ISSN 1563 – 0277
eISSN 2617 – 4871

ӘЛ-ФАРАБИ атындағы ҚАЗАҚ ҰЛТТЫҚ УНИВЕРСИТЕТІ

ХАБАРШЫ

Математика, механика, информатика сериясы

КАЗАХСКИЙ НАЦИОНАЛЬНЫЙ УНИВЕРСИТЕТ имени АЛЬ-ФАРАБИ

ВЕСТНИК

Серия математика, механика, информатика

AL-FARABI KAZAKH NATIONAL UNIVERSITY

Journal of Mathematics, Mechanics and Computer Science

№2 (126)

Алматы
«Қазақ университеті»
2025

Зарегистрирован в Министерстве информации и коммуникаций Республики Казахстан, свидетельство №16508-Ж от 04.05.2017 г. (Время и номер первичной постановки на учет №766 от 22.04.1992 г.). Язык издания: английский. Выходит 4 раза в год.
Тематическая направленность: теоретическая и прикладная математика, механика, информатика.

Индексируется и участвует:

Комитет по обеспечению качества в сфере образования и науки МОН РК
НАЦИОНАЛЬНЫЙ ЦЕНТР ГОСУДАРСТВЕННОЙ НАУЧНО-ТЕХНИЧЕСКОЙ ЭКСПЕРТИЗЫ
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Научное издание
Вестник КазНУ. Серия "Математика, механика, информатика", № 2 (126) 2025.
Редактор – А.Ғ. Шәкір. Компьютерная верстка – А.Ғ. Шәкір

ИБ N 15711
Формат 60 × 84 1/8. Бумага офсетная. Печать цифровая. Объем 13,7 п.л.
Заказ N 290. Издательский дом "Қазақ университетi"
Казахского национального университета им. аль-Фараби. 050040, г. Алматы, пр.аль-Фараби, 71, КазНУ.
Отпечатано в типографии издательского дома "Қазақ университетi".

1-бөлім Раздел 1 Section 1

Математика Математика Mathematics

IRSTI 27.31.21; 27.31.55 DOI: https://doi.org/10.26577/JMMCS2025126201

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HOMOGENIZATION OF ATTRACTORS TO THE REACTION-DIFFUSION SYSTEM IN A DOMAIN WITH ROUGH BOUNDARY

In this paper, we consider the homogenization problem in a micro inhomogeneous domain with a rapidly oscillating boundary. It is assumed that a system of nonlinear reaction–diffusion equations with rapidly oscillating terms and dissipation is considered in the domain. On the locally periodic oscillating part of the boundary, the third boundary condition with rapidly oscillating coefficients and a small parameter characterizing the oscillation of the boundary to some degree is imposed. Depending on the degree of the small parameter in the boundary condition, various homogenized (limit) problems are obtained and the convergence of the trajectory attractors of the given system to the attractors of the homogenized system is proved. Critical, subcritical and supercritical cases of attractor behavior as the small parameter tends to zero are carefully studied. The paper also considers problems in a domain with a random rapidly oscillating boundary. In this case, a homogenized system of reaction–diffusion equations with deterministic coefficients is obtained in the case of a statistically homogeneous random structure of the boundary. A theorem on the convergence of random trajectory attractors of the initial given system of reaction-diffusion equations to a deterministic attractor of the homogenized (limit) system of reaction–diffusion equations is also proved. The paper also proves the convergence of global attractors in the case of uniqueness of solutions, which in turn is proved for nonlinearity in a system of equations of a special type.

Key words: attractors, homogenization, reaction-diffusion equations, non-linear equations, weak convergence, rapidly oscillating boundary.

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Тез тербелмелі шекарасы бар аймақтардағы реакция-диффузия теңдеулерінің аттракторларының орташалауы

