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METHOD OF LINES FOR A LOADED PARABOLIC EQUATION

Loaded parabolic equations belong to a complex yet important class of differential equations and are widely applied in various scientific and engineering problems, as well as in ecology, epidemic propagation modeling, and biological systems. Special analytical and numerical methods are used to solve these equations, taking into account the influence of integral and functional loads. This article examines a two-point boundary value problem for loaded parabolic equations, defined in a closed domain. The solution is approached using the method of lines with respect to the variable x. As a result of this method, a discretized problem is formulated. The obtained discretized problem is represented in a vector-matrix form and is reduced to a two-point boundary value problem for a loaded system of differential equations. The parameterization method proposed by Professor Dzhumabaev is used to solve the boundary value problem. The efficiency of this method lies in the high accuracy of the numerical-analytical solution compared to the exact solution, as well as in the possibility of formulating the solvability conditions of the problem. As a theoretical justification of the method, an additional theorem is proven, based on which the solvability conditions of the problem are determined. The study explores the relationship between the original boundary value problem and its discretized form for the loaded parabolic equation. This relationship is substantiated using an additional theorem derived from the parameterization method.

Key words: loaded parabolic equations, two-point boundary value problem, method of lines, convergence, parameterization method.

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Жүктелген параболалық теңдеу үшін сынықтар әдісі

Жүктелген параболалық теңдеулер күрделі, бірақ маңызды теңдеулер класына жатады және олар әртүрлі ғылыми әрі инженерлік қолданбаларда, экологияда, эпидемиялардың таралуын модельдеуде және биологиялық жүйелерде кеңінен қолданылады. Мұндай теңдеулерді шешу үшін интегралдық және функционалдық жүктемелердің әсерін ескеретін арнайы аналитикалық және сандық әдістер қолданылады. Мақалады тұйық аймақта жүктелген параболалық теңдеулер үшін екі нүктелі шеттік есеп қарастырылады. Бұл есепті шешу мақсатында кеңістік x айнымалысы бойынша сынықтар әдісі қолданылады. Әдіс нәтижесінде дискреттелген есеп алынады. Алынған дискреттелген есеп вектор-матрицалық түрде өрнектеліп, жүктелген дифференциалдық теңдеулер үшін екі нүктелі шеттік есепке келтіріледі. Шеттік есепті шешу үшін профессор Жұмабаевтың параметрлеу әдісі қолданылады. Бұл әдістің тиімділігі – есептің сандық-аналитикалық шешімінің дәл шешімге жуықтау дәлдігінің жоғары болуында және есептің шешілімділік шарттарының алынуы болып табылады. Әдістің теориялық негіздемесі ретінде қосымша теорема дәлелденіп, есептің шешілімділік шарттары анықталады.

Зерттеу барысында жүктелген параболалық теңдеу үшін бастапқы шетті есеп пен оның дискреттелген есеп арасындағы байланыс қарастырылады. Бұл байланыс параметрлеу әдісі негізінде алынған қосымша теоремамен дәлелденеді.
Түйін сөздер: жүктелген параболалық теңдеулер, екі нүктелі шеттік есеп, сызықтар әдісі, жинақтылық, параметрлеу әдісі.

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Метод прямых для нагруженного параболического уравнения

Нагруженные параболические уравнения относятся к сложному, но важному классу дифференциальных уравнений и широко применяются в различных научных и инженерных задачах, а также в экологии, моделировании распространения эпидемий и биологических системах. Для их решения используются специальные аналитические и численные методы, учитывающие влияние интегральных и функциональных нагрузок. В данной статье рассматривается двухточечная краевая задача для нагруженных параболических уравнений, заданная в замкнутой области. Для ее решения применяется метод прямых по переменной x. В результате этого метода формулируется дискретизированная задача. Полученная дискретизированная задача представляется в векторно-матричной форме и сводится к двухточечной краевой задаче для нагруженной системы дифференциальных уравнений. Для решения краевой задачи используется метод параметризации, предложенный профессором Джумабаевым. Эффективность данного метода заключается в высокой точности численно-аналитического решения по сравнению с точным решением, а также в возможности формулирования условий разрешимости задачи. В качестве теоретического обоснования метода доказывается дополнительная теорема, на основе которой определяются условия разрешимости задачи. В исследовании изучается связь между исходной краевой задачей и ее дискретизированной формой для нагруженного параболического уравнения. Данная связь обосновывается с помощью дополнительной теоремы, полученной на основе метода параметризации.
Ключевые слова: нагруженные параболические уравнения, двухточечная краевая задача, метод прямых, сходимость, метод параметризации.

1 Introduction and preliminaries

The essence of the method of lines, which explains its name, is as follows: for example, in the case of a partial differential equation with respect to a function of two variables, constant values are assigned to one of these variables, and transforming the problem into an ordinary differential equation. Therefore, when addressing boundary and initial-boundary value problems for partial differential equations, existing methods for solving initial and boundary value problems for ordinary differential equations can be effectively applied. In this study, the problem for loaded parabolic equations is transformed into a problem for LDE.

A family of linear and nonlinear parabolic boundary value problems with the first boundary condition are addressed using the method of lines in [1]. It is demonstrated that there is a certain order of error in the approximate solutions produced by this method. The ease of solving the heat conduction equation receives special attention.

A thorough theoretical investigation aimed at proving the convergence and stability of solutions to one-dimensional parabolic equations with Dirichlet boundary conditions is presented in work [2] using the method of lines.

The work [3] approach the method of lines to solve certain quasilinear boundary value problems of parabolic type and establishes theorems proving the convergence and stability of the method.

The work [4] investigates boundary value problems for ordinary and partial differential equations with loading. Estimates for the solutions to both the differential and difference equations are established. These estimates ensure the stability and convergence of the difference schemes for the equations under consideration.

The work [5] analyzes the convergence of one-step schemes of the method of lines (MOL). The primary goal is to establish a general framework for convergence analysis applicable to nonlinear problems. The stability concepts used in this framework are based on the theory of nonlinear stiff ordinary differential equations. In this context, key notions include the norm of the logarithmic matrix and C-stability. To illustrate the proposed ideas, a nonlinear parabolic equation and the cubic Schredinger equation are considered.

The work [6] proposes a parameterization method for finding solutions to a system of ordinary differential equations. The work [7] examines a mixed boundary value problem for a linear parabolic equation with two independent variables. Using the method of lines, estimates for the solutions and their derivatives are obtained in terms of the equation's coefficients and the boundary conditions.

Loaded parabolic equations are widely encountered in mathematical biology, particularly in the mathematical modeling of transfer phenomena in living systems [8], [9]. Different types of boundary value problems for parabolic equations with loading have been explored in the works of T. Yuldashev, M.T. Dzhenaliev, M. I. Ramazanov, V.M. Abdullaev, K.R. Aida-zade and M. Dehghan [10]- [17].

This paper considers the following two-point boundary value problem for a loaded parabolic equation in $\Omega = [0,T]\times[0,\omega]$

$$\frac{\partial u}{\partial t} = a(t,x)\frac{\partial^2 u}{\partial x^2}+b(t,x)u(t,x)+\sum_{j=1}^{m} k_j(t,x)u(t_j,x)+f(t,x), \quad (t,x)\in\Omega=(0,T)\times(0,\omega), \tag{1}$$

$$B(x)u(0,x)+C(x)u(T,x)=\varphi(x), \qquad x\in[0,\omega], \tag{2}$$

$$u(t,0)=\psi_0(t), \quad u(t,\omega)=\psi_1(t), \quad t\in[0,T], \tag{3}$$

where $a(t,x)\ge\rho>0$, $b(t,x)\le 0$, $k_j(t,x)$, $f(t,x)$ - are continuous in t and Holder continuous in x. We assume that the functions $\varphi(x)$, $\psi_0(t)$, $\psi_1(t)$ are fully smooth and satisfy the following conditions: $B(0)\psi_0(0)+C(0)\psi_0(T)=\varphi(0)$, $B(\omega)\psi_1(0)+C(\omega)\psi_1(T)=\varphi(\omega)$.

The task is to find a function $u(t,x)$ which is continuously differentiable with respect to $t\in[0,T]$ and twice continuously differentiable with respect to $x\in[0,\omega]$, such that it satisfies equation (1) along with the conditions (2), (3).

By discretizing with respect to the spatial variable x, the problem (1)-(3) is transformed into a problem of LDE. The second derivative is approximated using the finite difference method. The finite difference methods are discussed in works [18]- [20]. An auxiliary problem for this system will be investigated, focusing on a two-point boundary value problem for

LDE employing the parameterization method with loaded interval partition points in $[0, T]$ [21]- [25]. A method for determining an approximate solution is proposed, together with sufficient conditions ensuring its convergence to the problem's unique solution. A method for numerically solving the problem in systems of LDE is presented [26]- [34].

2 Materials and methods

We consider $\forall x$ and discretize by setting $x_i = i\tau$, $i = \overline{0, N}$, $N\tau = \omega$, with $u_i(t) = u(t, i\tau)$, $a_i(t) = a(t, i\tau)$, $b_i(t) = b(t, i\tau)$, $k_i^j(t_j) = k_j(t, i\tau)$ and $f_i(t) = f(t, i\tau)$. The problem (1)-(3) is then reformulated in the following form:

$$\frac{du_i}{dt} = a_i(t)\frac{u_{i+1} - 2u_i + u_{i-1}}{\tau^2} + b_i(t)u_i + \sum_{j=1}^{m} k_i^j(t)u_i(t_j) + f_i(t), \qquad i = \overline{1, N-1}, \tag{4}$$

$$B_i u_i(0) + C_i u_i(T) = \varphi_i, \quad i = \overline{1, N-1}, \tag{5}$$

$$u_0(t) = \psi_0(t), \quad u_N(t) = \psi_1(t). \tag{6}$$

The solution to the discretized problem (4)-(6) is $\{u_1(t), u_2(t), \ldots, u_{N-1}(t)\}$ system, where is $u_i(t)$ an approximation to the value of the solution $u(t, x)$ at the spatial grid points x_i. It satisfies the system of equations (4), derived from (1) using finite difference approximations for the spatial derivatives. The conditions (5), (6) ensure that $u_i(t)$ adheres to the physical constraints of the problem.

Caused by the linear of the system, for every $\tau > 0$, there exists solution to problem (4)-(6) defined over the interval $[0, T] : \{u_1(t), u_2(t), \ldots, u_{N-1}(t)\}$.

The following statement holds true.

Theorem 1. *Let $a(t, x) \geq \rho > 0$, $b(t, x) \leq 0$, $k_j(t, x)$, $f(t, x)$ - are continuous in Ω, the functions $\varphi(x)$, $\psi_0(t)$, $\psi_1(t)$ are completely smooth and satisfy the matching conditions. Then the solution of the discretized problem (4)-(6) converges at a rate of $O(\tau^2)$ as $\tau \to 0$ approaches the solution of the two-point boundary value problem for a loaded parabolic equation (1)-(3).*

The main goal of this Theorem is to determine the solution to (4)-(6). Thus, we search the conditions for the existence of a solution to problem (4)-(6).

To do this, we write the discretized problem (4)-(6) in matrix-vector form:

$$\frac{dU}{dt} = A(t)U(t) + \sum_{j=1}^{m} M_j(t)U(t_j) + F(t), \quad U \in R^{N-1}, \tag{7}$$

$$BU(0) + CU(T) = \Phi, \quad t \in [0, T], \quad \Phi \in R^{N-1}. \tag{8}$$

Here, the $A(t), M_j(t)$, where $j = 1, \ldots, m$ are matrices of size $(N-1) \times (N-1)$ and $F(t)$ is a vector function of size $(N-1)$ that remains continuous on the interval $[0, T]$; the

B, C are matrices of size $(N-1)\times(N-1)$, where

$$A(t)=\begin{pmatrix} \frac{-2a_1(t)}{\tau^2}+b_1(t) & \frac{a_1(t)}{\tau^2} & 0 & \dots & 0\\ \frac{a_2(t)}{\tau^2} & \frac{-2a_2(t)}{\tau^2}+b_2(t) & \frac{a_2(t)}{\tau^2} & \dots & 0\\ 0 & \frac{a_3(t)}{\tau^2} & \frac{-2a_3(t)}{\tau^2}+b_3(t) & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \frac{-2a_{N-1}(t)}{\tau^2}+b_{N-1}(t)\end{pmatrix},$$

$$F(t)=\begin{pmatrix} \frac{a_1(t)\psi_0(t)}{\tau^2}+f_1(t)\\ f_2(t)\\ f_3(t)\\ \vdots\\ \frac{a_{N-1}(t)\psi_1(t)}{\tau^2}+f_{N-1}(t)\end{pmatrix},\quad M_j(t)=\begin{pmatrix} k_1^j(t) & 0 & 0 & \dots & 0\\ 0 & k_2^j(t) & 0 & \dots & 0\\ 0 & 0 & k_3^j(t) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & k_{N-1}^j(t)\end{pmatrix},$$

$$B=\begin{pmatrix} B_1 & 0 & \dots & 0\\ 0 & B_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & B_{N-1}\end{pmatrix},\quad C=\begin{pmatrix} C_1 & 0 & \dots & 0\\ 0 & C_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & C_{N-1}\end{pmatrix},\quad \Phi=\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_{N-1}\end{pmatrix}.$$

The solution to the problem (7),(8) is a vector function $U(t)$ that is continuously differentiable on $[0,T]$ that fulfills the system of LDE (7) and possesses values $U(0)$, and $U(T)$ at the points $t=0$, $t=T$ respectively, for which the equality (8) holds.

We define $C([0,T],R^{(N-1)})$ as the space of continuous functions $U:[0,T]\to R^{N-1}$, with $\|U\|_1=\max\limits_{t\in[0,T]}\|U(t)\|$, $\|\Phi\|=\max\limits_{i=\overline{1,N-1}}|\varphi_i|$,

$$\begin{aligned}&\|F\|_1\le\max_{t\in[0,T]}\|F(t)\|=\\ &=\max_{t\in[0,T]}\Big(\frac{\|a_1(t)\|\cdot\|\psi_0\|}{\tau^2}+\|f_1(t)\|,\max_{i=\overline{2,N-2}}\|f_i(t)\|,\frac{\|a_{N-1}(t)\|\cdot\|\psi_1\|}{\tau^2}+\|f_{N-1}(t)\|\Big)\le\\ &\qquad\le\max_{t\in[0,T]}\Big(\frac{\|a_1(t)\|\cdot\|\psi_0\|}{\tau^2},1,\frac{\|a_{N-1}(t)\|\cdot\|\psi_1\|}{\tau^2}\Big)+\max_{t\in[0,T]}\max_{i=\overline{1,N-1}}\|\|f_i(t)\|.\end{aligned}$$

The parametrization method developed by professor Dzhumabaev [21] is applied to solve problem (7),(8).

The given interval $[0,T]$ is divided by loading points as follows:

$[0,T)=\bigcup_{r=1}^{m+1}[t_{r-1},t_r)$, $0=t_0<t_1<t_2<\ldots<t_{m+1}=T$.

We define $C([0,T],t_r,R^{(N-1)(m+1)})$ as the space of function systems $U[t]=(U_1(t),\ldots,U_{m+1}(t))$, where the functions $U_r:[t_{r-1},t_r]\to R^{N-1}$ are continuous and have finite left-hand limits, i.e., $\lim\limits_{t\to t_r-0}U_r(t)$ exists for all $r=\overline{1,m+1}$, with $\|U[\cdot]\|_2=\max\limits_{r=\overline{1,m+1}}\sup\limits_{t\in[t_{r-1},t_r)}\|U_r(t)\|$.

The restriction of the function $U(t)$ to the $r-th$ interval $t\in[t_{r-1},t_r)$ is denoted as

$U_r(t) = U(t)$, $r = \overline{1, m+1}$. Consequently, we have

$$\frac{dU_r}{dt} = A(t)U_r(t) + \sum_{j=1}^{m} M_j(t)U_{j+1}(t_j) + F(t), \quad r = \overline{1, m+1}, \tag{9}$$

$$BU_1(0) + C \lim_{t \to T-0} U_{m+1}(t) = \Phi, \tag{10}$$

$$\lim_{t \to t_s - 0} U_s(t) = U_{s+1}(t_s), \quad s = \overline{1, m}. \tag{11}$$

By introducing parameters $\lambda_r = U_r(t_{r-1})$, $r = \overline{1, m+1}$, $\lambda_{m+2} = \lim\limits_{t \to T-0} U_{m+1}(t)$ and by replacing $U_r(t) = \widetilde{u}_r(t) + \lambda_r$ in each interval $[t_{r-1}, t_r)$, $r = \overline{1, m+1}$, we obtain problem with parameters

$$\frac{d\widetilde{u}_r}{dt} = A(t)(\widetilde{u}_r + \lambda_r) + \sum_{j=1}^{m} M_j(t)\lambda_{j+1} + F(t), \quad t \in [t_{r-1}, t_r), \tag{12}$$

$$\widetilde{u}(t_{r-1}) = 0, \qquad r = \overline{1, m+1}, \tag{13}$$

$$B\lambda_1 + C\lambda_{m+2} = \Phi, \tag{14}$$

$$\lambda_s + \lim_{t \to t_s - 0} \widetilde{u}_s(t) = \lambda_{s+1}, \quad s = \overline{1, m+1}. \tag{15}$$

The solution of the problem (12) – (15) is the pair $(\lambda, \widetilde{u}[t])$ with the elements:

$\lambda = (\lambda_1, \ldots, \lambda_{m+2})' \in R^{(N-1)(m+2)}$, $\widetilde{u}[t] = (\widetilde{u}_1(t), \ldots, \widetilde{u}_{m+1}(t))' \in C([0, T], t_r, R^{(N-1)(m+1)})$ here the functions $\widetilde{u}_r(t)$ are continuously differentiable on the $[t_{r-1}, t_r)$, $r = \overline{1, m+1}$, and the λ_r satisfy the system of ordinary differential equations (12) along with the conditions (13) – (15).

If the pair $(\lambda, \widetilde{u}[t])$, where $\lambda = (\lambda_1, \ldots, \lambda_{m+2})' \in R^{(N-1)(m+2)}$, $\widetilde{u}[t] = (\widetilde{u}_1(t), \ldots, \widetilde{u}_{m+1}(t))' \in C([0, T], t_r, R^{(N-1)(m+1)})$ is a solution to the problem (12) – (15), then the function $U(t)$ defined by the equalities $U_r(t) = \widetilde{u}_r(t) + \lambda_r$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, m+1}$, $U(T) = \lambda_{m+2}$, is the solution to the problem (7), (8). On the other hand, if $U^*(t)$ is the solution to problem (7), (8) then the pair $(\lambda^*, \widetilde{u}^*[t])$, where $\lambda^* = (U^*(t_0), U^*(t_1), \ldots, U^*(t_{m+1}))$, $\widetilde{u}^*[t] = (U^*(t) - U^*(t_0), U^*(t) - U^*(t_1), \ldots, U^*(t) - U^*(t_m))$ will serve as a solution to the problem (12) – (15).

The emergence of the initial conditions $\widetilde{u}_r(t_{r-1}) = 0$, $r = \overline{1, m+1}$, enables us to ascertain the functions $\widetilde{u}_r(t)$, $r = \overline{1, m+1}$, for constant $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m+2})$ derived from the Volterra integral equations of the second type:

$$\widetilde{u}_r(t) = \int_{t_{r-1}}^{t} A(\xi)(\widetilde{u}_r(\xi) + \lambda_r)\mathrm{d}\xi + \int_{t_{r-1}}^{t} \sum_{j=1}^{m} M_j(\xi)\mathrm{d}\xi\lambda_{j+1} + \int_{t_{r-1}}^{t} F(\xi)\mathrm{d}\xi, \quad r = \overline{1, m+1}. \tag{16}$$

In equation (16), by replacing $\widetilde{u}_r(\xi)$, $r=\overline{1,m+1}$, with the appropriate right-hand side, and by iterating this procedure $\nu, (\nu=1,2,\ldots)$ times, we acquire a depiction of the function $\widetilde{u}_r(t)$, $r=\overline{1,m+1}$, expressed in the form below:

$$\widetilde{u}_r(t)=D_{\nu r}(t)\lambda_r+\sum_{j=1}^{m}H_{\nu r}(t,M_j)\lambda_{j+1}+G_{\nu r}(t,\widetilde{u}_r)+\widetilde{F}_{\nu r}(t) \quad r=\overline{1,m+1}, \tag{17}$$

where

$$D_{\nu r}(t)=\int_{t_{r-1}}^{t}A(\xi_1)\mathrm{d}\xi_1+\int_{t_{r-1}}^{t}A(\xi_1)\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}A(\xi_\nu)\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1,$$

$$H_{\nu r}(t,M_j)=\int_{t_{r-1}}^{t}M_j(\xi_1)\mathrm{d}\xi_1+\int_{t_{r-1}}^{t}A(\xi_1)\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}M_j(\xi_\nu)\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1,$$

$$G_{\nu r}(t,\widetilde{u}_r)=\int_{t_{r-1}}^{t}A(\xi_1)\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}A(\xi_\nu)\widetilde{u}_r(\xi_\nu)\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1,$$

$$\widetilde{F}_{\nu r}(t)=\int_{t_{r-1}}^{t}F(\xi_1)\mathrm{d}\xi_1+\int_{t_{r-1}}^{t}A(\xi_1)\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}F(\xi_\nu)\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1.$$

From (17) we find

$$\lim_{t\to t_r-0}\widetilde{u}_r(t)=D_{\nu r}(t_r)\lambda_r+\sum_{j=1}^{m}H_{\nu r}(t_r,M_j)\lambda_{j+1}+G_{\nu r}(t_r,\widetilde{u}_r)+\widetilde{F}_{\nu r}(t_r), \quad r=\overline{1,m+1}.$$

Substituting the appropriate right-hand side from (17) into the conditions (14), (15) and multiplying (14) by $l=\max_s(t_s-t_{s-1})$, $s=\overline{1,m+1}$, we obtain the following:

$$B\lambda_1\cdot l+C\lambda_{m+2}\cdot l=\Phi\cdot l, \tag{18}$$

$$[I+D_{\nu s}(t_s)]\lambda_s+\sum_{j=1}^{m}H_{\nu s}(t_s,M_j)\lambda_{j+1}-\lambda_{s+1}=-G_{\nu s}(t_s,\widetilde{u}_s)-\widetilde{F}_{\nu s}(t_s), \quad s=\overline{1,m+1}. \tag{19}$$

where I is an identity matrix size of $((N-1)\times(N-1))$. Let $Q_\nu(l)$ represent the matrix relating to the left-hand side of the system (18), (19), we obtain

$$Q_\nu(l)\lambda=-\widetilde{F}_\nu(l)-G_\nu(\widetilde{u},l), \tag{20}$$

where $\widetilde{F}_\nu(l)=(-\Phi l,\widetilde{F}_{\nu 1}(t_1),\ldots,\widetilde{F}_{\nu m+1}(T))$,
$G_\nu(\widetilde{u},l)=(0,G_{\nu 1}(\widetilde{u}_1,t_1),\ldots,G_{\nu m+1}(\widetilde{u}_{m+1},T))$.

Therefore, to identify the unknown pairs $(\lambda,\widetilde{u}(t))$ that solve the problem (12) –(15) we possess a self-contained set of equations (16), (20). The pairs $(\lambda,\widetilde{u}(t))$ that solves the problem (12) – (15) results in sequences of pairs $(\lambda^k,\widetilde{u}^k(t)), k=0,1,2,\ldots$, determined by the subsequent algorithm:

Step 0: a) Supposing that for the selected $l\in R^+,\nu\in N$, the matrix $Q_\nu(l)$ is invertible, we establish the initial approximation concerning the parameters$\lambda^{(0)}=(\lambda_1^{(0)},\ldots,\lambda_{m+2}^{(0)})\in R^{(N-1)(m+2)}$ from the equation $Q_\nu(l)\lambda^{(0)}=-\widetilde{F}_\nu(l)$, producing $\lambda^{(0)}=-[Q_\nu(l)]^{-1}\widetilde{F}_\nu(l)$.

b) Utilizing the elements of the vector $\lambda^{(0)} \in R^{(N-1)(m+2)}$ and solving the Cauchy problems (12), (13) with $\lambda_r = \lambda_r^{(0)}$ on the $[t_{r-1}, t_r]$, we determine the functions $\widetilde{u}_r^{(0)}(t)$, $r = \overline{1, m+1}$.

Step 1: a) By inserting the obtained $\widetilde{u}_r^{(0)}(t), r = \overline{1, m+1}$, into the right-hand side of (20), we establish $\lambda^{(1)} = (\lambda_1^{(1)}, \dots, \lambda_{m+2}^{(1)}) \in R^{(N-1)(m+2)}$ from $Q_\nu(l)\lambda = -\widetilde{F}_\nu(l) - G_\nu(\widetilde{u}^{(0)}, l)$.
b) On the $[t_{r-1}, t_r]$, we address the Cauchy problems (12), (13) by using $\lambda_r = \lambda_r^{(1)}$ and determine the functions $\widetilde{u}_r^{(1)}(t)$, $r = \overline{1, m+1}$, etc.

Proceeding with the procedure, at the k - th step, we obtain a system of pairs $(\lambda^{(k)}, \widetilde{u}^{(k)}[t])$, $k = 0, 1, 2, \dots$. Observe that in point b), for constant values of the parameter λ_r, the solution to the Cauchy problem is determined individually for each interval $t \in [t_{r-1}, t_r]$, $r = \overline{1, m+1}$.

3 Conditions for Convergence of Algorithms and Unique Solution of the Problem (7), (8)

Suppose $\|A(t)\| \le \alpha = const$, $\|M_j(t)\| \le \beta_j = const$, $j = 1, \dots, m$. The requirement for the algorithm's convergence and the uniqueness of the solution to problem (7),(8) lead to the following statement.

