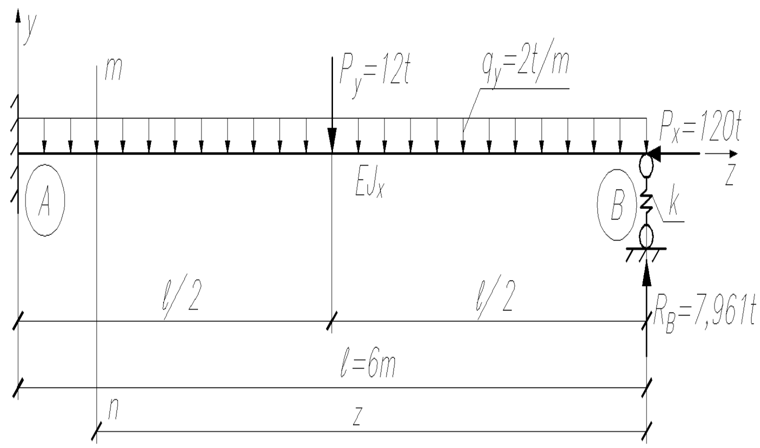

Figure 3: Design diagram of the beam for the method of initial parameters

$$y'(z) = y'(0)\cos\alpha z \; - \frac{Ky(0)}{\alpha^2 EJ}(1-\cos\alpha z\;) + \frac{R_B}{2EJ}z^2 - \frac{P_y}{2EJ}(z-3)^3 - \frac{q_y}{6EJ}z^3. \tag{22}$$

Based on condition (20), there are written down equations (21, 22)

$$0 = y(0) + y'(0)\cdot\frac{\sin\alpha\ell}{\alpha} - \frac{Ky(0)}{\alpha^3 EJ}(\alpha\ell - \sin\alpha\ell\;) + \frac{R_B}{6EJ}\ell^3 - \frac{P_y}{6EJ}(\ell-3)^3 - \frac{q_y}{24EJ}\ell^4. \tag{23}$$

$$0 = y'(0)\cos\alpha\ell \; - \frac{Ky(0)}{\alpha^2 EJ}(1-\cos\alpha\ell\;) + \frac{R_B}{2EJ}\ell^2 - \frac{P_y}{2EJ}(\ell-3)^3 - \frac{-q_y}{6EJ}\ell^3. \tag{24}$$

By solving (23, 24) together, there is obtained:

$$\alpha = \sqrt{31.58\cdot 10^{-3}} = 0.1777; \sin\alpha\ell\; = 0.860;\; \cos\alpha\ell\; = 0.4840;$$

Instead of (23, 24), there is obtained:

$$\begin{cases} -9.1933\cdot y(0) + 4.936\cdot y'(0) = 0.0426 \\ -4.9826\cdot y(0) + 0.4801\cdot y'(0) = 0.03316. \end{cases} \tag{25}$$

From here:

$$y(0) = -7.097\cdot 10^{-3}\cdot y'(0) = -4.588\cdot 10^{-3} \tag{26}$$

By substituting (26) into equations (23, 22), there is obtained

$$\begin{gathered} y(z) = 10^{-3}\cdot(-7.097 - 25.819\cdot\sin(0.1777z)\; + 382.78\,[0.1777z - \sin(0.1777z)\;]) + \\ + 0.52632(z-3)^3 - 0.02193z^4 \end{gathered} \tag{27}$$

- the equation of the compressed-bent rod deflections with elastic compliant support B.

$$y'(z) = 10^{-3} \cdot (-4.588\cos\alpha z \ + 68.017\,(1 - \cos\alpha z\ ) - 1.5789(z-3)^2 - 0.08772z^3 \quad (28)$$

- the equation for the angle of rotation.

Based on (27), let's build the diagram of deflections for a compressed-bent beam with elastic compliant support B (Fig. 1.f).

Table 2: ($\alpha = 0.1777$). Parameters of equations (27, 28)

z.м	$\sin\alpha z$	$\alpha z - \sin\alpha z$	αz	$\cos\alpha z$
0.75 (7 node)	0.130	0.003275	0.1333	0.991
1.5 (6 node)	0.257	0.0035	0.2666	0.9651
2.25 (5 node)	0.3890	0.010825	0.3998	0.9211
3.0 (4 node)	0.508	0.0251	0.5331	0.8612
3.75 (3 node)	0.6210	0.045	0.6664	0.785
4.5 (2 node)	0.717	0.08265	0.7997	0.6967
5.25 (1 node)	0.802	0.13025	0.9329	0.595

a) ($z = 0.75$) (7 node): $y_7 = 10^{-3}(-9.2068)$;
b) ($z = 1.5$) (6 node): $y_6 = 10^{-3}(-12.503774)$;
c) ($z = 2.25$) (5 node): $y_5 = 10^{-3}(-13.559)$;
d) ($z = 3.0$) (4 node): $y_4 = 10^{-3}(-11.2333)$;
e) ($z = 3.75$) (3 node): $y_3 = 10^{-3}(-10.4627)$;
f) ($z = 4.5$) (2 node): $y_2 = 10^{-3}(-4.7720)$;
g) ($z = 5.25$) (1 node): $y_1 = 10^{-3}(-0.3434)$;

$$K = 1.15 \cdot 10^3 \ (m)\,; \quad c = \frac{1}{K} = 0.87 \cdot \frac{10^{-3}m}{\text{т}}; \quad EJ_x = 3.8 \cdot 10^3 \ (tm^2)$$

Let's calculate the bending moments according to formulas ([18]. 2. 37. p. 461):

$$M(z) = -0.1777 \cdot 10^3 \cdot 4.588 \cdot 10^{-3}\sin\alpha z \ - \frac{1.15 \cdot 10^3 \cdot 7.097 \cdot 10^{-3}}{0.1777}\sin\alpha z \ -12\,(z-3) - z^2 \quad (29)$$

a) ($z = 6m$) (A node): $M_A = 42.0307 \cdot 0.877 - 12 \cdot 3 - 36 = -35.13067\ (tm)$;
b) ($z = 5.25m$) (1 node): $M_1 = -20.8539(tm)$;
c) ($z = 4.5m$ (2 node): $M_2 = 42.0307 \cdot 0.717 - 12 \cdot 1.5 - 20.25 = -8.114(tm)$;
d) ($z = 3.75m$) (3 node): $M_3 = 42.0307 \cdot 0.621 - 12 \cdot 0.75 - 14.0625 = 3.0386(tm)$;
e) ($z = 3.0m$ (4 node): $M_4 = 42.0307 \cdot 0.508 - 12 \cdot 0 - 9 = 12.3516(tm)$;
f) ($z = 2.25m$ (5 node): $M_5 = 42.0307 \cdot 0.389 - 5.0625 = 11.2874(tm)$;
g) ($z = 1.5m$) (6 node): $M_6 = 42.0307 \cdot 0.257 - 2.25 = 8.852(tm)$;
h) ($z = 0.75m$) (7 node): $M_7 = 42.0307 \cdot 0.13 - 0.5625 = 4.9015(tm)$;
i) $M_{\text{B}} = 0$ (hinge in the support).

$$Q(z) = 7.611\,(\cos\alpha z\ ) - 12 - 2z - P_x y'(z); (P_x = 120t) \quad (30)$$

$$Q_0 = 1.15 \cdot y(0).$$

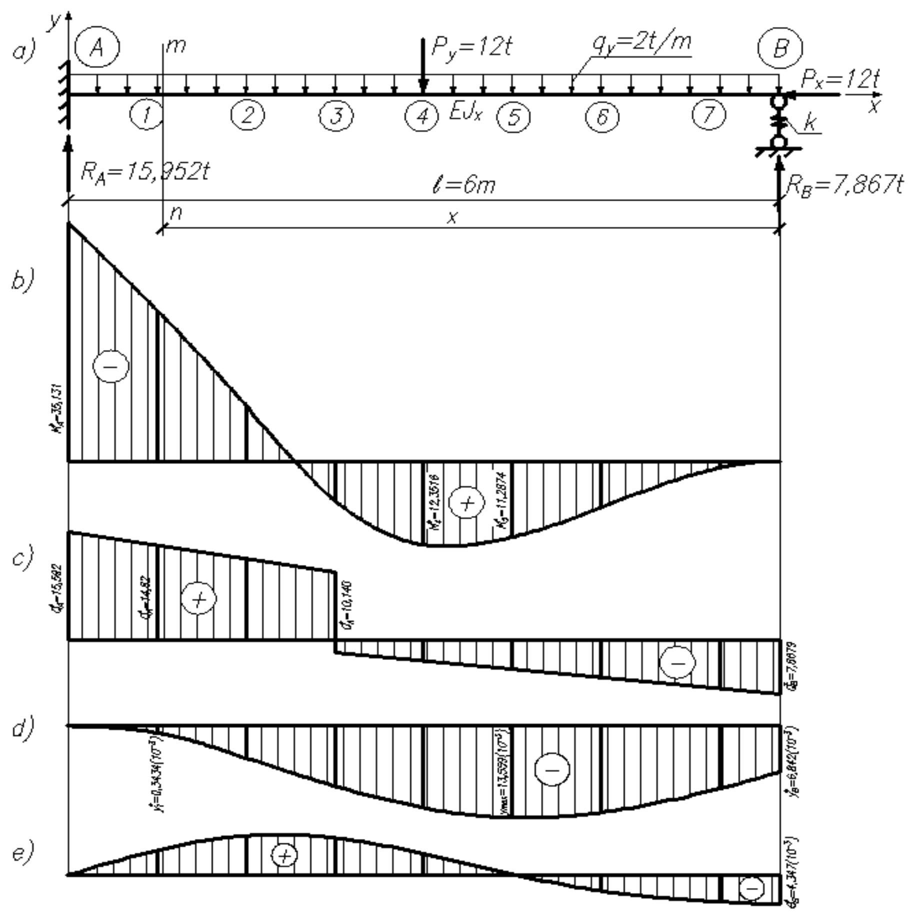

Figure 4: Designing a beam with the use of the initial parameters method: a) design diagram of the beam; b) design diagram of the moments taking into account the compliance of the support B (M_i^*,tm); c) design diagram of transverse forces Q_i^* (taking into account the axial force and compliance of the support B); d) design diagram of deflections (y_i, m) taking into account the settlement of the support B; e) design diagram of the rotation angles (y_i', rad) taking into account the settlement of the support B

Table 3: Parameters of equations (29, 30)

z	0.0	0.75	1.5	2.25	3.0	3.75	4.5	5.25	6.0
$y'(z) \cdot 10^3$	-4.347	-3.9776	-2.3501	0.1413	3.1211	5.508	5.8871	4.1305	0.0
$P_x y'(z)$	0.522	0.4496	0.266	-0.016	-0.353	-0.623	-0.666	-0.4676	0.0
$Q\,(z)$(т)	-4.389	-	-	-	-	-	-	-	-
cosαz	1.00	0.991	0.9651	0.9211	0.8612	0.785	0.6967	0.595	0.484

Based on [21], p. 45:

1) Expression for deflections $n = \sqrt{\frac{P_x}{EJ}} = \sqrt{\frac{120}{3.8 \cdot 10^3}} = 0.1777(\text{м}^{-1})$ – the axial force parameter, ($y_0.\theta_0.M_0.Q_0$ are the initial parameters of the method).

2) The resolving equation is:

$$y = y_0 + \frac{\sin nx}{n}\theta_0 + \frac{\cos nx - 1}{n^2 EJ}M_0 + \frac{\sin nx - nx}{n^3 EJ}Q_0 + \frac{1}{n^3 EJ}\sum_i^n P_i\left[n\left(x - a_i\right) - \sin n\left(x - a_i\right)\right] + \frac{q}{n^4 EJ}\left(\frac{n^2 x^2}{2} + \cos nx - 1\right). \quad (31)$$

where

$$\theta = \cos nx\theta_0 - \frac{\sin nx}{nEJ}M_0 + \frac{\cos nx - 1}{nEJ}Q_0 +$$

$$+\frac{1}{n^2 EJ}\sum_i^n P_i\left[1 - \cos n\left(x - a_i\right)\right] + \frac{q}{n^3 EJ}\left(\mathrm{nx} - \sin nx\right). \quad (32)$$

Based on Figure 3 and equations (31, 32) there is (the beginning of coordinates at the point B):

a) Initial conditions (at the point B):

$$M_0 = 0; \quad Q_0 = K \cdot y_0 = 1.15 \cdot 10^3 y_0, \quad (33)$$

$K = 1.15 \cdot 10^3 t/m$ is the coefficient of the support B rigidity, $y_0 = y_{B.}$

b) The boundary conditions (with $x = \ell = 6m$) (at the point A):

$$M\left(\ell\right) \neq 0; Q\left(\ell\right) \neq 0; y\left(\ell\right) = 0.\theta\left(\ell\right) = 0.\theta_0 = y'_B. \quad (34)$$

Taking into account (33, 34) and according to (31, 32) there is (along the m-n section (Figure 4), see Table 2)):

$$0 = y_0 + 5.6275\sin(1.0662)\theta_0 - P_y\frac{1}{(0.1777)^3 \cdot 3.8 \cdot 10^3}\times$$

$$\times\left\{0.1777\left(6 - 3\right) - \sin\left[(0.1777)\cdot(6 - 3)\right]\right\} + \frac{\sin\left(1.0662\right) - 0.1777 \cdot 6}{(0.1777)^3 \cdot 3.8 \cdot 10^3}\theta_0\times$$

$$\times(1.15 \cdot 10^3) - \frac{q}{(0.1777)^4 \cdot 3.8 \cdot 10^3}\left[\frac{(0.1777)^2 \cdot 6^2}{2} + \cos\left(1.0662\right) - 1\right]$$

$$0 = \cos\left(1.0662\right)\theta_0 - P_y\frac{1}{(0.1777)^2 \cdot 3.8 \cdot 10^3}\left[1 - \cos\left[(0.1777)\left(6 - 3\right)\right]\right] -$$

$$-\frac{q}{(0.1777)^3 \cdot 3.8 \cdot 10^3}\left(0.1777 \cdot 6 - \sin\left(1.0662\right)\right) + \frac{\cos\left(1.0662\right) - 1}{(0.1777)^2 \cdot 3.8 \cdot 10^3}\times$$

$\times(1.15 \cdot 10^3)y_0.$

or

$$\begin{cases} -9.323y_0 + 4.923\theta_0 = 0.041495. \\ -4.94526y_0 + 0.484\theta_0 = 0.031988. \end{cases} \Rightarrow \begin{cases} y_0 = -6.842 \ \cdot 10^{-3} \ (m) \\ \theta_0 = -4.347 \cdot 10^{-3}(rad). \end{cases}$$

Let's calculate the ordinates of the beam transverse force diagram Q_i^* (taking into account the axial force P_x and elastic compliance of the support B) (Fig. 4, c), ([21], formulas 24, 25, p.46) ($P_x = 120$т):

$$Q_x = \frac{dM}{dx} = n^2 EJ\text{cos}nx \ \theta_0 + \text{cos}nx \ \theta_0 + \ P_y \left(\text{cos}n\left(x - a_i\right) \ + + \frac{q_y}{n}\text{sin}nx \right) + P_x\theta(z);$$

$$Q_x = (0.1777)^2 \cdot 3.8 \cdot 10^3\text{cos}\left(1.0662\right)\left(-4.347 \cdot 10^{-3}\right) +$$

$$+\text{cos}\left(1.0662\right)\left(-1.15 \cdot 10^3 \cdot 6.842 \cdot 10^{-3}\right) + 12 \cdot \cos\left(0.5331\right) +$$

$$+11.255\text{sin}(1.0662) \ + 120 \cdot \theta(z);$$

a) $x = 6m$; $Q_{\text{A}} = -0.025246 - 3.808 + 10.334 + 9.693 + 0 = 15.953t$;
б) $x = 0$; $Q_B = -7.8679t$;
Checking the calculation (checking the equilibrium of forces on the Y axis): $\sum F_{ky} = 0$;

$$Q_A + Q_B - P_y - q_y \cdot \ell = 0; 15.952 + 7.8679 + 12 + 2.6 = 0;$$

$$23.8200 \approx 24.$$

The error is: $\delta = 0.750\% < [\delta = 5\%]$ (the calculation was made correctly).

The Q_i^* ordinates are calculated in Table 4 (taking into account the data of Tables 2, 3; the beginning of coordinates is the node B, Fig. 4, a).