Бұл жұмыста шекарасы тез тербелетін микро біртекті емес аймақта орташалау мәселесі қарастырылады. Аймақта жылдам тербелетін мүшелері мен диссипациясы бар сызықты емес реакция-диффузия теңдеулер жүйесі зерттелінген. Шекараның локальды тербелмелі бөлігінде жылдам тербелмелі коэффициенттері бар үшінші шекаралық шарт және шекараның белгілі бір дәрежеде тербелісін сипаттайтын шағын параметр белгіленеді. Шектік жағдайдағы кіші параметр дәрежесіне байланысты әртүрлі орташаланған (шектік) есептер алынды және бастапқы жүйенің траекториялық аттракторларының орташаланған жүйенің аттракторларына жинақталуы дәлелденді. Кіші параметр нөлге ұмтылған кезде аттракторлардың ерекшеліктері критикалық, субкритикалық және суперкритикалық жағдайлары мұқият зерттелінді. Мақалада сонымен қатар кездейсоқ, жылдам тербелетін шекарасы бар аймақтағы мәселелер қарастырылады. Бұл жағдайда шекараның статистикалық біртекті кездейсоқ құрылымы жағдайында детерминирленген коэффициенттері бар реакция-диффузия теңдеулерінің орташаланған жүйесі алынды. Реакция-диффузия теңдеулерінің бастапқы жүйесінің кездейсоқ траекториялық аттракторларының орташаланған (шектік) реакция-диффузия теңдеулер жүйесінің кездейсоқ емес есебінің аттракторына жинақталуы туралы теоремасы дәлелденген. Жұмыс сондай-ақ бірегей шешімдер жағдайында глобалды аттракторлардың жинақталуын дәлелденді, бұл жағдай сызықтық емес мүшелерге қосымша шарт қойылған кезде пайда болады.
Түйін сөздер: аттракторлар, орташалау, реакция-диффузия теңдеулері, сызықтық емес теңдеулер, әлсіз жинақтылық, тез тербелмелі шекара.

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Об усреднении аттракторов уравнений реакции-диффузии в области с шероховатой границей

В данной работе рассматривается задача усреднения в микро неоднородной области с быстро осциллирующей границей. Предполагается, что в области задана система нелинейных уравнений реакции–диффузии с быстро осциллирующими членами и диссипацией. На локально периодической осциллирующей части границы выставлено третье краевое условие с быстро осциллирующими коэффициентами и малым параметром, характеризующим осцилляцию границы, в некоторой степени. В зависимости от степени малого параметра в краевом условии получены различные усреднённые (предельные) задачи и доказана сходимость траекторных аттракторов исходной системы к аттракторам усреднённой системы. Аккуратно исследованы критический, субкритический и суперкритический случаи поведения аттракторов при стремлении малого параметра к нулю. В статье рассмотрены также задачи в области со случайной быстро осциллирующей границей. При этом получена усреднённая система уравнений реакции–диффузии с детерминированными коэффициентами в случае статистически однородной случайной структурой границы. Также доказана теорема о сходимости случайных траекторных аттракторов исходной системы уравнений реакции–диффузии к детерминированному аттрактору усреднённой (предельной) системы уравнений реакции–диффузии.В работе также доказана сходимость и глобальных аттракторов в случае единственности решений, которая в свою очередь доказана для нелинейности в системе уравнений специального вида.

Ключевые слова: аттракторы, усреднение, уравнения реакции-диффузии, нелинейные уравнения, слабая сходимость, быстро осциллирующая граница.

1 Introduction

In this paper, we present a review of results on homogenization of initial–boundary value problems for a system of reaction–diffusion equations in domains with a rapidly oscillating boundary (for detailed geometric formulations, see [3] and [40]). We consider nonlinear systems of reaction–diffusion equations in such a domain with a locally periodic and random rapidly oscillating boundary, and investigate the case of dissipative coefficients in the equations. We prove the existence of trajectory attractors, construct a limit (homogenized) system of reaction–diffusion equations both in the case of a locally periodic and in the case of a statistically homogeneous random boundary, and prove the convergence of the attractors of the original system as the small parameter characterizing the boundary oscillation, tends to zero, i.e. prove the Hausdorff convergence of the attractors of the original system to the attractors of the homogenized (limit) system as the small parameter tends to zero. In many purely mathematical works one can find asymptotic analysis of problems in domains with oscillating (rough) boundaries (see, for example, [1–10]). We also mention here the fundamental works on this topic [11–14], where one can find a detailed bibliography. A special feature of the second part of the work is the random geometry of the domain (see some examples in [37–39]). It is assumed that the random structure is statistically homogeneous. This fact allows us to obtain a deterministic limit problem (see [40]), which does not depend on the choice of an element of the probability space. Theoretical results on attractor averaging can be found, for example, in [15–17], and see references therein. Attractor averaging was also studied in [17–20] (see also [21–24]).

In this paper, we establish weak convergence (in the sense of "almost surely" in the probabilistic case, i.e. with probability one) of the trajectory attractor $\mathfrak{A}_\varepsilon$ of reaction–diffusion systems in domains with an oscillating boundary, for $\varepsilon \to 0$, to the trajectory attractors $\overline{\mathfrak{A}}$ of homogenized systems in some natural functional space. Here the small parameter ε characterizes the period and amplitude of the oscillations. The parameter ε also appears to some power in the third boundary condition on a part of the locally periodic boundary, and in the limit in the locally periodic case we obtain 3 different homogenized problems (critical, subcritical and supercritical cases) depending on the ratio between the powers of the small parameter. In the random formulation of the problem ε also characterizes the microinhomogeneity on the boundary.