Theorem 2. *Let the matrix $Q_\nu(l) : \mathbb{R}^{(N-1)(m+2)} \to \mathbb{R}^{(N-1)(m+2)}$ be invertible, for $l \in \mathbb{R}^+, \nu \in N$, and let the following conditions be satisfied:*

a) $\|[Q_\nu(l)]^{-1}\| \le \varepsilon_\nu(l)$,

b) $g_\nu(l) = \varepsilon_\nu(l) \cdot \Big[e^{\alpha l} - \sum\limits_{j=0}^{\nu} \dfrac{(\alpha l)^j}{j!} + \sum\limits_{j=1}^{m} \beta_j l \cdot \Big(e^{\alpha l} - \sum\limits_{k=0}^{\nu-1} \dfrac{(\alpha l)^k}{k!}\Big)\Big] < 1.$

Consequently, the two-point boundary value problem for the LDE (7), (8) has a unique solution $U^*(t)$ and the evaluation is just for its:

$$\|U^*\|_1 \le \widetilde{K}_\nu(l) \max(\|F\|_1, \|\Phi\|),$$

$$\widetilde{K}_\nu(l) = \Big\{\Big(e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^{m}\beta_j l\Big) \cdot \frac{\varepsilon_\nu(l)}{1 - g_\nu(l)} \cdot \frac{(\alpha l)^\nu}{\nu!} + \frac{\varepsilon_\nu(l)}{1 - g_\nu(l)} \cdot \frac{(\alpha l)^\nu}{\nu!} + 1\Big\} \times$$
$$\times \Big\{\Big(e^{\alpha l} - 1 + e^{\alpha l} \cdot \sum_{j=1}^{m}\beta_j l\Big)\varepsilon_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1}\frac{(\alpha l)^j}{j!}\Big) + e^{\alpha l}\Big\} l + \gamma_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1}\frac{(\alpha l)^j}{j!}\Big) l.$$

Proof. Under the assumptions of the theorem from step zero of the algorithm, we define and estimate $\lambda^{(0)}$:

$$\|\lambda^{(0)}\| = \max_{r=\overline{1,m+1}} \|\lambda_r^{(0)}\| \le \|[Q_\nu(l)]^{-1}\| \cdot \|\widetilde{F}_\nu(l)\| \le \varepsilon_\nu(l) \cdot \|\widetilde{F}_\nu(l)\|,$$

$$\|\widetilde{F}_\nu(l)\| \leq \max(\|\Phi\|l, \max_{r=\overline{1,m+1}} \|\widetilde{F}_{\nu r}(t_r)\|),$$

$$\|\widetilde{F}_{\nu r}(t_r)\| \leq \Big\| \int_{t_{r-1}}^{t_r} F(\xi_1)\mathrm{d}\xi_1 \Big\| + \Big\| \int_{t_{r-1}}^{t_r} A(\xi_1) \int_{t_{r-1}}^{\xi_1} F(\xi_2)\mathrm{d}\xi_2 d\xi_1 \Big\| + \ldots +$$
$$+ \Big\| \int_{t_{r-1}}^{t_r} A(\xi_1) \int_{t_{r-1}}^{\xi_1} A(\xi_2) \ldots \int_{t_{r-1}}^{\xi_{\nu-1}} F(\xi_\nu)\mathrm{d}\xi_\nu \ldots \mathrm{d}\xi_1 \Big\| \leq$$
$$\leq \|F\|_1 l + \alpha l \|F\|_1 l + \ldots + \frac{(\alpha l)^{\nu-1}}{(\nu-1)!}\|F\|_1 l = \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\|F\|_1 l.$$

Then

$$\|\lambda^{(0)}\| \leq \varepsilon_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\Big) \max(\|\Phi\|, \|F\|_1) l. \tag{21}$$

Functions $\widetilde{u}_r^{(0)}(t)$ we determine from the following integral systems equations
$\|\widetilde{u}_r^{(0)}(t)\| \leq \int_{t_{r-1}}^{t} \alpha \|\widetilde{u}_r^{(0)}\|\mathrm{d}\xi + \int_{t_{r-1}}^{t} \alpha \mathrm{d}\xi \|\lambda_r^{(0)}\| + \int_{t_{r-1}}^{t} \sum_{j=1}^{m} \beta_j \mathrm{d}\xi \|\lambda_{j+1}^{(0)}\| + \int_{t_{r-1}}^{t} \|F(\xi)\|\mathrm{d}\xi,$
adding $\|\lambda_r^{(0)}\|$, $r = \overline{1, m+1}$, and using the Gronwall-Bellman inequality, we obtain:
$\|\widetilde{u}_r^{0}(t)\| + \|\lambda_r^{(0)}\| \leq e^{\alpha(t-t_{r-1})} \cdot \max_{r=\overline{1,m+1}} \Big(\int_{t_{r-1}}^{t_r} \sum_{j=1}^{m} \beta_j \mathrm{d}\xi \|\lambda_{j+1}^{(0)}\| + \int_{t_{r-1}}^{t_r} \|F(\xi)\|\mathrm{d}\xi + \|\lambda_r^{(0)}\| \Big),$

$$\|\widetilde{u}^{(0)}[\cdot]\|_2 = \max_{r=\overline{1,m+1}} \sup_{t \in [t_{r-1}, t_r)} \|\widetilde{u}_r^{(0)}(t)\| \leq (e^{\alpha l} - 1)\|\lambda^{(0)}\| +$$
$$+ e^{\alpha l} \cdot \max_{r=\overline{1,m+1}} \Big(\int_{t_{r-1}}^{t_r} \sum_{j=1}^{m} (\beta_j \|\lambda_{j+1}^{(0)}\| + \|F(\xi)\|)\mathrm{d}\xi \Big).$$

Now we get

$$\|\widetilde{u}^{(0)}[\cdot]\|_2 \leq \Big(e^{\alpha l} - 1 + e^{\alpha l} \cdot \sum_{j=1}^{m} \beta_j l\Big)\|\lambda^{(0)}\| + e^{\alpha l}\|F\|_1 l. \tag{22}$$

from where, taking into account (21) we obtain:
$\|\widetilde{u}^{(0)}[\cdot]\|_2 \leq K_\nu(l) \max(\|\Phi\|, \|F\|_1),$
$K_\nu(l) = \Big(e^{\alpha l} - 1 + e^{\alpha l} \cdot \sum_{j=1}^{m} \beta_j l\Big)\varepsilon_\nu(l) \cdot \max\Big(1, \sum_{j=0}^{\nu-1} \frac{(\alpha l)^j}{j!}\Big) l + e^{\alpha l} l.$

Using the first step of the algorithm, we determine $\lambda^{(1)}$ and estimate the norm of the difference $\|\lambda^{(1)} - \lambda^{(0)}\|$:
$\|\lambda^{(1)} - \lambda^{(0)}\| \leq \gamma_\nu(l) \cdot \|G_\nu(t_r, \widetilde{u}^{(0)})\|,$
$\|G_\nu(t_r, \widetilde{u}^{(0)})\| \leq \max_{r=\overline{1,m+1}} \|G_{\nu r}(t_r, \widetilde{u}_r^{(0)})\| \leq \frac{(\alpha l)^\nu}{\nu!}\|\widetilde{u}^{(0)}[\cdot]\|_2,$

$\|\lambda^{(1)}-\lambda^{(0)}\| \leq \varepsilon_\nu(l)\cdot \frac{(\alpha l)^\nu}{\nu!}\|\widetilde{u}^{(0)}[\cdot]\|_2 \leq \varepsilon_\nu(l)\cdot \frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|,\|F\|_1)$,

Substituting $\lambda=\lambda^{(1)}$ to the right side of (16) and solving the Cauchy problem we determine $\widetilde{u}^{(1)}[t]=(\widetilde{u}_1^{(1)}(t),\widetilde{u}_2^{(1)}(t),\ldots,\widetilde{u}_{m+1}^{(1)}(t))\in C([0,T],t_r,R^{(N-1)(m+1)})$.

By persisting with the iterative process at the $k-th$ step, we derive a pair $(\lambda^{(k)},\widetilde{u}^{(k)}[t])$, $k=0,1,2,\ldots$.

$\widetilde{u}^{(k)}[t]=(\widetilde{u}_1^{(k)}(t),\widetilde{u}_2^{(k)}(t),\ldots,\widetilde{u}_{m+1}^{(k)}(t))\in C([0,T],t_r,R^{(N-1)(m+1)})$,

$\lambda^{(k)}=(\lambda_1^{(k)},\ldots,\lambda_{m+2}^{(k)})\in R^{(N-1)(m+2)}$.

Since $\lambda^{(k+1)}$, $\lambda^{(k)}$ are solutions to equation (20) with the corresponding right-hand sides, then their difference holds the following inequality:

$\|\lambda^{(k+1)}-\lambda^{(k)}\|\leq \varepsilon_\nu(l)\cdot\|G_\nu(t_r,\widetilde{u}^{(k)})-G_\nu(t_r,\widetilde{u}^{(k-1)})\|$,

$$G_{\nu r}(t_r,\widetilde{u}_r^{(k)}-\widetilde{u}_r^{(k-1)})= \\ =\int_{t_{r-1}}^{t_r}A(\xi_1)\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}A(\xi_{\nu-1})\int_{t_{r-1}}^{\xi_{\nu-1}}A(\xi_\nu)\Big(\widetilde{u}_r^{(k)}(\xi_\nu)-\widetilde{u}_r^{(k-1)}(\xi_{\nu-1})\Big)\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1. \quad (23)$$

Using the Gronwall-Bellman inequality again, we estimate the difference in solutions of the Cauchy problems through the difference in parameters:

$$\|\widetilde{u}_r^{(k)}(t)-\widetilde{u}_r^{(k-1)}(t)\|\leq (e^{\alpha(t-t_{r-1})}-1)\cdot\|\lambda_r^{(k)}-\lambda_r^{(k-1)}\|+ \\ +e^{\alpha(t-t_{r-1})}\int_{t_{r-1}}^{t_r}\sum_{j=1}^m\beta_j\mathrm{d}\xi\|\lambda_{j+1}^{(k)}-\lambda_{j+1}^{(k-1)}\|,\quad r=\overline{1,m+1} \quad (24)$$

Substituting (24) into the right side of (23) and calculating the repeated integrals, we obtain:

$$\|G_{\nu r}(t_r,\widetilde{u}^{(k)}-\widetilde{u}^{(k-1)})\|\leq\int_{t_{r-1}}^{t_r}\alpha\ldots\int_{t_{r-1}}^{\xi_{\nu-2}}\alpha\int_{t_{r-1}}^{\xi_{\nu-1}}\alpha\Big[(e^{\alpha\xi_\nu}-1)\cdot\|\lambda_r^{(k)}-\lambda_r^{(k-1)}\|+ \\ +e^{\alpha(t-t_{r-1})}\cdot\max_{r=\overline{1,m+1}}\int_{t_{r-1}}^{t_r}\sum_{j=1}^m\beta_j\mathrm{d}\xi\|\lambda_{j+1}^{(k)}-\lambda_{j+1}^{(k-1)}\|\Big]\mathrm{d}\xi_\nu\ldots\mathrm{d}\xi_1= \\ =\Big[e^{\alpha l}-1-\alpha l-\ldots-\frac{(\alpha l)^\nu}{\nu!}+e^{\alpha l}\sum_{j=1}^m\beta_j l-\sum_{j=1}^m\beta_j l-\ldots-\sum_{j=1}^m\beta_j\frac{(\alpha l)^{\nu-1}}{(\nu-1)!}\Big]\|\lambda^{(k)}-\lambda^{(k-1)}\|,$$

$$\|\lambda^{(k+1)}-\lambda^{(k)}\|\leq\varepsilon_\nu(l)\cdot\Big[e^{\alpha l}-\sum_{j=0}^\nu\frac{(\alpha l)^j}{j!}+\sum_{j=1}^m\beta_j l\cdot\Big(e^{\alpha l}-\sum_{\mu=0}^{\nu-1}\frac{(\alpha l)^\mu}{\mu!}\Big)\Big]\|\lambda^{(k)}-\lambda^{(k-1)}\|= \\ =g_\nu(l)\cdot\|\lambda^{(k)}-\lambda^{(k-1)}\|\quad k=1,2,\ldots \quad (25)$$

$$g_\nu(l)=\varepsilon_\nu(l)\cdot\Big[e^{\alpha l}-\sum_{j=0}^\nu\frac{(\alpha l)^j}{j!}+\sum_{j=1}^m\beta_j l\cdot\Big(e^{\alpha l}-\sum_{\mu=0}^{\nu-1}\frac{(\alpha l)^\mu}{\mu!}\Big)\Big]$$

$\|\lambda^{(k+1)}-\lambda^{(k)}\|\leq g_\nu(l)\cdot g_\nu(l)\cdot\|\lambda^{(k-1)}-\lambda^{(k-2)}\|\leq\ldots\leq g_\nu^k(l)\cdot\|\lambda^{(1)}-\lambda^{(0)}\|$,

$$\|\lambda^{(k+p)}-\lambda^{(k)}\|\leq\|\lambda^{(k+p)}-\lambda^{(k+p-1)}\|+\lambda^{(k+p-1)}-\lambda^{(k+p-2)}\|+\ldots+\|\lambda^{(k+1)}-\lambda^{(k)}\|\leq \\ \leq g_\nu^p(l)\cdot\|\lambda^{(k+1)}-\lambda^{(k)}\|+g_\nu^{p-1}(l)\cdot\|\lambda^{(k+1)}-\lambda^{(k)}\|+\ldots+g_\nu(l)\cdot\|\lambda^{(k+1)}-\lambda^{(k)}\|+\|\lambda^{(k+1)}-\lambda^{(k)}\|= \\ =(g_\nu^p(l)+g_\nu^{p-1}(l)+\ldots+g_\nu(l)+1)\|\lambda^{(k+1)}-\lambda^{(k)}\|.$$

Due to the condition $g_\nu(l) < 1$ and inequalities (22), (23) as $p \to \infty$ the sequence $\lambda^{(k)}$ converges to λ^*, sequence of systems of function $\widetilde{u}^{(k)}[t]$ by the norm of the space $C([0,T], t_r, R^{(N-1)(m+1)})$ converges to $\widetilde{u}^*[t]$ and the following estimates are valid:

$\|\lambda^* - \lambda^{(k)}\| \leq \frac{1}{1-g_\nu(l)}\|\lambda^{(k+1)} - \lambda^{(k)}\| \leq \frac{1}{1-g_\nu(l)} \cdot g_\nu^k(l)\gamma_\nu(l) \cdot \frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|, \|F\|_1)l,$

$$\|\widetilde{u}_r^{(k)}(t) - \widetilde{u}_r^{(k-1)}(t)\| \leq (e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^{m}\beta_j l) \cdot \|\lambda_r^{(k)} - \lambda_r^{(k-1)}\|,$$

$$\|\widetilde{u}^*[\cdot] - \widetilde{u}^{(k)}[\cdot]\|_2 \leq$$
$$\leq \left(e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^{m}\beta_j l\right)\frac{1}{1-g_\nu(l)}g_\nu^k(l)\varepsilon_\nu(l)\frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|, \|F\|_1) \quad k = 1, 2, \ldots$$

Using these inequalities for $k = 0$ and taking into account the established estimates (21), (22) we obtain:

$\|\lambda^* - \lambda^{(0)}\| \leq \frac{1}{1-g_\nu(l)}\varepsilon_\nu(l)\frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|, \|F\|_1),$

$$\|U^*\|_1 = \|\lambda^* + \widetilde{u}^*[\cdot]\|_2 \leq \|\lambda^* - \lambda^{(0)}\| + \|\widetilde{u}^*[\cdot] - \widetilde{u}^{(0)}[\cdot]\|_2 + \|\lambda^{(0)}\| + \|\widetilde{u}^{(0)}[\cdot]\|_2 \leq$$
$$\leq \frac{1}{1-g_\nu(l)}\varepsilon_\nu(l)\frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|, \|F\|_1) +$$
$$+ \left(e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^{m}\beta_j l\right)\frac{1}{1-g_\nu(l)}\varepsilon_\nu(l)\frac{(\alpha l)^\nu}{\nu!}K_\nu(l)\max(\|\Phi\|, \|F\|_1) +$$
$$+ \varepsilon_\nu(l)\max\left(1, \sum_{j=0}^{\nu-1}\frac{(\alpha l)^j}{j!}\right)\max(\|\Phi\|, \|F\|_1)l + K_\nu(l)\max(\|\Phi\|, \|F\|_1) = \widetilde{K}_\nu(l).$$

Uniqueness. Let $U^*(t)$, $U^{**}(t)$ two solutions to problem (7),(8). Then the corresponding systems of pairs $(\lambda^*, \widetilde{u}^*[t])$, $(\lambda^{**}, \widetilde{u}^{**}[t])$, where $\lambda^* = (U^*(t_0), U^*(t_1), \ldots, U^*(t_{m+1}))$, $\widetilde{u}^*[t] = (U^*(t) - U^*(t_0), U^*(t) - U^*(t_1), \ldots, U^*(t) - U^*(t_m))$, $\lambda^{**} = U^{**}(t_0), U^{**}(t_1), \ldots, U^{**}(t_{m+1}))$, $\widetilde{u}^{**}[t] = (U^{**}(t) - U^{**}(t_0), U^{**}(t) - U^{**}(t_1), \ldots, U^{**}(t) - U^{**}(t_m))$ are solutions to the boundary value problem with parameters (12) – (15) and satisfy relations (16), (20). Similar to estimates (24), (25) the following estimates are established:

$$\|\widetilde{u}^*[\cdot] - \widetilde{u}^{**}[\cdot]\|_2 \leq \left(e^{\alpha l} - 1 + e^{\alpha l}\sum_{j=1}^{m}\beta_j l\right) \cdot \|\lambda^* - \lambda^{**}\|,$$

$\|\lambda^* - \lambda^{**}\| \leq g_\nu(l)\|\lambda^* - \lambda^{**}\|.$

Since $g_\nu(l) < 1$ these estimates imply $\lambda^* = \lambda^{**}$, $\widetilde{u}^*[t] = \widetilde{u}^{**}[t]$ i.e. $U^*(t) = U^{**}(t)$ when $t \in [0, T]$.

The Theorem 2 is proved.

Let us now return to problem (4)-(6). Since there is a connection between problems (4)-(6) and (7)-(8), the following theorem holds.

Theorem 3. *Let $U^*(t)$ represent the solution to the problem (7)-(8) and $u^*(t) = (u_1^*(t), u_2^*(t), \ldots, u_{N-1}^*(t))$ represent the solution of the discretized problem (4)-(6). The following conditions hold:*

a) The coefficients $a(t,x)$, $b(t,x)$, $k_j(t,x)$, $f(t,x)$ are smooth and satisfy the constraints $a(t,x) \geq \rho > 0$, $b(t,x) \leq 0$ and Holder continuity in x,

b) The functions $\psi_0(t)$, $\psi_1(t)$, $\varphi(x)$ are sufficiently smooth and satisfy compatibility conditions,

c) The requirements of Theorem 2 are met.

Thus, the discretized problem (4)-(6) has a unique solution $u^*(t) = (u_1^*(t), u_2^*(t), \dots, u_{N-1}^*(t))$ and the assessment is fair for its:

$\|u_i^*(t)\| \leq \|U^*(t)\| \leq \widetilde{K}_\nu(l)\max(\|F\|_1, \|\Phi\|)$, $i = \overline{1, N-1}$.

Now we can prove Theorem 1. Let $u_i(t)$ - the solution of the discretized problem (4)-(6) and $u(t,x_i)$ - the solution at the grid points problem (1)-(3).

The solution $u(t,x)$ at x_i satisfies:

$\frac{\partial u(t,x_i)}{\partial t} = a(t,x_i)\frac{\partial^2 u}{\partial x^2}\Big|_{x=x_i} + b(t,x_i)u(t,x_i) + \sum_{j=1}^{m} k_j(t,x_i)u(t_j,x_i) + f(t,x_i)$

Using the finite difference approximation: $\frac{\partial^2 u}{\partial x^2}\Big|_{x=x_i} = \frac{u(t,x_{i+1}) - u(t,x_i) + u(t,x_{i-1})}{\tau^2} + O(\tau^2)$, we get

$$\frac{\partial u(t,x_i)}{\partial t} = a(t,x_i)\frac{u(t,x_{i+1}) - u(t,x_i) + u(t,x_{i-1})}{\tau^2} + b(t,x_i)u(t,x_i) + \\ + \sum_{j=1}^{m} k_j(t,x_i)u(t_j,x_i) + f(t,x_i) + O(\tau^2), \quad (26)$$

$$B(x_i)u(0,x_i) + C(x_i)u(T,x_i) = \varphi(x_i), \quad i = \overline{1, N-1}, \quad (27)$$

$$u_0(t) = \psi_0(t), \quad u_N(t) = \psi_1(t), \quad t \in [0,T]. \quad (28)$$

Subtracting the discretized problem (4)-(6) from (26)-(28) gives the error evolution equation $\delta_i(t) = u(t,x_i) - u_i(t)$:

$$\frac{\partial \delta_i}{\partial t} = a(t,x_i)\frac{\delta_{i+1} - \delta_i + \delta_{i-1}}{\tau^2} + b(t,x_i)\delta_i + \sum_{j=1}^{m} k_j(t,x_i)\delta(t_j) + R_i(t), \quad (29)$$

$$B(x_i)\delta(0) + C(x_i)\delta(T) = 0, \quad i = \overline{1, N-1}, \quad (30)$$

$$\delta_0(t) = 0, \quad \delta_N(t) = 0, \quad t \in [0,T]. \quad (31)$$

$R_i(t) = O(\tau^2)$ represents the truncation error from the finite difference approximation. (29)-(31) equation is similar to (4)-(6). This means we can use the estimate from Theorem 2:

$\max_i \|\delta_i(t)\| \leq \widetilde{K}_\nu(l)|R_i(t)|$, $i = \overline{1, N-1}$.

So, we have proven Theorem 1.

4 Conclusion

In this study, the method of lines is utilized to address the two-point boundary value problem for loaded parabolic equations. Assuming that the solution is sufficiently smooth to the initial problem, and according to Theorem 2, the interrelation between the two-point boundary value problem for loaded parabolic equations (1)-(3) and the discretized problem (4)-(6) is demonstrated. The stability and error of this problem will be studied in future works.

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ON THE LAPLACE-BELTRAMI OPERATOR IN STRATIFIED SETS COMPOSED OF PUNCTURED CIRCLES AND SEGMENTS

This paper discusses the introduction of local coordinates on the circle S^1 and the analysis of various classes of functions defined on it. It is proved that every smooth function on the circle corresponds to a smooth 2π -periodic function on the real axis. The Laplace-Beltrami operator on S^1 is introduced using the apparatus of exterior differential forms and the Hodge operator. Its explicit expression in local coordinates is calculated, and it is shown that it can be reduced to the double differentiation operator. Then, the spectral analysis of the Laplace-Beltrami operator is performed, its eigenvalues and the corresponding eigenfunctions expressed in terms of the Chebyshev polynomials of the first and second kind are found. Well-solved problems for the Laplace-Beltrami operator on a punctured circle are written out. In the final paragraph of the article "On the Laplace-Beltrami operator on stratified sets composed of punctured circles and segments"the eigenvalues and systems of eigenfunctions on one stratified set composed of two punctured circles and a finite interval are written out.
Key words: Laplace-Beltrami operator, one-dimensional punctured sphere, well-posed problems.

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Ойылған шеңберлер мен кесінділерден тұратын қабатты жиындардағы Лаплас-Бельтрами операторы туралы

Бұл мақалада S^1 шеңберіне локальді координаттарды енгізу және онда анықталған функциялардың әртүрлі кластарын талдау қарастырылады. Шеңбердегі әрбір тегіс функция нақты осьтегі тегіс 2π -периодтық функцияға сәйкес келетіні дәлелденді. S^1 бойындағы Лаплас-Бельтрами операторы сыртқы дифференциалдық формалар аппараты мен Ходж операторы арқылы енгізілген. Оның локальді координаттардағы айқын өрнегі есептеліп, оны екі еселі дифференциалдау операторына келтіруге болатыны көрсетілген. Одан әрі Лаплас-Бельтрами операторының спектрлік талдауы жүргізіліп, оның меншікті мәндері мен бірінші және екінші Чебышев көпмүшелері арқылы өрнектелген сәйкес меншікті функциялар табылады. Ойылған шеңбер бойынша Лаплас-Бельтрами операторына қисынды шешілетін есептер жазылған. "Ойылған шеңберлер мен кесінділерден тұратын қабатты жиындардағы Лаплас-Бельтрами операторы туралы"мақаланың соңғы бөлімінде екі ойылған шеңберден және ақырлы интервалдан тұратын бір қабатты жиындағы меншікті мәндер мен меншікті функциялардың жүйелері жазылған.
Түйін сөздер: Лаплас-Бельтрами операторы, бір өлшемді ауытқыған сфера, қисынды шешілетін есеп.

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Об операторе Лапласа-Бельтрами на стратифицированных множествах, составленных из проколотых окружностей и отрезков

В данной работе рассматривается введение локальных координат на окружности S^1 и анализ различных классов функций, определенных на ней. Доказывается, что каждая гладкая функция на окружности соответствует гладкой 2π -периодической функции на числовой оси. Вводится оператор Лапласа-Бельтрами на S^1 с использованием аппарата внешних дифференциальных форм оператора Ходжа. Вычисляется его явное выражение в локальных координатах, показывается, что он сводится к оператору двухкратного дифференцирования. Далее проводится спектральный анализ оператора Лапласа-Бельтрами, находятся его собственные значения и соответствующие собственные функции, выраженные через полиномы Чебышева первого и второго родов. Выписаны корректно разрешимые задачи для оператора Лапласа-Бельтрами на проколотой окружности. В заключительном параграфе статьи "Об операторе Лапласа-Бельтрами на стратифицированных множествах, составленных из проколотых окружностей и отрезков"выписаны собственные значения и системы собственных функции на одном стратифицированном множестве, составленных из двух проколотых окружностях и конечного интервала.
Ключевые слова: Оператор Лапласа-Бельтрами, возмущенная одномерная сфера, корректно разрешимые задачи.

1 Introduction

The circle S^1 is one of the simplest examples of manifolds studied in differential geometry and analysis. Despite its simplicity, it plays a key role in many areas of mathematics and physics, including spectral theory, harmonic analysis, and quantum mechanics.

One of the fundamental objects of study on manifolds is the Laplace-Beltrami operator, which generalizes the classical Laplace operator on Euclidean spaces. In the case of a circle, it is closely related to the theory of trigonometric series and the analysis of periodic functions.

In this paper, we consider local coordinates on S^1 and classes of functions defined on it. We define the Laplace-Beltrami operator and study its spectral structure.

The eigenfunctions of this operator form an orthonormal basis in the space of square integrable functions, which makes them an important tool for expanding functions in Fourier series. This fact has wide applications, from solving equations of mathematical physics to signal analysis in applied sciences.

Additional interest in the spectral properties of the Laplace-Beltrami operator on the circle is due to their connection with quantum mechanics and statistical physics. In particular, similar spectral problems arise in the study of string vibrations, heat conduction, and wave propagation. In addition, the circle serves as a model object for studying more complex manifolds with symmetries.

The goal of this paper is to conduct a detailed study of the circle S^1 as a differential manifold, describe its local coordinates, consider the main classes of functions defined on it, and study the spectral properties of the Laplace-Beltrami operator. The results obtained will allow a better understanding of the role of the circle in spectral geometry and its connections with various sections of analysis and mathematical physics.

2 Circle as a manifold of dimension one

In the two-dimensional space $R^2_{x^1x^2}$, consider the circle

$$S^1 = \left\{(x^1, x^2) \mid (x^1)^2 + (x^2)^2 = 1\right\}.$$

We introduce the local coordinates of the circle. It is not possible to introduce universal coordinates in the entire circle S^1, so the circle S^1 is represented as a union of two maps V_1 and V_2. Each map can define its own individual coordinates. For example, $V_1 = S^1 \backslash \{(1,0)\}$ is a punctured circle with coordinate t :

$$x^1 = \cos t, x^2 = \sin t,$$

where t runs through the interval $(0, 2\pi)$. On the map $V_2 = S^1 \backslash \{(-1,0)\}$ we enter the coordinate τ :

$$x^1 = \cos \tau, x^2 = \sin \tau,$$

where τ runs through the interval $(\pi, 3\pi)$. Note that the maps V_1 and V_2 intersect and their intersection

$$V_1 \cap V_2 = S^1_+ \cup S^1_-,$$

where S^1_+ and S^1_- are semicircles without intersections.

In S^1_+ the transition from t to τ is carried out by formula $\tau = t + 2\pi$, and in S^1_- the coordinates of t and τ coincide. Therefore, the maps V_1 and V_2 are consistent maps and define an atlas on S^1.

3 Function classes on S^1

Now we define the function classes $C^\infty(S^1)$. If the function $f \in C(S^1)$ is given, then $f = f(x^1, x^2)$. Then we define the restriction of f to S^1_+, which we denote by $f|_{S^1_+} = f_1(x^1, x^2)$. We can define the restriction of f to S^1_- in exactly the same way, that is, $f|_{S^1_-} = f_2(x^1, x^2)$. We denote by $f_1(\cos t, \sin t) = \hat{f}_1(t) \quad 0 < t < \pi$. The function $f_1(x^1, x^2)$ on S^1_+ can be represented as a function of t:

$$f_1(\cos t, \sin t) = \hat{f}_1(t), \quad 0 < t < \pi.$$

The same function $f_1(x^1, x^2)$ in S^1_+ can be written in terms of τ coordinates:

$$f_1(\cos \tau, \sin \tau) = \hat{\hat{f}}_1(\tau), 2\pi < \tau < 3\pi.$$

It is clear that

$$\hat{\hat{f}}_1(t + 2\pi) = \hat{f}_1(t).$$

If the restriction of $f(x^1, x^2)$ to S^1_- is denoted by $\hat{f}_2(t) = \hat{\hat{f}}_2(\tau)$, where $\tau = t$. Since $f(x^1, x^2) \in C(S^1)$, then the function $f(x^1, x^2)$ is continuous at the point $(1, 0)$. That is,

$$\lim_{(x^1,x^2)\to(1,0)} f(x^1, x^2) = f(1,0) \text{ or } \lim_{t\to 0} \hat{f}_1(t) = f(1,0)$$

or $\lim_{\tau\to 2\pi}\hat{f}_1(\tau)=f(1,0)=\hat{\hat{f}}(2\pi)\Rightarrow$

$$\Rightarrow \lim_{t\to 0}\hat{f}_1(t)=\hat{f}_1(2\pi).$$

Therefore, $\hat{f}_1(0) := \hat{f}_1(2\pi)$. Similarly, $f(x^1,x^2)$ is continuous at $(-1,0)$. That is, $\lim_{(x^1,x^2)\to(-1,0)} f(x^1,x^2)=f(-1,0)$ or

$$\lim_{t\to\pi}\hat{f}_1(t)=f(-1,0)=\lim_{\tau\to 3\pi}\hat{f}_1(\tau)=\hat{f}_1(\pi).$$

Therefore, $\hat{f}_1(3\pi)=\hat{f}_1(\pi)$. In the same way, we can extend $\hat{\hat{f}}_1(\tau)$ to the point $\tau=\pi$. For example,

$$\lim_{\substack{\tau\to\pi\\ \tau>\pi}}\hat{f}_1(\tau)=f(-1,0)=\hat{f}_1(\pi) \quad \text{or}$$
$$\hat{\hat{f}}_1(\pi+0)=\hat{f}_1(\pi)=\hat{f}_1(3\pi-0)$$

That is, $\hat{f}_1(\tau)$ can be defined at the points $\tau=\pi$ and $\tau=3\pi$. In other words, $\hat{\hat{f}}_1(t)$ is defined by $\tau\in[\pi,3\pi]$, and $\hat{\hat{f}}_1(\pi+0)=\hat{\hat{f}}_1(3\pi-0)$. Thus, the function $\hat{\hat{f}}_1(\tau)$ has a continuous 2π-periodic extension to the entire axis. In the same way, $\hat{f}_1(t)$ has a 2π-periodic continuous extension to the entire number axis. More of these extensions give the same periodic function

$$\hat{f}_1(t)=\hat{\hat{f}}_1(t), \quad \forall t\in R$$

Remark 1: if $f\in\mathbb{C}^\infty(S^1)$, then restriction $f|_{V_1}=\hat{f}(t)$ can be extended to the entire number line and the extension has the following properties:
1) 2π is periodic;
2) infinitely many times continuously differentiable.