Table 4: Ordinates of the Q_i^* diagram

No.	x, m	0.1777× (nx)	(x − 3) × (n_1x)	$(x-3)$ 0.1777 (nx)	sinnx	cosnx	cos(n_1x)	Q_i^*.т
A	6	1.0662	3.0	0.5331	0.86	0.484	0.8612	15.952
1	5.25	0.933	2.25	0.400	0.802	0.595	0.9211	14.6201
2	4.5	0.800	1.5	0.266	0.717	0.6967	0.9651	13.1398
3	3.75	0.666	0.75	0.133	0.621	0.785	0.9911	11.6735
4	3.0	0.5331	0.0	0.0	0.508	0.8612	1.0	-1.8608
5	2.25	0.400	-	-	0.389	0.9211	-	-3.3657
6	1.5	0.266	-	-	0.257	0.9651	-	-4.9386
7	0.75	0.1333	-	-	0.13	0.991	-	-6.4023
B	0.0	0.0	-	-	0.0	1.000	-	-7.8679

$$Q_x = -0.521613\cos(0.1777x) - 7.8683\cos(0.1777x) +$$

$$+12\cos[(0.1777(x-3)] + 11.255\sin(0.1777x) + 120 \cdot \theta(x).$$

$$Q_x = -8.3899\cos(0.1777x) + 12\cos[(0.1777(x-3)] +$$

$$+11.255\sin(0.177x) + 120 \cdot \theta(x).$$

Based on the values in Table 4, there was built the Q_i^* diagram (Fig. 4, c).

4 Conclusions

1) In this work, a study of the stress-strain state (SSS) of a single-span statically indeterminate compressed-bent beam was carried out that is under conditions of complex resistance, taking into account the elastic compliance of the hinge support B.

2) The initial diagram of bending moments that was constructed by the force method based on the equation of five support moments, taking into account the compliance coefficient of the support B with the value c c $= 0.87 \cdot 10^{-3} \left(\frac{m}{t}\right)$ (Fig. 1, c).

3) The diagram of beam deflections (Fig. 1,f). taking into account along with the transverse forces P_y, q_y and the action of the axial force P_x is constructed taking into account elastic compliance of support B with the use of the numerical finite difference method with a regular linear grid of density n=8.

4) In Figure 2, c, d, the calculated diagrams of bending moments and transverse forces are given, taking into account longitudinal-transverse bending and elastic-compliance of supports B.

5) In Figure 2 c, d, the calculated diagrams of bending moments and transverse forces are given. taking into account longitudinal-transverse bending and elastic-compliance of supports "B".

6) Based on the studies carried out, the following was established:

a) taking into account the action of the axial force P_x alongside with the transverse forces P_y, q_y, as well as taking into account the effect of the elastic compliance of support B (c $= 0.87 \cdot 10^{-3} \left(\frac{m}{t}\right)$ is the compliance coefficient) compared to the absolute rigidity of this support, significantly changes the nature and ordinate values of the bending moment diagram (compare Fig. 1,c and Fig. 4,b);

b) the diagrams of transverse forces do not significantly change the nature and magnitude of the ordinates of the diagram of transverse forces (compare Fig. 2,d and Fig. 4,d); the deflection diagrams (compare Fig. 1,e and Fig. 4,d) differ significantly. In this case, the magnitude of the deflections at point 5 is by (1,39) times greater when taking into account the settlement of the support B;

c) the diagram of rotation angles, taking into account the axial force P_x and the elastic settlement of the support B, has the largest value at node 2 (see Fig. 4,e); in this case, the angle of rotation at point 5 approaches zero; here the deflection reaches its maximum.

Figure 2 c,d shows the design diagrams of bending moments and transverse forces taking into account the longitudinal-transverse bending and elastic compliance of supports B.

7) The theoretical principles and applied results presented in this study can serve as the basis for calculations and design of core elements of various buildings and their structures that are used in the field of construction, mechanical engineering,. transport. mine and underground construction.

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MODELING AND INVESTIGATION OF THE INFLUENCE OF LOADING MODE ON THE DEFORMATION PROCESS OF ASPHALT CONCRETE MATERIALS

The article considers a problem of calculating the deformations of forward and reversed creep tension of rheonomic materials. The hereditary theory of creep by Yu.N. Rabotnov is used to describe the nonlinear deformation process. Proposed a method for determining the necessary material characteristics from forward and reversed creep data. During the experiments, there were tested two batches of asphalt concrete samples. Also, was shown behavior of asphalt concrete at cyclic increasing and constant loading modes. At a temperature $T = 24°C$ were tested 10 samples at cyclic increasing loading mode. Cyclic stresses were equal to 0.041; 0.074; 0.111; 0.148; 0.183 MPa and duration period between loading and relax period were chosen to be 570 seconds. At cyclic constant loading were investigated individual samples for the parameters of forward and reversed creep at stresses of 0.041; 0.117 MPa. Then, investigated the affect of reloading two reversed creep process of asphalt concrete samples. From first batch of samples were tested 9 samples before destruction with a period 65 seconds: forward creep of samples at $\sigma = 0.3053$ MPa for the following 5 seconds; reversed creep at $\sigma = 0$ for the following 60 seconds. From the second batch of samples were tested 11 samples of asphalt concrete before destruction with a period 70 seconds: forward creep $\sigma = 0.3053$ MPa for the following 10 seconds and reversed creep at $\sigma = 0$ for the following 60 seconds. Test results showed that the level of return of second batch increased than the first batch of samples.
In the work were tested 5 samples of asphalt concrete according to the direct tensile scheme until a complete failure. Test temperature was $T = 22-24°C$. For each level of loading with constant rate: $0,6519$; $0,4678$; $0,0580$; $0,0489$; $0,0055$ MPa^{s-1}. Using experimental data, were found parameters of the deformation at different constant rate of loading. As a result, all samples fractured brittle at small deformations.
Key words: cyclic loading, loading rate, asphalt concrete, forward creep, reversed creep, deformation, loading, unloading.

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Асфальтбетон материалдарының деформациялану процесіне жүктеме режимінің әсерін модельдеу және зерттеу

Мақалада реономды материалдардың созылуға сынау кезіндегі тура және кері жылжымалылық деформацияларын есептеу қарастырылады. Сызықтық емес деформациялану процесін сипаттауда Ю.Н. Работновтың жылжымалылық мұралық теориясы қолданылады. Тура және кері жылжымалылықтың тәжірибелік берілгендерінен материалдың қажетті сипаттамаларын анықтау әдістемесі ұсынылады. Нәтижесінде алынған анықтауыш теңдеулер асфальтбетон үлгілерін созылуға сынау кезінде тура және кері жылжымалылық деформациясы есептеулерінде қолданылды. Зерттеуде асфальтбетон үлгілерінің екі партиясы сыналды. Асфальтбетонның өсуі мен тұрақты циклдік жүктеме циклдік режиміндегі күйі көрсетілді. Циклдік өсу жүктемесінде $T = 24°C$ температурада 10 асфальтбетон үлгілері сыналды. Кернеулер мәні: $0,041$; $0,074$; $0,111$; $0,148$; $0,183$ МПа, ал жүктеу периоды мен тынығу периодының ұзақтығы 570 секундтқа тең болды. Циклдік тұрақты жүктемеде сынау кернеулері $0,041$; $0,117$ МПа асфальтбетонның жекелеген үлгілері тура және кері жылжымалылыққа зерттелді. Асфальтбетон үлгілерінің қайта жүктеу кезіндегі кері жылжымалылық процесіне әсері де қарастырылды. Үлгілердің бірінші партиясынан қирауға дейін 9 үлгі 65 секундтық периодымен сыналды: үлгілердің тура жылжымалылығы $\sigma = 0.3053$ МПа болғанда 5 секунд бойы; кері жылжымалылық $\sigma = 0$ болғанда 60 секунд бойы. Екінші партияның дайындамасынан 11 асфальтбетон үлгілері қирауға дейін 70 секундтық периодымен сыналды: тура жылжымалылығы $\sigma = 0.3053$ МПа болғанда 10 секунд бойы; кері жылжымалылық $\sigma = 0$ болғанда 60 секунд бойы. Сынақ нәтижелерінен екінші партия дайындама үлгілерінің қайту деңгейі бірінші партиядан алынған үлгілерді қайту деңгейімен салыстырғанда ұлғайғанын көрсетті.

Зерттеу жұмысында $22 - 24°C$ температурада асфальтбетонның 5 үлгісінен тұрақты жүктеу жылдамдығында қирауға дейін созылу схемасы бойынша сыналды: $0,6519$; $0,4678$; $0,0580$; $0,0489$; $0,0055$ МПа$^{\text{сек}-1}$. Әрбір жүктеу жылдамдықтарында тәжірибелік берілгендерді қолдана отырып, жылжымалылық деформациясы мәнінің параметрлері табылды, нәтижесінде барлық үлгілер аз деформацияда морт қирағанын көрсетті.

Түйін сөздер: циклдік жүктеме, жүктелу жылдамдығы, асфальтбетон, тура жылжымалылық, кері жылжымалылық, деформация, жүктеме, жүксіздеу.

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Моделирование и исследование влияния режима нагружения на процесс деформирования асфальтобетонных материалов

В статье рассматривается задача расчета деформаций прямой и обратной ползучести при растяжении реономных материалов. Для описания процесса нелинейного деформирования используется наследственная теория ползучести Ю.Н. Работнова. Предлагается методика определения необходимых характеристик материала из данных прямой и обратной ползучести. Полученные в результате определяющие уравнения апробированы на задаче расчета деформаций прямой и обратной ползучести при растяжении асфальтобетона. Было испытано две партии образцов асфальтобетона. Показаны поведение асфальтобетона в режиме циклически возрастающей и постоянной нагрузке. Схемы испытании в режиме циклически возрастающей нагрузке испытаны 10 образцов асфальтобетона при температуре $T = 24°C$. Напряжения в циклах были равными $0,041$; $0,074$; $0,111$; $0,148$; $0,183$ МПа, а продолжительность периода нагружения и периода отдыха были выбраны равными 570 секунд. Схемы испытании в режиме циклически постоянной нагрузке исследованы отдельные образцы асфальтобетона на параметры прямой и обратной ползучести при напряжениях $0,041$; $0,117$ МПа. Также исследуется влияния повторного нагружения на процесс обратной ползучести образцов асфальтобетона. Из первой партии заготовок испытаны 9 образцов до разрушения с периодом 65 секунд: прямая ползучести образцов при $\sigma = 0.3053$ МПа в течении 5 секунд;

обратная ползучесть при $\sigma = 0$ в течении 60 секунд. Из второй партии заготовок были испытаны 11 образцов до разрушения с периодом 70 секунд: прямая ползучесть образцов асфальтобетона при $\sigma = 0.3053$ МПа в течении 10 секунд; обратная ползучесть образцов при $\sigma = 0$ в течении 60 секунд. Результаты испытаний показали, что уровень возврата образцов второй партии выросли по сравнению с уровнем возврата образцов из первой партии заготовок.
В работе по 5 образцов асфальтобетона испытаны по схеме прямого растяжения до разрушения при температуре $22 - 24°C$ в условиях на каждой нагружения с постоянной скоростью: $0,6519$; $0,4678$; $0,0580$; $0,0489$; $0,0055$ МПа$^{\text{сек}-1}$. При различных скоростях нагружения, используя экспериментальные данные, были найдены параметры значения деформации ползучести. В результате все образцы разрушились хрупко при малых деформациях.
Ключевые слова: циклическое нагружение, скорость нагружения, асфальтобетон, прямая ползучесть, обратная ползучесть, деформация, нагрузка, разгрузка.

1 Introduction

In everyday life transports with different weights move along the highway. The intensity of traffic causes the accumulation of rapid fatigue damage of asphalt concrete coatings and an increase in the corresponding cycles. The characteristic of the development of fatigue damage in the material affects the magnitude of the cyclic load [1-3].

It can be seen from the research of many authors [4-7] that with cyclic loading, there is an increase in recovery during the rest period of an asphalt concrete over a long period of time. Therefore, the experimental study of deformation and destruction of an asphalt concrete under cyclic loading is becoming increasingly important.

This work presents the results of tensile creep tests of asphalt concrete samples until a complete failure, conducted in Kazakhstan Highway Research Institute. The purpose of this research is to study the deformation of asphalt concrete material under forward and reversed creep, as well as at a constant loading rate of asphalt concrete.

2 Determining relation for ergonomic material

2.1 Even if the deformation $\varepsilon(t)$ in the interval $(0, t_1)$ (t – time, $t_1 > 0$) is a non-decreasing function of time, $\dfrac{d\varepsilon(t)}{dt} > 0$. In this case, we take the defining relation as

$$\varepsilon(t) = \varphi[\sigma(t)] + \int_0^t K(t-\tau)\varphi[\sigma(\tau)]d\tau, \tag{1}$$

where σ is the conditional stress, $\varphi(\sigma)$ is the function of conditionally instantaneous loading; $K(t-\tau)$ – kernel of forward creep.

2.2 Even if the deformation $\varepsilon(t)$ in the interval $(t_1, t)(t > t_1)$ be a decreasing function of time $\dfrac{d\varepsilon}{dt} < 0$.

The defining relation for processes with decreasing deformation is written as

$$\varepsilon(t,t_1) = \varphi_1[\sigma(t)] - \int_{t_1}^{t} K_1(t-\tau)\varphi_1[\sigma(\tau)]d\tau, \tag{2}$$

where $\varphi_1(\sigma)$ is the function of conditionally instantaneous unloading; $K_1(t-\tau)$ is kernel of reversed creep.

3 A method for determining the necessary material characteristics from data of forward and reversed creep

3.1 Tension of samples at $\sigma = const$ and constant temperature $T = const$

Creep kernel

$$K(t-\tau) = \delta(t-\tau)^{-\alpha}, \tag{3}$$

where $\alpha \in (0,1)$; $\delta > 0$.

Considering equation (3) from (1), we will obtain the equation of simple creep:

$$\varepsilon_m(t,\sigma) = \varphi[\sigma(0)]\left(1 + \frac{\delta}{1-\alpha}t^{1-\alpha}\right), \tag{4}$$

where $\varphi[\sigma(0)] = \varepsilon_0^m(\sigma)$ – conditionally instantaneous deformation, $\varepsilon_m(t,\sigma)$ – calculated values of creep deformation of the material.

Equation (4) contains three unknown parameters $\varepsilon_0^m(\sigma), \alpha$ and δ. Following [8-9], the parameter α will be considered as known from the interval (0,1), and the unknown parameters $\varepsilon_0^m(\sigma)$ and δ are determined using the least squares method:

$$\begin{gathered} \varepsilon_0^m = \frac{\sum\limits_{i=1}^{m}\varepsilon_e(t_i)\sum\limits_{i=1}^{m}t_i^{2(1-\alpha)} - \sum\limits_{i=1}^{m}\varepsilon_e(t_i)t_i^{(1-\alpha)}\sum\limits_{i=1}^{m}t_i^{(1-\alpha)}}{m\sum\limits_{i=1}^{m}t_i^{2(1-\alpha)} - \left[\sum\limits_{i=1}^{m}t_i^{(1-\alpha)}\right]^2}, \\ \delta = \frac{\sum\limits_{i=1}^{m}\left(\dfrac{\varepsilon_e(t_i)}{\varepsilon_0^m} - 1\right)t_i^{(1-\alpha)}}{\dfrac{1}{1-\alpha}\sum\limits_{i=1}^{m}t_i^{2(1-\alpha)}}, \end{gathered} \tag{5}$$

where $\varepsilon_e(t)$ – creep strain values determined experimentally; m – creep strain number.

Following [9], we accept that

$$\varepsilon_0^m(\sigma) \approx \varepsilon_0^e(\sigma), \tag{6}$$

where $\varepsilon_0^e(\sigma)$ – the value of the conditionally instantaneous deformation determined experimentally.

Considering equation (6) from (5), we will obtain

$$1 = \frac{\sum_{i=1}^{m} K_e(t_i) \sum_{i=1}^{m} t_i^{2(1-\alpha)} - \sum_{i=1}^{m} t_i^{(1-\alpha)} \sum_{i=1}^{m} K_e(t_i) t_i^{(1-\alpha)}}{m \sum_{i=1}^{m} t_i^{2(1-\alpha)} - \left[\sum_{i=1}^{m} t_i^{(1-\alpha)}\right]^2}, \tag{7}$$

$$\delta = \frac{\sum_{i=1}^{m} (K_e(t_i) - 1)}{\frac{1}{1-\alpha} \sum_{i=1}^{m} t_i^{2(1-\alpha)}}. \tag{8}$$

Here

$$K_e(t, \sigma_\xi, T) = \frac{\varepsilon_e(t, \sigma_\xi, T)}{\varepsilon_0^e(\sigma_\xi, T)}, \tag{9}$$

$\sigma_\xi(\xi = 1 - n)$, n is the number of loadings; $K_e(t)$is the experimental rheological creep parameter [8].