In the second section of the paper one can find the main preliminary results on attractors and random sets, the third section is devoted to homogenization in the locally periodic case. In the fourth section we present the results of homogenization when the boundary has a random structure.

2 Preliminary information.

2.1 Trajectory attractors of evolution equations

This section is devoted to the construction of trajectory attractors to autonomous evolution equations (see details in [17]).

Consider an autonomous evolution equation of the form for $t>0$

$$\frac{\partial u}{\partial t} = A(u). \tag{1}$$

Here $A(\cdot): E_1 \to E_0$ is a nonlinear operator, E_1, E_0 are Banach spaces and $E_1 \subseteq E_0$. As an example one can consider $A(u) = \lambda\Delta u - a(\cdot)f(u) + h(\cdot)$.

We study weak solutions $u(t)$ to (1) as functions $t \in \mathbb{R}_+$ as a whole. The set of solutions of (1) is said to be *a trajectory space* $\mathcal{K}^+$ of equation (1). Now, we describe the trajectory space $\mathcal{K}^+$ in detail.

Consider solutions $u(t)$ of (1) defined on $[t_1, t_2] \subset \mathbb{R}$. We consider solutions to problem (1) in a Banach space $\mathcal{F}_{t_1,t_2}$. The space $\mathcal{F}_{t_1,t_2}$ is a set $f(s), s \in [t_1, t_2]$ satisfying $f(t) \in E$ for almost all $t \in [t_1, t_2]$, where E is a Banach space, satisfying $E_1 \subseteq E \subseteq E_0$.

For instance, $\mathcal{F}_{t_1,t_2}$ can be considered as the intersection spaces $C([t_1, t_2]; E)$, or $L_p(t_1, t_2; E)$, for $p \in [1, \infty]$. Suppose that $\Pi_{t_1,t_2}\mathcal{F}_{\tau_1,\tau_2} \subseteq \mathcal{F}_{t_1,t_2}$ and $\|\Pi_{t_1,t_2} f\|_{\mathcal{F}_{t_1,t_2}} \leq C(t_1, t_2, \tau_1, \tau_2)\|f\|_{\mathcal{F}_{\tau_1,\tau_2}}$ $\forall f \in \mathcal{F}_{\tau_1,\tau_2}$. Here $[t_1, t_2] \subseteq [\tau_1, \tau_2]$ and Π_{t_1,t_2} denotes the restriction operator onto $[t_1, t_2]$, constant $C(t_1, t_2, \tau_1, \tau_2)$ does not depend on f.

Denote by $S(\tau)$ for $\tau \in \mathbb{R}$ the translation operator $S(\tau)f(t) = f(\tau + t)$. It is easy to see, that if the argument t of $f(\cdot)$ belongs to the segment $[t_1, t_2]$, then the argument t of $S(\tau)f(\cdot)$ belongs to $[t_1 - \tau, t_2 - \tau]$ for $\tau \in \mathbb{R}$. Suppose that the mapping $S(\tau)$ is an isomorphism from F_{t_1,t_2} to $F_{t_1-\tau,t_2-\tau}$ and $\|S(\tau)f\|_{\mathcal{F}_{t_1-\tau,t_2-\tau}} = \|f\|_{\mathcal{F}t_1,t_2}$ $\forall f \in \mathcal{F}_{t_1,t_2}$. It is easy to see that this assumption is natural.

Suppose that if $f(t) \in \mathcal{F}_{t_1,t_2}$, then $A(f(t)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{D}_{t_1,t_2}$ is a Banach space, which is larger, $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The derivative $\frac{\partial f(t)}{\partial t}$ is a distribution with values in E_0, $\frac{\partial f}{\partial t} \in D'((t_1, t_2); E_0)$ and we suppose that $\mathcal{D}_{t_1,t_2} \subseteq D'((t_1, t_2); E_0)$ for all $(t_1, t_2) \subset \mathbb{R}$. A function $u(t) \in \mathcal{F}_{t_1,t_2}$ is a *solution* of (1), if $\frac{\partial u}{\partial t}(t) = A(u(t))$ in the sense of $D'((t_1, t_2); E_0)$.