Now consider an arbitrary continuous 2π - periodic function on the entire number line.

$$\hat{f}(t)=\hat{f}(t+2\pi),\forall t\in R$$

Let $(x^1,x^2)\in S^1\backslash\{1,0\}=V_1$. Find a unique $t\in(0,2\pi)$ such that $x=\cos t, x^2=\sin t$. Since $\hat{f}-2\pi$ is a continuous periodic function, then $\hat{f}(t)=\frac{a_0}{2}+\sum(a_k\cos kt+b_k\sin kt)=$

$$=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos(k\,\mathrm{arcsos}\,x^1)+b_k\sin(k\arccos x^1)\right)$$
$$=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_kT_K(x^1)+b_kx^2U_k(x^1)\right)=f(x^1,x^2)$$

where $T_k(.)$ and $U_k(.)$ are Chebyshev polynomials of the first and second kind.

Remark 2: Thus, for each smooth $2\pi-$ periodic function $\hat{f}(t)$, we can uniquely construct a smooth function $f(x^1,x^2)$, defined on the circle S^1.

4 Laplace-Beltrami operator on the circle

We denote by $C^{\infty}(S^1)$ the set of infinitely continuously differentiable functions on S^1. By $\Lambda^1(S^1)$ we denote the exterior differential forms of the first order in the circle S^1, that is

$$\omega^1(x) = \alpha_1(x)dx^1 + \alpha_2(x)dx^2, \quad x = \left(x^1, x^2\right) \in S^1,$$

where $\alpha_1, \alpha_2 \in C^{\infty}(S^1)$.

By $\Lambda^2(S^1)$ we denote the exterior differential forms of the second order in the circle S^1, that is

$$w^2(x) = \alpha(x)dx^1 \wedge dx^2,$$

where $\alpha \in C^{\infty}(S^1)$. Let us recall how the Hodge operator acts on the basis elements:

$$*dx^1 = dx^2, *dx^2 = -dx^1, *dx^1 \wedge dx^2 = 1.$$

Take a scalar function $h \in C^{\infty}(S^1)$. Calculate its differential $dh(x) = \frac{\partial h}{\partial x^1}dx^1 + \frac{\partial h}{\partial x^2}dx^2$. Now apply the Hodge operator

$$*dh(x) = \frac{\partial h}{\partial x^1}dx^2 - \frac{\partial h}{\partial x^2}dx^1.$$

Let's calculate the differential of 1-form

$$d * dh(x) = \frac{\partial}{\partial x^1}\left(\frac{\partial h}{\partial x^1}dx^1 + \frac{\partial h}{\partial x^2}dx^2\right) \wedge dx^2 - \frac{\partial}{\partial x^2}\left(\frac{\partial h}{\partial x^1}dx^1 + \frac{\partial h}{\partial x^2}dx^2\right) \wedge dx^1 =$$

$$= \left(\frac{\partial^2 h}{\left(\partial x^1\right)^2} + \frac{\partial^2 h}{\left(\partial x^2\right)^2}\right) dx^1 \wedge dx^2$$

It remains to apply the Hodge operator, as a result we obtain the Laplace-Beltrami operator on the circle

$$\Delta h \equiv *d * dh(x) = \frac{\partial^2 h}{\left(\partial x^1\right)^2} + \frac{\partial^2 h}{\left(\partial x^2\right)^2}$$

The Laplace-Beltrami operator has physical and geometric meanings, therefore, this definition of the Laplace-Beltrami operator has an invariant description. Indeed, the Laplace-Beltrami operator is defined through exterior differential forms and operations on forms that are invariant with respect to the choice of local coordinates.

Now we calculate the Laplace-Beltrami operator in local coordinates. Let $x = (x^1, x^2)$ belong to the map V_1, that is, $x^1 = \cos t, x^2 = \sin t, t \in (0, 2\pi)$. According to the results of point 2, the scalar function $h(x) \in C^{\infty}(S^1)$ in local coordinates has the form $h(x) = \hat{h}(\cos t, \sin t) = \hat{h}(t)$ for $t \in (0, 2\pi)$. Moreover, $\hat{h}(t) - 2\pi$-periodically extends to the entire real axis, and the extension $\hat{h}(t)$ is an infinitely differentiable function. Calculate the differential

$$dh(x) = d\hat{h}(t) = \frac{d\hat{h}(t)}{dt}dt$$

Apply the Hodge operator, then $*dh(x) = *d\hat{h}(t) = \frac{d\hat{h}}{dt}$. Hence $\Delta h = *d * dh(x) = *d * d\hat{h}(t) = \frac{d^2\hat{h}(t)}{dt^2}$. Therefore, the Laplace-Beltrami operator on the real axis represents the double differentiation operator.

5 Spectral analysis of the Laplace-Beltrami operator on the circle

In this section, we calculate the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on the circle. In the previous paragraph, the Laplace-Beltrami operator is defined by the formula

$$\Delta = *d * d$$

where d is the exterior differentiation operator, and * is the Hodge operator.

Since the Laplace-Beltrami operator is defined in invariant form, its eigenvalues do not depend on the choice of local coordinates on the circle.

The eigenvalues of the Laplace-Beltrami operator are determined from the equation.

$$-\Delta u(x) = \lambda u(x), x \in S^1. \qquad (1)$$

In this case, the complex number λ will be an eigenvalue of the Laplace-Beltrami operator if equation (1) has a non-zero solution for the corresponding λ.

To find the eigenvalues λ of the Laplace-Beltrami operator, equation (1) can be considered in local coordinates, since the eigenvalues are invariant with respect to local coordinates. Therefore, we write equation (1) on the local map V_1.

The role of the local coordinate in V_1 is played by the variable t, which was introduced in point 1. Then, according to the results of point 3, equation (1) takes the form

$$-\hat{u}''(t) = \lambda \hat{u}(t), \quad t \in \mathbb{R}, \qquad (2)$$

where $\hat{u}(t) - 2\pi$ is a periodic function on $\mathbb{R}$.

Thus, we need to find λ, for which equation (2) has non-trivial 2π periodic solutions.

The solution to this problem is known [1]:

the numbers $\lambda = 0, 1, 4, 9, \ldots$ are eigenvalues, and the corresponding eigenfunctions take the form

$$\widehat{u}_0(t) \equiv 1, \hat{u}_{+\sqrt{\lambda}}(t) = \cos\sqrt{\lambda}t, \widehat{u}_{-\sqrt{\lambda}}(t) = \sin\sqrt{\lambda}t$$

Thus, $\lambda = 0$ is a simple eigenvalue, and all non-zero eigenvalues have multiplicity equal to two.

Now we rewrite the eigenfunctions $\widehat{u}_{\pm\sqrt{\lambda}}(t)$ in the variables $x = (x^1, x^2) \in S^1$. To do this, we need to use Chebyshev polynomials.

$$u_{+\sqrt{\lambda}}(x) = T_{\sqrt{\lambda}}(x^1), \quad u_{-\sqrt{\lambda}}(x) = x^2 \cdot U_{\sqrt{\lambda}}(x^1))$$

where $T_{\sqrt{\lambda}}(x^1)$ and $U_{\sqrt{\lambda}}(x^1)$ are Chebyshev polynomials of genus 1 and 2.

Thus, the system of eigenfunctions of the Laplace-Beltrami operator on the circle has the form

$$\left\{1, T_k(x^1), x^2 U_k(x^1), k = 1, 2, 3, \ldots\right\}, \quad \text{where} \quad x = (x^1, x^2) \in S^1.$$

6 Inverse operator to the Laplace-Beltrami operator on the circle

It follows from the results of section 4 that the equation $(I-\Delta)u(x)=f(x)$ for $x\in S^1$ has a unique solution for any $f\in L_2\left(S^1\right)$, and

$$\begin{aligned}
\hat{u}(t)=&-\int_0^t \hat{f}(t)\operatorname{sh}(t-\tau)d\tau- \\
&-\frac{\operatorname{ch} t}{2-2\operatorname{ch}2\pi}\cdot\int_0^{2\pi}\hat{f}(\tau)\begin{vmatrix}\operatorname{sh}2\pi & \operatorname{sh}(2\pi-\tau)\\ \operatorname{ch}2\pi-1 & \operatorname{ch}(2\pi-\tau)\end{vmatrix}d\tau+ \\
&+\frac{\operatorname{sh} t}{2-2\operatorname{ch}2\pi}\int_0^{2\pi}\hat{f}(\tau)\begin{vmatrix}\operatorname{ch}2\pi-1 & \operatorname{sh}(2\pi-\tau)\\ \operatorname{sh}2\pi & \operatorname{ch}(2\pi-\tau)\end{vmatrix}d\tau,
\end{aligned}\tag{3}$$

where $\hat{f}(t)=f\left(x^1,x^2\right)\big|_{V_1}$, $\quad \hat{u}(t)=u\left(x^1,x^2\right)\big|_{V_1}$.

In order to write out the formula for the solution $u(x^1,x^2)$, in the last formula we need to go from the local coordinate t to $(x^1,x^2)\in V_1$. For this we need the following auxiliary statement.

Lemma 1. For any smooth 2π - periodic function $\hat{F}(t)$ the integral identity holds

$$\int_0^t \hat{F}(\tau)d\tau=\int_{\gamma_x}F(\xi^1,\xi^2)(\xi^1d\xi^2-\xi^2d\xi^1),$$

where γ_x is a positively oriented arc of S^1 connecting the point $(1,0)$ with the point (x^1,x^2).

Proof of Lemma 1. The identity holds

$$\int_0^t \hat{F}(\tau)d\tau=\int_0^t \hat{F}(t)(\cos^2(t)+\sin^2(t))dt=$$

$$=\int_0^t \hat{F}(t)(\cos t d\sin t-\sin t d\cos t)=$$
$$=\int_{\gamma_x}F\left(x^1,x^2\right)\left(x^1dx^2-x^2dx^1\right).$$

Lemma 1 is completely proved.

We expand the function $\operatorname{sh}t$ and $\operatorname{ch}t$ on $(0,2\pi)$ into trigonometric Fourier series, that is,

$$cht=\frac{c_0}{2}+\sum_{k=1}^{\infty}\left(c_k\cos kt+d_k\sin kt\right),$$
$$sht=\frac{s_0}{2}+\sum_{k=1}^{\infty}\left(s_k\cos kt+r_k\sin kt\right),$$

where $\{c_k\},\{d_k\},\{s_k\},\{r_k\}$ are the corresponding Fourier coefficients in the trigonometric system.

We introduce two functions by the formulas

$$\mathbb{C}(x) = \mathbb{C}\left(x^1, x^2\right) = \frac{c_0}{2} + \sum_{k=1}^{infty} \left(c_k T_k\left(x^1\right) + d_k x^2\left(U_k\left(x^1\right)\right)\right),$$

$$\mathbb{S}(x) = \mathbb{S}\left(x^1, x^2\right) = \frac{s_0}{2} + \sum_{k=1}^{infty} \left(s_k T_k\left(x^1\right) + r_k x^2 U_k\left(x^1\right)\right).$$

On the circle S^1 we introduce the concept of convolution of two functions $f(x)$ and $g(x)$ for $x \in S^1$. We choose a fixed point $a = (\cos t_0, \sin t_0) \in S^1$. Then the convolution of two functions f and g at point a is defined by the integral

$$(f *_a g)(a) = \int_0^{t_0} \hat{f}(t)\hat{g}(t_0 - t)dt,$$

where $\hat{f}(t) = f(\cos t, \sin t)$, $\quad \hat{g}(t) = g(\cos t, \sin t)$.

Then from representation (3) taking into account Lemma 1 we have the relation

$$u(x) = -\left(f *_x \mathbb{S}\right)(x) -$$

$$-\frac{\operatorname{sh} 2\pi}{2 - 2\operatorname{ch} 2\pi}\begin{vmatrix} \mathbb{C}(x) & \left(f *_\eta \mathbb{S}\right)(\eta) \\ \mathbb{S}(x) & \left(f *_\eta \mathbb{C}\right)(\eta) \end{vmatrix} - \frac{1}{2}\begin{vmatrix} \mathbb{S}(x) & \left(f *_\eta \mathbb{S}\right)(\eta) \\ \mathbb{C}(x) & \left(f *_\eta \mathbb{C}\right)(\eta) \end{vmatrix},$$

where $\eta = (1, 0) \in S^1$.

Thus, the inverse operator to the Laplace-Beltrami operator has the form

$$(I - \Delta)^{-1} f(x) = -\left(f *_x \mathbb{S}\right)(x) -$$

$$-\frac{\operatorname{sh} 2\pi}{2 - 2\operatorname{ch} 2\pi}\begin{vmatrix} \mathbb{C}(x) & \left(f *_\eta \mathbb{S}\right)(\eta) \\ \mathbb{S}(x) & \left(f *_\eta \mathbb{C}\right)(\eta) \end{vmatrix} - \frac{1}{2}\begin{vmatrix} \mathbb{S}(x) & \left(f *_\eta \mathbb{S}\right)(\eta) \\ \mathbb{C}(x) & \left(f *_\eta \mathbb{C}\right)(\eta) \end{vmatrix}.$$

From this it is clear that the inverse operator is a linear integral operator. Denote by $G(x, \xi)$ the kernel of the inverse operator to the Laplace-Beltrami operator on the circle.

7 Well-solvable restrictions of the Laplace-Beltrami operator on a punctured circle

Choose an arbitrary point $x_0 \in S^1$. Denote by S_0^1 the punctured circle $S^1 \backslash \{x_0\}$. Consider the equation

$$(I - \Delta)w(x) = f(x), x \in S_0^1 \qquad (4)$$

Note that the inhomogeneous equation (4) for any right-hand side $f \in L_2(S^1)$ has infinitely many solutions. Indeed, let $u(x)$ be a solution to the inhomogeneous equation $(I - \Delta)w(x) = f(x)$ for $\forall x \in S^1$. In the previous paragraph it was proved that such a solution exists. Let us choose an arbitrary number $\alpha \in \mathbb{R}$ and consider the expression

$$w(x) = u(x) + \alpha G(x, x_0), \quad \text{for} \quad x \in S_0^1.$$

Since $x \neq x_0$, then

$$(I - \Delta)G(x, x_0) = 0.$$

Therefore, for any $\alpha \in \mathbb{R}$ the function $W(x) = u(x) + \alpha G(x, x_0)$ for $x \in S_0^1$ satisfies the inhomogeneous equation (4).

Thus, with respect to the inhomogeneous equation (4), the following question arises:

How many and what additional conditions should be added to the inhomogeneous equation (4) so that for $\forall f \in L_2(S^1)$ equation (4) has a unique solution.

To answer this question, we introduce two linear functionals. Let $x_0 = (\cos t_0, \sin t_0)$, where t_0 is a fixed number.

$$U_0(w) = \lim_{\delta \to +0} \left(\hat{w}(t_0 + \delta) - \hat{w}(t_0 - \delta)\right). \qquad (5)$$

The functional $U_0(\cdot)$ can be rewritten in another form. We will write $x'' < x_0 < x'$, if $x'', x_0, x' \in S^1$ and they preserve positive ordering on the circle S^1. Then

$$U_0(w) = \lim_{x'' < x_0 < x' x' \to x^0 x'' \to x^0} \left[w(x'') - w(x')\right].$$

We introduce another linear functional in the same way.

$$U_1(w) = \lim_{x'' < x_0 < x' x' \to x^0 x'' \to x^0} \left[\frac{\partial}{\partial \tau} w(x'') - \frac{\partial}{\partial \tau} w(x')\right],$$

where $\frac{\partial}{\partial \tau}$ is the derivative in the tangent direction.

Now we can formulate one of the main results of this article.

Theorem 1. For any function $f \in L_2(S^1)$ and any numbers γ_0 and γ_1, the inhomogeneous equation (4) is supplemented by the conditions

$$U_0(w) = \gamma_0, \quad U_1(w) = \gamma_1 \qquad (6)$$

has a unique solution.

The proof of Theorem 1 is proved by repeating the arguments given in the works [2, 3]. Theorem 1 can also be proved by simpler arguments.

Indeed, equation (4) is equivalent to the equation

$$\hat{w}(t) - \hat{w}''(t) = \hat{f}(t), \quad 0 \leq t \leq 2\pi, \quad t \neq t_0, \qquad (7)$$

with the boundary conditions

$$\begin{cases} \hat{w}(t_0 + 0) - \hat{w}(t_0 - 0) = \gamma_0, \\ \hat{w}'(t_0 + 0) - \hat{w}'(t_0 - 0) = \gamma_1 \end{cases} \qquad (8)$$

and the periodicity conditions

$$\begin{cases} \hat{w}(0) = \hat{w}(2\pi), \\ \hat{w}'(0) = \hat{w}'(2\pi). \end{cases} \qquad (9)$$

Problems (7)-(9) have a unique solution, since $ch 2\pi \neq 1$. Thus, Theorem 1 is completely proven.

From Theorem 1 it follows that

Corollary 1 Let $\gamma_0(\cdot), \gamma_1(\cdot)$ be arbitrary linear continuous functionals on $L_2(S^1)$. Then for any function $f \in L_2(S^1)$ the following boundary value problem applies.

$$\begin{cases} (I-\Delta)w(x) = f(x), \quad x \in S_0^1, \\ U_0(w) = \gamma_0((I-\Delta)w), \qquad (10) \\ U_1(w) = \gamma_1((I-\Delta)w) \end{cases}$$

has a unique solution.

By choosing the linear functionals $\gamma_0(\cdot)$ and $\gamma_1(\cdot)$ in a special way, we can refine Corollary 1 as follows.

We choose two functions $\alpha_0(x)$ and $\alpha_1(x)$ for $x \in S^1$ such that $(I-\Delta)\alpha_j(x) = 0, \quad \forall x \in S^1, \quad j = 0,1$. Let the linear functionals γ_0 and γ_1 be chosen according to the Riesz theorem in the form $\gamma_j(f) = \int_{S^1} f(x)\overline{\alpha_j(x)}dS_x^1, j = 0,1$. Then the boundary conditions (10) can be rewritten as

$$\begin{cases} U_0(w) + \overline{\hat{\alpha}_0'}(t_0)U_0(w) - \overline{\hat{\alpha}_0}(t_0)U_1(w) = A_0 w(0) + B_0 w'(0), \\ U_1(w) + \overline{\hat{\alpha}_1'}(t_0)U_0(w) - \overline{\hat{\alpha}_1}(t_0)U_1(w) = A_1 w(0) + B_1 w'(0), \end{cases}$$

where $A_0 = \hat{\alpha_0}'(2\pi) - \hat{\alpha_0}'(0), \quad B_0 = \overline{\hat{\alpha_1}'(2\pi) - \hat{\alpha_1}'(0)}$.

Thus, the following statement is true.

Corollary 2 Let $\alpha_0(x)$ and $\alpha_1(x)$ for $x \in S^1$ be a solution of the equation $(I-\Delta)\alpha_j(x) = 0, x \in S^1, j = 0,1$. Then for any function $f \in L_2(S^1)$ the following boundary value problem applies.

$$(I-\Delta)w(x) = f(x), \quad x \in S_0^1,$$

$$\begin{cases} U_0(w) + \overline{\hat{\alpha}_0'}(t_0)U_0(w) - \overline{\hat{\alpha}_0}(t_0)U_1(w) = A_0 w(0) + B_0 w'(0), \\ U_1(w) + \overline{\hat{\alpha}_1'}(t_0)U_0(w) - \overline{\hat{\alpha}_1}(t_0)U_1(w) = A_1 w(0) + B_1 w'(0), \end{cases}$$

has a unique solution.

Similar problems on punctured balls and punctured spheres can be found in [4]- [13].

8 Spectral analysis of the Laplace-Beltrami operator on one stratified set

In this section, we consider a stratified set consisting of two punctured circles and one segment connecting these circles. Consider the eigenvalue problem on the stratified set $\mathbb{S} = \{X^1, X^2, X^3, A, B\}$, where A and B are two points on X^1 and X^2, respectively. That is, consider the system of equations

$$\begin{cases} (I-\Delta_1)w_1(x_1) = \lambda w_1(x_1), x_1 \in X^1, x_1 = (x_1^1, x_1^2) \\ (I-\Delta_2)w_2(x_2) = \lambda w_2(x_2), x_2 \in X^2, \qquad (11) \\ w_3(x_3) - w"_3(x_3) = \lambda w_3(x_3), x_3 \in X^3 \equiv (0,1), \end{cases}$$

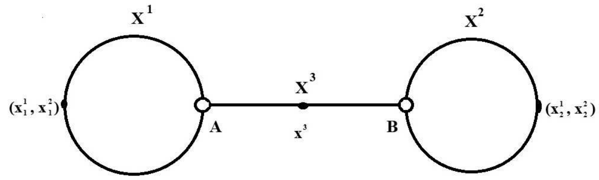

Рис. 1: Stratified set

with boundary conditions

$$\begin{cases} U_0^1(w_1) = w_3(0), \\ U_1^1(w_1) = w_3'(0), \\ U_0^2(w_2) = w_3(1), \\ U_1^2(w_2) = w_3'(1), \\ w_3(0) = 0, \\ w_3(1) = 0 \end{cases} \qquad (12)$$

Here, the functionals $U_0^j(\cdot)$ and $U_1^j(\cdot)$ for $j = 1, 2$ are defined the same way as the functionals U_0 and U_1 were defined in step 6 for the punctured circle S_0^1..

Now we calculate the eigenvalues and eigenfunctions of problem (11)-(12). The eigenvalues of problem (11)-(12) consist of eigenvalues of two types:

1. Each eigenvalue $\lambda_n(D)$ of the Dirichlet problem in the interval $(0,1)$: $y(t) - y"(t) = \lambda y(t), t \in (0,1), y(0) = 0, y(1) = 0$ are also eigenvalues of the original problem (11)-(12). If $\lambda_n(D)$ corresponds to an eigenfunction $y_n(t)$, then the eigenvector function $(w_{1n}(x_1), w_{2n}(x_2), w_{3n}(x_3))$ of problem (11)-(12) has the form

$$w_{3n}(x_3) = y_n(x_3), x_3 \in X^3,$$

for $j = 1, 2$ the function $w_{jn}(x_j)$ coincides with the solution of the problem

$$(I - \Delta_j)w_{jn}(x_j) = \lambda_n(D))w_{jn}(x_j), \quad U_0^j(w_{jn}) = \gamma_{0j}, \quad U_1^j(w_{jn}) = \gamma_{1j},$$

where $\gamma_{01} = y_n(0), \gamma_{21} = y_n'(0), \gamma_{02} = y_n(1), \gamma_{12} = y_n'(1)$.

2. Each eigenvalue $\lambda_m(S^1)$ problems

$$(I - \Delta_{S^1})u(x) = \lambda_m(S^1)u(x), x \in S^1$$

is also an eigenvalue of the original problem (11)-(12). If $\lambda_m(S^1)$ corresponds to an eigenfunction $u_m(x)$, then the eigenvector function $(w_{1m}(x_1), w_{2m}(x_2), w_{3m}(x_3))$ of problem (11)-(12) has the form

$$w_{1m}(x_1) = u_m(x_1),$$

$$w_{2m}(x_2) = \varepsilon u_m(x_2),$$

$$w_{3m}(x_3) \equiv 0,$$

where ε can be equal to 0 and/or 1.
The eigenvalue problem studied in this section is analogous to spectral problems on graphs [14,15].

9 Conclusion

In this paper, a detailed characterization of the circle S^1 as a differential manifold was carried out, local coordinates and classes of functions defined on it were considered. Particular attention was paid to the Laplace-Beltrami operator, its spectral properties and connection with harmonic analysis.

The results obtained confirm the fundamental role of the circle in spectral theory and analysis of periodic functions. The study of the spectrum of the Laplace-Beltrami operator demonstrates its connection with trigonometric functions, which is the basis for many applications in mathematical physics and signal theory.

Thus, the circle S^1 remains an important object of mathematical analysis, and further study of its properties can lead to new results in related areas, such as geometric analysis and representation theory.

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CONDITIONS FOR SOLVABILITY AND COERCIVENESS OF A FOURTH-ORDER DIFFERENTIAL EQUATION WITH AN INTERMEDIATE COEFFICIENT

The article considers a three-term fourth-order differential equation with unbounded coefficients. The coefficient of the intermediate term of the equation is assumed to be a smooth and rapidly increasing function at infinity. This intermediate term, as an operator, does not obey the differential operator formed by the extreme terms of the equation. This is precisely what makes the work unique. Using functional methods, sufficient conditions are obtained for a generalized solution of the equation to exist, be unique and maximally regular. These conditions characterize the relationship between intermediate and small coefficients. The differential equation under consideration is caused by problems of practical processes of stochastic analysis, shaft oscillations, etc. The article uses such methods as obtaining an a priori estimate of the solution, reducing the problem above to the problem of invertibility of a third-order differential operator with a potential of constant sign, and constructing a pseudo-resolvent using correct local operators. In general, the article substantiates an effective method for solving the main problems posed for differential equations on an infinite interval in the case of a new equation with an unbounded intermediate coefficient. Although the coefficients are assumed to be smooth, the work does not impose restrictions on the variation of their derivatives. This, in turn, allows us to cover a wide class of fourth-order equations. The methods developed in the work and the results obtained can be used in the study of the qualitative properties of higher-order differential equations.

Key words: differential equation, variable coefficient, strong solution, correctness, regularity.

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Аралық коэффициенті бар төртінші ретті бір дифференциалдық теңдеудің шешілу және коэрцитивтілік шарттары

Мақалада коэффициенттері шенелмеген төртінші ретті үшмүшелі дифференциалдық теңдеу қарастырылған. Теңдеудің аралық мүшесінің коэффициенті тегіс және шексіздікте жылдам өсетін функция деп есептеледі. Бұл аралық мүше оператор ретінде теңдеудің шеткі мүшелерінен құралған дифференциалдық операторға бағынбайды. Жұмыстың ерекшелігі осында. Біз функционалдық әдістерді қолдана отырып, теңдеудің жалпыланған шешімінің табылуы, жалғыз және максималды регулярлы болуы үшін жеткілікті шарттар алдық. Бұл шарттар аралық және кіші коэффициенттердің өзара байланысын сипаттайды. Қарастырылған дифференциалдық теңдеуге стохастикалық талдаудың, біліктің тербелісінің және т.б. практикалық процестердің мәселелері алып келеді. Мақалада шешімнің априорлық бағасын алу, қойылған есепті потенциалы тұрақты таңбалы бір үшінші ретті дифференциалдық оператордың қайтымдылық мәселесіне келтіру, корректілі локальды операторлар арқылы псевдорезольвентаны тұрғызу сияқты тың амалдар қолданылды. Жалпы, мақала шексіз аралықта берілген дифференциалдық теңдеулер үшін қойылатын негізгі есептерді жаңа, аралық коэффициенті шенелмеген теңдеу жағдайында шешудің бір тиімді әдісін негіздейді. Коэффициенттер тегіс деп есептелсе де, олардың туындыларының өзгеруіне жұмыста шектеулер қойылмайды. Бұл, өз кезегінде, зерттеумен төртінші ретті теңдеулердің кең класын қамтуға мүмкіндік береді. Жұмыста жасалған әдістер мен алынған нәтижелерді жоғарғы ретті дифференциалдық теңдеулерді сапалық зерттеу кезінде пайдалануға болады.

Түйін сөздер: дифференциалдық теңдеу, айнымалы коэффициент, қатаң шешім, қисынды-лық, регулярлық.