The analysis of the relation (7) – (9) shows that finding the values of the entire set of parameters determining the relation is not unique. There are three types of creep curves.

I. If $K_e(t)$ is particular independent of value of the stresses, then (7) and (8) have a unique solution of the form:

$$\alpha(T) = const; \quad \delta(T) = const. \tag{10}$$

In this case, the creep curves will be similar.

II. If $K_e(t, \sigma_\xi)(\xi = 1 - n)$ depend on the value of the σ_ξ stresses, then (7) and (8) will have n+1 solutions of the form:

$$\alpha(\sigma_\xi, T) = const; \quad \delta(\sigma_\xi, T) = const, \tag{11}$$

$$\alpha(\overline{K}_e, T) = const; \quad \delta(\overline{K}_e, T) = const, \tag{12}$$

where

$$\overline{K}_e(t, T) = \frac{1}{n} \sum_{\xi=1}^{n} K_e(t, \sigma_\xi, T). \tag{13}$$

In this case, creep curves will be considered almost similar, if they can be described by a single set of forward creep parameters.

III. Creep curves are not similar to each other, if they cannot be described by a single set of forward creep parameters.

The calculated rheological creep parameter is determined by the formula:

$$K_m(t, T) = 1 + \frac{\delta}{1-\alpha} t^{1-\alpha}, \tag{14}$$

where $\alpha(T) \in (0,1)$; $\delta(T) > 0$, $t \in [0, t_1]$.

The similarity coefficient.

$$K_m(t_s, T) = 1 + \frac{\delta}{1-\alpha} t_s^{1-\alpha}, \tag{15}$$

where $t_s \in [0, t_1]$.

Then $\varepsilon_0^m(\sigma_\xi, T)$ is defined by the formula:

$$\varepsilon_0^m(\sigma_\xi, T) = \frac{1}{m} \sum_{s=1}^{m} \frac{\varepsilon_e(t_s, \sigma_\xi, T)}{K_m(t_s, T)}, \tag{16}$$

where $\varepsilon_e(t)$ – experimental values of creep deformation of the material; $K_m(t_s)$ – defined by (15).

3.2 Tension of samples at $\dot{\sigma} = const$ and constant temperature $T = const$.

Put in

$$\sigma(t) = \dot{\sigma} t, \tag{17}$$

$$\varphi[\sigma(t)] = a\sigma^\gamma(t), \tag{18}$$

where $a > 0$; $\gamma \geq 1$.

Considering equations (3), (17) and (18) from (1) we will obtain:

$$\varepsilon_m(t, T) = a(\dot{\sigma} t)^\gamma \left[1 + \overline{\delta} \frac{\Gamma(1+\gamma)\Gamma(1-\overline{\alpha})}{\Gamma(2+\gamma-\overline{\alpha})} t^{(1-\overline{\alpha})}\right], \tag{19}$$

where $\dot{\sigma} > 0$; $\Gamma(\cdot)$ – gamma function; $t \in [0, t_1]$; $\overline{\alpha} \in (0,1)$.

Using the least squares method, we will obtain $\overline{\delta} = \overline{\delta}(\overline{\alpha}, T)$:

$$\overline{\delta}(\overline{\alpha}, T) = \frac{\Gamma(2+\gamma-\overline{\alpha})}{\Gamma(1+\gamma)\Gamma(1-\overline{\alpha})} \cdot \frac{1}{\sum_{i=1}^{m} t_i^{2(\gamma+1-\overline{\alpha})}} \cdot \left[\frac{1}{a(\dot{\sigma})^\gamma} \sum_{i=1}^{m} \varepsilon_e(t_i, \dot{\sigma}) t_i^{(\gamma+1-\overline{\alpha})} - \sum_{i=1}^{m} t_i^{(2\gamma+1-\overline{\alpha})}\right], \tag{20}$$

where $\varepsilon_e(t, \dot{\sigma})$ – creep strain values determined experimentally at $\dot{\sigma} = const$.

3.3 Determination of material characteristics in case of reversed creep

We take reversed creep kernel in the form:

$$K_1(t-\tau) = \delta_1 (t-\tau)^{-\alpha_1}, \tag{21}$$

where $\alpha_1 \in (0,1)$; $\delta_1 > 0$.

Considering equation (21) from (2), we will obtain

$$\varepsilon_m(t, t_1) = \varepsilon_*^m(\sigma) \left(1 - \frac{\delta_1}{1-\alpha_1} (t-t_1)^{(1-\alpha_1)}\right), \tag{22}$$

where $t \geq t_1$; $\varepsilon_*^m(\sigma)$ – calculated values of conditionally instantaneous deformation during unloading process.

Put in

$$\varepsilon_*^m(\sigma) \approx \varepsilon_*^e(\sigma), \tag{23}$$

here $\varepsilon_*^e(\sigma)$ – experimental values of conditionally instantaneous deformation during unloading.

We introducing the notation:

$$\Gamma_e(t, t_1, \sigma_\xi) = \frac{\varepsilon_e(t, t_1, \sigma_\xi)}{\varepsilon_*^e(\sigma_\xi)}, \tag{24}$$

$$\Gamma_m(t, t_1) = 1 - \frac{\delta_1}{1-\alpha_1}(t - t_1)^{1-\alpha_1}, \tag{25}$$

here $\Gamma_e(t, t_1)$ – experimental rheological parameter of the reversed creep of material; $\Gamma_m(t, t_1)$ – the calculated rheological parameter of the reversed creep of material.

Using the least squares method, we determine the α_1 and δ_1 parameters:

$$\begin{gathered} 1 = \left\{ \sum_{i=1}^{m} \Gamma_e(t_i) \sum_{i=1}^{m} (t_i - t_1)^{2(1-\alpha_1)} - \sum_{i=1}^{m} (t_i - t_1)^{(1-\alpha_1)} \sum_{i=1}^{m} \Gamma_e(t_i)(t_i - t_1)^{(1-\alpha_1)} \right\} \cdot \\ \cdot \left\{ m \sum_{i=1}^{m} (t_i - t_1)^{2(1-\alpha_1)} - \left[\sum_{i=1}^{m} (t_i - t_1)^{(1-\alpha_1)} \right]^2 \right\}^{-1}, \end{gathered} \tag{26}$$

$$\delta_1 = \frac{\sum_{i=1}^{m} (1 - \Gamma_e(t_i))}{\frac{1}{1-\alpha_1} \sum_{i=1}^{m} (t_i - t)^{2(1-\alpha_1)}}, \tag{27}$$

where $t > t_1$; $\Gamma_e(t_i, t_1)$ – determined by the relation (24); when deducing (26) and (27), (23) were considered. Based on (26) and (27), we will analyze the reversed creep curves. There are three types of reversed creep curves.

I. If $\Gamma_e(t_i, t_1)$ practically does not depend on the value of the stresses, then (26) and (27) have a unique solution of the form (10):

$$\alpha_1(T) = const; \quad \delta_1(T) = const.$$

In this case, the reversed creep curves will be similar.

II. If $\Gamma_e(t_i, t_1, \sigma_\xi)(\xi = 1 - n)$ depend on the value of the σ_ξ stresses, then (26) and (27) will have n+1 solutions of the form (11) and (12). Reversed creep curves will be considered almost – similar, if they can be described by a single set of α_1 and δ_1 parameters.

III. If the reversed creep curves cannot be described by a single set of α_1 and δ_1 parameters, then they are not similar to each other.

From (25) at $t = t_s$, we determine the similarity coefficient

$$\Gamma_m(t_s, t_1) = 1 - \frac{\delta_1}{1-\alpha_1}(t_s - t_1)^{1-\alpha_1}, \tag{28}$$

where $t_s \geq t_1$; $\alpha_1 \in (0,1)$; $\delta_1 > 0$; the unknown $\varepsilon_*^m(t_1, \sigma_\xi)$ is determined by the formula:

$$\varepsilon_*^m(t_1, \sigma_\xi) = \frac{1}{m}\sum_{s=1}^{m}\frac{\varepsilon_e(t_s, t_1, \sigma_\xi)}{\Gamma_m(t_s, t_1)}, \tag{29}$$

where $\varepsilon_e(t, t_1)$ – experimental values of the reversed creep deformation of the material; $\Gamma_m(t_s, t_1)$ – defined by (28).

4 Experimental verification of the defining equation for asphalt concrete during loading and unloading

4.1 Behavior of asphalt concrete in cyclic loading

The test scheme in the increasing cyclic loading mode is shown in figure 1. Asphalt concrete 10 samples were tested at a temperature of $T = 24°C$. The experimental average values of the forward creep deformation $\varepsilon_e(t, \sigma)$ of an asphalt concrete are presented in table 1. According to table 1, we calculate:

$$\begin{array}{ll} K_e(570, 0.041) = 3.13; & K_e(570, 0.074) = 3.24; \\ K_e(570, 0.111) = 2.92; & K_e(570, 0.148) = 2.77; \\ K_e(570, 0.183) = 3.22. & \end{array} \tag{30}$$

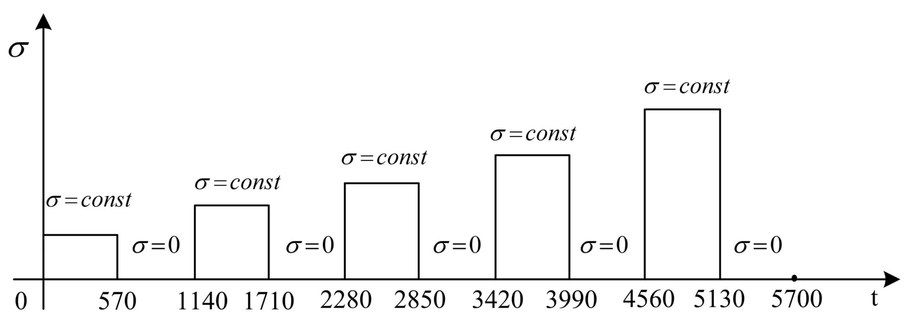


Figure 1: Test scheme in the increasing cyclic loading mode

In the work [8] we will obtain:

$$\alpha(T = 24°C) = 0.3; \quad \delta(T = 24°C) = 0.0153s^{\alpha-1}.$$

Similarity coefficients:

$$K_m(t_s, T) = 1 + 0.0218t_s^{0.7}, \tag{31}$$

where $t_s \in [0, 570]$; t – time in seconds.

Based on the analysis of data (30) and (31), we conclude that in the segment [0,570] the forward creep curves of asphalt concrete are almost similar.

Considering equation (31) and (16), using the formula (4), we compute the calculated values of the creep strain $\varepsilon_m(t, \sigma)$ of an asphalt concrete, they are presented in table 2.The coincidence is satisfactory.

Table 1: Experimental deformations of forward and reversed creep of an asphalt concrete samples

t, s	0	90	210	330	450	570	Number of cycles
$\sigma = 0.041$ MPa	0.0475	0.0849	0.1099	0.1255	0.1415	0.1488	1
0	0.1414	0.1389	0.1380	0.1287	0.1282	0.1268	
0.074	0.0543	0.0932	0.1232	0.1457	0.1627	0.1759	2
0	0.1565	0.1432	0.1358	0.1315	0.1287	0.1259	
0.111	0.0805	0.1338	0.1643	0.1937	0.2158	0.2349	3
0	0.1996	0.1762	0.1632	0.1572	0.1497	0.1489	
0.148	0.1313	0.1782	0.2484	0.2839	0.3293	0.3632	4
0	0.3049	0.2783	0.2471	0.2350	0.2286	0.2174	
0.183	0.1782	0.2855	0.3710	0.4482	0.5105	0.5741	5
0	0.4852	0.4176	0.3808	0.3567	0.3357	0.3308	

Table 2: Calculated values of the deformation of the forward reversed creep of asphalt concrete samples

t, s	0	90	210	330	450	570	Number of cycles
$\sigma = 0.041$ MPa	0.0522	0.0787	0.1002	0.1181	0.1341	0.1488	1
0	0.1446	0.1384	0.1346	0.1316	0.1291	0.1269	
0.074	0.0593	0.0894	0.1139	0.1342	0.1624	0.1691	2
0	0.1670	0.1478	0.1385	0.1319	0.1264	0.1217	
0.111	0.0824	0.1243	0.1582	0.1865	0.2117	0.2350	3
0	0.1994	0.1765	0.1654	0.1575	0.1509	0.1453	
0.148	0.1274	0.1922	0.2447	0.2883	0.3273	0.3633	4
0	0.3021	0.2673	0.2506	0.2385	0.2286	0.2201	
0.183	0.1892	0.2854	0.3634	0.4282	0.4861	0.5395	5
0	0.4560	0.4036	0.3782	0.3601	0.3451	0.3323	

4.2 Analysis of asphalt concrete behavior during unloading

The experimental average values of the reversed creep deformation of asphalt concrete are presented in table 1. According to table 1, the parameters of reversed creep are determined:

$\Gamma_e(t, 0.041)$; $\alpha_{11} = 0.43$; $\delta_{11} = 0.0019\ s^{\alpha-1}$;
$\Gamma_e(t, 0.074)$; $\alpha_{12} = 0.5641$; $\delta_{12} = 0.0054\ s^{\alpha-1}$;
$\Gamma_e(t, 0.111)$; $\alpha_{13} = 0.5863$; $\delta_{13} = 0.0079\ s^{\alpha-1}$;
$\Gamma_e(t, 0.148)$; $\alpha_{14} = 0.4394$; $\delta_{14} = 0.0047 s^{\alpha-1}$;
$\Gamma_e(t, 0.183)$; $\alpha_{15} = 0.5535$; $\delta_{15} = 0.0087 s^{\alpha-1}$;
$\overline{\Gamma}_e(t)$; $\alpha_{16} = 0.5351$; $\delta_{16} = 0.0066\ s^{\alpha-1}$.

The analysis of the data is obtained above shows that the reversed creep curves of asphalt concrete are not similar in the segment [0, 570]. The reversed creep curve at $\sigma = 0.041$ is described by the equation

$$\Gamma_m(t) = 1 - 0.0033t^{0.57}, \tag{32}$$

where $t \in [0, 570]$.

The remaining curves of the reversed creep of asphalt concrete are modeled by the relation

$$\Gamma_m(t) = 1 - 0.0142t^{0.4649}, \tag{33}$$

where $t \in [0, 570]$.

Considering expressions (32) and (33) the equation (29), using the formula (22), calculated values of the reversed creep deformation $\varepsilon_m(t)$ of asphalt concrete, they are presented in table 2. The coincidence is satisfactory. It was shown in [10-11] that microstresses cause reversed creep of materials. The degree of reversed creep of the asphalt concrete material shows that the level of microstress values increases during cyclic loading.The source of micro-damage is the micro-stresses arising in the material [12].The root cause of the appearance of the microstress field in asphalt concrete is the microinhomogeneity and microanisotropy of the material structure. In the loading mode, the deformation of asphalt concrete samples is determined by the expression

$$\varepsilon_m(t, \sigma) = \varepsilon_m^c(t, \sigma) + \varepsilon_m^d(t, \sigma), \tag{34}$$

here $\varepsilon_m^c(t, \sigma)$ – failure of the material due to creep; $\varepsilon_m^d(t, \sigma)$ – failure of the material due to damage.

Since the forward creep curves of asphalt concrete are almost similar, we conclude from (34):

$$\varepsilon_m^d(t, \sigma) << \varepsilon_m^c(t, \sigma). \tag{35}$$

Considering equation (35), in cyclic loading, together (34) we take

$$\varepsilon_m(t, \sigma) = \varepsilon_m^c(t, \sigma). \tag{36}$$

For the cyclic loading scheme (figure 1) we have:

$$\begin{aligned}
&\varepsilon_m(t, \sigma) = 0.0522(1 + 0.0218t^{0.7}) + 0.1446[1 - 0.0033(t - 570)^{0.57}] + \\
&+0.0593[1 + 0.0218(t - 1140)^{0.7}] + 0.1670[1 - 0.0142(t - 1710)^{0.4649}] + \\
&+0.0824[1 + 0.0218(t - 2280)^{0.7}] + 0.1994[1 - 0.0142(t - 2859)^{0.4649}] + \\
&+0.1274[1 + 0.0218(t - 3420)^{0.7}] + 0.3021[1 - 0.0142(t - 3990)^{0.4649}] + \\
&+0.1892[1 + 0.0218(t - 4560)^{0.7}] + 0.4560[1 - 0.0142(t - 5130)^{0.4649}],
\end{aligned} \tag{37}$$

where $t \in [0, 5700]$, t – time in seconds.