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Условия разрешимости и коэрцитивности одного дифференциального уравнения четвертого порядка с промежуточным коэффициентом

В статье рассматривается трехчленное дифференциальное уравнение четвертого порядка с неограниченными коэффициентами. Коэффициент промежуточного члена уравнения предполагается гладкой и быстро возрастающей функцией на бесконечности. Этот промежуточный член, как оператор, не подчиняется дифференциальному оператору, образованному крайними членами уравнения. Именно в этом и заключается уникальность работы. Используя функциональные методы, получены достаточные условия для того, чтобы обобщенное решение уравнения существовало, было единственным и максимально регулярным. Эти условия характеризуют взаимосвязь промежуточного и младшего коэффициентов. К рассматриваемому дифференциальному уравнению приводят проблемы практических процессов стохастического анализа, колебаний вала и др. В статье используются такие методы, как получение априорной оценки решения, сведение поставленного вопроса к задаче обратимости дифференциального оператора третьего порядка со знакопостоянным потенциалом и построение псевдорезольвенты с использованием корректных локальных операторов. В целом, в статье обоснован эффективный метод решения основных задач, поставленных для дифференциальных уравнений на бесконечном интервале, в случае нового уравнения с неограниченным промежуточным коэффициентом. Хотя коэффициенты предполагаются гладкими, работа не накладывает ограничений на изменение их производных. Это, в свою очередь, позволяет охватить исследованием широкий класс уравнений четвертого порядка. Разработанные в работе методы и полученные результаты могут быть использованы при исследовании качественных свойств дифференциальных уравнений высших порядков.
Ключевые слова: дифференциальное уравнение, переменный коэффициент, сильное решение, корректность, регулярность.

1 Introduction

Let's consider the equation

$$L_0 y = y^{(4)} + r(x)y' + q(x)y = F(x), \tag{1}$$

where we assume that $x \in \mathbb{R} = (-\infty, \infty)$, $r(x) > 0$, $r(x) \in C^{(1)}_{\text{loc}}(\mathbb{R})$, $q(x)$ is a continuous function, and $F(x) \in L_2(\mathbb{R})$. Let L denote the closure in the $L_2(\mathbb{R})$ norm of the operator L_0, defined by the equality $L_0 y = y^{(4)} + r(x)y' + q(x)y$ on the set $C^{(4)}_0(\mathbb{R})$ of continuously differentiable functions up to the fourth order with compact support. An element $y \in D(L)$ satisfying the equality $Ly = F$ is called a solution to equation (1). The coefficients of equation (1) can grow infinitely. It is known that fourth-order differential equations with variable coefficients are of great importance in physics and engineering [1, 2]. The features of their research and the obtained results in the case of singular coefficients are presented in the article [3]. The solvability conditions and maximal regularity of various differential equations with intermediate coefficients in an infinite interval are considered in the works [4-9].

2 Research method

The paper uses methods such as obtaining an a priori estimate for the solution of a fourth-order differential equation with variable coefficients, reducing the problem to the problem of invertibility of a third-order differential operator with a potential of constant sign, and constructing a pseudo-resolvent using some correct local operators.

3 Case $q(x) = 0$

We take a binomial operator $L_0 y = y^{(4)} + r(x)y'$ with $D(L_0) = C_0^{(4)}(\mathbb{R})$, and denote its closure in $L_2(\mathbb{R})$ by L. We introduce the following notation for continuous functions $\rho(t)$ and $v(t) \neq 0$:

$$\alpha_{\rho,v}(x) = \sup_{x>0} \|\rho\|_{L_2(0,x)} \|v^{-1}\|_{L_2(x,\infty)}, \quad \beta_{\rho,v}(x) = \sup_{x<0} \|\rho\|_{L_2(x,0)} \|v^{-1}\|_{L_2(-\infty,x)},$$

$$\gamma_{\rho,v} = \max(\alpha_{\rho,v}(x), \beta_{\rho,v}(x)).$$

Lemma 1. If the following conditions are satisfied for the coefficient $r(x)$:

$$r(x) \geq 1, \quad \gamma_{1,\sqrt{r}} < \infty, \tag{2}$$

then the operator L is invertible and for $y \in D(L)$, the inequality

$$\|\sqrt{r} y'\|_2 + \|y\|_2 \leq C\|Ly\|_2 \tag{3}$$

is true.

Proof. Transforming the functional $(L_0 y, y')$, we obtain

$$\|\sqrt{r} y'\|_2 \leq \left\| \frac{L_0 y}{\sqrt{r}} \right\|_2 .$$

According to condition (2) and the results of [5],

$$\|\sqrt{r} y'\|_2 + \|y\|_2 \leq (1 + 2\gamma_{1,\sqrt{r}}) \left\| \frac{L_0 y}{\sqrt{r}} \right\|_2 .$$

Closing this inequality, we obtain (3). The lemma is proved.

Let us now consider equation

$$ly = y^{(4)} + r(x)y' = f(x). \tag{4}$$

The following statement follows directly from (3).

Lemma 2. Let $r(x)$ satisfy the conditions of Lemma 1. Then the solution of equation (4) is a unique .

Suppose that the conditions of Lemma 1 are satisfied. From estimate (3) we obtain the relation $\sqrt{r}\, y' \in L_2(\mathbb{R})$ for each $y \in D(L)$. According to conditions (2) and estimate (3), $y' \in L_2(\mathbb{R})$ and $\|y'\|_2 \leq C\|Ly\|_2$. If we make the notation $y' = z$, then in view of (4), we have:

$$z^{(3)} + r(x)z = f(x). \tag{5}$$

Let L be the closure in the space $L_2(\mathbb{R})$ of the operator $L_0 : L_0 v = v^{(3)} + r(x)v$, $D(L_0) = C_0^{(3)}(\mathbb{R})$. A solution of equation (5) is a function $z \in D(L)$ satisfying the equality $Lz = f$. If the conditions of Lemma 1 are satisfied and a solution to equation (4) exists, then it is clear that equation (5) also has a solution. The converse is also true, namely:

Lemma 3. Suppose that $r(x)$ satisfies the conditions of Lemma 1 and equation (5) has a solution. Then equation (4) is also solvable.

Proof. If $z \in L_2(\mathbb{R})$ is a solution of equation (5), then there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset C_0^{(3)}(\mathbb{R})$ such that $\|z_n - z\|_2 \to 0$ and $\|L_0 z_n - f\|_2 \to 0$ as $n \to \infty$. Let us take the functional (Lz_n, z_n). Repeating the method of Lemma 1, we will see that $\|\sqrt{r}\, z_n\|_2 \leq \|L_0 z_n\|_2$. Let $y_n(x)$ be a function such that $y_n' = z_n$. Then y_n is four times continuously differentiable, and, according to [5], from the inequality $\|\sqrt{r}\, y_n'\|_2 \leq \|L_0 z_n\|_2$, we obtain the estimate

$$\|y_n\|_2 \leq \|L_0 z_n\|_2, \quad z_n \in C_0^{(3)}(\mathbb{R}). \tag{6}$$

That is, $y_n \in L_2(\mathbb{R})$. Then, since $y_n' \in C_0^{(3)}(\mathbb{R})$, there is a number $a > 0$ such that the equality $y_n(x) = 0$ holds for all $|x| > a$ (otherwise the relation $y_n \in L_2(\mathbb{R})$ is violated). Therefore, $y_n(x) \in C_0^{(4)}(\mathbb{R}) = D(L_0)$. From estimate (6) it follows that there exists an element $\bar{y} \in L_2(\mathbb{R})$ and the relations $\|y_n - \bar{y}\|_2 \to 0$, $\|L_0 y_n - f\|_2 \to 0$ as $n \to \infty$ are satisfied. Consequently, $\bar{y}$ is a solution to equation (4). The lemma is proven.

4 Separability of a third-order differential operator

Let us assume that the function $r(x)$, in addition to the conditions of Lemma 1, also satisfies the relation

$$\sup_{x,\eta \in \mathbb{R}, |x-\eta| \leq 1} \frac{r(x)}{r(\eta)} < \infty. \tag{7}$$

We choose sequences of intervals $\{\Delta_j\}_{j=1}^{\infty}$, $\{\Omega_j\}_{j=1}^{\infty}$, and functions $\varphi_j(x) \in C_0^{\infty}(\Omega_j)$ as follows:
(a) $\Delta_j = [j, j+1)$, $\Omega_j = (j - \frac{1}{2}, j + \frac{3}{2})$ for $j \in \mathbb{Z}$,
(b) $0 \leq \varphi_j \leq 1$, $\quad \varphi_j(x) = 1\, \forall x \in \Delta_j$ for $j \in \mathbb{Z}$, $\sup\limits_{j \in \mathbb{Z}} \max\limits_{x \in \Omega_j} (|\varphi_j'(x)|, |\varphi_j''(x)|, |\varphi_j^{(3)}|) \leq M$.
Then:

$$|\Omega_j| = 2, \quad \overline{\Delta_j} \subset \Omega_j \subset \Delta_{j-1} \cup \Delta_j \cup \Delta_{j+1}, \quad \Delta_j \cap \Delta_k = \emptyset \quad (j \neq k), \quad \bigcup_{j=-\infty}^{\infty} \Delta_j = \mathbb{R},$$

$$\Omega_j \cap \Omega_m = \emptyset \quad (|j - m| \geq 2), \quad \sum_j \varphi_j(x) \chi_{\Delta_j}(x) = 1,$$

where χ_{Δ_j} is the characteristic function of Δ_j. Recall that the sequence $\{\varphi_j(x)\}_{j=1}^{\infty}$ satisfying relations (b) always exists.

We extend the restriction of the function $r(x)$ to the interval Ω_j on all $\mathbb{R}$ so that the resulting extension $r_j(x)$ (for $j \in \mathbb{Z}$) turns out to be a continuously differentiable function and satisfies the inequalities

$$\frac{1}{2} \inf_{z \in \Omega_j} r(z) \leq r_j(x) \leq 2 \sup_{z \in \Omega_j} r(z), \quad x \in \mathbb{R}.$$

According to (7), such an extension $r_j(x)$ exists. Let $\theta_{\lambda j}(\lambda \geq 0)$ denote the closure in $L_2(\mathbb{R})$ of the differential operator

$$\theta_{0\lambda j} z = z^{(3)} + [r_j(x) + \lambda] z(x), \quad D(\theta_{0\lambda j}) = C_0^{(3)}(\mathbb{R}).$$

Then, it is easy to see that $D(\theta_{\lambda j}) = W_2^3(\mathbb{R})$. Therefore, for an element $z \in D(\theta_{\lambda j})$, the relations $z \in C^{(2)}(\mathbb{R})$ and $z(-\infty) = z(+\infty) = z^{(k)}(-\infty) = z^{(k)}(+\infty) = 0$ $(k = 1, 2)$ are satisfied. Taking these equalities into account, we transform the scalar product $(\theta_{\lambda j} z, z)$ $(z \in D(\theta_{\lambda j}))$. Then we have

$$\|z\|_2 \leq \frac{2}{\delta + 2\lambda} \|\theta_{\lambda j} z\|_2.$$

Consequently, $R(\theta_{\lambda j})$ is a closed set.

Lemma 4. If the function $r(x)$ satisfies conditions (2) and (7), then $R(\theta_{\lambda j}) = L_2(\mathbb{R})$ for $\lambda \geq 0$.

Proof. If $R(\theta_{\lambda j}) \neq L_2(R)$, then there exists an element $w \in L_2(R) \setminus R(\theta_{\lambda j})$, $w \neq 0$, such that the equality

$$\theta_{\lambda j}^* w = -w''' + (r_j(x) + \lambda) w = 0 \tag{8}$$

holds, where $\theta_{\lambda j}^*$ is the operator conjugate to $\theta_{\lambda j}$. Since the function $r_j(x)$ is bounded, by (8) and condition (2), $w \in W_2^3(R)$. Consequently, $w \in C^2(R)$ and $w(-\infty) = w(+\infty) = w^{(k)}(-\infty) = w^{(k)}(+\infty) = 0$ $(k = 1, 2)$. Taking these equalities into account and transforming the functional $(\theta_{\lambda j}^* w, w)$, we obtain the estimate $\delta \|w\|_2 \leq 2 \|\theta_{\lambda j}^* w\|_2$. According to (8), $w = 0$. The lemma is proved.

If the conditions of Lemma 4 are met, then, similarly to the proof of Lemma 1, we obtain the inequality

$$\|z\|_2 \leq \frac{2}{\delta + 2\lambda} \|\theta_{\lambda j} z\|_2 \quad (j \in \mathbb{Z}, z \in D(\theta_{\lambda j}), \lambda \geq 0). \tag{9}$$

Consider the following operator $L_{0\lambda} = L_0 + \lambda E$, $D(L_{0\lambda}) = C_0^3(R)$, where $\lambda \in \mathbb{R}_+ = [0, +\infty)$, and E is the identity operator. Denote the closure of $L_{0\lambda}$ in $L_2(R)$ as L_λ. Repeating the method of Lemma 1, we obtain the estimate

$$\sqrt{1 + \lambda} \|z\|_2 \leq \|L_\lambda z\|_2 \tag{10}$$

for each $z \in D(L_\lambda)$. Therefore, there exists an inverse operator L_λ^{-1} $(\lambda \geq 0)$.

Lemma 5. Let the coefficient r satisfy conditions (2) and (7). Then the operator L_λ is continuously invertible, and for each $z \in D(L_\lambda)$ the following inequality is true:

$$\|z'''\|_2 + \|rz\|_2 + \lambda \|z\|_2 \leq C \|L_\lambda z\|_2. \tag{11}$$

Proof. Let $K^2 = \sup\limits_{x,t \in \Omega_j, |x-t| \leq 1} \frac{r(x)}{r(t)}$. For $z \in D(\theta_{\lambda j})$, we have

$$(i)\ \|z'''\|_2 \leq \left(3 + 2 \sup_{x,t \in \Omega_j, |x-t| \leq 1} \frac{r(x)}{r(t)}\right) \|\theta_{\lambda j} z\|_2 \leq (3 + 2K^2) \|\theta_{\lambda j} z\|_2.$$

Using (9) and simple calculations, we get

$$(ii)\ \|z'\|_2 \le \left(\frac{2}{\delta+2\lambda}\right)^{2/3}(3+2K^2)^{1/3}\|\theta_{\lambda j} z\|_2,$$

$$(iii)\ \|z''\|_2 \le \left(\frac{2}{\delta+2\lambda}\right)^{1/3}(3+2K^2)^{2/3}\|\theta_{\lambda j} z\|_2, \quad z\in D(\theta_{0\lambda j}) = C_0^3(R).$$

Let $f\in C_0^3(R)$, and M_λ and B_λ be operators acting according to the following formulas:

$$M_\lambda f = \sum_j \phi_j \theta_{\lambda j}^{-1}(\chi_{\Delta_j} f),$$

$$B_\lambda f = \sum_j \phi_j^{(3)}\theta_{\lambda j}^{-1}(\chi_{\Delta_j} f) + 3\sum_j \phi_j''(\theta_{\lambda j}^{-1}(\chi_{\Delta_j} f))' + 3\sum_j \phi_j'(\theta_{\lambda j}^{-1}(\chi_{\Delta_j} f))''.$$

It is easy to see that the equality

$$L_\lambda(M_\lambda f) = (B_\lambda + E) f \tag{12}$$

holds. Considering that $\Omega_j\cap\Omega_k=\emptyset$ if $|j-k|\ge 2$, we obtain the following inequality:

$$\|B_\lambda f\|_2^2 \le 9M^2\left(\sum_j \|\theta_{\lambda j}^{-1}(\chi_{\Delta_j} f)\|_2^2 + \sum_j \|9(\theta_{\lambda j}^{-1}(\chi_{\Delta_j} f))'\|_2^2 + \sum_j \|9(\theta_{\lambda j}^{-1}(\chi_{\Delta_j} f))''\|_2^2\right).$$

According to $(i),(ii),(iii)$, we have the following estimates:

$$\|\theta_{\lambda j}^{-1}(\chi_{\Delta_j} f)\|_2^2 \le \left(\frac{2}{\delta+2\lambda}\right)^2\|\chi_{\Delta_j} f\|_2^2,$$

$$\|(\theta_{\lambda j}^{-1}(\chi_{\Delta_j} f))'\|_2^2 \le \left(\left(\frac{2}{\delta+2\lambda}\right)^{2/3}(3+2K^2)^{1/3}\right)^2\|\chi_{\Delta_j} f\|_2^2,$$

$$\|(\theta_{\lambda j}^{-1}(\chi_{\Delta_j} f))''\|_2^2 \le \left(\left(\frac{2}{\delta+2\lambda}\right)^{1/3}(3+2K^2)^{2/3}\right)^2\|\chi_{\Delta_j} f\|_2^2 \quad (j\in\mathbb{Z}).$$

Hence,

$$\|B_\lambda f\|_2^2 \le 9M^2\left(\frac{2}{\delta+2\lambda} + 9\left(\frac{2}{\delta+2\lambda}\right)^{2/3}(3+2K^2)^{1/3} + 9\left(\frac{2}{\delta+2\lambda}\right)^{1/3}(3+2K^2)^{2/3}\right)^2\|f\|_2^2. \tag{13}$$

If we choose the number λ_0 so that

$$9M^2\left(\frac{2}{\delta+2\lambda_0} + 9\left(\frac{2}{\delta+2\lambda_0}\right)^{2/3}(3+2K^2)^{1/3} + 9\left(\frac{2}{\delta+\lambda_0}\right)^{1/3}(3+2K^2)^{2/3}\right)^2 \le \beta^2,$$

$(0 < \beta < 1)$ then $\|B_\lambda\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} \le \beta$ $(\lambda \ge \lambda_0)$, where $\|\cdot\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})}$ is the operator norm. So, for any $\lambda \ge \lambda_0$, the operator $E + B_\lambda$ is boundedly invertible, and its inverse $(E+B_\lambda)^{-1}$ satisfies the estimates

$$(1+\beta)^{-1} \le \|(E+B_\lambda)^{-1}\| \le (1-\beta)^{-1} \quad (\lambda \ge \lambda_0). \tag{14}$$

From (12) it follows that

$$L_\lambda^{-1} = M_\lambda (E+B_\lambda)^{-1}, \quad \lambda \ge \lambda_0. \tag{15}$$

By (10), $\|z\|_2 \le \|L_\lambda z\|_2$ for $z \in D(L_\lambda)$ and $\lambda \ge 0$. Then, according to the well-known statement [10] (p. 350), the operator L_λ is continuously invertible for any $\lambda \ge 0$.

Let us prove estimate (11). It suffices to show that $\|(r+\lambda)z\|_2 \le C\|L_\lambda z\|_2$ $(\lambda \ge \lambda_0 > 0, z \in D(L_\lambda))$. By (14) and (15), $\|(r+\lambda)L_\lambda^{-1}\| \le (1-\beta)^{-1}\|(r+\lambda)M_\lambda\|$ $(\lambda \ge \lambda_0)$, and

$$\|(r+\lambda)M_\lambda f\|_2^2 \le 3\sum_{j=-\infty}^{\infty}\left(\sup_{x\in\Omega_j}(r(x)+\lambda)^2\int_{-\infty}^{\infty}|\phi_j(x)\theta_{\lambda j}^{-1}(\chi_{\Delta_j}f)|^2\,dx\right).$$

Taking into account the property (b) of the sequence $\{\phi_j(x)\}_{j=1}^{\infty}$, we have

$$\|(r+\lambda)M_\lambda f\|_2^2 \le 12(K^2+1)\|f\|_2^2 \quad (\lambda \ge \lambda_0). \tag{16}$$

If $z \in D(L_\lambda)$, $L_\lambda z = f$ $(\lambda \ge \lambda_0)$, then $z = L_\lambda^{-1} f$. Therefore, according to (15) and (16),

$$\|(r+\lambda)z\|_2 \le 2\sqrt{3}(K^2+1)\|(E+B_\lambda)^{-1}\|\|f\|_2 \le 2\sqrt{3}(K^2+1)(1-\beta)^{-1}\|f\|_2. \tag{17}$$

Then

$$\|z'''\|_2 \le \left(2\sqrt{3}(K^2+1)(1-\beta)^{-1}+1\right)\|f\|_2. \tag{18}$$

By (18) and (17), $\|z'''\|_2 + \|rz\|_2 + \|\lambda z\|_2 \le \left(6\sqrt{3}(K^2+1)(1-\beta)^{-1}+1\right)\|f\|_2$. The lemma is proved.

Thus, if conditions (2) and (7) are satisfied, then, according to Lemma 3, the following estimate is valid for the solution y of equation (4):

$$\|y^{(4)}\|_2 + \|ry'\|_2 \le \left(6\sqrt{3}(K^2+1)(1-\beta)^{-1}+1\right)\|f\|_2. \tag{19}$$

Remark 1. The statement of Lemma 5 remains true if condition (2) is replaced by $r(x) \ge \delta > 0$.

5 Main Result

Theorem 1. If the functions $r(x)$ and $q(x)$ satisfy conditions (2), (7), and $\gamma_{q,r} < \infty$, then for any $f \in L_2(\mathbb{R})$, there exists a unique solution y of equation (1). Furthermore, the following inequality holds for y:

$$\|y^{(4)}\|_2 + \|ry'\|_2 + \|qy\|_2 \le C\|f\|_2. \tag{20}$$

Proof. Let us replace $x = t/a$ $(a > 0)$ in (1). Denoting $\tilde{y}(t) = y(a^{-1}t)$, $\tilde{r}(t) = r(a^{-1}t)$, $\tilde{q}(t) = q(a^{-1}t)$, $\tilde{F}(t) = a^{-4}F(a^{-1}t)$, equation (1) becomes

$$\tilde{L}_{0a}\tilde{y} = \tilde{y}^{(4)}(t) + a^{-3}\tilde{r}(t)\tilde{y}'(t) + a^{-4}\tilde{q}(t)\tilde{y}(t) = \tilde{F}(t). \tag{21}$$

Denote the closure of the operator $l_{0a}\tilde{y} = \tilde{y}^{(4)}(t) + a^{-3}\tilde{r}(t)\tilde{y}'(t)$, $\tilde{y} \in C_0^{(4)}(\mathbb{R})$, in the norm of $L_2(\mathbb{R})$ as l_a. From the condition $\tilde{r}(t) \geq 1$, it follows that $1 \geq \tilde{r}^{-1} \geq \tilde{r}^{-2}$. The coefficient $a^{-3}\tilde{r}(t)$ of the operator l_a satisfies the conditions of Lemma 3. Consequently, l_a is a continuously invertible operator, and

$$\|\tilde{y}^{(4)}\|_2 + \|a^{-3}\tilde{r}\tilde{y}'\|_2 \leq C_a\|l_a\tilde{y}\|_2, \quad \tilde{y} \in D(l_a). \tag{22}$$

It is straightforward to calculate that $\gamma_{a^{-4}\tilde{q},a^{-3}\tilde{r},1} = \gamma_{q,r,1} < \infty$. Therefore,

$$\|a^{-4}\tilde{q}\tilde{y}\|_2 \leq 2\gamma_{a^{-4}\tilde{q},a^{-3}\tilde{r},1}\|a^{-3}\tilde{r}\tilde{y}'\|_2 \leq 2\gamma_{a^{-4}\tilde{q},a^{-3}\tilde{r},1}C_a\|l_a\tilde{y}\|_2. \tag{23}$$

Using the substitution $x = a^{-1}\tau$, we obtain that $\gamma_{a^{-4}\tilde{q},a^{-3}\tilde{r},1} = \tilde{\gamma}_{a^{-4}\tilde{q},a^{-3}\tilde{r},1}$. Let us prove the equality

$$\lim_{a\to\infty} \tilde{\gamma}_{(a^{-4}\tilde{q},a^{-3}\tilde{r},1)}(a) = 0. \tag{24}$$

Let $\tilde{y} \in D(l_a)$. Since l_a is a closed operator, there exists a sequence $\{\tilde{y}_n\}_{n=1}^{\infty} \subseteq C_0^{(4)}(\mathbb{R})$ such that $\|\tilde{y}_n - \tilde{y}\|_2 \to 0$, $\|l_a\tilde{y}_n - l_a\tilde{y}\|_2 \to 0$ $(n \to \infty)$. Let $\{\tilde{y}_n\}_{n=1}^{\infty}$ and a number N_0 be such that $\operatorname{supp}\tilde{y}_n \subseteq [-N_0, N_0]$. Denote

$$\tilde{q}_{N_0}(t) = \begin{cases} \tilde{q}(t), & t \in [-N_0, N_0], \\ 0, & t \notin [-N_0, N_0], \end{cases} \quad \tilde{r}_{N_0}(t) = \begin{cases} \tilde{r}(t), & t \in [-N_0, N_0], \\ 0, & t \notin [-N_0, N_0]. \end{cases}$$

According to (23), for each $\tilde{y} \in C_0^{(4)}[-N_0, N_0]$, we obtain the estimate:

$$\|a^{-4}\tilde{q}_{N_0}\tilde{y}\|_{L_2[-N_0,N_0]} \leq 2\gamma_{a^{-4}\tilde{q}_{N_0},a^{-3}\tilde{r}_{N_0},1}(a)\|a^{-3}\tilde{r}_{N_0}\tilde{y}'\|_{L_2[-N_0,N_0]}.$$

Further,

$$\alpha_{a^{-4}\tilde{q}_{N_0},a^{-3}\tilde{r}_{N_0}}(a) = \sup_{x>0}\left(\int_0^{a^{-1}x} a^{-8}\tilde{q}_{N_0}^2(t)\,dt\right)^{1/2}\left(\int_{a^{-1}x}^{N_0} \frac{a^6}{\tilde{r}_{N_0}^2(t)}\,dt\right)^{1/2} =$$

$$= \sup_{0<x\leq N_0}\left(\int_0^{a^{-1}x} a^{-7}q^2(t)\,dt\right)^{1/2}\left(\int_{a^{-1}x}^{N_0} \frac{a^7}{r^2(t)}\,dt\right)^{1/2} \leq \alpha_{q,r} < \infty. \tag{26}$$

Therefore,

$$\lim_{a\to\infty}\sup_{0<x\leq N_0}\left(\int_0^{a^{-1}x} q^2(t)\,dt\right)^{1/2}\left(\int_{a^{-1}x}^{N_0} \frac{1}{r^2(t)}\,dt\right)^{1/2} =$$

$$= \sup_{0<x\leq N_0}\lim_{a\to\infty}\left(\int_0^{a^{-1}x} q^2(t)\,dt\right)^{1/2}\left(\int_{a^{-1}x}^{N_0} \frac{1}{r^2(t)}\,dt\right)^{1/2} = 0.$$

Similarly, we have:

$$\lim_{a\to\infty} \sup_{-N_0\le x<0} \left(\int_{a^{-1}x}^{0} q^2(t)\,dt\right)^{1/2} \left(\int_{-N_0}^{a^{-1}x} \frac{1}{r^2(t)}\,dt\right)^{1/2} = 0.$$

From the last two equalities follows (24). Therefore, there exists $a_0 > 0$ such that $4C_a\gamma_{(a^{-4}\tilde{q},a^{-3}\tilde{r},1)}(a) \le 1$ for all $a \ge a_0$. From (23), it follows that

$$\|a^{-4}\tilde{q}\tilde{y}\|_2 \le \frac{1}{2}\|l_a\tilde{y}\|_2 \quad (a \ge a_0). \tag{24}$$

By theorem on small perturbations of linear operators, the operator $\tilde{L}_a\tilde{y} = l_a\tilde{y} + a^{-4}\tilde{q}(t)\tilde{y}(t)$ is closed and boundedly invertible. Thus, for each $\tilde{F}(t) \in L_2(\mathbb{R})$, equation (21) has a unique solution $\tilde{y}$. It remains to show the validity of estimate (20). It is easy to see that $\|l_a\tilde{y}\|_2 \le 2\|\tilde{L}_a\tilde{y}\|_2$. According to (22) and (24), $\|\tilde{y}^{(4)}\|_2 + \|a^{-3}\tilde{r}\tilde{y}'\|_2 + \|a^{-4}\tilde{q}\tilde{y}\|_2 \le (C_a + 1/2)\|l_a\tilde{y}\|_2$. From the last two inequalities we have $\|\tilde{y}^{(4)}(t)\|_2 + \|a^{-3}\tilde{r}\tilde{y}'\|_2 + \|a^{-4}\tilde{q}\tilde{y}\|_2 \le (2(C_a+1)\|\tilde{L}_a\tilde{y}\|_2$, or equivalently,

$$\|a^{-4}y^{(4)}(a^{-1}t)\|_2 + \|a^{-4}r(a^{-1}t)y'(a^{-1}t)\|_2 + \|a^{-4}q(a^{-1}t)y(a^{-1}t)\|_2 \le (2C_a+1)\|a^{-4}F(a^{-1}t)\|_2.$$

Putting $t = ax$, we obtain inequality (20), where $C = 2C_a + 1$. The theorem is proved.

6 Conclusion

The paper studies one fourth order three-term differential equation (1) with an intermediate term. A special case is considered where the intermediate term as an operator does not obey the differential operator formed by the extreme terms of the equation. Sufficient conditions for the existence of a strong solution to the equation, its uniqueness, and maximal regularity are shown. For this purpose, such methods as obtaining an a priori estimate of the solution, reducing the problem to the study of properties of one of the third-order differential operator with potential of constant sign, and estimation using local operators were used. The obtained conditions are specified in the form of an integral relation between the intermediate and small coefficients of the equation and allow us to cover a wide class of differential equations of the fourth order. The methods developed in the paper and the results obtained can be used in a qualitative study of singular differential equations of higher orders.

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SPECTRUM OF THE GENERALIZED CESÀRO OPERATOR ON LORENTZ SPACES

The aim of this paper is to investigate the boundedness and spectrum of generalized Cesàro operators defined on Lorentz spaces over a finite interval and the positive half-line. When $\beta = 1$, these operators coincide with the classical Cesàro operator. In this paper, we extend the results obtained for Sobolev spaces in [5] to Lorentz spaces. The primary tools employed in this work are C_0-groups, C_0-semigroups, and their generators. C_0-groups and C_0-semigroups are used to demonstrate the boundedness of the generalized Cesàro operator. Since the spectrum of the bounded linear operators is non-empty, we investigate the spectrum of the generalized Cesàro operator. The generators of these C_0-groups and C_0-semigroups are utilized to analyze the spectral properties of the generalized Cesàro operator. We study the spectra of the generators and determine the spectra of the generalized Cesàro operators using the spectral mapping theorem. Additionally, we provide results on the point spectrum of generalized Cesàro operators defined on Lorentz spaces over a finite interval.

Key words: Generalized Cesàro operator, spectrum, Lorentz $L_{p,q}$ spaces, C_0-group, C_0-semigroup.

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Лоренц кеңістіктерінде анықталған жалпыланған Чезаро операторының спектрі

Бұл мақалада ақырлы аралықта және оң жарты өсінде анықталған Лоренц кеңістігіндегі жалпыланған Чезаро операторларының шенелгендігі мен спектрі зерттеледі. $\beta = 1$ болған жағдайда, бұл операторлар классикалық Чезаро операторына сәйкес келеді. Бұл зерттеуде біз [5]-дағы Соболев кеңістіктеріне арналған нәтижелерді Лоренц кеңістігіне кеңейтеміз. Бұл жұмыста негізгі қолданылатын құралдар C_0-топтар, C_0-жартылай топтар және олардың туындатушы операторлары болып табылады. C_0-топтар мен C_0-жартылай топтар жалпыланған Чезаро операторының шенелгендігін дәлелдеуде қолданылады. Шенелген сызықтық операторлардың спектрі бос емес болғандықтан, біз жалпыланған Чезаро операторының спектрін зерттейміз. Осы C_0-топтар мен C_0-жартылай топтардың туындатушы операторлары жалпыланған Чезаро операторының спектрлік қасиеттерін талдауда пайдаланылады. Біз туындатушы операторлардың спектрлерін зерттеп, спектрлік бейнелеу теоремасы арқылы жалпыланған Чезаро операторларының спектрін анықтаймыз. Сонымен қатар, біз ақырлы аралықта анықталған Лоренц кеңістігінде жалпыланған Чезаро операторларының нүктелік спектрі бойынша нәтижелерді ұсынамыз.

Түйін сөздер: Жалпыланған Чезаро оператор, спектр, Лоренц $L_{p,q}$ кеңістіктер, C_0-топ, C_0-жартылай топ.

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Спектр обобщенного оператора Чезаро на пространствах Лоренца

В данной работе исследуются ограниченность и спектр обобщенных операторов Чезаро, определенных на пространствах Лоренца на конечном интервале и положительной полуоси. В случае, когда $\beta = 1$, эти операторы совпадают с классическим оператором Чезаро. В данном исследовании мы расширяем результаты, полученные для пространств Соболева в [5], на пространства Лоренца. Основными инструментами, используемыми в данной работе, являются C_0-группы, C_0-полугруппы и их порождающие операторы. C_0-группы и C_0-полугруппы используются для демонстрации ограниченности обобщенных операторов Чезаро. Поскольку спектр ограниченных линейных операторов не является пустым, мы изучаем спектр обобщенного оператора Чезаро. Порождающие операторы этих C_0-групп и C_0-полугрупп применяются для анализа спектральных свойств обобщенного оператора Чезаро. Мы изучаем спектры порождающих операторов и определяем спектры обобщенных операторов Чезаро с помощью теоремы спектрального отображения. Кроме того, мы представляем результаты по точечному спектру обобщенных операторов Чезаро, определенных на пространствах Лоренца на конечном интервале.

Ключевые слова: Обобщенный оператор Чезаро, спектр, пространства Лоренца $L_{p,q}$, C_0-группа, C_0-полугруппа.

1 Introduction

Let $\mathbb{R}_+ = (0, \infty)$. For $\beta > 0$, $1 < p < \infty$, and $1 \le q \le \infty$, the generalized Cesàro operators C^1_β and C^∞_β are defined on $L_{p,q}(0,1)$ and $L_{p,q}(\mathbb{R}_+)$, respectively, with the same formulas

$$(C^1_\beta f)(t) = \frac{\beta}{t^\beta} \int\limits_0^t (t-s)^{\beta-1} f(s) ds, \ t \in (0,1) \tag{1}$$

and

$$(C^\infty_\beta f)(t) = \frac{\beta}{t^\beta} \int\limits_0^t (t-s)^{\beta-1} f(s) ds, \ t \in \mathbb{R}_+. \tag{2}$$

The generalized Cesàro operator C^∞_β was first studied in [5] on Sobolev spaces which are contained in the Lebesgue spaces $L_p(\mathbb{R}_+)$. This work demonstrated the boundedness and spectral properties of the generalized Cesàro operator. Boundedness of the generalized Cesàro operator in L_p spaces confirmed by the following generalized Hardy inequality in [11]:

$$\left(\int\limits_0^\infty \left| \frac{\beta}{t^\beta} \int\limits_0^t (t-s)^{\beta-1} f(s) ds \right|^p dt \right)^{\frac{1}{p}} \le \frac{\Gamma(\beta+1)\Gamma(\frac{1}{p})}{\Gamma(\beta+\frac{1}{p})} \|f\|_{L_p}, \tag{3}$$

for $1 < p < \infty$, where Γ denotes the Gamma function. The discrete version of this operator was studied in [10]. In the special case when $\beta = 1$, this operator coincides with the classical Cesàro operator. For the boundedness and other properties such as spectrum of the Cesàro operator in different spaces, we refer the reader to [8], [9], [12], [13], [14], [15], [17], [18], [19].

The aim of this paper is to study boundedness and the spectrum of the generalized Cesàro operators C^∞_β and C^1_β on Lorentz spaces $L_{p,q}(\mathbb{R}_+)$ and $L_{p,q}(0,1)$, respectively. The main tools are so-called the C_0-group and C_0-semigroup, which are denoted by $\{T(t)\}_{t\in\mathbb{R}}$ and $\{S(t)\}_{t\in\mathbb{R}_+}$, and given by

$$(T(t)f)(s) = e^{-\frac{p}{t}} f\left(e^{-t}s\right), \ t \in \mathbb{R} \tag{4}$$

and

$$(S(t)f)(s) = e^{-\frac{p}{t}} f(e^{-t}s), \ t \in \mathbb{R}_+. \tag{5}$$

The idea comes from the papers [5], [10], where authors studied similar problems in Sobolev spaces.

The outline of the paper is as follows. In Section 2, we introduce a notion on the spectrum of linear bounded operators in (quasi-)Banach spaces and give definitions of Lorentz spaces as well as definitions of general C_0-group and C_0-semigroup with generators. In Section 3, we study the spectrum of generators of the C_0-group and C_0 -semigroup. Finally, in Section 4, we present the main results that include the boundedness and the spectrum of the generalized Cesàro operators on Lorentz spaces $L_{p,q}(\mathbb{R}_+)$ and $L_{p,q}(0,1)$, respectively.

2 Preliminaries

In this section, we give main definitions and properties of the spectral theory of linear operators, Lorentz spaces and operator semigroups. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+$ be the set of positive real numbers. As usual, $\mathbb{C}$ is the set of complex numbers, $\mathbb{C}_+$ and $\mathbb{C}_-$ are sets of complex numbers with positive and negative real parts, respectively. Throughout this paper, the closure of a set Ω is indicated by $\overline{\Omega}$.

2.1 Spectrum of linear operators and Lorentz spaces

Let X be a Banach space and $B(X)$ be the algebra of all bounded linear operators on X.

Definition 1 *[1] Let $A \in B(X)$. The resolvent set of A, denoted by $\rho(A)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ has a bounded linear inverse. For each $\lambda \in \rho(A)$, the resolvent operator is defined as*

$$R(\lambda, A) = (\lambda I - A)^{-1}.$$

The spectrum of A, denoted by $\sigma(A)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ does not have a bounded linear inverse.

One can define the different parts of the spectrum as follows:

Definition 2 *[1] Let $A \in L(X)$ a linear operator. The point spectrum, continuous and residual spectrum are defined as*

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \textit{ such that } \lambda I - A \textit{ is not injective}\},$$
$$\sigma_c(A) = \{\lambda \in \mathbb{C} \textit{ such that } \lambda I - A \textit{ is injective, } \overline{\mathrm{Im}(\lambda I - A)} = X, \textit{ but } \mathrm{Im}(\lambda I - A) \neq X\},$$
$$\sigma_r(A) = \{\lambda \in \mathbb{C} \textit{ such that } \lambda I - A \textit{ is injective, } \overline{\mathrm{Im}(\lambda I - A)} \neq X\}.$$

Clearly, $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$ are disjoint, and

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

In order to define the Lorentz spaces, we need the following notions of the distribution function and the decreasing rearrangement of a given measurable function. Let f be a Lebesgue measurable function defined on Ω with the Lebesgue measure ν, where Ω is either $\mathbb{R}_+$ or $(0,1)$.

Definition 3 *The function $\mu_f : [0;\infty) \to [0;\infty]$ defined by*

$$\mu_f(\lambda) = \nu\left\{t \in \mathbb{R} : |f(t)| > \lambda\right\}, \quad \lambda \geq 0$$

is called the distribution function of f.

Definition 4 *The function $f^* : [0;\infty) \to [0;\infty]$ defined by*

$$f^*(t) = \inf\left\{\lambda \geq 0 : \mu_f(\lambda) \leq t\right\}, \quad t \geq 0$$

is called the decreasing rearrangement of f.

Definition 5 *The function $f^{**} : (0,\infty) \to [0,\infty]$ is defined as*

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

We now present the main objective of this paper, the Lorentz spaces $L_{p,q}(\Omega)$.

Definition 6 *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. The Lorentz space $L_{p,q}(\Omega)$ is the set of all Lebesgue measurable functions f such that the functional $\|f\|_{L_{p,q}(\Omega)} < \infty$, where*

$$\|f\|_{L_{p,q}(\Omega)} = \begin{cases} \left(\int\limits_0^{\nu(\Omega)} \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 1 \leq p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t), & \text{if } 1 \leq p \leq \infty,\ q = \infty. \end{cases}$$

Note that for finite q the space $L_{\infty,q}(\Omega)$ is trivial. Furthermore, the Lorentz space $L_{p,q}(\Omega)$ is the generalization of the Lebesgue space $L_p(\Omega)$, which is quasi-Banach in general, and Banach for $1 \leq q \leq p < \infty$ or $p = q = \infty$, see for example, [2, IV.Theorem 4.2]. If $p = q$, then $L_{p,q}(\Omega)$ coincides with $L_p(\Omega)$ and

$$\|f\|_{L_{p,p}} = \|f\|_{L_p}, \quad f \in L_p(\Omega).$$

Definition 7 *For any $f \in L_{p,q}(\Omega)$ the norm $\|\cdot\|^*_{L_{p,q}(\Omega)}$ is defined by*

$$\|f\|^*_{L_{p,q}(\Omega)} = \begin{cases} \left(\int\limits_0^{\nu(\Omega)} \left(t^{\frac{1}{p}} f^{**}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 1 < p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^{**}(t), & \text{if } 0 < p \leq \infty,\ q = \infty. \end{cases}$$

According to [2], if $1 < p < \infty$, $1 \leq q \leq \infty$ or $p = q = \infty$, then $\| \cdot \|^*_{L_{p,q}(\Omega)}$ is a norm on $L_{p,q}(\Omega)$. This means that $(L_{p,q}(\Omega), \| \cdot \|^*_{L_{p,q}(\Omega)})$ is a Banach space. Moreover, the quasi-norms $\| \cdot \|_{L_{p,q}}$ and $\| \cdot \|^*_{L_{p,q}(\Omega)}$ are equivalent, as shown by the Hardy inequality:

$$\|f\|_{L_{p,q}} \leq \|f\|^*_{L_{p,q}} \leq \frac{p}{p-1}\|f\|_{L_{p,q}}.$$

If $p = \infty$, then we understand that $p/(p-1) = 1$.

A measurable locally bounded function $\omega : \mathbb{R} \to \mathbb{R}$ is a weight if it satisfies $\omega(t) \geq 1$ and $\omega(t + s) \leq \omega(t)\omega(s)$ for all $t, s \in \mathbb{R}$. The weight $\omega(t)$ is called non-quasianalytic if

$$\int_{\mathbb{R}} \frac{\ln \omega(t)}{1+t^2} dt < \infty.$$

Definition 8 *Let ω be a non-quasianalytic weight function on $\mathbb{R}$. The Beurling algebra $L^1_\omega(\mathbb{R})$ is the space of all integrable functions $f \in L_1(\mathbb{R})$ satisfying*

$$\|f\|_\omega = \int_{\mathbb{R}} |f(t)|\omega(t)dt < \infty.$$

2.2 Strongly continuous semigroup

Definition 9 *A family $T = \{T(t)\}_{t\in\mathbb{R}_+}$ in $B(X)$ is called a C_0-semigroup (or strongly continuous semigroup) if the following properties are satisfied:*

(i) $T(0) = I$, *where I is the identity operator on X;*

(ii) $T(t + s) = T(t)T(s)$, *for every $t, s \in \mathbb{R}_+$;*

(iii) $\lim\limits_{t\to 0} \|T(t)x - x\| \to 0$, *for all $x \in X$.*

If $s, t \in \mathbb{R}$ then $T = \{T(t)\}_{t\in\mathbb{R}}$ is called a C_0-group (or strongly continuous group).

The generator of $T = \{T(t)\}_{t\in\mathbb{R}_+}$ (or $\{T(t)\}_{t\in\mathbb{R}}$) is the linear operator A defined by

$$\mathcal{A}x = \lim_{h\to 0} \frac{T(h)x - x}{h}, x \in D(A),$$

where $D(A)$ is domain of operator $\mathcal{A}$.

Definition 10 *Let A be the generator of T. The spectral bound of A is defined by*

$$s(A) = \sup\{Re\lambda : \lambda \in \sigma(A)\}.$$

Definition 11 *Let $T = \{T(t)\}_{t\in\mathbb{R}_+}(\{T(t)\}_{t\in\mathbb{R}})$ be the strongly continuous semigroup(group), then*

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} : \exists M_\omega \geq 1 \text{ such that } \|T(t)\| \leq M_\omega e^{\omega t} \ \forall t \in \mathbb{R}_+ \,(\mathbb{R})\}$$

is called the growth bound of T.

The growth bound of the semigroup can also be determined by following formula

$$\omega_0(T) = \lim_{t\to\infty} \frac{\log \|T(t)\|}{t}.$$

Definition 12 *The open sector of angle ω is defined as*

$$S_\omega := \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \omega\},\ 0 < \omega \le \pi,$$
$$S_0 := (0,\infty),\ \omega = 0.$$

Definition 13 *Let $0 \le \omega < \pi$, the operator A on a Banach space X is called sectorial of angle ω if $\sigma(A) \subset S_\omega$ and*

$$\sup\left\{\|\lambda(\lambda - A)^{-1}\| : \lambda \notin \overline{S_{\omega'}}\right\} < \infty,$$

for all $\omega < \omega' < \pi$.

The following lemma is a key role in the calculations in the study of the spectrum of generators.

Lemma 1 *If $1 < p < \infty, 1 \le q \le \infty$, then*

(i) $t^\gamma \notin L_{p,q}(\mathbb{R}_+)$ *for* $\gamma \in \mathbb{C}$,

(ii) $(\alpha + t)^{-\gamma} \in L_{p,q}(\mathbb{R}_+)$ *for* $\mathrm{Re}\gamma > \dfrac{1}{p}$ *and* $\alpha > 0$,

(iii) $t^\gamma \in L_{p,q}(0,1)$ *for* $\mathrm{Re}\gamma > -\dfrac{1}{p}$.

Proof. First, we prove (i). By using the property of decreasing rearrangement, we have

$$((t^\gamma)^*)^q = (|t^\gamma|^q)^* = (t^{q\mathrm{Re}\gamma})^*.$$

Since $q \ge 1$, there are two possible cases for $\mathrm{Re}\gamma$, when $\mathrm{Re}\gamma \le 0$ and $\mathrm{Re}\gamma > 0$. Let us consider each situation separately. First, consider the case $\mathrm{Re}\gamma \le 0$. In this case, the function $t^{q\mathrm{Re}\gamma}$ is non-increasing. Its decreasing rearrangement is given by $(t^{q\mathrm{Re}\gamma})^* = t^{q\mathrm{Re}\gamma}$. Then

$$\|f\|^q_{L_{p,q}(\mathbb{R}_+)} = \int_0^\infty (t^{\frac{1}{p}}(t^{\mathrm{Re}\gamma})^*)^q \frac{dt}{t} = \int_0^\infty t^{\frac{q}{p}-1} t^{q\mathrm{Re}\gamma} dt = \int_0^\infty t^{\frac{q}{p}-1+q\mathrm{Re}\gamma} dt = \infty.$$

It means that $t^{\mathrm{Re}\gamma} \notin L_{p,q}(\mathbb{R}_+)$ with $\mathrm{Re}\gamma \le 0$.

If $\mathrm{Re}\gamma > 0$, then $t^{\mathrm{Re}\gamma}$ is increasing and $\mu_f(\lambda) = \infty$ for all $\lambda \ge 0$. Therefore, $t^{\mathrm{Re}\gamma} \notin L_{p,q}(\mathbb{R}_+)$.

Now, let us prove (ii). Here we also consider two cases when $\mathrm{Re}\gamma \ge 0$ and $\mathrm{Re}\gamma < 0$. First, let $\mathrm{Re}\gamma \ge 0$, then the function $(a+t)^{-q\mathrm{Re}\gamma}$ is non-increasing and

$$(((a+t)^{-\beta})^*)^q = (|(a+t)^{-\gamma}|^q)^* = ((a+t)^{-q\mathrm{Re}\gamma})^* = (a+t)^{-q\mathrm{Re}\gamma}.$$

Moreover, one has

$$\|f\|^q_{L_{p,q}(\mathbb{R}_+)} = \int\limits_0^\infty t^{\frac{q}{p}-1}(a+t)^{-q\mathrm{Re}\gamma}dt$$
$$= \int\limits_0^1 t^{\frac{q}{p}-1}(a+t)^{-q\mathrm{Re}\gamma}dt + \int\limits_0^\infty t^{\frac{q}{p}-1}(a+t)^{-q\mathrm{Re}\gamma}dt.$$

The integral converges under certain conditions on the parameters p, q and $\mathrm{Re}\gamma$. To determine when it converges, let us to analyse the behaviour of integral at the endpoints.

First, let $t \to 0^+$, then near $t = 0$, the term $(a+t)^{-\mathrm{Re}\gamma}$ approaches to $a^{-\mathrm{Re}\gamma}$, so the integral behaves as $\int\limits_0^\infty t^{\frac{q}{p}-1}dt$, and this integral converges if $\dfrac{q}{p} > 0$. Second, let $t \to \infty$, then for large t, the term $(a+t)^{-q\mathrm{Re}\gamma}$ behaves like $t^{-q\mathrm{Re}\gamma}$, so the integral behaves as $\int\limits_1^\infty t^{\frac{q}{p}-1-q\mathrm{Re}\gamma}$. This integral converges if $\dfrac{q}{p} - 1 - q\mathrm{Re}\gamma < -1$, that is, $\mathrm{Re}\gamma > \dfrac{1}{p}$.

In the case when $\mathrm{Re}\gamma < 0$, the function $(a+t)^{-q\mathrm{Re}\gamma}$ is increasing and $\mu_f(\lambda) = \infty$ for all $\lambda \geq 0$. Then $(a+t)^{-\mathrm{Re}\gamma} \notin L_{p,q}(\mathbb{R}_+)$.

Finally, we need to prove (iii). We consider two cases: $\mathrm{Re}\gamma \leq 0$ and $\mathrm{Re}\gamma > 0$.

First, let again $\mathrm{Re}\gamma \leq 0$. Here, the function $t^{\mathrm{Re}\gamma}$ is non-increasing, so $(t^{\mathrm{Re}\gamma})^* = t^{\mathrm{Re}\gamma}$ and the integral

$$\int\limits_0^1 t^{\frac{q}{p}-1}t^{\mathrm{Re}\gamma}dt = \int\limits_0^1 t^{\frac{q}{p}-1+q\mathrm{Re}\gamma}dt = \frac{t^{\frac{q}{p}+q\mathrm{Re}\gamma}}{q(\frac{1}{p}+\mathrm{Re}\gamma)}\Bigg|_0^1 < \infty, \text{ when } \mathrm{Re}\gamma > -\frac{1}{p}.$$

Now, let $\mathrm{Re}\gamma > 0$. In this case the decreasing rearrangement of $t^{\mathrm{Re}\gamma}$ is $f^*(t) = (1-t)^{\mathrm{Re}\gamma}, t \in (0,1)$. Then holds

$$\int\limits_0^1 t^{\frac{q}{p}-1}((t^{\mathrm{Re}\gamma})^*)^q dt = \int\limits_0^1 t^{\frac{q}{p}-1}(1-t)^{\mathrm{Re}\gamma}dt < \infty.$$

Thus, the function t^γ belongs to $L_{p,q}(0,1)$ if and only if $\mathrm{Re}\gamma > -\dfrac{1}{p}$.

3 Spectrum of generators of the C_0-group and C_0 -semigroup

The main result of this section is the following theorem.

Theorem 1 *For $1 < p < \infty$ and $1 \leq q \leq \infty$, the family of operators $T = \{T(t)\}_{t\in\mathbb{R}}$ is C_0-group of isometries on the space $L_{p,q}(\mathbb{R}_+)$. The generator $\mathcal{A}$ of this C_0-group is given by the following form:*

$$\mathcal{A}f(s) = -sf'(s) - \frac{1}{p}f(s),$$

with the domain $D(\mathcal{A}) = \{f \in L_{p,q}(\mathbb{R}_+) : tf' \in L_{p,q}(\mathbb{R}_+)\}$.

Similarly, the family $S = \{S(t)\}_{t\in\mathbb{R}_+}$ *is* C_0*- semigroup on the space* $L_{p,q}(0,1)$*. The generator* $\mathcal{B}$ *of this* C_0*-semigroup is given by*

$$\mathcal{B}f(s) = -sf'(s) - \frac{1}{p}f(s),$$

with domain $D(\mathcal{B}) = \{f \in L_{p,q}(0,1) : tf' \in L_{p,q}(0,1)\}$.

Proof. We begin with the checking that the operators $\{T(t)\}_{t\in\mathbb{R}}$ are isometries. The following equation holds:

$$\begin{aligned}\|T(t)f\|_{L_{p,q}(\mathbb{R}_+)} &= \left(\int_0^\infty (s^{\frac{1}{p}}(T(t)f)^*(s))^q \frac{ds}{s}\right)^{\frac{1}{q}} = \left(\int_0^\infty (s^{\frac{1}{p}} e^{-\frac{t}{p}} f^*(e^{-t}s))^q \frac{ds}{s}\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty ((e^s u)^{\frac{1}{p}} e^{-\frac{s}{p}} f^*(u))^q \frac{du}{u}\right)^{\frac{1}{q}} = \|f\|_{L_{p,q}(\mathbb{R}_+)}.\end{aligned}$$

In the case, when $q = \infty$, we have

$$\begin{aligned}\|(T(s)f)(t)\|_{L_{p,\infty}(\mathbb{R}_+)} &= \sup_{t>0} t^{\frac{1}{p}}((T(s)f)(t))^* = \sup_{t>0} t^{\frac{1}{p}} e^{-\frac{s}{p}} f^*(e^{-s}t) \\ &= \sup_{u>0} u^{\frac{1}{p}} f^*(u) = \|f\|_{L_{p,\infty}(\mathbb{R}_+)}.\end{aligned}$$

To show that the family of operators $\{T(t)\}_{t\in\mathbb{R}}$ forms a C_0-group, we have to show that

$$\lim_{t\to 0} \|T(t)f - f\|_{L_{p,q}(\mathbb{R}_+)} = 0.$$

for each $f \in L_{p,q}(\mathbb{R}_+)$.

We first verify this property for $C_c^\infty(\mathbb{R}_+)$ by proving the next

$$\begin{aligned}\lim_{t\to 0} \|T(t)f - f\|_\infty &= \lim_{t\to 0} \sup_{x>0} |(T(t)f)(x) - f(x)| \\ &= \lim_{t\to 0} \sup_{x>0} |e^{-\frac{t}{p}} f(e^{-t}x) - f(x)| = 0.\end{aligned}$$

Since $C_c^\infty(\mathbb{R}_+)$ is dense in $L_{p,q}(\mathbb{R}_+)$ [4, Theorem 3.3], it follows that

$$\lim_{t\to 0} \|T(t)f - f\|_{L_{p,q}(\mathbb{R}_+)} = 0.$$

By definition of generator of the C_0-group for every $f \in D(A)$ we get following

$$\begin{aligned}\mathcal{A}f(s) &= \lim_{t\to 0} \frac{T(t)f(s) - f(s)}{t} = \lim_{t\to 0} \frac{e^{-\frac{t}{p}} f(e^{-t}s) - f(s)}{t} \\ &= -sf'(s) - \frac{1}{p}f(s),\end{aligned}$$

with $D(\mathcal{A}) = \{f \in L_{p,q}(\mathbb{R}_+) : tf' \in L_{p,q}(\mathbb{R}_+)\}$.

Next, we will prove that the family of operators $\{S(t)\}_{t\in\mathbb{R}_+}$ is C_0-semigroup. First, we show that it is bounded for each t in the following

$$\|S(t)f\|_{L_{p,q}(0,1)} = \left(\int_0^1 (s^{\frac{1}{p}}(S(t)f)^*(s))^q \frac{ds}{s}\right)^{\frac{1}{q}} = \left(\int_0^1 (s^{\frac{1}{p}}e^{-\frac{t}{p}}f^*(e^{-t}s))^q \frac{ds}{s}\right)^{\frac{1}{q}}$$
$$= \left(\int_0^{e^{-t}} ((e^s u)^{\frac{1}{p}}e^{-\frac{s}{p}}f^*(u))^q \frac{du}{u}\right)^{\frac{1}{q}} \leq \|f\|_{L_{p,q}(0,1)}.$$

Using a similar argument as above, it follows that this semigroup is a C_0-semigroup, and its generator is given by

$$\mathcal{B}f(s) = -sf'(s) - \frac{1}{p}f(s),$$

with domain $D(\mathcal{B}) = \{f \in L_{p,q}(0,1) : tf' \in L_{p,q}(0,1)\}$.

In the following proposition, we find the spectrum of the generators of the C_0-group and C_0-semigroup.

Proposition 1 *Let $1 < p < \infty$ and $1 \leq q \leq \infty$, then*

i) $\sigma_p(\mathcal{A}) = \emptyset, \quad \sigma(\mathcal{A}) = i\mathbb{R}$.

ii) $\sigma_p(\mathcal{B}) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}, \quad \sigma(\mathcal{B}) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\}$.