4.3 Investigation of asphalt concrete creep in the constant cyclic loading mode

The test scheme of the N68 asphalt concrete sample in the constant cyclic loading mode is shown in figure 2. The experimental values of forward and reversed creep deformation of the N68 sample are shown in table 3. Based on the analysis of the data in table 3:

1) the forward creep of sample N68 is described by equation (31);
2) the reversed creep of sample N68 is modeled by equation (33).

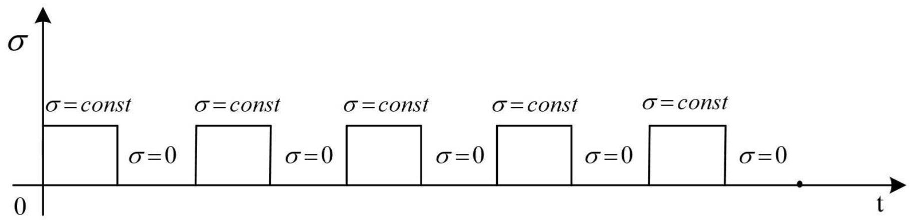


Figure 2: Test scheme in the constant cyclic loading mode

Table 3: Experimental values of forward and reversed creep deformation of sample N68

t, s	0	30	60	90	150	270	Number of cycles
$\sigma = 0.117$ MPa	0.0976	0.1259	0.1477	0.1651			1
0	0.1412	0.1337	0.1294	0.1266	0.1228	0.1157	
$\sigma = 0.117$ MPa	0.0802	0.1001	0.1160	0.1300			2
0	0.1020	0.0899	0.0852	0.0815	0.0768	0.0681	
$\sigma = 0.117$ MPa	0.0790	0.0970	0.1107	0.1231			3
0	0.0899	0.0808	0.0749	0.0712	0.0659	0.0560	
$\sigma = 0.117$ MPa	0.0808	0.0973	0.1098	0.1228			4
0	0.0855	0.0771	0.0715	0.0675	0.0612	0.0513	
$\sigma = 0.117$ MPa	0.0836	0.0998	0.1122	0.1253			5
0	0.0871	0.0777	0.0709	0.0662	0.0600	0.0516	

Table 4: Calculated values of forward and reversed creep deformation of sample N68

t, s	0	30	60	90	150	270	Number of cycles
$\sigma = 0.117$ MPa	0.1019	0.1259	0.1409	0.1537			1
0	0.1437	0.1338	0.1300	0.1272	0.1227	0.1162	
$\sigma = 0.117$ MPa	0.0831	0.1027	0.1149	0.1254			2
0	0.0909	0.0846	0.0823	0.0805	0.0776	0.0735	
$\sigma = 0.117$ MPa	0.0816	0.1008	0.1128	0.1231			3
0	0.0783	0.0729	0.0708	0.0693	0.0669	0.0633	
$\sigma = 0.117$ MPa	0.0814	0.1006	0.1126	0.1228			4
0	0.0738	0.0687	0.0668	0.0653	0.0630	0.0596	
$\sigma = 0.117$ MPa	0.0831	0.1026	0.1148	0.1253			5
0	0.0728	0.0678	0.0659	0.0644	0.0622	0.0589	

The calculated values of forward and reversed creep deformation of sample N68 asphalt concrete are presented in table 4. The coincidence of tables 3 and 4 is satisfactory.

4.4 Investigation the influence of structure of individual asphalt concrete samples on the parameters of forward and reversed creep

The experimental values of forward and reversed creep deformation of 10 individual asphalt concrete samples are shown in table 5. From the data in table 5, it can be seen that data at a stress $\sigma = 0.041$ MPa $\varepsilon_0(0.041)$ and $\varepsilon_*(0.041)$ of all 10 samples does not match

(ε_0 – conditionally instantaneous deformation of samples during loading; ε_* – conditionally instantaneous deformation of samples during unloading). Table 1 shows the average values of forward and reversed creep deformation for the tested 10 asphalt concrete samples. According to table 1, we found the parameters of forward and reversed creep:

1) forward creep is described by equation (31);

2) the reversed creep is modeled by equation (32).

Considering equation (31) and (32), we find the calculated values of forward and reversed creep deformation of all 10 asphalt concrete samples. The data obtained from the calculated values completely coincided with the data presented in table 5.

Table 5: Experimental values of forward and reversed creep deformation of individual asphalt concrete samples

t, s	0	90	210	330	450	570	Sample number
$\sigma = 0.041$ MPa	0.0498	0.0845	0.1122	0.1285	0.1445	0.1561	252
0	0.1476	0.1454	0.1415	0.1402	0.1380	0.1372	
$\sigma = 0.041$ MPa	0.0475	0.0849	0.1099	0.1254	0.1415	0.1488	253
0	0.1414	0.1389	0.1380	0.1287	0.1282	0.1268	
$\sigma = 0.041$ MPa	0.0355	0.0785	0.1033	0.1193	0.1311	0.1425	254
0	0.1299	0.1222	0.1205	0.1189	0.1176	0.1171	
$\sigma = 0.041$ MPa	0.0332	0.0588	0.0777	0.0899	0.1001	0.1099	255
0	0.0989	0.0945	0.0929	0.0902	0.0899	0.0887	
$\sigma = 0.041$ MPa	0.0233	0.0452	0.0622	0.0732	0.0853	0.0945	257
0	0.0822	0.0777	0.0737	0.0730	0.0720	0.0710	
$\sigma = 0.041$ MPa	0.0262	0.0503	0.0672	0.0795	0.0897	0.0974	258
0	0.0907	0.0846	0.0820	0.0817	0.0813	0.0801	
$\sigma = 0.041$ MPa	0.0375	0.0656	0.0736	0.0812	0.0897	0.0966	259
0	0.0845	0.0796	0.0770	0.0754	0.0747	0.0714	
$\sigma = 0.041$ MPa	0.0486	0.0912	0.1215	0.1456	0.1633	0.1758	260
0	0.1634	0.1552	0.1522	0.1499	0.1496	0.1485	
$\sigma = 0.041$ MPa	0.0546	0.0997	0.1293	0.1543	0.1723	0.1872	261
0	0.1754	0.1687	0.1645	0.1633	0.1630	0.1623	
$\sigma = 0.041$ MPa	0.1096	0.1623	0.1933	0.2142	0.2304	0.2433	262
0	0.2283	0.2203	0.2178	0.2160	0.2146	0.2136	

The above samples were from the same batch of samples. Now let's consider the forward and reversed creep of sample N76 from another batch of samples. The experimental values of forward and reversed creep deformation of sample N76 are shown in table 6. Considering equations (31), (32) and (33), the calculated data of forward and reversed creep deformation of sample N76 are found and presented in table 7. The coincidence of table 6 and table 7 is satisfactory.

4.5 Investigation of the reloading effect on the reversed creep process of asphalt concrete samples

The sample testing scheme is shown in Figure 2. From the first batch of samples, 9 samples were tested,until moment a failure with a period of 65 seconds: forward creep of the samples at $\sigma = 0.3053$ MPa for the following 5 seconds; reversed creep at $\sigma = 0$ for the following 60 seconds.

Table 6: Experimental values of forward and reversed creep deformation of sample N76

t, s	0	90	210	330	450	570	Number of cycles
$\sigma = 0.074$ MPa	0.0668	0.1324	0.1792	0.2099	0.2348	0.2584	1
0	0.2388	0.2345	0.2242	0.2223	0.2205	0.2196	
$\sigma = 0.1448$ MPa	0.0932	0.1661	0.2289	0.2761	0.3171	0.3428	2
0	0.3149	0.2950	0.2838	0.2767	0.2708	0.2643	
$\sigma = 0.2232$ MPa	0.1519	0.2581	0.3574	0.4426	0.5224	0.5755	3
0	0.5261	0.4823	0.4559	0.4404	0.4295	0.4255	

Table 7: Calculated values of forward and reversed creep deformation of sample N76

t, s	0	90	210	330	450	570	Number of cycles
$\sigma = 0.074$ MPa	0.0878	0.1324	0.1686	0.1987	0.2256	0.2504	1
0	0.2449	0.2344	0.2279	0.2229	0.2186	0.2149	
$\sigma = 0.1448$ MPa	0.1123	0.1694	0.2157	0.2541	0.2885	0.3202	2
0	0.3493	0.3091	0.2897	0.2758	0.2644	0.2545	
$\sigma = 0.2232$ MPa	0.1861	0.2808	0.3573	0.4211	0.4781	0.5307	3
0	0.5608	0.4963	0.4651	0.4428	0.4244	0.4086	

The average experimental values of the creep strain of 9 samples $\varepsilon_e(t)$ are shown in table 8. According to table 8, were found:

$$\alpha = 0.3; \quad \delta = 1.1204; \quad K_m(t) = 1 + 1.6005t^{0.7}, \tag{38}$$

where $t \in [0, 5]$, t – time in seconds. Considering equation(38), the average calculated values of forward creep strain of 9 samples $\varepsilon_m(t)$ are calculated and presented in table 8. Data analysis of table 8 shows:

1) creep occurs with the hardening of the material;
2) creep of the samples is modeled by one rheological parameter (38);
3) creep rate of the samples is determined by the rated stress, i.e. $\varepsilon^d(t) << \varepsilon^c(t)$;
4) reloading significantly affect the conditionally instantaneous deformations of samples $\varepsilon_e\,(t = 0)$;
5) on the segment [1, 5], the coincidence of $\varepsilon_m(t)$ and $\varepsilon_e(t)$ is satisfactory.

All tested samples fractured brittle under small deformations (table 8). In this case, fractured brittle of asphalt concrete samples occurs as a result of local accumulation of micro-damages around a weak section.

Table 8: Translations into Kazakh for different types of sentences and assessment of translation errors

t, s	0	1	2	3	4	5	Number of cycles
$\varepsilon_e, \%$	0.0751	0.1953	0.2880	0.3562	0.4033	0.4379	1
$\varepsilon_m, \%$	0.0772	0.2008	0.2780	0.3439	0.4034	0.4585	
$\varepsilon_e, \%$	0.0203	0.1226	0.2041	0.2454	0.2800	0.3178	2

$\varepsilon_m, \%$	0.0518	0.1347	0.1865	0.2307	0.2706	0.3076	
$\varepsilon_e, \%$	0.0214	0.1237	0.2040	0.2518	0.2970	0.3281	3
$\varepsilon_m, \%$	0.0518	0.1347	0.1865	0.2307	0.2706	0.3076	
$\varepsilon_e, \%$	0.0264	0.1495	0.2347	0.2630	0.2976	0.3201	4
$\varepsilon_m, \%$	0.0593	0.1542	0.2135	0.2641	0.3098	0.3521	
$\varepsilon_e, \%$	0.0125	0.1414	0.2142	0.2664	0.3060	0.3331	5
$\varepsilon_m, \%$	0.0577	0.1500	0.2076	0.2569	0.3013	0.3425	
$\varepsilon_e, \%$	0.0322	0.1811	0.2628	0.3241	0.3732	0.4273	6
$\varepsilon_m, \%$	0.0718	0.1866	0.2584	0.3196	0.3749	0.4261	
$\varepsilon_e, \%$	0.0172	0.1735	0.2393	0.2796	0.3112	0.3414	7
$\varepsilon_m, \%$	0.0626	0.1628	0.2254	0.2789	0.3271	0.3718	
$\varepsilon_e, \%$	0.0318	0.1707	0.2501	0.2987	0.3368	0.3480	8
$\varepsilon_m, \%$	0.0642	0.1670	0.2311	0.2859	0.3354	0.3812	
$\varepsilon_e, \%$	0.0242	0.1602	0.2236	0.2799	0.3223	0.3515	9
$\varepsilon_m, \%$	0.0615	0.1599	0.2213	0.2739	0.3212	0.3652	
$\varepsilon_e, \%$	0.0207	0.1829	0.2933	0.3499	0.4052	0.4483	10
$\varepsilon_m, \%$	0.0767	0.1994	0.2761	0.3416	0.4006	0.4554	
$\varepsilon_e, \%$	0.0402	0.2402	0.3286	0.3797	0.4324	0.4648	11
$\varepsilon_m, \%$	0.0863	0.2219	0.3072	0.3800	0.4458	0.5067	
$\varepsilon_e, \%$	0.0588	0.2045	0.3347	0.3657	0.4587	0.4941	12
$\varepsilon_m, \%$	0.0849	0.2209	0.3057	0.3783	0.4437	0.5044	

After that, forward creep process of 9 samples is considered separately. Computed deformation values are calculated using the formula (38) of the forward creep of individual samples (N366, N367, N368, N369, N370, N371, N372, N373, N374) of asphalt concrete. As a result, the experimental strain values coincided with the results of the calculated strain values on the segment [1, 5].

When the samples were unloading the following values of elastic deformation were found:

$$N366 - \varepsilon^{ee} = 0.0160; \quad N367 - \varepsilon^{ee} = 0.0081; \quad N368 - \varepsilon^{ee} = 0.0108;$$

$$N369 - \varepsilon^{ee} = 0.0108; \quad N370 - \varepsilon^{ee} = 0.0186; \quad N371 - \varepsilon^{ee} = 0.0155;$$

$$N372 - \varepsilon^{ee} = 0.0075; \quad N373 - \varepsilon^{ee} = 0.0078; \quad N374 - \varepsilon^{ee} = 0.0249.$$

Rheological parameters of reversed creep (1 cycle) were found from the tested asphalt concrete samples of experimental values of reversed creep deformation:

$$\alpha_1 = 0.65; \quad \delta_1 = 0.0187; \quad \Gamma_m(t) = 1 - 0.0535t^{0.35}, \tag{39}$$

where $t \in [0, 60]$, t – time in seconds. Considering equation (39), the calculated values of reversed creep deformation of individual samples are found. As a result, coincidence $\varepsilon_m(t)$ and $\varepsilon_e(t)$ on the segment [0, 60] are good.

The average experimental values of reversed creep deformation of all 9 asphalt concrete samples were found values of rheological parameters of reversed creep:

$$\begin{array}{l} 2,3-cycles; \quad \alpha_1=0.65; \quad \delta_1=0.0327; \\ 4,5,6,7-cycles; \quad \alpha_1=0.65; \quad \delta_1=0.0467; \\ 8,12-cycles; \quad \alpha_1=0.65; \quad \delta_1=0.0502; \\ 9,11-cycles; \quad \alpha_1=0.65; \quad \delta_1=0.0572; \\ 10-cycle; \quad \alpha_1=0.65; \quad \delta_1=0.0537. \end{array} \tag{40}$$

The average calculated values of reversed creep deformation of asphalt concrete samples are calculated using formulas (39) and (40). As a result, coincidence $\varepsilon_m(t)$ and $\varepsilon_e(t)$ are satisfactory. After the dependence of $\dfrac{\varepsilon_e(t)}{\varepsilon_e(0)}$ on the reloading of asphalt concrete samples was found, they are shown in figure 3. From the analysis of the data and figure 3 it follows:

1) the reversed creep curves of asphalt concrete have horizontal asymptotes;

2) non-loaded asphalt concrete samples continue to decline, this phenomenon is called a return;

3) the level of return of asphalt concrete samples significantly depends on the loading history;

4) the maximum return level reaches 70% of the level of forward creep of asphalt concrete;

5) the level of microstress in the sample increases with increasing reloading.

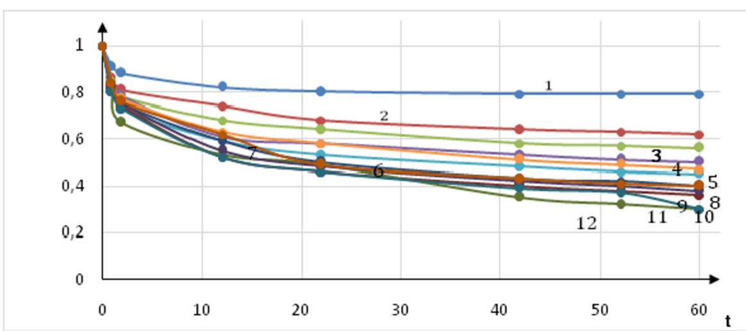


Figure 3: Recovery of the asphalt concrete strain after removing of the stress in different cycles

From the second batch, 11 samples were tested before destruction with a period of 70 seconds: forward creep of asphalt concrete samples at $\sigma = 0.3053$ MPa for the following 10 seconds; reversed creep of samples at $\sigma = 0$ for the following 60 seconds. According to the experimental values deformation of forward creep samples, the following values were found:

$$\alpha = 0.3; \quad \delta = 0.4227; \quad K_m(t) = 1 + 0.6038t^{0.7}, \tag{41}$$

where $t \in [0, 10]$, t – time in seconds. Considering equation (41), the calculated values of the creep deformation $\varepsilon_m(t)$ are calculated. The coincidence of $\varepsilon_m(t)$ with $\varepsilon_e(t)$ on the segment [1, 10] is satisfactory. From the co MParison (41) and (38) it follows that the rate of forward creep of asphalt concrete samples from the second batch of samples is almost three times lower than the rate of forward creep of asphalt concrete samples from the first batch of samples.