Proof. i) Let $\lambda \in \mathbb{C}$. The equation $\mathcal{A}f = \lambda f$ is equivalent to the differential equation

$$tf'(t) + (\lambda + \frac{1}{p})f(t) = 0.$$

Its non-zero solutions are given by $f(t) = ct^{-(\lambda+\frac{1}{p})}$ with $c \neq 0$. According to Lemma 1, these solutions do not belong to $L_{p,q}(\mathbb{R}_+)$. Therefore, $\mathcal{A}$ has no eigenvalues, and the point spectrum is empty: $\sigma_p(\mathcal{A}) = \emptyset$.

Since each $T(s)$ is an invertible isometry, its spectrum is confined to the unit circle:

$$\sigma(T(t)) \subseteq \{z \in \mathbb{C} : |z| = 1\}.$$

By the spectral mapping theorem for C_0-group (see [6, IV. Theorem 3.6]), the relation $e^{t\sigma(\mathcal{A})} \subseteq \sigma(T(t))$ holds. Hence, if $\eta \in \sigma(\mathcal{A})$, it follows that $e^{t\eta} \in \{z \in \mathbb{C} : |z| = 1\}$, implying $\sigma(\mathcal{A}) \subseteq i\mathbb{R}$.

Assume $\xi \in i\mathbb{R}$ and that $\xi \in \rho(\mathcal{A})$. Let $\eta = \xi + \frac{1}{p}$. According to Lemma 1, the function $f(t) = (1+t)^{-\eta-1}$ lies in $L_{p,q}(\mathbb{R}_+)$. Since the resolvent operator $R(\xi, \mathcal{A})$ is bounded, the function $g(t) = R(\xi, \mathcal{A})f(t)$ also belongs to $L_{p,q}(\mathbb{R}_+)$. This implies that $g(t)$ satisfies the differential equation

$$\eta g(t) + tg'(t) = f(t).$$

Solving this equation yields the general solution

$$\widetilde{g}(t) = ct^{-\eta} + \frac{1}{\eta}(1+t)^{-\eta},$$

where c is a constant. However, as in Lemma 1, it can be confirmed that $\widetilde{g}(t) \notin L_{p,q}(\mathbb{R}_+)$. Thus $\xi \in \sigma(\mathcal{A})$.

ii) In this case, we first examine the equation $\mathcal{B}f = \lambda f$. The solution to this equation is given by $f(s) = s^{-(\lambda - \frac{1}{p})}$. We can see by Lemma 1 that the function $f(s) = s^{-(\lambda + \frac{1}{p})}$ belongs to $L_{p,q}(0,1)$ if and only if $\mathrm{Re}\lambda < 0$.

We know from [6, Corollary 1.13] that

$$-\infty \leq s(\mathcal{B}) \leq \omega_0(S) < \infty.$$

By definition,

$$\omega_0(S) = \lim_{t\to\infty} \frac{\log \|S(t)\|}{t} = 0.$$

Considering these facts, the spectrum is given by

$$\sigma(\mathcal{B}) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq 0\}.$$

4 Spectrum of Generalized Cesàro operators on Lorentz spaces

In this section, we establish the spectrum of the generalized Cesàro operator on Lorentz spaces.

4.1 The case $\mathbb{R}_+$

The following result shows the boundedness of the generalized Cesàro operator C_β^∞ on $L_{p,q}(\mathbb{R}_+)$ spaces.

Theorem 2 *Let $\beta > 0$, $1 < p < \infty$ and $1 \leq q \leq \infty$, then the operator C_β^∞ is bounded on $L_{p,q}(\mathbb{R}_+)$.*

If $f \in L_{p,q}(\mathbb{R}_+)$, then

$$C_\beta^\infty f(t) = \beta \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})}(s) f(t) ds. \tag{6}$$

Proof. Let us first demonstrate the equality (6). By changing the variable $\tau = te^{-s}$, we obtain the following

$$C_\beta^\infty f(t) = \frac{\beta}{t^\beta} \int_0^t (t-\tau)^{\beta-1} f(\tau) ds = \beta \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} T(s) f(t) ds.$$

Considering the density of simple functions in the $L_{p,q}(\mathbb{R}_+)$ space and utilizing the properties of the Bochner integrable functions, we can observe that the operator C_β^∞ is well-defined and bounded on $L_{p,q}(\mathbb{R}_+)$. When $1 \le q \le p < \infty$, $p \ne 1$, then, we have

$$\begin{aligned}\|C_\beta^\infty f\|_{L_{p,q}(\mathbb{R}_+)} &\le \beta \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} \|T(s)f\| ds \\ &= \beta \|f\|_{L_{p,q}(\mathbb{R}_+)} \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} ds = \beta \|f\|_{L_{p,q}(\mathbb{R}_+)} \int_0^1 (1-u)^{\beta-1} u^{1-\frac{1}{p}} \frac{du}{u} \\ &= \beta \|f\|_{L_{p,q}(\mathbb{R}_+)} \int_0^1 (1-u)^{\beta-1} u^{1-\frac{1}{p}-1} du = \|f\|_{L_{p,q}(\mathbb{R}_+)} \frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p})}{\Gamma(\beta+1-\frac{1}{p})}.\end{aligned}$$

Here, the Beta function is applied to evaluate the integral. In general case, when $1 < p < q \le \infty$ by [18], we get that C_β^∞ is bounded on $L_{p,q}(\mathbb{R}_+)$ with respect to $\|\cdot\|^*_{L_{p,q}}$. Therefore, we have

$$\|C_\beta^\infty f\|_{L_{p,q}(\mathbb{R}_+)} \le c_{\beta,p} \|f\|_{L_{p,q}(\mathbb{R}_+)},$$

where $c_{\beta,p} > 0$ is a constant depending only on β and p.

The first main result is the following theorem.

Theorem 3 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $\beta > 0$. For the operator C_β^∞ on $L_{p,q}(\mathbb{R}_+)$ we have*

$$\sigma(C_\beta^\infty) = \overline{\left\{ \frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p}+it)}{\Gamma(\beta+1-\frac{1}{p}+it)} : t \in \mathbb{R} \right\}}.$$

Proof. In the previous theorem, we demonstrated that the operator C_β^∞ can be expressed in terms of the semigroup $T(t)$, i.e.,

$$C_\beta^\infty f(t) = \beta \int_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} T(s) f(t) ds = \int_{-\infty}^\infty g_{\beta,p}(s) T(s) f(t) ds,$$

where $g_{\beta,p}(s) = \chi_{[0,\infty)}(s)\beta(1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}$ for $s \in \mathbb{R}$. According to [7], if the function $g_{\beta,p}$ belongs to the space $L^1_\omega(\mathbb{R})$, then it follows that

$$\sigma(C_\beta^\infty) = \overline{\widehat{g_{\beta,p}}(\sigma(i\mathcal{A}))},$$

where $\widehat{g_{\beta,p}}$ is the Fourier transform of the function $g_{\beta,p}$. In our case, the non-quasianalytic weight is equal to 1. Therefore, it is straightforward to verify that $g_{\beta,p} \in L^1_1(\mathbb{R})$ due to the properties of the Beta function.

Then, for $t \in \sigma(i\mathcal{A}) = \mathbb{R}$ (see Proposition 1) we have

$$\widehat{g_{\beta,p}}(\lambda) = \beta \int_0^\infty e^{-i\lambda s}(1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}ds = \beta \int_0^1 (1-u)^{\beta-1}u^{1-\frac{1}{p}+it-1}du$$
$$= \beta B(\beta, 1-\frac{1}{p}+it) = \frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p}+it)}{\Gamma(\beta+1-\frac{1}{p}+it)}.$$

4.2 The case $(0,1)$

Let $L_{p,q}(0,1)$. In contrast to the previous subsection, we now describe the spectrum of the generalized Cesàro operator on $L_{p,q}(0,1)$. The main result of this section is given in the following theorem.

Theorem 4 *Let $\beta > 0$, $1 < p < \infty$ and $1 \le q \le \infty$, then the operator C_β^1 is bounded on $L_{p,q}(0,1)$.*

If $f \in L_{p,q}(0,1)$, then

$$C_\beta^1 f(t) = \beta \int_0^\infty (1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}S(s)f(t)ds. \tag{7}$$

Proof. We apply the change of variable $\tau = te^{-s}$ to obtain the following

$$C_\beta^1 f(t) = \frac{\beta}{t^\beta}\int_0^t (t-\tau)^{\beta-1}f(\tau)ds = \beta\int_0^\infty (1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}S(s)f(t)ds.$$

It proves the equality (7).

Note that, due to this equality, C_β^1 is well-defined and acts as a bounded operator on $L_{p,q}(0,1)$ for $1 \le q \le p < \infty$, $p \ne 1$, then

$$\begin{aligned}
\|C_\beta^1 f\|_{L_{p,q}(0,1)} &\le \beta\int_0^\infty (1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}\|S(s)f\|_{L_{p,q}(0,1)}ds \\
&\le \beta\|f\|_{L_{p,q}(0,1)}\int_0^\infty (1-e^{-s})^{\beta-1}e^{-s(1-\frac{1}{p})}ds = \beta\|f\|_{L_{p,q}(0,1)}\int_0^1 (1-u)^{\beta-1}u^{1-\frac{1}{p}}\frac{du}{u} \\
&= \beta\|f\|_{L_{p,q}(0,1)}\int_0^1 (1-u)^{\beta-1}u^{1-\frac{1}{p}-1}du = \|f\|_{L_{p,q}(0,1)}\frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p})}{\Gamma(\beta+1-\frac{1}{p})}.
\end{aligned}$$

As in Theorem 4, in general case when $1 < p < q \le \infty$, we have

$$\|C_\beta^1 f\|_{L_{p,q}} \le c_{\beta,p}\|f\|_{L_{p,q}},$$

where $c_{\beta,p} > 0$ is a constant depending only on β and p.

Theorem 5 *Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $\beta > 0$. For the operator C^1_β on $L_{p,q}(0,1)$ we have*

$$\sigma_p(C^1_\beta) = \left\{ \frac{\Gamma(\beta+1)\Gamma(\lambda+1-\frac{1}{p})}{\Gamma(\beta+\lambda+1-\frac{1}{p})} : \lambda \in \mathbb{C}_+ \right\}$$

and

$$\sigma(C^1_\beta) = \overline{\left\{ \frac{\Gamma(\beta+1)\Gamma(\lambda+1-\frac{1}{p})}{\Gamma(\beta+\lambda+1-\frac{1}{p})} : \lambda \in \mathbb{C}_+ \cup i\mathbb{R} \right\}}.$$

Proof. Define the function

$$h_\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}, \ \ \gamma \in \mathbb{C}.$$

The functions h_γ are eigenfunctions of the operator C^1_β, it means that

$$(C^1_\beta h_\gamma)(t) = \frac{\beta}{\Gamma(\gamma)t^\gamma} \int\limits_0^t (t-s)^{\beta-1} s^{\gamma-1} ds = \frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} h_\gamma(t).$$

According to Lemma 1, the function h_γ belongs to $L_{p,q}(0,1)$ if and only if $\mathrm{Re}\gamma - 1 > -\frac{1}{p}$. It follows that the point spectrum of the operator C^1_β in $L_{p,q}(0,1)$ is the set

$$\sigma_p(C^1_\beta) = \left\{ \frac{\Gamma(\beta+1)\Gamma(\lambda+1-\frac{1}{p})}{\Gamma(\beta+\lambda+1-\frac{1}{p})} : \lambda \in \mathbb{C}_+ \right\}.$$

Next, we consider the Hille-Phillips functional calculus for the generator $\mathcal{B}$ of the semigroup $S = \{S(t)\}_{t\geq 0}$. According to Theorem 4, we can write $C^1_\beta = \mathcal{L}(g)(-\mathcal{B})$ that is

$$C^1_\beta f = \beta \int\limits_0^\infty (1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} S(s) f(t) ds = \int\limits_0^\infty g_{\beta,p}(t) S(t) f dt = \mathcal{L}(g_{\beta,p})(-\mathcal{B}) f,$$

where $g_{\beta,p}(t) = \beta(1-e^{-t})^{\beta-1} e^{-t(1-\frac{1}{p})}$ and $\mathcal{L}$ is the Laplace transform.

$$\begin{aligned} \mathcal{L}(g_{\beta,p})(z) &= \beta \int\limits_0^\infty e^{-zs}(1-e^{-s})^{\beta-1} e^{-s(1-\frac{1}{p})} ds \\ &= \frac{\Gamma(\beta+1)\Gamma(z+1-\frac{1}{p})}{\Gamma(\beta+z+1-\frac{1}{p})} = h_{\beta,p}(z), \ \ z \in \overline{\mathbb{C}_+}. \end{aligned}$$

By [10, p. 1458], the function $h_{\beta,p}$ satisfies Spectral Mapping Theorem [16, Theorem 2.7.8]. Since $-\mathcal{B}$ is a sectorial operator of angle $\frac{\pi}{2}$ and $\mathcal{B}$ is injective $(0 \notin \sigma_p(\mathcal{B}))$, we have

$$\sigma(C^1_\beta) = \sigma(h_{\beta,p}(-\mathcal{B})) = \overline{h_{\beta,p}(\sigma(-\mathcal{B}))} = \overline{\left\{ \frac{\Gamma(\beta+1)\Gamma(\lambda+1-\frac{1}{p})}{\Gamma(\beta+\lambda+1-\frac{1}{p})} : \ \lambda \in \mathbb{C}_+ \cup i\mathbb{R} \right\}}.$$

4.3 Conclusion

In this paper, we studied the boundedness and spectral properties of the generalized Cesàro operators C_β^1 and C_β^∞ defined on the Lorentz spaces $L_{p,q}(0,1)$ and $L_{p,q}(\mathbb{R}_+)$, respectively. Using tools such as the C_0-group $\{T(t)\}_{t\in\mathbb{R}}$ and the C_0-semigroup $\{S(t)\}_{t\in\mathbb{R}_+}$, we analyzed the boundedness and spectrum of these operators. The spectral properties of the generators of these groups and semigroups were studied, which played a central role in determining the spectrum of the generalized Cesàro operators. The main results demonstrated that the generalized Cesàro operators are bounded on Lorentz spaces and provided a detailed characterization of their spectra.

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NUMERICAL MODELING OF OPTIONS IN DIFFUSION (B, S) STOCK MARKETS

This article examines the calculation of the option price $V(t, x)$, the stock price $x(t)$, and the optimal stopping (execution) time τ; $(\equiv t)$ over both finite and infinite time horizons. It then delves into determining a fair value for American-style options, leveraging the optimal stopping time within the framework of diffusion processes in stock markets, represented by (B, S). Additionally, the article explores the pricing of European-style options, starting with the buyer's perspective and then transitioning to the seller's viewpoint. The problems are solved either analytically, when the optimal stopping time is pre-determined, or numerically using methods like the sweep method and finite element techniques.These methods are applied by reducing the problem to Stefan's problem, where $Y^*(t, x)$ represents the rational option value, τ_T^* indicates the rational execution time, and $x^*(t)$ corresponds to the rational stock price.
Key words: option prices, stock prices, equity diffusion markets, options of American and European types, Stefan's problem, numerical modeling.

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Диффузиондық (B, S) акциялар нарығындағы опциондарды сандық модельдеу

Бұл мақалада опцион бағасын $V(t, x)$, акция бағасын $x(t)$ және тиімді тоқтату (орындау) уақытын τ $(\equiv t)$ ақырлы және шексіз уақыт аралықтарында есептеудің ерекшеліктері талқыланады. Кейінірек диффузиялық (B, S) – нарықтарында тиімді тоқтату мезетін ескере отырып, Американдық типтегі опциондардың әділ бағасын анықтау мәселесі қарастырылады. Одан әрі, Европалық типтегі опциондардың бағасын дұрыс есептеу мәселесі зерттеледі. Алдымен, опцион сатып алушының көзқарасынан қарастырылып, оның опционы талданады, содан соң сатушының опционы қарастырылады. Барлық есептер, егер тиімді тоқтату уақыты алдын ала белгілі болса, дәл шешілуі мүмкін немесе сандық әдістер арқылы — қуалау және ақырлы элементтер әдісін пайдаланып, Стефан есебіне келтіріліп, $Y^*(t, x)$ – опционның рационалды құны, τ_T^* – тиімді орындау уақыты және $x^*(t)$ – акция бағасының рационалды мәні бойынша шешілуі мүмкін.
Түйін сөздер: опцион бағасы, акция бағасы, шындық диффузиялық нарық, Американдық және Европалық типті опциондар, Стефан есебі, сандық модельдеу.

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Численное моделирование опционов на диффузионных (B, S)-рынках акций

В данной статье рассматриваются особенности вычисления стоимости опциона $V(t, x)$, цены акции $x(t)$ и оптимального момента остановки (или исполнения) τ $(\equiv t)$ как для конечных, так и для бесконечных временных интервалов. Далее изучается задача определения справедливой стоимости опционов американского типа на основе оптимального момента остановки, в контексте диффузионных рынков акций (B, S). Затем обсуждается проблема определения рациональной цены опционов Европейского типа. Вначале рассматривается ситуация с точки зрения покупателя опциона, после чего анализируется ситуация с точки зрения продавца. Все поставленные задачи решаются точно, если заранее найден оптимальный момент остановки, либо численно — с использованием методов прогонки и конечных элементов, путем преобразования их в задачу Стефана, где $Y^*(t, x)$ представляет собой рациональную цену опциона, τ_T^* – оптимальный момент исполнения, а $x^*(t)$ – рациональную цену акции.
Ключевые слова: цена опциона, цена акции, справедливый диффузионный рынок, опционы Американского и Европейского типов, задача Стефана, численное моделирование.

1 Introduction

Building on the work in [1], this paper investigates various aspects of calculating the option price $V(t,x)$, the stock price $x(t)$, and the optimal stopping (execution) time τ $(\equiv t)$ over finite and infinite time intervals. In then explores the determination of a fair price for American-style options, utilizing the optimal stopping time within diffusion-based (B, S)-stock market models. The discussion proceeds to address the pricing of European-style options, starting with an analysis on the buyer's perspective, particularly the call option, followed by a focus on the put option. The problems are solved either exactly when the optimal stopping time is predetermined or numerically by reformulating them into the Stefan problem. In mathematical physics, the Stefan problem arises in the study of physical processes associated with the phase transformation of matter and consists in finding a function $u = u(t,x)$ that describes the temperature regime of the phases and the separation boundary $x = x(t),\ t \geq 0$ of these phases.

In the case of standard buyer and seller options, a two-phase situation also takes place – when searching for optimal stopping rules, we can restrict ourselves to considering only two simply connected phases: the area of continuation of observations C^T and the area D^T.

All problems can be solved analytically if the optimal stopping time is known beforehand or numerically if it is not.

The results of numerical modeling of the Stefan's problem by the sweep method and the finite element method for standard call and ask options are presented. As well as a comparative analysis of the numerical results by the sweep method and the finite element method (FEA).

Statement of the problem from the book [1]. Standard American-type buyer and seller options and optimal process stopping are considered in the works [2], [3], [4], [5], [6], [7]. Numerical methods for solving the Stefan problem and other numerical methods for solving stochastic (diffusion) partial differential equations are considered in the works [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18].

2 Numerical solution of the Stefan's problem for the call option

In the area $C^T = \Big\{ \big(t,\, x\big) :\ x < x^*(t),\ t \in \big[0,\, T\big) \Big\}$ consider the equation

$$-\frac{\partial Y^*(t,\, x)}{\partial t} + \beta Y^*(t,\, x) = \mathrm{L}Y^*(t,\, x), \tag{1}$$

where $\beta = \lambda + r,\ \mathrm{L}Y^*(t,\, x) = rx\dfrac{\partial Y^*(t,\, x)}{\partial x} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 Y^*(t,\, x)}{\partial x^2}$ and in the area $D^T \cup \Big\{ \big(T,\, x\big) :$ $x \in E \Big\}$ consider

$$Y^*(t,\, x) = g(x) \tag{2}$$

at the boundary $x^* = x^*(t),\ 0 \le t < T$, the section of "two phases the Dirichlet condition is fulfilled:

$$Y^*\big(t,\, x^*(t)\big) = g\big(x^*(t)\big); \tag{3}$$

and the Neumann condition:

$$\left.\frac{\partial Y^*(t,\, x)}{\partial x}\right|_{x \uparrow x^*(t)} = \left.\frac{dg(x)}{dx}\right|_{x \downarrow x^*(t)}. \tag{4}$$

Let us discretize the phase domain $C^T = \Big\{ (t,\, x) :\ x < x^*(t),\ t \in \big[0,\, T\big) \Big\}$ with t respect to the step τ, $t_n = n\tau,\ n = 0,\, 1,\, 2,\, \ldots,\, N,\ \tau = \frac{T}{N}$, with x respect to the step h, $x_i = ih,\ i = 0,\, 1,\, \ldots$. We also discretize the area $D^T \cup \Big\{ \big(T,\, x\big) :\ x \in E \Big\}$. Omit index * above $Y^*(t,\, x)$). We approximate (1) by an implicit scheme, and for the discrete domain C^T_{ni} we obtain the difference equation

$$\alpha_i Y^{n+1}_{i+1} + \beta_i Y^{n+1}_i + \gamma_i Y^{n+1}_{i-1} = -\frac{1}{\tau} Y^n_i, \tag{5}$$

where $\alpha_i = -\Big(\frac{rx_i}{h} + \frac{\sigma^2 x_i^2}{2h^2}\Big)$, $\beta_i = \Big(\frac{rx_i}{h} + \frac{\sigma^2 x_i^2}{h^2} + \beta - \frac{1}{\tau}\Big)$, $\gamma_i = -\frac{\sigma^2 x_i^2}{2h^2}$. In the discrete domain D^T_{ni}, we write the Dirichlet condition (3):

$$Y^n_i = g_i \ \text{ or } \ Y^n\big(x^*_i\big) = g(x^*_i\big) \tag{6}$$

and the Neumann condition (4):

$$\left.\frac{Y^n_{i+1} - Y^n_i}{h}\right|_{x_i = x^*_i + 0} = \left.\frac{g_{i+1} - g_i}{h}\right|_{x_i = x^*_i - 0}. \tag{7}$$

If the condition $\beta > \frac{1}{\tau}$, is satisfied, equation (5) can be solved, for instance, using the sweep method. To solve (5) we ensure that at the boundary $\left(x^*\right)_i^n$, the two-phase conditions (6) and (7) are met. At each step, we verify that the "front" $\left(x^*\right)_i^n$ is defined at a grid point. If not, we can adjust the step sizes τ and h.

Next, issues of numerical modeling of the Stefan's problem are considered. [8], [9], [10], [17], [18].

3 Numerical modeling by the method of double-sweep method the Stefan's problem for standard call options

As a first example for a call option, we consider an American call option with financial variables $K = 10$, $\sigma = 0.6$, $r = 0.25$, $\delta = 0.2$, $x_0 = 10$ and $T = 1$. For these data, proposes a value of $Y^*\big(0,\,K\big) = 2.18728$, which corresponds to the variant shown in following Figure 1.

The selected values for the test data $\big(K = 10,\ \sigma = 0.6,\ r = 0.25,\ \delta = 0,\ T = 1\big)$ are shown in Table 1 and visualized in Figure 2.

Table 1: Values $Y^*(0,x)$ of the American call option at K=10, σ=0.6, r=0.25, δ=0, T=1.

m (grid dimension), $Y^*_{\max}$	Sweep method	Finite element method (FEM)
100	2.181171	2.186701
200	2.186031	2.187181
400	2.186941	2.187251
800	2.187191	2.187271
1600	2.187261	2.187281

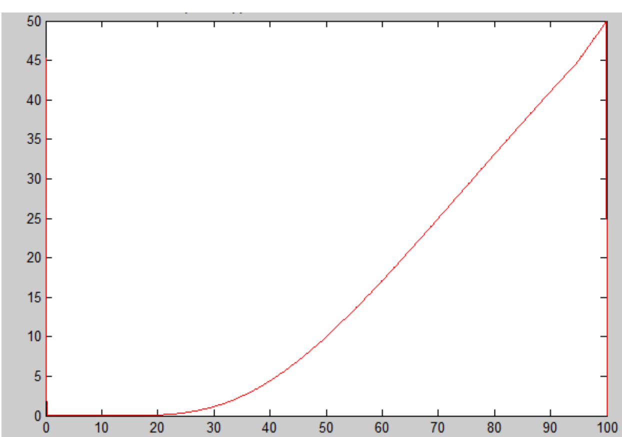

Figure 1: American call option pricing for $K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $T = 5/12$ (grid dimension 1600×1600).

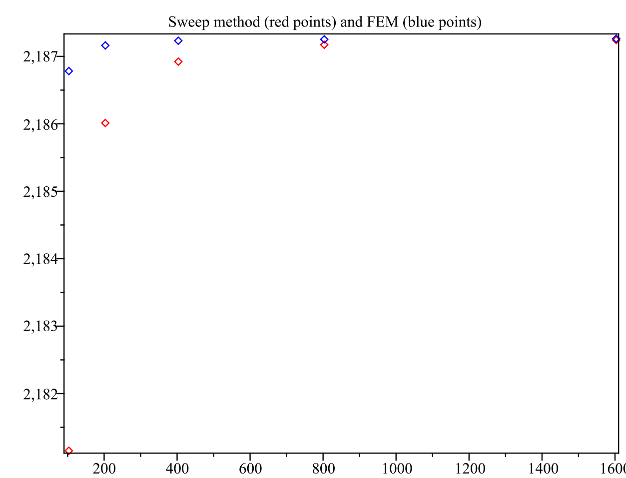

Figure 2: Convergence of the price an American call option $Y^*(0,\,x)$ $\big(K = 10,\ \sigma = 0.6,\ r = 0.25,\ \delta = 0.2,\ T = 1\big)$ with grid dimension.

As a second example for call option, we consider an American call option with financial variables $K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $x_0 = 50$ and $T = 5/12$. For these data proposes a value of $Y^*\big(0,\,K\big) = 21.28638$, which corresponds to the variant shown in Figure 3.

The selected values for the test data $\big(K = 50,\ \sigma = 0.4,\ r = 0.1,\ \delta = 0,\ T = 5/12\big)$ are shown in Table 2 and visualized in Figure 4.

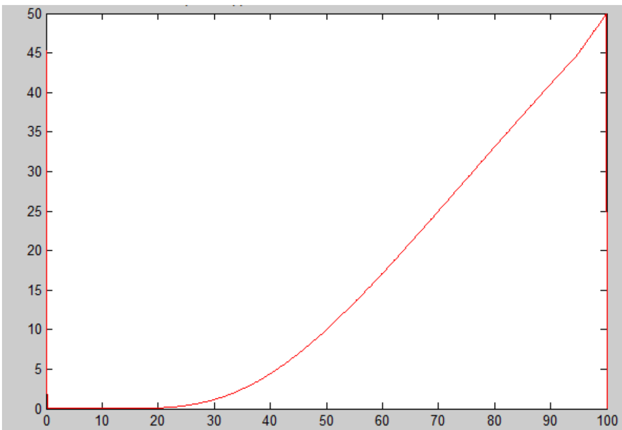

Figure 3: American call option pricing for $K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $T = 5/12$ (grid dimension 1600×1600).

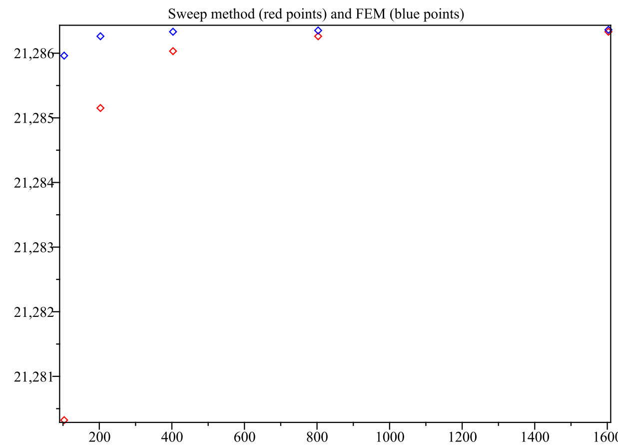

Figure 4: Convergence of the price of an American call option $Y^*(0, x)$ $\big(K = 50, \sigma = 0.4, r = 0.1,\ \delta = 0,\ T = 5/12\big)$ with increasing grid dimension.

Table 2: Values $Y^*(0, x)$ of the American call option at K=50, σ=0.4, r=0.1, δ=0, T=5/12.

m (grid dimension), $Y^*_{\max}$	Sweep method	Finite element method (FEM),
100	21.280341	21.285981
200	21.285171	21.286281
400	21.286051	21.286351
800	21.286281	21.286371
1600	21.286351	21.286381

4 Numerical modeling by the method of running the Stefan's problem for standard put options

As the first example for a put option, we consider an American put option with financial variables $K = 10$, $\sigma = 0.6$, $r = 0.25$, $\delta = 0.2$, $x_0 = 10$, $T = 1$, which is shown in Figure 5.

This curve, shown in Figure 6, defines the option's early exercise strategy.

As a second example for a put option, we consider an American put option with financial variables K=50, σ=0.4, r=0.1, δ=0, x_0=50 and T=5/12, which is shown in Figure 7. We also calculate the point $x^*(0)$ for early exercise of the put option. Numerical results are given in Table 3 and illustrated in Figure 8.

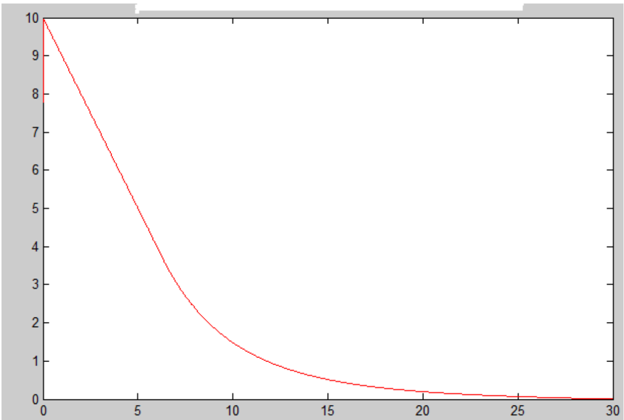

Figure 5: American put option pricing function $Y^*(0, x)$ for $K = 10$, $\sigma = 0.6$, $r = 0.25$, $\delta = 0.2$, $x_0 = 10$, $T = 1$ (grid dimension 1600×1600).

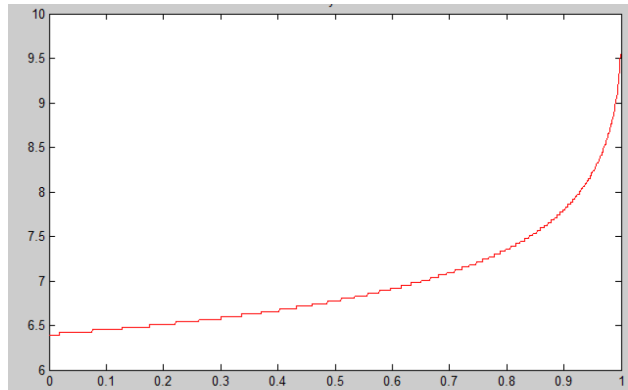

Figure 6: Time structure of the early exercise boundary $x^*(t)$ of a put option $K = 10$, $\sigma = 0.6$, $r = 0.25$, $\delta = 0$, $T = 1$.

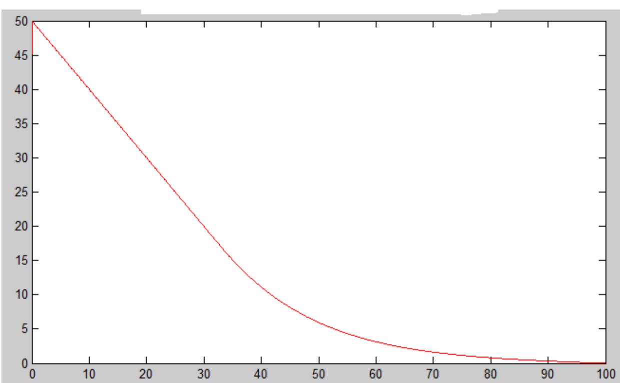

Figure 7: The pricing function $Y^*(0, x)$ of an American put option at $K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $T = 5/12$ (grid dimension 1600×1600).

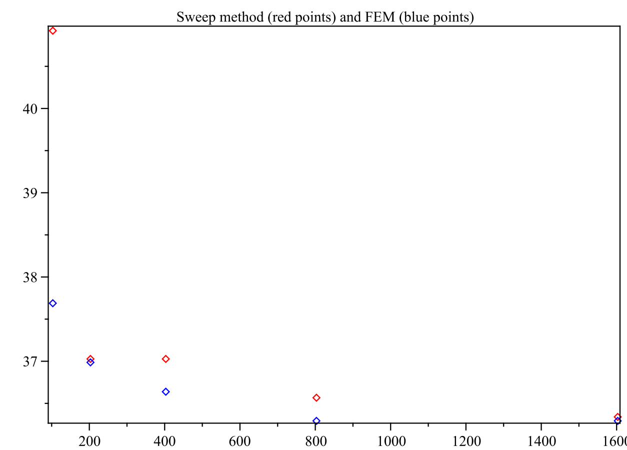

Figure 8: Convergence of the price of an American put option $\big(K = 50$, $\sigma = 0.4$, $r = 0.1$, $\delta = 0$, $T = 5/12\big)$ with increasing grid dimension.

Table 3: The boundary $x^*(0)$ of the early exercise of the K=50, σ=0.4, r=0.1, δ=0, T=5/12.

m (grid dimension), $Y^*_{\max}$	Sweep method	Finite element method (FEM)
100	40.93651	37.70211
200	37.04091	37.00011
400	37.04091	36.65201
800	36.58081	36.30571
1600	36.35291	36.30571

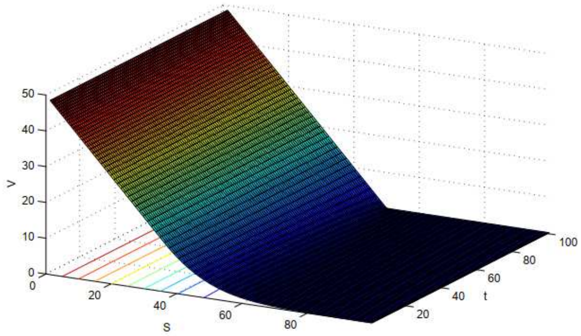

Figure 9: Structure $Y^*(t,\, x)$ of an American put option.

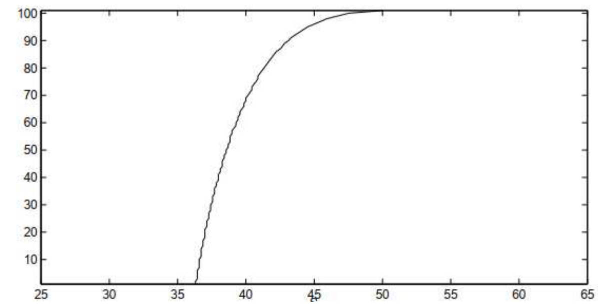

Figure 10: Structure of the free margin $x^*(t)$ of an American put option.

5 Conclusion

The advantage of finite difference methods (the sweep method) compared to the finite element method (FEA), in particular, the solution by the sweep method provides knowledge about the development of the option value function for each time step, i.e. the entire term structure of an American put option can easily visualize. See Figure 9. The accuracy of the sweep method is higher than FEA, see Figures 2, 4 and 8. At present $t = 0$, which is the "leading edge" of the surface, the shape of the cost function $Y^*(0,\, x)$ can be clearly seen, as shown in Figure 9. By selecting the $Y^*(t,\, x)$ functions for each t during the life of the option, one can obtain a complete term structure, and by the maturity date T approaches the non-smooth payoff function $Y^*\big(K - x\big)^+$. Of the entire surface of options, of particular interest is the development of the high contact point over time, namely $x^*(t)$, which is shown in Figure 10. This curve can be obtained by projecting the upper contact point at each time step onto the $x^*(t) - t$-plane, and it determines the option's early exercise strategy.

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2-бөлім **Раздел 2** **Section 2**

Информатика **Информатика** **Computer Science**

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IMPROVED DEEP LEARNING MODEL FOR CATTLE IDENTIFICATION USING MUZZLE IMAGES

Using traditional methods such as ear tags, branding, and tattoos for cattle identification requires continuous human involvement and demands significant time and effort. Although radio-frequency identification (RFID) methods are widely used today, they also have certain disadvantages. The RFID devices used must be constantly installed, which can cause discomfort to the animals, and during their movements, the devices may become damaged or lost, leading to further issues. By using biometric features for cattle identification, these disadvantages are eliminated. In this method, animals are identified using their unique biometric features, such as iris patterns, skin textures, muzzle prints, and facial features. This article focuses specifically on identifying cattle based on their muzzle images. In this study, a total of 4923 images of 268 cattle were used. The architectures of eight different models were improved and selected for the training process, and the training was conducted. According to the results, the DenseNet-121, WideResNet-50, and Inception V3 models achieved the highest accuracy rates, with 99.2%, 99.1%, and 99.1%, respectively. These results demonstrate the effectiveness of the proposed architecture.

Key words: Cattle muzzle images, pattern recognition, feature extraction, biometric features, cattle identification, deep learning models, model architecture.

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Тұмсығы кескіндері пайдалану арқылы ірі қара малды сәйкестендіру үшін терең оқытудың жетілдірілген моделі

Ірі қара малды сәйкестендірудің дәстүрлі әдістерін, мысалы ретінде, құлақ сырғаларын, таңбалауды және татуировкаларды қолдану үнемі адамның араласуын, сондай-ақ уақыт пен күштің айтарлықтай шығынын талап етеді. Қазіргі уақытта радиожиілікті сәйкестендіру (RFID) әдістерінің кеңінен қолданылуына қарамастан, олардың да белгілі бір кемшіліктері бар. Бұл үшін қолданылатын RFID құрылғылар үнемі орнатылып тұруы қажет, бұл жануарларды мазасыздандыруы мүмкін, сонымен қатар олар қозғалған кезде құрылғылар зақымдалып немесе жоғалып кетуі мүмкін, бұл қосымша мәселелерге әкеледі. Ірі қара малды сәйкестендіруде биометриялық сипаттамаларды қолдану осы кемшіліктерді жояды. Бұл әдісте жануарлар олардың бірегей биометриялық сипаттамалары, мысалы, көздің радужка суреттері, тері текстурасы, мұрын іздері және бет ерекшеліктері бойынша анықталады. Бұл әдісте жануарлар олардың бірегей биометриялық сипаттамалары, мысалы, көздің радужка суреттері, тері текстурасы, мұрын іздері және бет ерекшеліктері бойынша анықталады. Бұл мақала ірі қара малды мұрын іздері негізінде сәйкестендіруге арналған. Бұл зерттеуде жалпы саны 4923 суреттен тұратын 268 ірі қара малдың бейнесі пайдаланылды.

Сегіз түрлі модельдің архитектуралары жақсартылып, оқыту процесі үшін таңдалып, оқытылды. Зерттеу нәтижелері бойынша DenseNet-121, WideResNet-50 және Inception V3 модельдері ең жоғары дәлдікке, сәйкесінше 99,2%, 99,1% және 99,1% жетті. Бұл нәтижелер ұсынылған архитектураның тиімділігін көрсетеді.
Түйін сөздер: ірі қара малдың тұмсығы кескіндері, үлгіні тану, ерекшеліктерді алу, биометриялық белгілер, ірі қара малды сәйкестендіру, терең оқыту модельдері, модель архитектурасы.

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Улучшенная модель глубокого обучения для идентификации крупного рогатого скота с использованием изображений морды

Использование традиционных методов идентификации крупного рогатого скота, таких как ушные бирки, клеймение и татуировки, требует постоянного участия человека, а также значительных затрат времени и усилий. Несмотря на широкое применение радиочастотных методов идентификации (RFID) в наши дни, они также имеют определенные недостатки. RFID устройства, используемые для этого, должны постоянно устанавливаться, что может вызывать беспокойство у животных, и во время их передвижения устройства могут быть повреждены или потеряны, что приводит к дальнейшим проблемам. Использование биометрических характеристик для идентификации крупного рогатого скота устраняет эти недостатки. В этом методе животных идентифицируют по их уникальным биометрическим характеристикам, таким как рисунки радужки, текстуры кожи, отпечатки носа и черты лица. Данная статья посвящена идентификации крупного рогатого скота на основе отпечатков носа. В этом исследовании было использовано в общей сложности 4923 изображения 268 голов крупного рогатого скота. Архитектуры восьми различных моделей были улучшены и выбранных для процесса обучения и обучение было проведено. По результатам исследования модели DenseNet-121, WideResNet-50 и Inception V3 достигли наивысшей точности, составив 99,2%, 99,1% и 99,1% соответственно. Эти результаты демонстрируют эффективность предложенной архитектуры.
Ключевые слова: изображения морд крупного рогатого скота, распознавание образов, извлечение признаков, биометрические признаки, идентификация крупного рогатого скота, модели глубокого обучения, архитектура модели.

1 Introduction

Currently, artificial intelligence is one of the most important directions in modern technology and is widely applied in various fields. Artificial intelligence is aimed at implementing the capabilities of human intelligence through computers. Artificial Intelligence approaches human reasoning through a productive structure that encompasses data processing, calculation of algorithms, and machine learning among other things. Such a development enables extensive data to be examined and well-informed decisions to be rendered based on the information available. Another area of artificial intelligence that has attracted interest is classification which is the process of grouping information into categories or sets. It is an essential step in managing, analyzing and predicting of data. Classifying is applicable in a broad spectrum of industries such as health, finance, marketing and even livestock management. Extensive research has of late been done by scientists in these industries [1–4]. In the work of classification there is an objective of arranging the information according to set attributes. It involves the application of many methods and algorithms including deep learning models which are mostly machine-learning models. Object identification is also a part of this classification problem. In identification, each object is treated as a unique class. This article is focused on applying such an identification method to recognize cattle.

Cattle identification is one of the key aspects of effectively managing and monitoring the livestock industry. Cattle identification is the process of recognizing each individual cattle in a livestock farm. Currently, several methods are used for identification. These methods can be divided into contact-based and non-contact-based categories [5]. Contact-based methods include branding, freeze branding, ear tags, tattoos, and other techniques [6]. These identification methods always require human involvement, as well as time and effort. Additionally, these methods have their disadvantages. The procedures can be distressing for the animals, ear tags may be detached, and tattoos can deteriorate or become illegible over time. However, these methods are more cost-effective than others [7]. Another widely used contact-based method is the radio frequency identification (RFID) system [8]. This identification technique entails placing tag-based RFID with microchips on livestock ears. The radio frequency identification RFID tags have a unique code that can be picked by an appropriate reader enabling identification of livestock. Although this method seems to be expensive during initial installation within farms, it will be the most frequently used method within mechanized agriculture. Since all devices include chips that are believed to be unremovable, it has its demerits including bringing pain to the livestock, and chances of the chips being either lost or broken during movement [9]. Other non-contact methods use the animal's biometrics thus reducing the pain an animal goes through. Biometric features include iris images, skin patterns, muzzle images, and facial recognition [10].

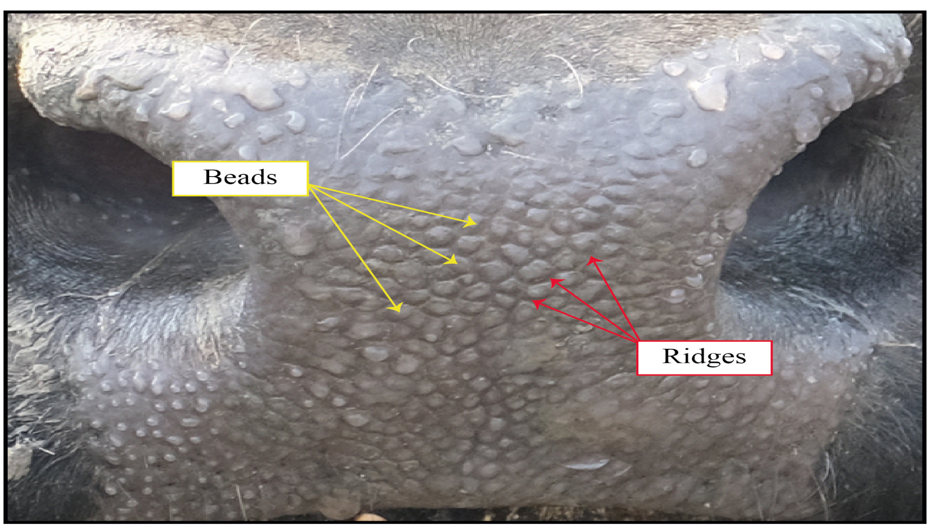

Figure 1: Biometric features in the muzzle image of cattle: The beads are marked in yellow, and the ridges are marked in red

Among the mentioned biometric features, identification using muzzle prints is a simple method, and interest in research in this area has been increasing recently [11–15]. The muzzle print of cattle contains unique features, much like human fingerprints [16]. There are two types of distinctive biometric features of the cattle's muzzle: beads and ridges (Figure 1). The beads have an uneven structure and resemble islands, while the ridges resemble rivers running between the islands. These beads and ridges serve as unique biometric identifiers for recognizing cattle. Research has confirmed that muzzle prints are accurate and remain unchanged over time, making them a reliable biometric identifier. This identifier has been studied since 1921 [17].

The aim of this study is to collect muzzle images of cattle, create a dataset, and apply various deep learning methods to identify the cattle based on the dataset. The results will be analyzed using metrics such as image processing speed, training speed, and accuracy levels.

2 Materials and Methods

2.1 Image Collection

In order to form a dataset of cattle muzzle images, images from previous studies [6] were used. The dataset contains 4,923 images of 268 cattle [18]. All images are high-resolution, RGB color images, taken with a digital camera equipped with a 70–300 mm F4-5.6 focus lens. The muzzle sections were cropped from the cattle images in the dataset (Figure 2).

Figure 2: Samples of images from the dataset consisting of cattle muzzle prints

It is important to note that the color and texture of cattle muzzle prints may appear very similar, but when analyzing the beads and ridges, significant differences become noticeable. The images in the dataset vary in size, and they need to be standardized before being fed into the training model. Based on the analysis of previous research and experimental results, the image size was set to 300 x 300. It was found that training at this size operates faster. While reducing the image size to 250 x 250 accelerates training speed, it may lead to the loss of finer features. However, the 300 x 300 resolution was retained for this study, as it facilitates faster processing while preserving more detailed information crucial for cattle identification based on muzzle print characteristics.

In this work, the applied image augmentation techniques such as shifting, rotating, color manipulation and blurring images, which all helped to increase the number of sample images, were used in order to obtain better model accuracy. Also, as a result of carrying out data augmentation, the accuracy of the model is expected to be improved in the future due to various repetitions of the existing images.

2.2 Deep Learning Models

After conducting a comprehensive review of global research on the subject, eight deep-learning models were chosen for image classification tasks. The selected models include AlexNet, GoogleNet, DenseNet, WideResNet, MobileNet V2, MobileNet V3, ShuffleNet V2, and Inception V3. Table 1 provides a detailed summary of these models along with their key characteristics.

Table 1: Characteristics of deep learning models used in the study

No.	Model name	Number of parameters (million)	Model size (MB)
1	Alexnet	62	233
2	GoogleNet	13	50
3	DenseNet-121	8	30
4	WideResNet-50-2	69	262
5	MobileNet V2	3.5	13
6	MobileNet V3 Large	5.5	21
7	ShuffleNet V2	2.2	8.6
8	Inception V3	27	103.67

2.3 Training Process

With the assistance of transfer learning, the models have been implemented using the Pytorch framework. This approach involved utilizing models pre-trained on the ImageNet dataset [19]. The model's fully connected layer was modified to tailor it for the classification of the current dataset. The convolutional network weights remained pre-trained, while the final layer was fine-tuned specifically for cattle identification tasks. This strategy enhances training efficiency. Proper selection of hyperparameters and configurations helps achieve high accuracy levels. Based on the acquired results, the subsequent parameters and configurations were determined: the training process is set to a maximum of 50 epochs, utilizing the Adam optimization algorithm, and employing the Cross-Entropy loss function. The optimizer's peak learning rate is established at 1e-4, with a gradient clipping threshold of 0.1 implemented to mitigate potential gradient explosion. Additionally, a regularization coefficient of 0 was applied to minimize overfitting risks. Early stopping is incorporated, halting the training process after 7 epochs in the absence of any improvement in the accuracy metric.

Throughout the training phase, accuracy, loss, and additional metrics were evaluated at each epoch. The progression of accuracy throughout training is illustrated in Figure 3. The duration of each training epoch, as well as the overall training time, was meticulously documented. Based on the outcomes obtained from the experiment, the hyperparameters' optimal values were determined.

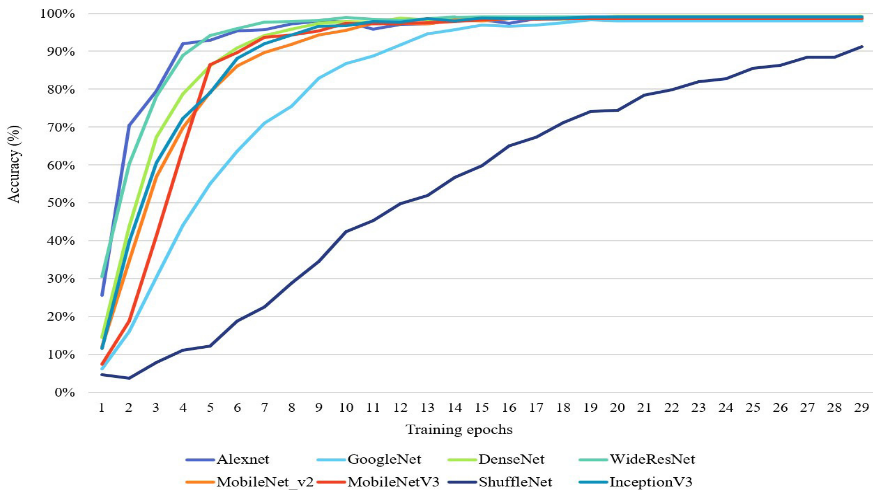

Figure 3: The Accuracy of deep learning models

The dataset comprises 268 distinct objects, each represented by 4 to 70 images. Objects with fewer images introduce challenges in object recognition, primarily due to class imbalance. To resolve this challenge two approaches were implemented, the Weighted Cross-Entropy (WCE) loss function [20] and data augmentation techniques [21]. Augmentation techniques aimed at increasing the dataset size and diversity. These strategies were applied across all models and enabled the determination of optimal accuracy and computational efficiency. Among these metrics, accuracy was prioritized as the most critical factor.

The WCE loss function assigns greater weights to cattle classes with a smaller number of images. This weighting is calculated using the following approach:

$$WCE_{\text{loss}} = -\sum_{i=1}^{C} w_i t_i \log(p_i) \tag{1}$$

where $p_i \in \mathbb{R}^{268}$ refers to the probabilities assigned to each of the 268 cattle classes in the output of the LogSoftMax layer. C represent the total number of cattle, while t_i indicates the actual probability for the cattle, which is calculated as follows:

$$t_i = \begin{cases} 1, & \text{if } i = \text{true} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

w_i – the individual weight assigned to the i-th cattle is calculated as follows:

$$w_i = \left(\frac{N_{\max}}{N_i} \right) \tag{3}$$

N_i denotes number of images for the i-th cattle, N_max denotes number of images devoted to one cattle in the dataset (which in this dataset equals 70).

70% of the dataset was used for training and the remaining 30% for testing. The image channel pixel color intensities were rescaled to the interval [0,1], which increases the effectiveness of image recognition, as noted by [22].

The model training was conducted on a computing machine with the following specifications: Intel i7-11800H 2.3GHz – 4.6GHz, 32 GB DDR4 RAM, RTX 3070 8 GB GDDR6.

During the training process, modifications were made to the architecture of the selected models. Instead of using the original fully connected output layer, a new architecture was developed. In the new architecture, the number of outputs in the last layer matches the number of classes in the dataset, i.e., 268 outputs. A linear layer was added to the output part of the initially trained model. This layer had an input size of *inputs* from the previous model and an output size of 512 for the next layer. The number of neurons in this linear layer was determined to be optimal for our case based on the results of the experiments, with 512 neurons selected. Between this linear layer and the final output layer, a ReLU activation function and a dropout layer were added to prevent overfitting. The final layer is a linear layer, and the architecture concludes with the LogSoftMax function. The proposed architecture is illustrated in Figure 4c.

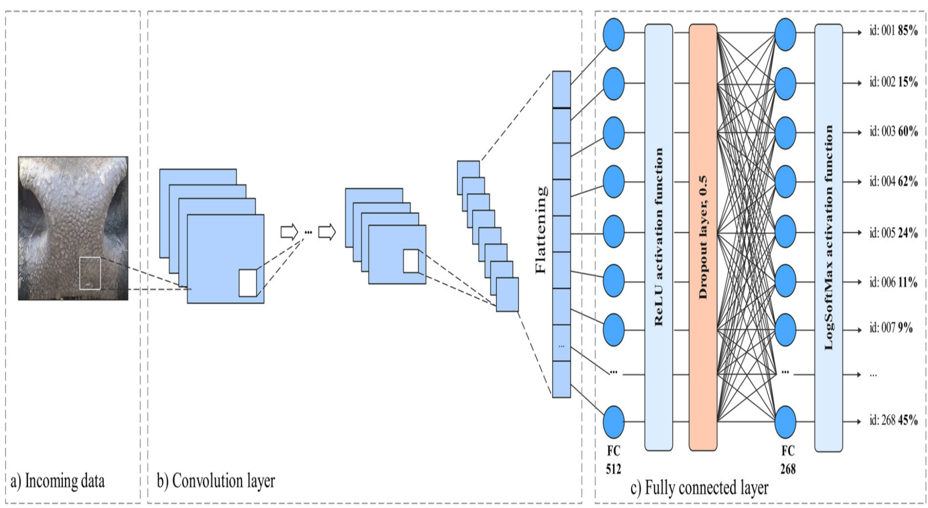

Figure 4: General architecture of the training model: (a) input data, (b) convolutional layers of each model, (c) improved fully connected linear layer

3 Results

Eight optimized models were trained using the previously described settings, hyperparameters, and architecture. A maximum of 50 training epochs was set, but early stopping was applied if accuracy did not change over a period of time. Consequently, high accuracy was achieved between epochs 13 and 29. There was an average of 33 to 67 minutes spent on each model training process. Table 2 presents the training results for the models. Model accuracy ranged from 91% to 99% according to the results. DenseNet-121 is the model with the highest accuracy, while GoogleNet has the lowest accuracy.

This study achieved high performance with enhanced models. The DenseNet-121 model achieved 99.2% accuracy, while the WideResNet-50 and Inception V3 models achieved 99.1%

accuracy. This result can be compared with other studies focused on identifying cattle based on the muzzle image [6, 15, 23, 24]. The improved model comparison Muzzle image cattle identification was performed in [6] and our enhanced model demonstrated better performance. Comparison results are shown in Table 2.

Table 2: Training results of the models and comparison with other studies

No.	Model name	Epochs	Accuracy (%)	Training Time (minutes)	Comparison with other studies [6]
1	Alexnet	19	98.8	34	96.5
2	GoogleNet	20	97.9	39	59.4
3	DenseNet-121	20	99.2	67	93.0
4	WideResNet-50	15	99.1	38	89.6
5	MobileNet V2	16	98.6	34	-
6	MobileNet V3 Large	18	98.5	33	95.9
7	ShuffleNet V2	29	91.2	51	1.2
8	Inception V3	19	99.1	41	81.7

4 Conclusion

The study established the usefulness of the pre-trained models regarding cattle identification. Using a large dataset with pre-trained models not only improves model performance in terms of accuracy but also cuts down training time massively. The process of identification was done from muzzle images and eight previously known models were chosen as the subject of testing. The Modifying changes done to the final output layer of the models resulted in the proposal of an alternative structure. The new structure outperformed previous training standards for models with DenseNet-121, WideResNet-50 and Inception V3 reporting accuracy rates of about 99.2%, 99.1% and 99.1% respectively. These results support the effectiveness of the new architecture. Even when there are differences in the types of cattle, the conditions of data collection and the architectures of the different models, all the experiments reached an accuracy rate of about 90% confirming the effectiveness of the proposed method.

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ALGORITHM FOR CLUSTERING DIFFERENT TYPES OF DRUGS AFFECTING BLOOD PRESSURE

This article presents the development of an algorithm and software for grouping different types of drugs that affect blood pressure in humans. The results are experimentally tested on a cross-section of more than 1,100 drugs that affect human blood pressure. Using the proposed algorithm, the training sample is formed and the problem of clustering is solved. The training set consists of ten classes. Selection symbols are given in different types, they consist of nominal and value symbols. In the article, each object is examined, and the importance of the object in the sample is assessed using a criterion. This criterion contributes to the formation of the studied class of the object. The developed algorithm works taking into account both types of features. If the similarity of the object under study with any class is high, this object is transferred to this class. This process is performed sequentially several times for all objects of the class. The process stops when the position of objects remains unchanged and the degree of similarity exceeds the required percentage. The accuracy of data set classification by object classes was experimentally verified using an algorithm and software package based on neural networks.