From the second batch of samples, the experimental values of reversed creep deformation of the samples are found:

$$\alpha_1 = 0.65; \quad \delta_1 = 0.0257; \quad \Gamma_m(t) = 1 - 0.0735t^{0.35}, \tag{42}$$

where $t \in [0, 60]$, t – time in seconds. Using the formula (42), we compute the calculated values of the reversed creep deformation of samples from the second batch of samples. The coincidence of $\varepsilon_m(t)$ and $\varepsilon_e(t)$ is good. According to the experimental data, the dependence of $\dfrac{\varepsilon_e(t)}{\varepsilon_e(0)}$ on reloading was found for samples N275 and N279, they are shown in figure 4 and figure 5. These figures show that the level of sample return has increased co MPared to the level of sample return from the first batch of samples (figure 3). All samples of the second batch of samples were fractured brittle with small deformations.

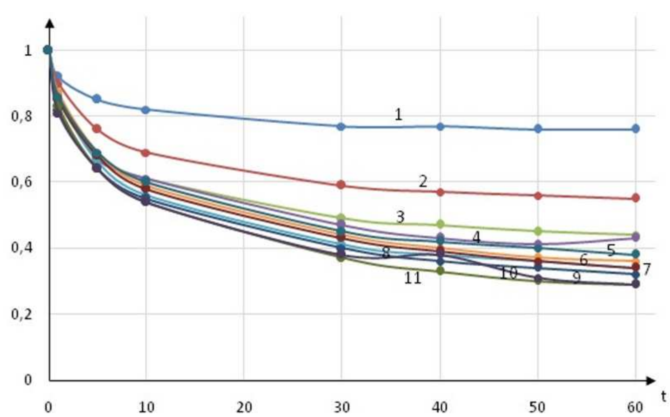


Figure 4: Recovery of the asphalt concrete strain after removing the stress in different cycles (Sample No. 275)

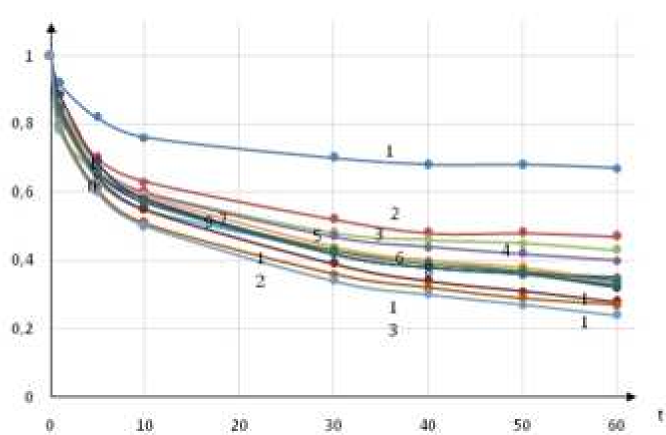
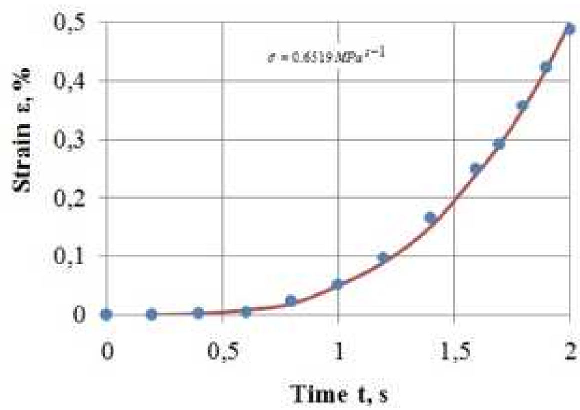


Figure 5: Recovery of the asphalt concrete strain after removing the stress in different cycles (Sample No. 279)

5 Analysis of the creep of asphalt concrete samples at $\dot{\sigma} = const$ until moment a failure at test temperature $T = 22°C - 24°C$

5.1. Loading rate $\dot{\sigma} = 0.6519$ MPa^{s-1}. Asphalt concrete 5 samples were tested. All samples were fractured brittle with small deformations. The experimental average values of creep deformation of asphalt concrete samples are shown in figure 6. Using experimental data, we will obtain:

$$a = 0.196; \quad \gamma = 3.3; \quad \overline{\alpha} = 0.9; \quad \overline{\delta} = 0.0073. \tag{43}$$



(●) – experiment, (-) - calculation

Figure 6: Graphs of strain variation in time at various loading rates $\dot{\sigma} = 0.6519$ MPa^{s-1}

Substituting (43) into (19), the calculated creep strain values $\varepsilon_m(t)$ are calculated at $\dot{\sigma} = 0.6519$ MPa^{s-1}, they are shown in figure 6. The coincidence is satisfactory. Creep parameters ($\overline{\alpha}$, $\overline{\delta}$) calculated by the above method and the correlation coefficients of asphalt concrete samples at the remaining loading rates:

$$\dot{\sigma} = 0.4678 \text{ MPa}^{s-1}: a = 0.297; \quad \gamma = 3.3; \quad \overline{\alpha} = 0.9; \quad \overline{\delta} = 0.0124.$$

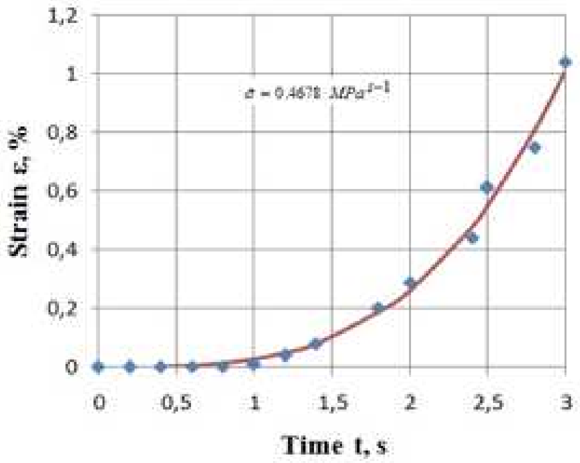
$$\dot{\sigma} = 0.0580 \text{ MPa}^{s-1}: a = 0.754; \quad \gamma = 1.6; \quad \overline{\alpha} = 0.9; \quad \overline{\delta} = 0.0004.$$

$$\dot{\sigma} = 0.0489 \text{ MPa}^{s-1}: a = 1.851; \quad \gamma = 1.8; \quad \overline{\alpha} = 0.9; \quad \overline{\delta} = 0.0032.$$

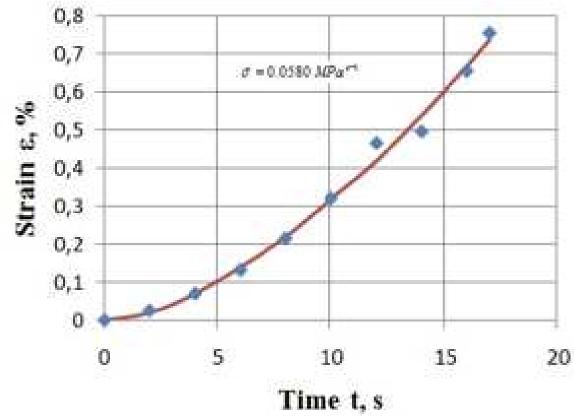
$$\dot{\sigma} = 0.0055 \text{ MPa}^{s-1}: a = 1.069; \quad \gamma = 1.01; \quad \overline{\alpha} = 0.6; \quad \overline{\delta} = 0.1517.$$

Asphalt concrete 5 samples were tested at each constant loading rate. Graphs of the deformation change over time at the found constant loading rates using the above parameters are shown in figures 7.

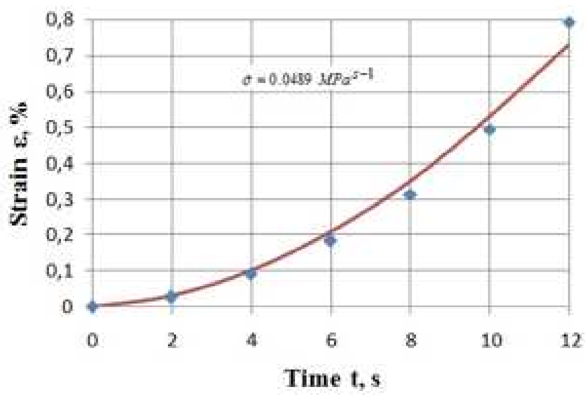
From the analysis of the constructed graphs, it can be seen that the calculated creep curves of asphalt concrete samples at each constant loading rate at a high level coincide with the corresponding experimental values.



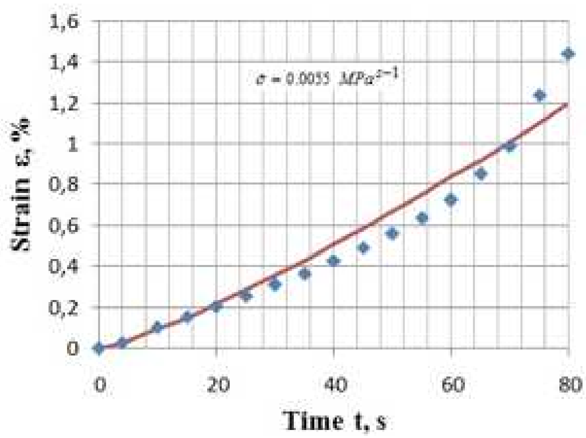
a) (•) – experiment, (–) – calculation



b) (•) – experiment, (–) – calculation



c) (•) – experiment, (–) – calculation



d) (•) – experiment, (–) – calculation

Figure 7: Graphs of strain variation in time at various loading rates

6 Conclusion

A method is proposed for determining the necessary material characteristics from the data of forward and reversed creep, as well as the tension of samples at a constant loading rate ($\dot{\sigma} = const$) and constant temperature ($T = const$) creep of rheonomic materials. The behavior of asphalt concrete in the increasing and constant cyclic loading is shown. The reloading effect on the reversed creep process of asphalt concrete samples has also been investigated. The analysis of the obtained results from experiments and calculations is conducted. In the course of the experiments, all samples were fractured brittle with small deformations. Until moment of failure, microcracks were observed in all samples. The source of microcracks was the local accumulation of microstress. At the same time, the level of microstress in the tested samples increased with increasing reloading. As a result of all the investigations, the result of the above equations fully corresponds to the results of the conducted experiments.

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USING OF MICROCONTROLLER FOR STUDENT LEARNING PROCESS

Use of the latest achievements in the field of microcontroller programming, such as the Arduino platform, allows to qualitatively change the educational process, makes it more intense, increases student motivation, and makes it possible to implement an individual approach, which is important. And this, in turn, improves the efficiency and quality of microcontroller programming. The purpose of this study is to propose an effective methodology for using Arduino Atmega 328 microcontrollers for teaching students and evaluate the effectiveness of teaching programming based on the use of Arduino Atmega 328 microcontrollers based on the Kirkpatrick model. The paper presents a broad review of works that consider the interaction of a person and microcontrollers. In addition, the impact of this approach on the process of learning and teaching is being evaluated. More than 95 students took part in this experiment. First, during the semester, students were taught programming using Arduino Atmega 328 microcontrollers, after which they evaluated this learning. The evaluation was carried out at three levels of the Kirkpatrick model [1], and as a result, the second and third levels showed almost the same results with an error of 3 percent. This study concluded that such teaching methodology is very important in the process of student learning. Interaction and collaboration in the field of microcontroller programming has also been used to introduce non-traditional curricula, including courses in robotics as a tool for addressing the social aspects of robotics and artificial intelligence.

Key words: Programming, Microcontrollers, Arduino, Effectivity, Methodology.

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Студенттердің оқу процесі үшін микроконтроллерді пайдалану

Ардуино платформасы сияқты микроконтроллерлерді бағдарламалау саласындағы соңғы жетістіктерді пайдалану оқу процесін сапалы өзгертуге мүмкіндік береді, оны көбірек етеді, студенттердің ынтасын арттырады және жеке көзқарасты жүзеге асыруға мүмкіндік береді, бұл маңызды. Ал бұл, өз кезегінде, микроконтроллерді бағдарламалаудың тиімділігі мен сапасын арттырады. Бұл зерттеудің мақсаты студенттерді оқыту үшін Arduino Atmega 328 микроконтроллерлерін пайдаланудың тиімді әдістемесін ұсыну және Киркпатрик моделі негізінде Arduino Atmega 328 микроконтроллерлерін пайдалану негізінде бағдарламалауды оқытудың тиімділігін бағалау болып табылады. Жұмыста адам мен микроконтроллерлердің өзара әрекеттесуін қарастыратын жұмыстардың кең шолуы берілген.

Сонымен қатар, бұл тәсілдің оқу мен оқыту үдерісіне әсері де бағалануда. Бұл экспериментке 95-тен астам оқушы қатысты. Біріншіден, семестр барысында студенттерге Arduino Atmega 328 микроконтроллері арқылы бағдарламалау үйретілді, содан кейін олар осы оқуды бағалады. Бағалау Киркпатрик моделінің үш деңгейінде жүргізілді [1], нәтижесінде екінші және үшінші деңгейлер 3 пайыздық қателікпен бірдей дерлік нәтиже көрсетті. Бұл зерттеуде мұндай оқыту әдістемесі студенттердің оқу процесінде өте маңызды деген қорытындыға келді. Микроконтроллерді бағдарламалау саласындағы өзара әрекеттесу және ынтымақтастық сонымен қатар дәстүрлі емес оқу жоспарын, соның ішінде робототехника мен жасанды интеллекттің әлеуметтік аспектілерін шешу құралы ретінде робототехника курстарын енгізу үшін пайдаланылды.
Түйін сөздер: Бағдарламалау, Микроконтроллерлер, Arduino, тиімділік, әдістеме.

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Использование микроконтроллеров в процессе обучения студентов

Использование новейших достижений в области программирования микроконтроллеров, таких как платформа Arduino, позволяет качественно изменить учебный процесс, делает его более интенсивным, повышает мотивацию учащихся, дает возможность реализовать индивидуальный подход, что немаловажно. А это, в свою очередь, повышает эффективность и качество программирования микроконтроллеров. Целью данного исследования является предложить эффективную методику использования микроконтроллеров Arduino Atmega 328 для обучения студентов и оценить эффективность обучения программированию на основе использования микроконтроллеров Arduino Atmega 328 на основе модели Киркпатрика. В статье представлен широкий обзор работ, рассматривающих взаимодействие человека и микроконтроллеров. Кроме того, оценивается влияние такого подхода на процесс обучения и преподавания. В эксперименте приняли участие более 95 студентов. Сначала в течение семестра студентов обучали программированию с использованием микроконтроллеров Arduino Atmega 328, после чего они оценивали полученные знания. Оценка проводилась на трех уровнях модели Киркпатрика [1], в результате второй и третий уровни показали практически одинаковые результаты с ошибкой в 3 процента. В этом исследовании сделан вывод, что такая методика преподавания очень важна в процессе обучения студентов. Взаимодействие и сотрудничество в области программирования микроконтроллеров также используются для внедрения нетрадиционных учебных программ, в том числе курсов по робототехнике как инструмента решения социальных аспектов робототехники и искусственного интеллекта.
Ключевые слова: программирование, микроконтроллеры, Arduino, эффективность, методология.

1 Introduction

Over the past few years, we have seen a huge need for educational systems to equip students with competencies that are increasingly required in the labor market [2], such as innovation, collaboration, problem solving, critical thinking and digital literacy [3]. This encourages academic institutions to work on approaches to teaching and learning, to promote the development of such competencies, and to promote pedagogical innovation and digital learning [4] Microcontroller lesson material is needed in order to improve students' readiness to work. Almost all equipment in the industry has used automation in the production process.