Key words: Drugs, pattern recognition, symbols, clustering issue, algorithm and software.

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Қан қысымына әсер ететін дәрілердің түрлі түрлерін кластерлеу алгоритмі

Мақалада адамдардың қан қысымына әсер ететін дәрілердің әртүрлі түрлерін топтастыруға арналған алгоритм мен бағдарламалық пакетті жасау зерттеледі. Нәтижелер адамның қан қысымына әсер ететін 1100-ден астам дәрі-дәрмектің көлденең қимасында эксперименталды түрде тексерілді. Ұсынылған алгоритмді пайдалана отырып, кластерлеу мәселесі шешіліп, оқыту үлгісі қалыптастырылды. Білім беру іріктеу он сыныптан тұрады. Таңдау белгілері әртүрлі түрде беріледі, олар номиналды және мәндік белгілерден тұрады. Мақалада әрбір объект зерттеліп, іріктемедегі объектінің маңыздылығы критерий арқылы бағаланады. Бұл критерий объектінің зерттелетін класын қалыптастыруға ықпал етеді. Әзірленген алгоритм белгілердің екі түрін де ескере отырып жұмыс істейді. Егер зерттелетін объектінің қандай да бір класқа ұқсастығы жоғары болса, онда бұл объект осы класқа көшіріледі. Бұл процесс кластың барлық объектілері үшін кезекпен бірнеше рет орындалады. Нысандардың орналасуы өзгермей, ұқсастық дәрежесі қажетті пайыздан асқан кезде үдеріс тоқтатылады. Оқыту объектілерінің дұрыс жіктелуі нейрондық желіге негізделген алгоритм және бағдарламалық пакетті қолдану арқылы эксперименталды түрде тексеріледі.

Түйін сөздер: дәрілер, үлгіні тану, белгілер, кластерлеу мәселесі, алгоритм және бағдарламалық қамтамасыз ету.

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Алгоритм кластеризации различных видов препаратов, влияющих на артериальное давление

В статье исследуется разработка алгоритма и программного комплекса для группировки лекарственных средств, выраженных в различных типах, влияющих на кровяное давление человека. Результаты были апробированы на более чем 1100 препаратах, влияющих на кровяное давление человека. С помощью предложенного алгоритма решена задача кластеризации и сформирована наборы данных. Наборы данных состоит из десяти классов. Признаки выборки были представлены различными типами, состоящими из номинальных и стоимостных признаков. В статье исследуется каждый объект, и важность объекта в выборке оценивается с помощью критерия. Этот критерий способствует формированию исследуемого класса объекта. Разработанный алгоритм работает с учетом обоих типов признаков. Если сходство исследуемого объекта с каким-либо классом высокое, этот объект переводится в этот класс. Этот процесс выполняется последовательно несколько раз для всех объектов класса. Процесс останавливается, когда положение объектов остается неизменным и степень сходства превышает требуемый процент. Правильность определения классами объектов наборы данных была экспериментально проверена с использованием алгоритма и программного комплекса, основанного на нейронных сетях.
Ключевые слова: Лекарственные препараты, распознавание образов, символы, проблема кластеризации, алгоритм и программное обеспечение.

1 Introduction

Clustering different types of antihypertensive drugs involves classifying them based on similarities in their mechanisms of action, chemical structures, or therapeutic effects. This task is critical for medical professionals to optimize treatment plans, improve patient understanding, and facilitate research. The necessary technologies and their analysis in the construction of algorithms and software for clustering drugs affecting blood pressure can be found in [1-6]. Research works [12-19] serve as the main sources of data clustering methods, algorithms and software used in medicine.

Medicines are divided into groups based on how they affect blood pressure.

Angiotensin-converting enzyme inhibitors (Inhibitory angiotensin converting enzyme (IACE)). Angiotensin-converting enzyme inhibitors (IACE) are a class of drugs used primarily to treat high blood pressure (hypertension), treat heart failure, prevent stroke, and protect the kidneys in people with diabetes [7].

Angiotensin-II receptor blockers (BRA). Angiotensin-**II** receptor blockers (ARB) are a class of drugs used to treat high blood pressure (hypertension), heart failure, diabetic nephropathy, and certain other conditions. They work by blocking the effects of angiotensin-**II**, a hormone that causes blood vessels to constrict and increase blood pressure. By blocking these receptors, ARB help relax blood vessels, thereby lowering blood pressure and reducing the workload on the heart.

Beta-blockers (BB). ($\boldsymbol{\beta}$-adrenoblocker, BB). Beta-blockers (BB) are a class of drugs used primarily to treat arrhythmias, prevent recurrent myocardial infarction, treat hypertension, heart failure, certain types of tremors, migraines, and glaucoma. These drugs can be used to treat high blood pressure, certain types of heart arrhythmias, and migraines.

Calcium channel blockers (CCB). Calcium channel blockers (CCBs) are a class of drugs used to treat a variety of cardiovascular conditions, including hypertension (high blood pressure), angina (chest pain due to reduced blood flow to the heart muscle), and some forms of arrhythmia [9].

Diuretics: Diuretics are drugs that help to remove more fluids from the body. They work by affecting the work of the kidneys, resulting in increased urine production and excretion. Diuretics are used to treat various diseases, including heart failure, high blood pressure, swelling (edema), and others. Diuretics help control the amount of fluid in the body and can reduce swelling and pressure in the blood vessels [8-9].

Peripheral vasodilators. Peripheral vasodilators are a group of drugs that widen peripheral blood vessels, reduce the resistance of the vessel walls, and thereby lower blood pressure.

Peripheral vasodilators are used in combination with other medications to better control blood pressure and overall cardiovascular health. They can cause side effects such as dizziness, swelling, facial flushing, headache, and reflex tachycardia, so their use should be under strict medical supervision.

Selective sinus node If-channel inhibitors. Selective sinus node If-channel inhibitors are a group of drugs often used to control sinus rhythm. If channels, also called "phase inhibitors play an important role in the control of electrical impulses in cardiac myocytes. These channels are more common in myocytes in the sinus node, and by inhibiting them, sinus rhythm can be slowed, which reduces cardiac output and may be useful in controlling arrhythmias.

If-channel inhibitors can be used to ease the heart's workload in certain heart conditions, including high blood pressure, heart failure, and atrial fibrillation. One of the famous representatives of these drugs is ivabradine. Ivabradine targets only the If channels, which makes it safe and effective for heart rhythm control because it does not affect other cardiac channels and therefore has few side effects.

If-channel inhibitors work primarily by slowing sinus rhythm and reducing the pressure the heart exerts on the circulatory system. One of the advantages of these drugs is their ability to lower the heart rate, which is especially useful for patients with respiratory problems.

Medicines that improve myocardial metabolism. Medicines used to improve the metabolism of the myocardium are often used to improve the blood circulation of the heart and increase the performance of the heart muscle. Such drugs are mainly prescribed for cardiovascular diseases, such as ischemic heart disease, angina pectoris and myocardial infarction.

Antiplatelet agents. Antiplatelet agents are a group of drugs used to prevent or reduce the clumping of blood platelets. They are mainly used in the prevention and treatment of thrombosis, myocardial infarction, stroke and other vascular diseases. Antiplatelet drugs inhibit the activity of blood platelets and prevent excessive blood clotting and the formation of blood clots [10-11].

2 Clustering of given objects using symbols of different types

Let's assume that in the space of N-dimensional nominal and value symbols, given drugs of various types $x_i \in X$, $i = \overline{1, M}$, i.e., objects. So, $x_i = (x_i^1,\ x_i^2,\ \ldots,\ x_i^N)$, $i = \overline{1, M}$, drug is given in the space of N-dimensional nominal and value characters. And all of them are X collection objects. We call it the common sample objects, and the set to which they belong is denoted by X.

Task. It is required to form educational selections on the basis of the given general selection. That is, it is required to form classes expressed in the following form $x_{p1},\ x_{p2},\ \dots,\ x_{pm_p} \in X_p,\ p=\overline{1,r}$. Where x_{pi} is read as the i- object of class X_p in the N-dimensional nominal and valued symbol space, and it is written in the following form in the N-dimensional nominal and valued symbol space $x_{pi}=\left(x_{pi}^1,\ \ x_{pi}^2,\ \dots x_{pi}^N\right),\ i=\overline{1,m_p}$ viewed in the space of N-dimensional nominal and value symbols, $X=\bigcup_{p=1}^{r} X_p$ consisting of m_p objects $x_{p1},\dots,\ x_{p_{m_p}})$ in class X_p . In [7-14], this issue is referred to as the object clustering issue.

Let's introduce the following dimensions and designations

Let the quantity indicating the similarity of objects in the space of nominal symbols be determined by $\rho^j(x_{pi,}x_{pq})$ and calculated by (1), i.e.

$$\rho_{pi}^j\left(x_{pi},x_{pq}\right)=\begin{cases} 1,\ \text{if } \left(x_{pi}^j-x_{pq}^j\right)=0; \\ \\ 0, \qquad\qquad \text{otherwise} \end{cases} \tag{1}$$

Let the quantity indicating the similarity of objects in the space of numerical symbols be determined by $\rho^j(x_{pi,}x_{pq})$ and calculated by (2), i.e.

$$\rho_{pi}^j\left(x_{pi},x_{pq}\right)=\begin{cases} 1,\ \text{if} \quad |x_{pi}^j-x_{pq}^j|\le\varepsilon^j \\ \\ 0, \qquad\qquad \text{otherwise} \end{cases} \tag{2}$$

where $p=\overline{1,r};\ i\ne q=\overline{1,m_p};j=\overline{1,N}$; The expressed quantities (1) and (2) are the parameters of the vector, which is expressed in the following form $\rho_{pi}\left(x_{pi,}x_{pq}\right)=\left(\rho_{pi}^1\left(x_{pi,}x_{pq}\right),\rho_{pi}^2\left(x_{pi,}x_{pq}\right),\dots,\rho_{pi}^N\left(x_{pi,}x_{pq}\right)\right)$. So, if the considered j-symbol is nominal, the j-symbol of the vector $\rho_{pi}\left(x_{pi,}x_{pq}\right)$ is calculated using (1), otherwise, that is, if the symbol is numerical, then this symbol of the vector is calculated by (2) . In this case, the ε^j- threshold values corresponding to the j-character are performed by the following formula for each character of the class objects:

$$\varepsilon^j=\frac{1}{M-1}\sum_{i=1}^{M-1}\left|x_{pi}^j-x_{pi+1}^j\right|,$$

where $j=\overline{1,N};p=\overline{1,r};i=\overline{1,M-1};$.

We present the following steps for solving the above-mentioned clustering problem:

1. Drug data is preprocessed into $x_i=(x_i^1,\ x_i^2,\ \dots,\ x_i^N)\in X,\ i=\overline{1,M}$ In this case, missing data are filled in, anomalous data are replaced by the mean value, and quantitative signs are normalized. Standardization is carried out for all quantitative signs based on the following formula:

$$x_i^j=\frac{x_i^j-\min_i x_i^j}{\max_i x_i^j\ -\min_i x_i^j}.\ i=\overline{1,M};\ j=\overline{1,N};$$

2. Then, drugs $x_i \in X$, $i = \overline{1, M}$, objects are arbitrarily divided into r classes. That is,arbitrary are divided into $x_{p1},\ x_{p2},\ \ldots,\ x_{pm_p} \in X_p,\ p=\overline{1,r}$ classes;

3. Based on the equations (1) and (2) given above, all $p = \overline{1, r};\ i \neq q = \overline{1, m_p};\ j = \overline{1, N}$ for all parameters of the vector$\rho_{pi}\left(x_{pi,}x_{pq}\right)$. That is $\rho_{pi}\left(x_{pi,}x_{pq}\right) = \left(\rho_{pi}^1\left(x_{pi,}x_{pq}\right), \rho_{pi}^2\left(x_{pi,}x_{pq}\right), \ldots, \rho_{pi}^N\left(x_{pi,}x_{pq}\right)\right.$ vector symbols are calculated for $p = \overline{1, r};\ i \neq q = \overline{1, m_p};\ j = \overline{1, N}$;

4. The position of the i- object in the optional p-class in the remaining set of $m_p - 1$ objects of this class is evaluated as follows:

$$\Gamma_{pi}\left(x_{pi,}X_p\right) = \frac{1}{m_p - 1} \sum_{q=1}^{m_p - 1} \sum_{j=1}^{N} \rho^j\left(x_{pi,}x_{pq}\right),\ p=\overline{1,r}\ ;i=\overline{1,m_p};i\neq q.$$

5. The general grade of the arbitrary p-class is calculated based on the criterion $\Gamma_p\left(X_p\right) = \frac{1}{m_p}\sum_{i=1}^{m_p}\Gamma_{pi}\left(x_{pi,}X_p\right)$, $p=\overline{1,r}$. The degree of similarity of their objects is evaluated as follows

$$\nu_p\left(X_p\right) = = \frac{\Gamma_p\left(X_p\right) \quad *100\%}{N},\ p=\overline{1,r}.$$

6. Also, the i -object in an arbitrary $p-$class is evaluated by other X_q, $q=\overline{1,r-1}$ class objects: $\Gamma_{pi}\left(x_{pi,}X_q\right) = \frac{1}{m_q}\sum_{k=1}^{m_q}\sum_{j=1}^{N}\rho^j\left(x_{pi,}x_{qk}\right), p=\overline{1,r}\ ;i=\overline{1,m_p};i\neq q.$ And the i-object in the p-class is transferred to the class which is highly valued by the objects of the class. If the highest marks are tied, they are kept in their own class. If the upper values are equal in two classes other than in the p-class, then this object is moved to the smaller q index.

7. Usually, the degree of similarity of objects in the formed classes is required to be

$$\nu_{\mathrm{p}}\left(\mathrm{X_p}\right) \geq \delta \geq 55\%.$$

8. These calculations are performed for all objects $x_{pi,}\ p=\overline{1,r}\ ;i=\overline{1,m_p};$ and the resulting new $x_{p1},\ x_{p2},\ \ldots,\ x_{pm_p} \in X_p,\ p=\overline{1,r}$ classes gives a true clustered training sample.

Based on these clustering steps, 1116 different types of blood pressure drugs were clustered using the algorithm and software. One nominal character and nine quantitative characters were studied. They are given in the table below:

Table 1

Information about drugs that affect blood pressure (antihypertensive)

№	The name of the drug	Percentage of active substance in mg	Melting point %	Color	Form of production	Blood concentration (max)	Distribution density	Technological indicators of prepared compositions			
								Spreadability, 10^{-3} kg/s	Spreading density, kg/m^3	Metabolism in the liver %	Decomposition
1	x_1	0,10	96	yellow	10	64	3,50	4,51	20,72	90	7,41
2	x_2	0,10	85	white	20	78	4,20	7,85	14,36	75	8,56
3	x_3	0,50	88	pale yellow	25	20	4	3,87	25,48	95	5,69
4	x_4	0,03	89	white	50	72	0,20	16,87	15,64	75	7,96
5	x_5	0,20	91	white	15	60	3,20	19,20	25,63	85	5,98
6	x_6	0,50	93	white	25	57	3,90	4,89	48,65	89	6,85
7	x_7	0,20	85	white	10	105	4,24	3,56	13,56	74	6,47
8	x_8	16	95	yellow	50	29	4,80	7,86	14,65	96	4,12
1111	x_{1111}	0,4	75	pink	30	96	3,5	6,52	12,69	85	7,89
1112	x_{1112}	0,2	78	gray	20	97	3,6	10,42	12,25	80	8,54
1113	x_{1113}	0,5	76	white	25	86	4,01	7,41	13,25	78	9,65
1114	x_{1114}	0,25	74	gray	14	93	4,5	3,41	10,26	60	12,48
1115	x_{1115}	0,5	72	white	28	94	3,6	4,85	12,48	85	9,85
1116	x_{1116}	0,25	76	white	30	85	3,9	4,63	7,96	86	8,69

The following table provides information about the digitization of the nominal values of drugs, that is, their colors:

Table 2

brown	1
blue	2
gray	3
red	4
black	5
white	6
fire color	7
pale red	8
pale pink	9
pale yellow	10
pink	11
yellow	12

3 Algorithm for clustering given objects in different types of character space

The algorithm for finding a solution to the problem described in the article is presented below. The algorithm consists of six clauses, and it is appropriate to apply the problem of symbol determination only to objects of a class taken separately.

First step. Drug data is preprocessed into $x_i = (x_i^1,\ x_i^2,\ \ldots,\ x_i^N) \in X$, $i = \overline{1, M}$. In this case, missing data are filled in, anomalous data are replaced by the mean value, and quantitative indicators are normalized.

Second step. Subjects of educational selection are included in the database. The initial database is formed on the intersection of all X_p, $p = \overline{1, r}$ class;

Third step. The magnitude (1) indicating the similarity of objects in the space of nominal symbols and the magnitude (2) indicating the similarity of objects in the space of numerical symbols, which are used to determine the contribution of objects of class X_p to the formation of their class, are calculated based on the formula;

Fourth step. The position of the i- object of the arbitrary p-class in the selection marks in the set of $m_p{-}1$ objects of the same class is calculated based on the formula in step 3 of solving the above-mentioned clustering problem;

Fifth step. The evaluation of the i- object in the optional p-class by other X_q, $q{=}\overline{1{,}r-1}$ class objects in the selection marks is calculated based on the formula in the step 4 of solving the above-mentioned clustering problem;

Sixth step. In selection symbols, the i- object of the arbitrary p-class is transferred to the class which is highly evaluated by the objects of the class. If the highest marks are tied, they are kept in their own class. If the upper values are equal in two classes other than the p-class, then this object is shifted to the smaller q index.

Based on the proposed theoretical research, algorithm, we will solve the problem described above. Cross-clustering of blood pressure (antihypertensive) drugs is reflected in the following table.

Table 3

Information about drugs that affect blood pressure (antihypertensive)

№	The name of the drug	Percentage of active substance in mg	Melting point %	Color	Form of production	Blood concentration (Cmax)	Distribution density	**Technological indicators of prepared compositions**				
								Spreadability, 10^{-3} kg/s	Spreading density, kg/m^3	Metabolism in the liver %	Decomposition	Class
1	x_1	0,10	96	12	10	64	3,50	4,51	20,72	90	7,41	1
2	x_2	0,10	85	6	20	78	4,20	7,85	14,36	75	8,56	1
3	x_3	0,50	88	10	25	20	4	3,87	25,48	95	5,69	7
4	x_4	0,03	89	6	50	72	0,20	16,87	15,64	75	7,96	10
5	x_5	0,20	91	6	15	60	3,20	19,20	25,63	85	5,98	10
6	x_6	0,50	93	6	25	57	3,90	4,89	48,65	89	6,85	10
7	x_7	0,20	85	6	10	105	4,24	3,56	13,56	74	6,47	1
8	x_8	16	95	12	50	29	4,80	7,86	14,65	96	4,12	10
...												
1111	x_{1111}	0,4	75	11	30	96	3,5	6,52	12,69	85	7,89	10
1112	x_{1112}	0,2	78	3	20	97	3,6	10,42	12,25	80	8,54	10
1113	x_{1113}	0,5	76	6	25	86	4,01	7,41	13,25	78	9,65	10
1114	x_{1114}	0,25	74	3	14	93	4,5	3,41	10,26	60	12,48	10
1115	x_{1115}	0,5	72	6	28	94	3,6	4,85	12,48	85	9,85	10
1116	x_{1116}	0,25	76	6	30	85	3,9	4,63	7,96	86	8,69	10

4 A neural network-based clustering algorithm using linear regression

On the result obtained by the clustering algorithm, an experimental test was conducted on the belonging of the object to the class using a neural network. The neural network is constructed as follows:

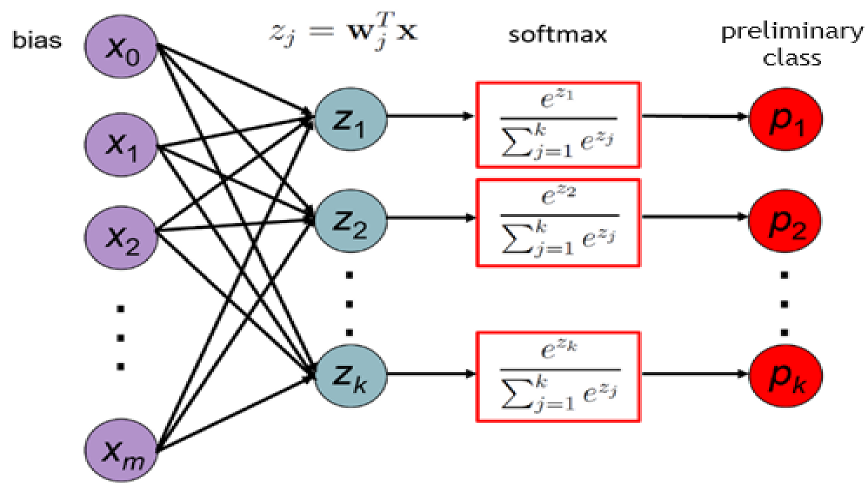

Figure 1: An exemplary structure of a neural network

A neural network based on linear regression was built for this problem.

$$z = W\overrightarrow{x} + \overrightarrow{b}$$

where W is a **weight matrix** initially filled with a set of random numbers, b is a **bias** vector initially filled with a set of random numbers. x is the **training sample** in the data set.

Activation functions are presented as follows:

$$\hat{y} = \sigma(\overrightarrow{z})$$

where $\hat{y}$ is the prediction value from the neural network. The activation functions ReLU and Softmax were used in this network.

Back-propagation and gradient descent were used to reduce errors in calculations:

$$L = -\sum (y - \log\hat{y}\ ) + (1 - y)(log1 - \hat{y})$$

$$W' = W - \alpha\frac{\partial L}{\partial W}$$

Where W'- this is a **weighting matrix** that brings the prediction closer to the true value at each step**.**

Based on the data and parameters, neural network training was carried out using the gradient descent method. The algorithm of this neural network is as follows:

First step. Import the necessary libraries and define the main parameters of the neural network.

Second step. Selection of activation functions and error elimination functions (sparse cross-entropy).

Third step. Downloading data from a file and preparing it for training and experiments.

Fourth step. Initialize neural network weights and offsets with random values.

Fifth step. Neural network training using stochastic gradient descent (SGT) and mini-batch (mini-batch) on a specific epoch (NUM_EPOCHS).

Sixth step. Class prediction for test data using a trained network.

Seventh step. Calculation of accuracy of predictions based on test data.

Eighth step. Display and visualize the accuracy of predictions on the screen.

A block diagram of this algorithm is fully described in Figure 2:

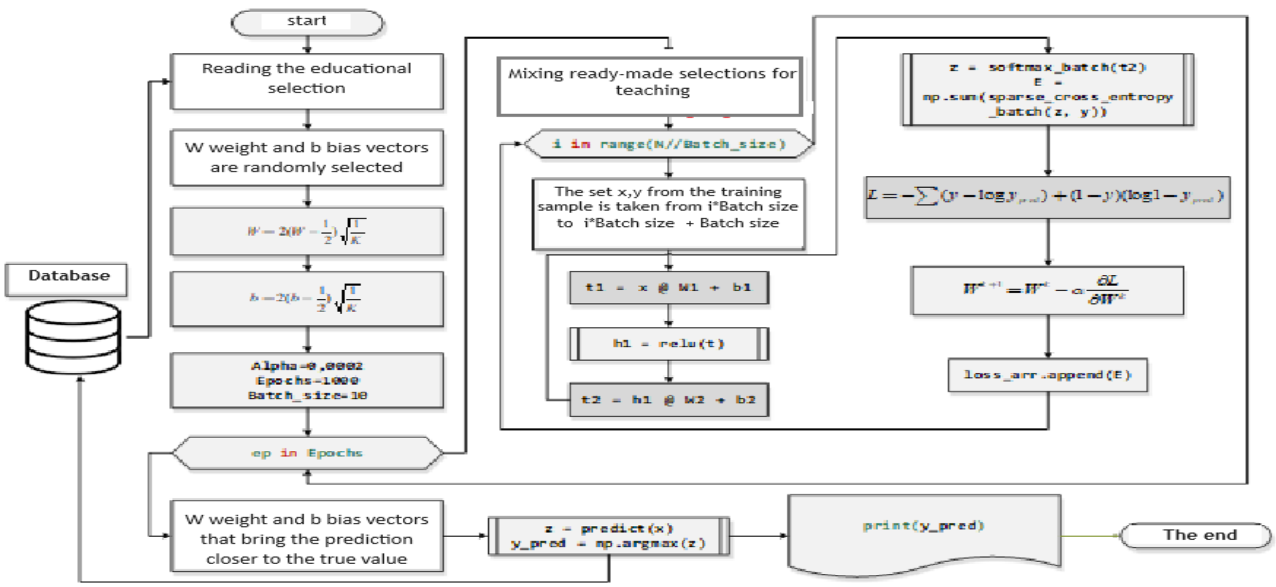

Figure 2: Block diagram of a neural network algorithm

The experimental results of the constructed neural network predicted the class of the new object with 81 - 87% accuracy.

5 A clustering model built on the KNIME platform

In addition, experiments were conducted in the **KNIME** environment, which is intended for building neural networks in the form of a **scratch-block**. A neural network model in the **KNIME** environment is shown in figure 3.

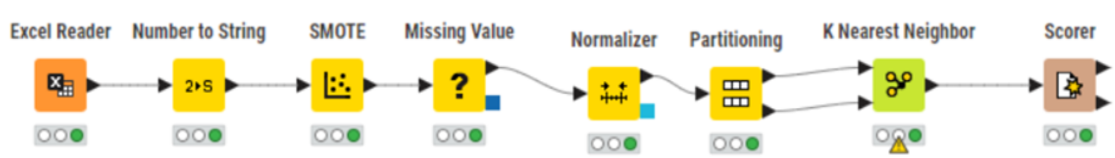

Figure 3: K-nearest neighbors model in KNIME

The functions of each **scratch block** in this model are as follows:

1. **Excel Reader** - reading and downloading data from an Excel file.
2. **Number to String** - convert numeric data to string data.
3. **SMOTE** - reproduce data of many different types using the following methods: "Synthetic Minority Over-sampling Technique".
4. **Missing Value** - filling in or removing missing data.
5. **Normalizer** - normalizing data using specific methods.
6. **Partitioning** - splitting data into test and training samples.
7. **K Nearest Neighbor** - data classification or regression with KNN algorithm.
8. **Scorer** - print summary results of model accuracy metric evaluations.

Through this model, 90% of the training sample was allocated to training and 10% to testing, and the level of clustering as a result of the model was estimated in the range of 88-94%.

The fourth table below presents a comparative analysis of the three studied algorithms.

Table 4

Class	**Recognition accuracy by neural network**	**Recognition accuracy by model generated in KNIME**	**Algorithm accuracy of classification of given objects in different types of character space**
1	82.62	94.23	95.52
2	81.4	93.48	90.18
3	84.8	88.03	97.36
4	86.23	90.26	88.07
5	82.02	94.11	95.19
6	83.79	91.48	90.27
7	81.83	93.49	94.13
8	85.61	88.43	93.47
9	87.32	94.13	94.06
10	82.55	90.88	96.47

According to the analysis, an experiment-test was conducted based on the classification of the given objects in the space of various types of symbols. Therefore, it was noteworthy that the sample was divided into clusters based on the proposed algorithm.

The first column of the Table 4 lists the names of 10 classes formed on the basis of clustering. The second column shows the accuracy of recognition by the neural network model, the third column shows the accuracy of recognition by the model created in KNIME, and the fourth column shows the accuracy of the object classification algorithm in different types of character space.

6 Conclusion

In conclusion, the article proposes a new approach to solving the problem of clustering through the character space of the objects of the sample class and an algorithm based on neural networks to correctly find the classes of the objects of the training sample.

In classification, clustering, pattern recognition and intelligent data analysis, reducing the set of symbols in pre-processing and solving the problems proposed based on them is a very important research.

An algorithm and software complex developed to clusterize drugs affecting human blood pressure has been researched. The results of the study covered more than 1,100 drugs. The tests confirmed the effectiveness of the algorithm and the ability to help.

The proposed algorithm works in solving the problem of clustering, taking into account nominal and value symbols. The selection consists of ten classes, and the information of each class is organized from nominal and value symbols. As a result of using the algorithm, a software package based on neural networks and providing the correct identification of classes was developed.

The proposed algorithm and software complex achieved high results in the clustering of drugs that affect human blood pressure, ensuring their accurate and effective grading. Techniques based on neural nets provide an opportunity to work with both nominal and value characters in mind, and this in turn increases the accuracy and efficiency of clustering.

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