Meanwhile, to study microcontroller material better, microcontroller trainer media is needed. Currently, Arduino is one of the most convenient platforms for developing control devices on microcontrollers [5]. The Arduino board contains: Atmel's ATmega microcontroller, reset circuits, a quartz resonator, a built-in power supply voltage stabilizer, a USB adapter that provides communication with a personal computer, a built-in programmer, tools for in-circuit programming [6]. Arduino's programming language is based on C/C++, but has a simplified syntax and is relatively easy to learn [7]. The Arduino platform makes it relatively easy to develop applications based on AVR microcontrollers and has a number of advantages over other platforms in terms of learning and mastering the technology of developing microcontroller devices: Low cost; Cross platform; Simple and clear programming environment; Extensible open source software; Arduino modules are expandable hardware with open circuit diagrams [8].

The above advantages can be decisive when choosing an object of study and research, study and research of development technologies based on microcontrollers with a limited time volume of training courses on microprocessor technology [9]. Such time limitations are typical during the transition to a bachelor's degree, as well as for specialties in which electronics and microprocessor technology are introductory courses. It is necessary to indicate one more distinctive feature of the Arduino platform in terms of use in the educational process. This is relatively cheap compared to industrial "brand"laboratory equipment. For example, the NI MyRIO-based laboratory setup from National Instruments [10] relies on the Lab View software environment. The cost of NI MyRIO with the Lab View software environment, depending on the composition of the modules, can reach several hundred thousand rubles [11, 12]. The Arduino board is ideal for the first steps in this area, because has a compact size and simple circuitry.

2 Literature Review

The use of microcontrollers in the education process, not only for students of computer science, computer engineering and related specialties, but also for physicists, chemists, mathematicians, networkers and others [13] is one of the most necessary in the modern educational environment [14]. From the basics of the electromechanical device of modern digital devices to the level of the assembler of these devices, programming both individual components [15] and the entire device based on visual aids and debuggers allows the student to gain valuable practical skills and abilities that are generally aimed at systematization of knowledge and skills. It is also important to note the widespread use of Arduino microcontrollers not only in higher education institutions, but also for teaching senior students of secondary schools, gymnasiums, lyceums, as well as colleges.

Learning using microcontrollers for a teacher can be accompanied by a number of problems, such as installing an accompanying programming platform, mastering a programming language, a large array of sensors and circuits on a board, teaching methods [5] and presenting educational material. The use of Arduino Atmega 328 microcontrollers allows the student to understand the real situation in the context of circuits, experiment, invent, make mistakes, correct mistakes. Although the process of real-world context to microcontroller context may be accompanied by some problems of learners [16], in the process of sequential learning they can be overcome.

In teaching programming in laboratory classes, the results of a 2021 study by [17] showed that students as interesting and exciting characterized the use of Arduino microchips. A survey was also conducted, the results of which revealed that microcontrollers have a visually positive effect on the perception of educational material. In general, this and many other [18] studies confirm the importance and interest in using Arduino Atmega 328 microcontrollers among students. The formation of design competence is implemented with strict consistent implementation of the instructions of the rules for working with microcontrollers.

An analysis of the effectiveness of the Arduino Atmega 328 microcontroller was carried out in the work of [19]. The magnitude of the effect obtained was more than 60%, which confirmed the effectiveness of using these devices. Features such as the type of school, the presence of a course in the curriculum, the peculiarity of the student, the chosen programming language, the time of study, the number of hours per week and the contingent of students were taken into account in the calculation of efficiency. However, it is important to note that this study was conducted in schools, among students.

The importance of using microcontrollers is also noted in higher education institutions. For example, Arduino has been used in a biological analysis process for glucose detection [20], in information security tasks, in a security and energy efficient home automation system, as well as data logging related to solar panel use . Arduino has also been used in weather monitoring, in gesture control, it was integrated into the digital signal processing system from the well , as well as IoT-based air quality monitoring using Arduino sensors and MQ series with dataset analysis.

3 Research method

The purpose of this paper is to describe the methodology of using Arduino Atmega 328 microcontrollers for effective student learning. To achieve this goal, the following tasks should be performed:

1) To justify the choice of the necessary programming languages for their step-by-step study from the point of view of the simplicity of their development and at the same time the completeness of functionality for use in robotics;

2) Give recommendations on the study of microcontrollers of the software element base at the university.

This device contains at the same time a microcontroller (usually from Atmel), a programmer, a quartz resonator, a power stabilizer, and much more that is necessary for comfortable use. This device is programmed from the USB port. For Arduino Atmega 328, there is a special development environment Arduino IDE, written on the Java virtual platform. The Arduino IDE has a C++ dialect that will make it easier for the student to understand. Arduino IDE is a free environment. The authors of the article consider it the most relevant for today's application for teaching students.

Arduino serves to solve one difficult problem; it is how to teach students how to create electronic devices. Arduino is a flexible tool for designing automated and automatic control systems at the physical and software levels [15].

4 Materials and methods

4.1 Research context

This study was conducted at the Faculty of Information Technology of the Kazakh National University named after al-Farabi for 3rd year students of the specialties "Computer Engineering"and "Computer Engineering and Software". The program of the course with the optional component "Design and Embedded Multiprocessors"implies 1 lecture and 2 laboratory classes, in total 45 hours or 5 credits. Training and research was carried out in the laboratory of "Intelligent Programmable Systems"for 5 years. The prerequisites of this course are the disciplines "Physics "Integrated Circuits". Participants of the study, 3rd year students who do not have experience with Arduino microcontrollers. Teachers are concerned about the effectiveness of using microcontrollers, namely Arduino, assessing the work of a teacher based on student surveys, student performance and the process of developing education.

4.2 Procedures

The effectiveness of training using Arduino microcontrollers was carried out on the basis of the classical approach is the Kirkpatrick model. The four-level model contains the following steps: learner's reaction or learners opinion and feelings about learning; learning is an indicator of the growth of education and skills; behavior is improvement of skills and abilities, usually based on the assessment of the teacher; the result is the effect of learning.

All of these measurements are recommended for a complete and meaningful assessment of the learning process. To implement the assessment, a study was conducted based on a survey of students. The questionnaire consists of 20 questions assessing response and learning, that is, the first three stages of the Kirkpatrick model. 95 students took part in the survey, Table 1 shows the characteristics of the surveyed groups.

Questions for the survey were compiled on the basis of a review of a large volume of research and analysis. For this study, the 2 stages of the Kirkpatrick model are more informative and important. Therefore, the main part of the questions was compiled in these two sections. Several questions are devoted to determining views on the accessibility of the course: understandability, complexity, importance in the future, novelty, competence of the teacher.

Further, it is proposed to evaluate the relevance, applicability, organization of the educational process on a scale. The rating scale consists of 5 points. Participants anonymously, in an independent form, could mark the necessary points.

The next group of questions is devoted to assessing the acquired theoretical knowledge and practical skills on the basis of multiple choice test questions. This group of questions is based on three levels of difficulty: easy, medium and difficult questions. Easy options contain, for example, questions to test knowledge of the name of the components of the Arduino board, followed by questions to determine the knowledge of the function and their capabilities, and complex questions, mainly with mathematical and physical calculations.

Further, the implementation of the third stage of the Kirkpatrick model took place on the basis of the teacher's assessment for three boundary controls of the academic semester. The maximum score that a student could receive abroad is 100%, that is, three milestone weeks make up 300%, where scores above 70 indicate good mastery of the material, and above 90 indicate excellent academic performance. According to statistics, about 25 thousand visually impaired citizens live in Kazakhstan, so it is very important to make it clear to students that there is such a problem and, most importantly, it can be eliminated.

5 Results

All respondents found the content of the course to be very simple and understandable. Moreover, 50.5% of the respondents rated this item on a five-point scale at 4 points, and the rest 49.5 percent at 5 (Fig. 1). That is, the students had practically no problems in mastering the course.

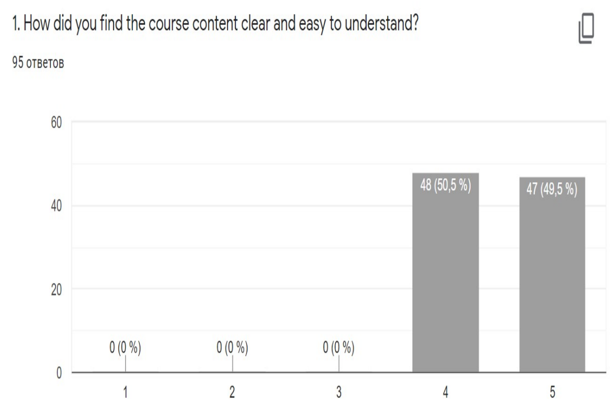

Figure 1: Evaluation of comprehensibility and simplicity of the course

The following questions to assess the relevance and applicability of this course showed very good results, almost all respondents (98.9%) believe that the proposed course is relevant and the knowledge gained from the course can be used in production (Fig. 2).

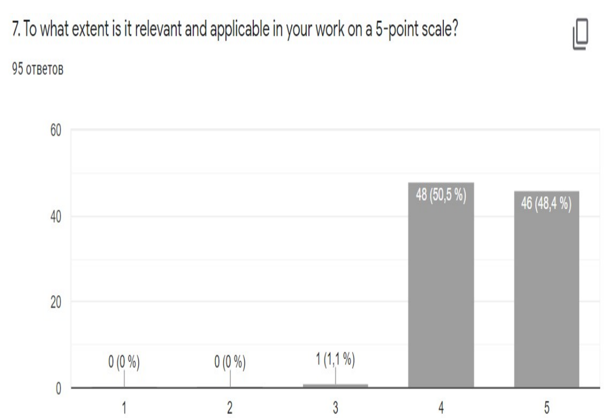

Figure 2: Relevance and applicability of the course

The organization of training received a rating of four points from 22.1% of respondents, when 7.9 believe that this item deserves the highest rating (Fig. 3).

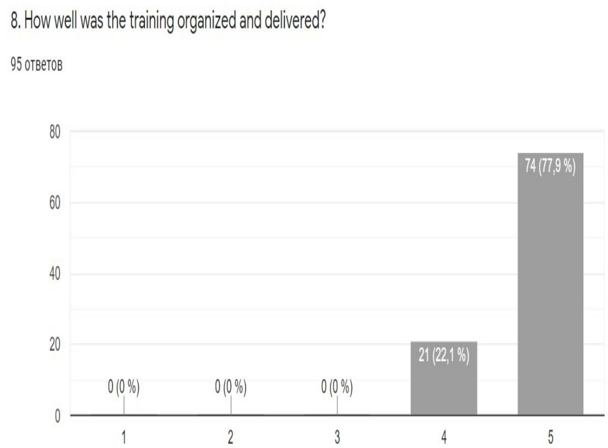

Figure 3: Evaluation of course organization

Interesting suggestions were made by learners to improve the course content. More than 50 percent of students suggest adding more practice tasks, while 32 percent believe that no improvement is needed, the rest answered that they were satisfied with all the content (Fig.4).

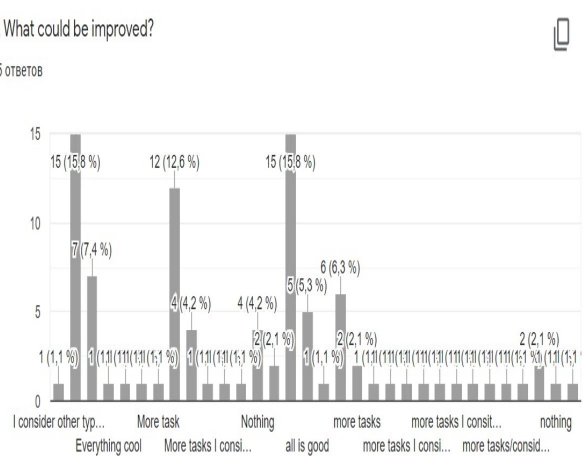

Figure 4: Participants' proposals

Further, test questions were organized to assess the knowledge gained and progress in this kus. On average, students showed excellent results on test questions. More than 93.7 percent of those surveyed showed excellent results. The assessment of the teacher and the assessment of the testing conducted, in principle, showed the same results. The error was only 3.17 points (Fig. 5).

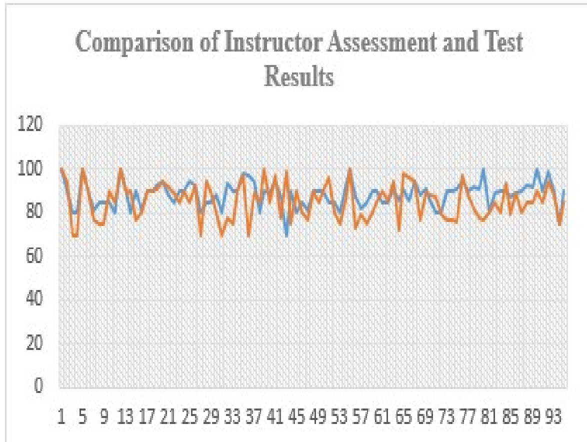

Figure 5: Instructor evaluation

6 Conclusion

This article discusses one of the methodology for teaching students programming based on the use of Arduino microcontrollers. The work on the article was aimed at sharing the experience gained over several years of teaching students the basics of programming. In the practical part, it was interesting to observe the development of the material, the receipt and interpretation of the results that they received.

In general, this study is the beginning of further study of the process of teaching students using not only Arduino microcontrollers, but also other types. The results obtained showed that this type of integration had a good effect on student satisfaction, however, further intensive development of information technology requires the same progress from the educational process.

Understanding the principles of operation of microcontrollers is the basis for the effective professional activity of specialists in this area. The solution to this problem can be the use of the Arduino debug board in the process of teaching students of the specialty 5B071900 - Radio engineering, electronics and telecommunications, 6B06103 - Computer engineering, 5B070400 - Computer engineering and software programming microcontrollers.

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HIERARCHICAL MODEL FOR BUILDING COMPOSITE WEB SERVICES

The current evolution of disseminated computer program frameworks is characterized by a growing adherence to the principles of service-oriented engineering (SOA). Simultaneously, these frameworks are becoming more intricate, with an increasing number of components and more complex data connections between them. This situation underscores the importance of employing mechanisms to unify artifacts in the process of developing composite web services, which govern, among other aspects, the architectural plane of the frameworks under construction. A model for constructing composite web services is suggested as a suitable tool, implemented in accordance with a hierarchical approach, intended for use in designing distributed systems. Model is constructed over an assumption that coordination of the components of a composite web service is carried out in a centralized manner ? in accordance with the orchestration model. To implement formalization and obtain, based on analytical representations, the corresponding software implementations, it has been decided to use the DEVS mathematical apparatus. The aspect of software implementation is considered pivotal in determining the feasibility of automating the acquisition of composite web services that operate within the orchestration model. Obtained research results has been interpreted as a confirmation of the effectiveness of this approach on the basis of the scenario of querying the database. Resulting artifacts have been represented with UML notation. The relationship between analytical representations and corresponding software implementations has also been demonstrated. Usage of the DEVS Suite tools has made it possible to visualize the process of simulation - to obtain estimated values of the indexes of the resulting solutions.

Key words: distributed system, composite web service, DEVS, UML.

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Композиттік веб-қызметтерді құрудың иерархиялық моделі

Бөлінген бағдарламалық жүйелерді дамытудың қазіргі деңгейін сервистік-бағдарланған архитектура (СБА) ережелерін ұстану барған сайын кең таралған тәжірибеге айналатын деңгей ретінде сипаттауға болады. Сонымен қатар, мұндай жүйелердің күрделілік деңгейі қатысатын құрамдас бөліктердің саны бойыншада, осы құрамдас бөліктер арасында орнатылған ақпараттық байланыстардың күрделілігі бойынша да арта түсуде. Бұл жағдай, өз кезегінде, құрылатын жүйелердің архитектуралық құрамдас бөлігін реттейтін құрама веб-қызметтерді әзірлеу процесінде артефактілерді біріктіру тетіктерін пайдаланудың маңыздылығын анықтайды. Тиісті құрал ретінде иерархиялық тәсілге сәйкес жүзеге асырылатын композиттік веб-қызметтерді құру моделі ұсынылады.

Модель бөлінген жүйені жобалау кезеңінде пайдалануға арналған. Модель композиттік веб-қызметтің құрамдас бөліктерін үйлестіру орталықтандырылған - оркестрлік модельге сәйкес жүзеге асырылады деген болжамға негізделген. Ресімдеуді жүзеге асыру және аналитикалық ұсыну негізінде сәйкес бағдарламалық қамтамасыз етуді енгізу үшін DEVS математикалық аппаратын пайдалану туралы шешім қабылданды. Бағдарламалық қамтамасыз етуді енгізу, өз кезегінде, оркестрлік модель бойынша жұмыс істейтін композиттік веб-қызметтерді алу процесін автоматтандыру мүмкіндігін анықтайтын фактор ретінде қарастырылады. Зерттеу нәтижелері дерекқорға сұраныстарды орындау сценарийінің мысалын пайдалана отырып, бұл тәсілдің тиімділігін растады. Алынған артефактілер UML экспрессивті құралдары арқылы ұсынылды. Сондай-ақ аналитикалық көріністермен сәйкес бағдарламалық қамтамасыз етуді енгізу арасындағы байланыс көрсетілді. DEVS Suite құралдарының мүмкіндіктерін пайдалану, басқалармен қатар, модельдеу процесін визуализациялауға – жасалатын шешімдердің көрсеткіштерінің болжалды мәндерін алуға мүмкіндік берді.

Түйін сөздер: үлестірілген жүйе, композиттік веб-сервис, DEVS, UML

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Иерархическая модель построения составных веб-сервисов

Текущий уровень развития распределенных программных систем можно охарактеризовать как такой, при котором следование положениям сервис-ориентированной архитектуры (СОА) становится все более повседневной практикой. Вместе с тем уровень сложности таких систем продолжает возрастать – как с позиции количества задействованных компонентов, так и с позиции комплексности информационных связей, устанавливаемых между данными компонентами. Такое положение вещей, в свою очередь, обуславливает важность использования в процессе разработки составных веб-сервисов механизмов унификации артефактов, регламентирующих, в том числе, архитектурную составляющую создаваемых систем. В качестве соответствующего инструмента предлагается модель построения составных веб-сервисов, реализованная согласно иерархическому подходу. Модель предназначена к использованию на этапе проектирования распределенной системы. Модель построена на допущении, что координирование компонентов составного веб-сервиса осуществляется централизованно – согласно модели оркестровки. Для проведения формализации и получения на основе аналитических представлений соответствующих программных реализаций принято решение задействовать математический аппарат DEVS. Программная реализация, в свою очередь, адресована в качестве фактора, обуславливающего возможность автоматизации процесса получения составных веб-сервисов, функционирующих согласно модели оркестровки. Результаты проведенных исследований подтвердили действенность такого подхода на примере сценария выполнения запросов к базе данных. Получаемые при этом артефакты были представлены с использованием выразительных средств UML. Также была продемонстрирована связь между аналитическими представлениями и соответствующими программными реализациями. Использование возможностей инструментария DEVS Suite позволило, в том числе, визуализировать процесс имитационного моделирования – для получения оценочных значений показателей создаваемых решений.

Ключевые слова: распределенная система, композитный веб-сервис, DEVS, UML.

1 Introduction

Today, the concept of reuse, which underlies service-oriented architecture (SOA), is one of the defining concepts considered when creating distributed web applications. The reason for this could be to save money on the development process. Web services are parts of SOA-based systems. One big advantage of web services is that they are not tightly connected. This makes the system better at working together and communicating with its different parts. The group of online services in a system is often called a composite web service, and the individual parts of the system are called atomic web services.

Given that composite web services (CWS) can embody systems of varying complexity, it is prudent to conceptualize a CWS as a stratified system. This approach facilitates the process of specification, verification, and validation, ensuring that the synthesized CWS operates correctly through a series of automated steps. Research [1] highlights a deficiency in addressing verification and validation (V&V) issues during development. Synthesizing a CWS involves employing diverse methods to create a CWS with the necessary functional (F) and non-functional (NF) characteristics.

Having a detailed plan in writing, like a set of rules or instructions. A detailed plan for how atomic web services interact with each other is needed to automatically create CWS. This condition is needed because machines need to clearly understand the specification when they are doing automated tasks. Proposed to use TLA (Temporal Logic of Actions) formalism by L. Lamport [2] is a way to explain how something works. The selection of this formalism is substantiated by several distinguishing characteristics: firstly, the utilization of the Model Checking verification method (TLC, TLA Checker) is seamlessly integrated into the corresponding TLA Toolbox software. Secondly, the incorporation of the "behavior"concept enables the description of acceptable scenarios for the operation of the system under examination. Through the Model Checking approach, it becomes feasible to automatically verify permissible system states, acceptable parameter values for specifications, and to detect potential "deadlocks". One notable advantage of adopting the Model Checking approach for verification lies in its potential for complete automation.

The verification process is tasked with determining whether the formal specification of the Composite Web Service (CWS) has been accurately constructed. This entails confirming the correctness of the specification. A supporting statement from [3] reinforces this notion: "Finding methods to ensure that the developed hardware and software meet their specifications is a core challenge in computer science."Conversely, the validation procedure for CWS seeks to answer the question, "Are we developing the appropriate system with CWS?"This process focuses on validating the suitability of the CWS, ensuring its compliance with predefined standards for both functional (F) and non-functional (NF) attributes. It is recommended to evaluate the practicality of the synthesized CWS for a specific instance, with predetermined criteria for its F and NF characteristics.

2 Formulation of the problem

It is imperative to identify the functional (F) and non-functional (NF) properties of a newly conceptualized Composite Web Service (CWS) during the planning phase of its creation [4]. The automated synthesis of CWS is proposed to proceed systematically through the stages of conceptualization, specification, and verification and validation (V&V). This sequence, referred to as "conceptualization/specification/V&V,"entails developing a formal model specification, designing a CWS model, validating the model, and assessing its adequacy. To streamline the specification stage, it is recommended to structure the model of the CWS during the conceptualization phase. This involves creating a formal, machine-interpretable definition of acceptable scenarios for the operation of the CWS. The verification and validation (V&V) stage ensures alignment between the developed model and specification, while also confirming the adequacy of the model by verifying that the planned CWS's F and NF characteristics meet the required specifications. As a result, this work undertakes the investigation of the proposed sequence of steps in the automated synthesis process of CWS, along with the technologies and tools utilized in its implementation.

3 Conceptualization of CWS

Consider the Composite Web Service (CWS) as a hierarchical structure, which simplifies the subsequent specification process. Complex hierarchical systems can be organized as follows [5, 6]: initially, a conceptual model of the system is crafted, with each layer representing a different level of the subsystem hierarchy. The hierarchical modeling approach is demonstrated by creating models of system components that are then integrated into the overall system model.

The system under investigation is denoted as "coordinator/computers,"representing a specific instance of CWS. The "Controller"design pattern [7] can aptly describe the behavior of such a system. According to this pattern, "a controller should typically delegate tasks to other objects and manage their activities rather than executing tasks themselves."Examples of existing system components aligning with the "Controller"pattern include elements of the Grid infrastructure (CE/WNs, Computing Element & Working Nodes), where the CE component assumes the role of the controller (coordinator). These proposed abstractions resonate with the composition model outlined in the WS-BPEL standard [8], known as "orchestration a model for centrally coordinating web services within a composition (CWS). The function of the synthesis process coordinator is performed by the BPEL Engine component, implemented as part of the corresponding tools (Oracle BPEL Process Manager, ActiveBPEL, Eclipse BPEL Designer, etc.). Let's denote the BPEL Engine component as CRD (Coordinator, Controller).

To describe the specification formalism, we use a set-theoretic approach. Let us denote the set of atomic web services as $AWS = \left\{aws_i \;\middle|\; i = \overline{1, m}\right\}$, $m \in N$, where $aws_i \in AWS$ – atomic web services available for use. Sets of some necessary for the implementation of the F-characteristics of CWS will be represented as subsets of the set AWS: $\left\{C_j \middle|\; j = \overline{1,\ n}\right\}$, $n \in N$, where $C_j \subseteq AWS$ – subset of atomic web services required for implementation j-th F-characteristics of CWS.$C_j = \left\{aws_k \middle|\; k = \overline{1, p}\right\}$, $p \in N$, and $p \leq m$.

Let everyone aws_i characterized by a pair $(af_i,\ anf_i)$, where af_i and anf_i – Φ- and HΦ-characteristics aws_i, respectively. By af_iwill understand some F-transformation, performed on a set of input data vec_i: $af_i\ = f_i(vec_i)$. Conceptually under aws_iwe can understand some abstract entity that implements a function $f_i(vec_i)$.

Let the NF characteristic anf_i determined by three (r_i, t_i, c_i), where r_i(response) – response time aws_i; t_i (throughput) – the capacity of the network channel formed by the sender node of the request and the recipient node (on which some aws_i); c_i(cost) – function execution cost value $f_i(vec_i)$. In this case, we will assume that the value of the element $r_i$equals the sum of the times spent on transmitting the request (from the sending node to the receiving node) and on implementing the F-characteristic to some aws_i, deployed on the recipient node.

Let's assume that the client request specifies requirements for F-(F_req) and NF characteristics (NF_req)CWS. NF_req, wherein, are determined by three (r_req, t_req, c_req), where the elements of the triple represent the response time, link capacity, and cost requirements of CWS, respectively.

A positive answer to the question "Does some NF characteristic of CWS satisfy the requirements of the client request?" it is proposed to give if the corresponding inequalities are true:

$$r\_req \geq \sum_{i=1}^{n} r_i. \tag{1}$$

$$t\_req \leq \min(t_i).$$

$$c\_req \geq \sum_{i=1}^{n} c_i.$$

If we view the interactions among certain aws_k, $aws_{k+1} \in C_j$ entities through the lens of CWS as sequential exchanges between computing processes dispersed geographically, facilitated by asynchronous exchange of structured messages, it seems plausible to consider a formalism grounded in the principle of function superposition as an apt means to depict the functional characteristic of CWS. This approach can be justified by Charles Hoare's theory of interacting sequential processes [9] and specific aspects of message exchange mechanisms among distributed computer system components outlined in the SOAP protocol [10]. This technique serves as a natural method for attaining the requisite functional characteristic of CWS aggregation, as illustrated in (Fig. 1).

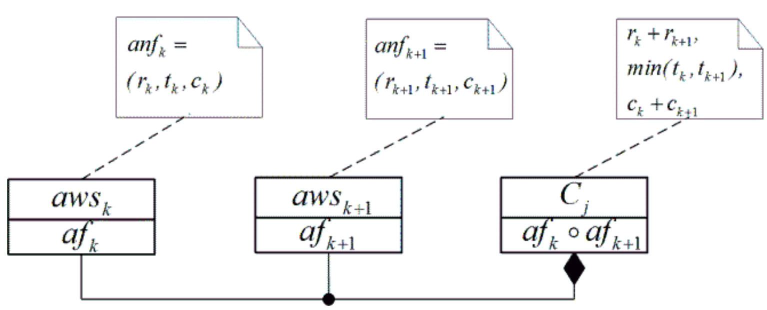

Figure 1: j-th F-characteristic of CWS aggregation scheme

Let's separate the "coordinator/computers"system into two strata: St_0 and St_1 (Table 3).

Because the function is to coordinate atomic web services as part of CWS; by the coordination procedure we mean the execution of calls to some $aws_k \in C_j$ in a given sequence. CRD and aws_k In this case, we will call them elements of the corresponding strata.

Losses	Purpose of the component	Formal notation
St_0	coordinator	CRD
St_1	calculators	$C_j = \{aws_k\}$

Utilizing the introduced formalism, we offer a structural UML diagram depicting the stratification of the CWS "coordinator/computers"(fig. 2). Here, the term "refines,"denoted by the operation (operator), signifies the execution of the coordination procedure.

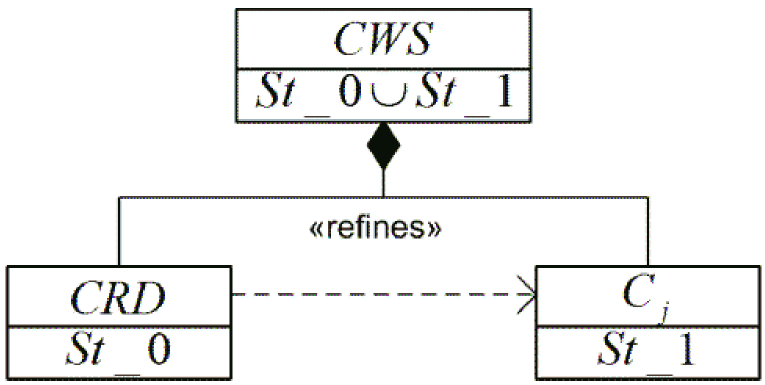

Figure 2: CWS stratification scheme

Let the procedure for coordinating elements C_j is implemented within a certain subroutine. In the theory of interacting sequential processes proposed by Charles Hoare, it is recommended to regard the entire system under examination as a process. Here, the behavior of this process is delineated by the behaviors of its constituent subprocesses. These subprocesses' behavior, in turn, is contingent upon the frequency and order of events. Consequently, alterations in the states of the system in question transpire upon the incidence of events of three distinct types: "boundary", "challenge", "result". Let us represent these types of events in the form of corresponding sets:

$$REQ = \{req,\ resp\}\,,$$

whereREQ – many boundary events, and req– coordinator receiving event CRDrequest with requirements for F and NF characteristics of CWS (we will consider req as initial event);$resp$– final event – sending the result of the CWS work;

$$INVOKE = \{invoke_k\}\,,$$

where$INVOKE$ – many call events from the coordinator CRDelements $aws_k \in C_j$;

$$RES = \{res_k\}\,,$$

whereRES – set of receiving events by coordinator CRDresults of element's operation $aws_k \in C_j$.

Some event $invoke_k \in INVOKE$ we will consider as a stimulus the following type of display:

$$f_k : vec_k \mapsto res_k.$$

We aim to delineate subprocesses that unveil the functional characteristics of CWS via suitable scenarios. Drawing from Charles Hoare's formalism, we propose documenting events using a protocol—a predefined sequence of notations linked to events. We advocate substituting the term "protocol"with the notion of "scenario."This adjustment aligns better with the intricacies of the system under consideration, as the orchestration model delineates a centralized approach to orchestrating the coordination process. Let us denote by a set of scenarios describing the dynamics CWS's (F-characteristics):

$$S = \left\{ s_j^{CRD} \right\}.$$

$$s_j^{CRD} =< invoke_k, ..., \; res_l >, \; l = \overline{1, \; p}. \tag{2}$$

Those every s_j^{CRD} describes a method (scenario) for implementing some CWS F-characteristic based on coordination of elements C_j.The initial entry of the script corresponds to some event of the "call" type, and the final entry corresponds to an event of the "result" type. It's obvious that $|S|$ (cardinality of the set S) equal to the number of F-characteristics of CWS.

The item under investigation (system) first takes part in an event, and then it behaves exactly like a process (subprocess), according to C. Hoare's theory of interacting sequential processes. Formally, it is proposed to write it like this:$x \rightarrow P$, where $x, \; P$ – some event and process (as a sequence of events), respectively; ' ' – follow operator; reads like "P for x". Let's modify this recording method by including a selected type of boundary events into consideration. To do this, let us denote by s_{j+1}^{CRD} some alternative scenario specifying an alternative CWS's F-characteristic. The alternative will be designated as ."The following characteristics apply to acceptable CWS speakers:

$$req \rightarrow (s_j^{CRD} | \; s_{j+1}^{CRD}) \rightarrow resp. \tag{3}$$

One could think about this method of defining the CWS dynamics (3) as an expansion of (2).

It is also important to note that [11] suggests an alternative method of documenting events (instead of laying out a timeline). The concept of "process history"(h) is utilized in place of "protocol."Synopsis h is carried out in the manner: $e \xrightarrow{h} e'; \; e, \; e' \in E$, where E– several incidents, '$\rightarrow$'indicates the changes between occurrences, h – an arrangement of transitional events from E.

From the perspective of streamlining the process for interpreting a script into a formal TLA specification, we believe that setting the sequence of events using scripts is a more acceptable method.

4 A system with CWS example

Now, let's delve into a specific scenario. Let's suppose we're examining a system equipped with Composite Web Services (CWS). This system can be perceived as a modified version of the "coordinator/computers"system. We introduce an additional actor named "Client"into this system, denoting a source of boundary events: request generation is represented by event *req*; while receiving the CWS output is depicted by event *resp*.

As an example domain scenario, let's consider the process of generating queries to a Database Management System (DBMS). The significance of this scenario is underscored by the prevalence of corresponding web-based software systems (eBay, newegg, etc.). Because Oracle or MySQL solutions are usually used as a DBMS; let the set of query generation functions be presented as the following set: {*select*, *delete*, *update*}, where the elements denote the functions for generating queries for selecting, deleting and modifying table records, respectively. Let the specified functions be implemented by atomic web services $aws_1,\ aws_2,\ aws_3$, respectively.

To modify (delete) the required record of a table, you must first generate a query to make sure that exactly the required record is selected; then, depending on the end goal being pursued, execute either the request *delete*, or request *update*. As a consequence, we see that a possible way to automate this procedure is the synthesis of CWS, the functioning of which can be carried out according to two scenarios:

$$req \rightarrow (s_1^{CRD} |\ s_2^{CRD}) \rightarrow resp.$$

Scenarios s_1^{CRD} and s_2^{CRD} reveal the F-characteristics of CWS:

$$AWS = \{aws_1,\ aws_2,\ aws_3\}\,;$$

$$C_1 = \{aws_1,\ aws_2\};\ C_2 = \{aws_1,\ aws_3\}\,;$$

$$s_1^{CRD} =< invoke\_1,\ res\_1,\ invoke\_2,\ res\_2 >;$$

$$s_2^{CRD} =< invoke\_1,\ res\_1,\ invoke\_3,\ res\_3 > .$$

Let us present the described scenarios in the form of a UML interaction sequence diagram (fig. 3).

5 Specification, V & V

Interpreting scenarios s_1^{CRD} and s_2^{CRD} into a formal TLA specification involves envisioning scenario records as sequences of Composite Web Service (CWS) states. To achieve this, we establish rules for specifying event occurrences:initialization of variables corresponding to events involves assigning elements of the set; '0' denotes the event did not occur, while any other value represents occurrence; a modifier (") indicates the value of a variable specifying event occurrence at a subsequent point in time.

To define CWS states and their sequence: utilize the conjunction operator ($\wedge$) to connect the current state to the previous one and set variable values based on events within the CWS state; employ the disjunction operator ($\vee$) to indicate alternation in scenarios; the 'UNCHANGED' modifier denotes that a variable's value in the current state remains unchanged from the previous state.

The TLA specification for the given cases ($s_1^{CRD}$and s_2^{CRD}), is then created using the interpretation rules outlined above. Listing for formal CWS TLA demonstrated (Fig. 4).

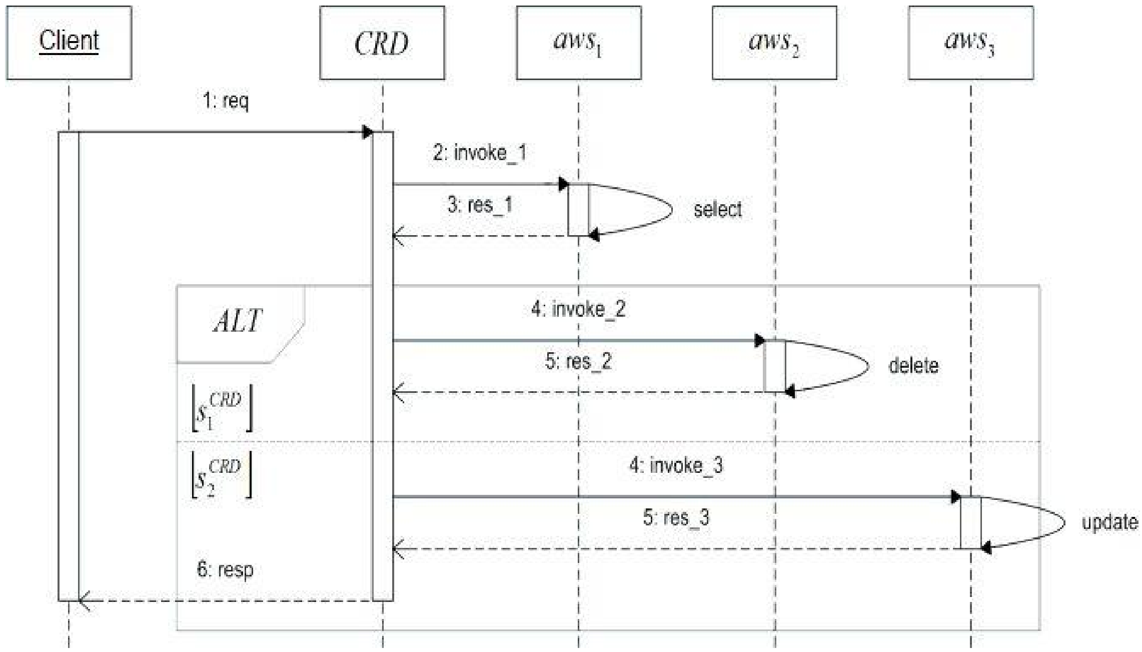

Figure 3: Scenarios for the operation of a system with CWS

Listing 1 defines valid CWS states according to the following conventions: Init, OnReq, OnInvoke_1,..., OnRes3, OnResp. The correctness of the specification was checked using the Model Checking method (TLC, TLA Checker), integrated into the TLA Toolbox development environment.

Analyzing the proposed specification method, one can note some cumbersomeness (syntactic redundancy) of the resulting CWS TLA specification. As an opposite (positive) point, we can point out the clarity and structure of the TLA specification obtained through the use of the proposed set of translation rules. The indicated advantages and disadvantages can also characterize wsdl (Web Services Description Language) descriptions of atomic web services.

The next step is to implement the validation procedure. In our case, the validation procedure consists of conducting discrete-event simulation modeling in the DEVS Suite environment. A distinctive feature of the DEVS formalism is the concept of "atomic model" [12]. This concept is preferable in that it allows one to naturally represent the hierarchical connections (relationships) of component models within a system model with a CWS.

Let us denote by $am\_1, ..., am\_3$ atomic web service models $aws_1, ..., aws_3$, respectively. Coordinator Model CRD let's denote it as am_CRD. We will represent the "Client" component of a system with CWS in the form of an atomic model of a scenario generator (s_1^{CRD}, or s_2^{CRD}), which are then sent to the model's input ports am_CRD. Let us denote the model of the "Client" component as am_Gen.

Let's include models of atomic components as part of the system model with CWS (cm_CWS). We will record the moments when messages appear on the input and output ports (in and out) models cm_CWS as moments of the onset of boundary events $req$and $resp$, respectively.

```
\* operate with natural numbers
EXTENDS Naturals
\* variables denoting events
VARIABLES   req, resp,
       invoke_1, invoke_2, invoke_3,
       res_1, res_2, res_3
\* setting acceptable values
Def ==  /\ req\in{0,1}
/\ resp \in{0,1}
/\ invoke_1 \in{0,1}
/\res_1 \in {0,1}
/\ invoke_2 \in{0,1}
/\ res_2 \in {0,1}
/\ invoke_3 \in {0,1}
/\ res_3 \in {0,1}
\* CWS state specification:
\* 1 - none of the events happened
\*
Init == /\ req=0  /\ invoke_1=0/\res_1=0/\resp=0/\invoke_2=0/\ res_2=0/\invoke_3=0 /\ res_3=0
\* 2 - receipt of a request
\* from the client
OnReq == /\ req' = 1 - req
         /\ UNCHANGED<<resp>>
/\ UNCHANGED<<invoke_1, invoke_2, invoke_3>>
/\ UNCHANGED<<res_1, res_2, res_3>>
\* 3 - call by coordinator aws_1
OnInvoke_1 == /\ OnReq
       /\ invoke_1'=1-invoke_1
      /\ UNCHANGED<<req, resp>>
/\ UNCHANGED<<invoke_2, invoke_3>>
/\ UNCHANGED<<res_1, res_2, res_3>>
\* 4 - receipt by coordinator
\* call result aws_1
OnRes_1 == /\ OnInvoke_1
       /\ res_1' = 1 - res_1
      /\ UNCHANGED<<req, resp>>
/\ UNCHANGED<<invoke_1, invoke_2, invoke_3>>
/\ UNCHANGED<<res_2, res_3>>
\* 5 - call by coordinator aws_2
OnInvoke_2 == /\ OnRes_1
       /\ invoke_2' = 1 - invoke_2
      /\ UNCHANGED<<req, resp>>
/\ UNCHANGED<<invoke_1, invoke_3>>
/\ UNCHANGED<<res_1, res_2, res_3>>
\* 6 - receipt by coordinator
\* call result aws_2
OnRes_2 == /\ OnInvoke_2
       /\ res_2' = 1 - res_2
      /\ UNCHANGED<<req, resp>>
/\ UNCHANGED<<invoke_1, invoke_2, invoke_3>>
/\ UNCHANGED<<res_1, res_3>>
\* 5 - call by coordinator aws_3
OnInvoke_3 == /\ OnRes_1
       /\ invoke_3' = 1 - invoke_3
      /\ UNCHANGED<<req, resp>>
/\ UNCHANGED<<invoke_1, invoke_2>>
/\ UNCHANGED<<res_1, res_2, res_3>>
\* 6 - receipt by coordinator
\* call result aws_3
OnRes_3 == /\ OnInvoke_3
       /\ res_3' = 1 - res_3
      /\ UNCHANGED<<req, resp>>
/\ UNCHANGED<<invoke_1, invoke_2, invoke_3>>
/\ UNCHANGED<<res_1, res_2>>
\* 7 - sending to client
\* work result CWS,
\* task of alternativeness
\* scenarios
OnResp == (OnRes_2 \/ OnRes_3) /\ resp' = 1 - resp
Spec == Init /\ [OnResp]_<<req, resp, invoke_1, invoke_2, invoke_3,
res_1, res_2, res_3>>
```

Figure 4: Fragment of specification

Now let's model the system we are studying. As NF features of the system model's constituent parts with CWS, we select the response time, ms:$anf_1.r_1 = 30$, $anf_2.r_2 = 40$, $anf_3.r_3 = 35$. Model response time am_Gen set equal to 10 ms, and the time spent on implementing the coordination procedure by the model am_CRD – 50 ms.

Let the requirements for SF characteristics CWS $NF\_req.r\_req = 200$ ms.Our task is to check through simulation whether the CWS model satisfies the given NF requirements. Satisfaction of the requirements for the CWS F-characteristics is confirmed by the correct functioning of the model.

Consider the case when the input port of the coordinator model am_CRD script arrived s_1^{CRD} (fig.5).

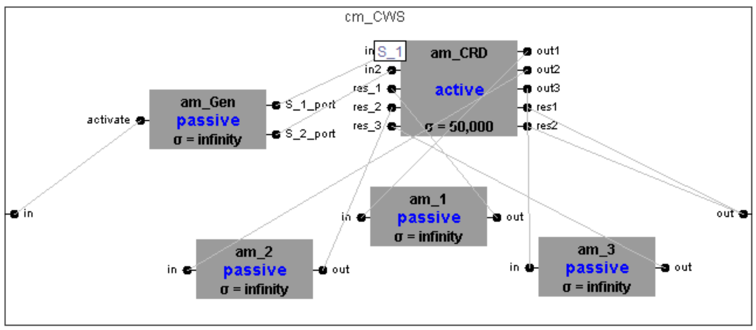

Figure 5: Block diagram of a system with CWS

A fragment of time diagrams of the modeling process is shown in fig. 6.

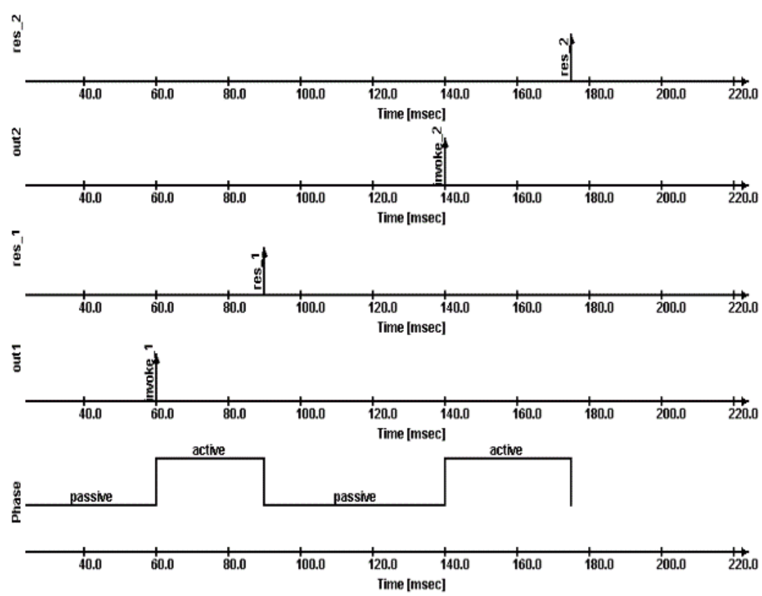

Figure 6: Operation time intervals am_1, am_2

The results of the simulation show that the total value of the component's NF characteristics cm_CWS sums to 225 milliseconds, which is insufficient to meet inequality

(1). For clarity purposes, we have incorporated an illustration of the proposed conceptual model in Figure 7, specifically pertaining to the scenario under examination.

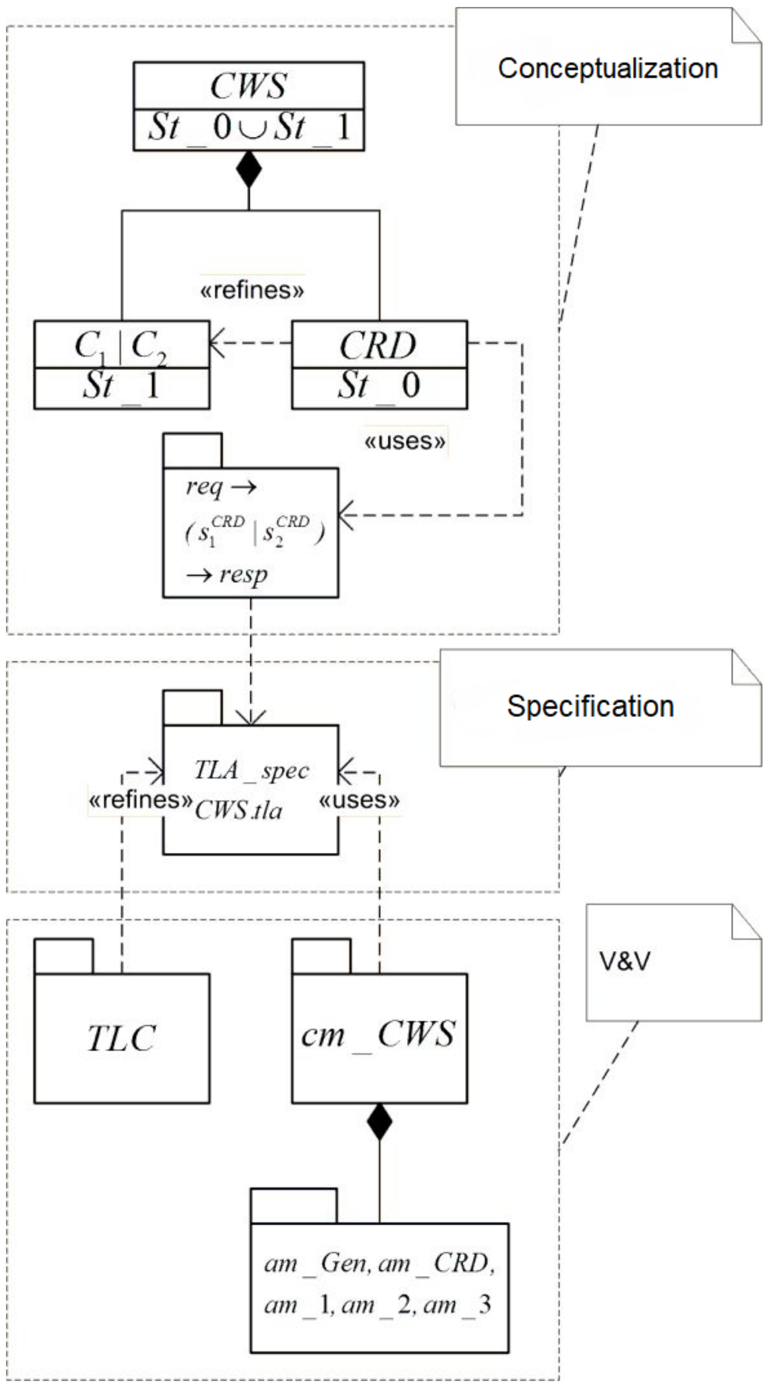

Figure 7: CWS automated synthesis process conceptual model

6 Conclusion

Henceforth, the automated synthesis of CWS entails three consecutive stages: conceptualization, specification, and V&V.

In the conceptualization phase, the composite web service model was stratified, drawing upon Charles Hoare's theory of interacting sequential processes to establish conditions facilitating the subsequent specification stage.

Guidelines for formalizing ideas from the conceptualization stage into a TLA specification are proposed during the specification step.

During the V&V phase, a discrete-event simulation model of CWS was developed within the DEVS Suite environment, and the accuracy of the TLA specification was validated. A domain scenario instance was examined, involving the formulation of queries for a database management system.

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