Let us define the space $\mathcal{F}_+^{loc} = \{f(t),\, t \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(t) \in \mathcal{F}_{t_1,t_2}, \quad \forall\, [t_1, t_2] \subset \mathbb{R}_+\}$. For instance, if $\mathcal{F}_{t_1,t_2} = C([t_1, t_2]; E)$, then $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$ and if $\mathcal{F}_{t_1,t_2} = L_p(t_1, t_2; E)$, then $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(t) \in \mathcal{F}_+^{loc}$ is a solution of (1), if $\Pi_{t_1,t_2}u(t) \in \mathcal{F}_{t_1,t_2}$ and $u(t)$ is a solution of (1) for every $[t_1, t_2] \subset \mathbb{R}_+$.

Let $\mathcal{K}^+$ be a set of solutions to (1) from $\mathcal{F}_+^{loc}$. Note, that $\mathcal{K}^+$ in general is not the set of *all* solutions from $\mathcal{F}_+^{loc}$. The set $\mathcal{K}^+$ consists on elements, which are *trajectories* and the set $\mathcal{K}^+$ is the *trajectory space* of the equation (1).

Suppose that the trajectory space $\mathcal{K}^+$ is *translation invariant*, i.e., if $u(t) \in \mathcal{K}^+$, then $u(\tau + t) \in \mathcal{K}^+$ for every $\tau \geq 0$.

Consider the translation operators $S(\tau)$ in $\mathcal{F}_+^{loc}: S(\tau)f(t) = f(\tau+t), \tau \geq 0$. It is easy to see that the map $\{S(\tau), \tau \geq 0\}$ forms a semigroup in $\mathcal{F}_+^{loc}:\ S(\tau_1)S(\tau_2) = S(\tau_1+\tau_2)$ for $\tau_1, \tau_2 \geq 0$ and in addition $S(0)$ is the identity operator. The *translation semigroup* $\{S(\tau), \tau \geq 0\}$ maps the trajectory space $\mathcal{K}^+$ to itself: $S(\tau)\mathcal{K}^+ \subseteq \mathcal{K}^+$ for all $\tau \geq 0$.

We investigate attracting properties of the translation semigroup $\{S(\tau)\}$ acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. Next step is to define a topology in the space $\mathcal{F}_+^{loc}$.

Let some metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ be defined on $\mathcal{F}_{t_1,t_2}$ for every $[t_1, t_2] \subset \mathbb{R}$. Suppose that

$$\rho_{t_1,t_2}\left(\Pi_{t_1,t_2} f, \Pi_{t_1,t_2} g\right) \leq D(t_1, t_2, \tau_1, \tau_2)\rho_{\tau_1,\tau_2}(f, g) \quad \forall f, g \in \mathcal{F}_{\tau_1,\tau_2},\ [t_1, t_2] \subseteq [\tau_1, \tau_2],$$

$$\rho_{t_1-\tau,t_2-\tau}(S(\tau)f, S(\tau)g) = \rho_{t_1,t_2}(f, g) \quad \forall f, g \in \mathcal{F}_{t_1,t_2},\ [t_1, t_2] \subset \mathbb{R},\ \tau \in \mathbb{R}.$$

Now, we denote by Θ_{t_1,t_2} metric spaces on $\mathcal{F}_{t_1,t_2}$. For instance, ρ_{t_1,t_2} is metric associated with the norm $\|\cdot\|_{\mathcal{F}_{t_1,t_2}}$ of $\mathcal{F}_{t_1,t_2}$. On the other hand, in application ρ_{t_1,t_2} generates the topology Θ_{t_1,t_2} that is weaker than the strong one of the $\mathcal{F}_{t_1,t_2}$.

The *projective limit* of the spaces Θ_{t_1,t_2} defines the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$, that is, by definition, a sequence $\{f_k(t)\} \subset \mathcal{F}_+^{loc}$ tends to $f(t) \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\rho_{t_1,t_2}(\Pi_{t_1,t_2} f_k, \Pi_{t_1,t_2} f) \to 0$ as $k \to \infty$ for all $[t_1, t_2] \subset \mathbb{R}_+$. It is possible to show that the topology Θ_+^{loc} is metrizable. For this aim we use, for example, the Frechet metric

$$\rho_+(f_1, f_2) := \sum_{m\in\mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1, f_2)}{1 + \rho_{0,m}(f_1, f_2)}. \tag{2}$$

The translation semigroup $\{S(\tau)\}$ is continuous in Θ_+^{loc}. This statement follows from the definition of Θ_+^{loc}.

We also define the following Banach space $\mathcal{F}_+^b := \{f(t) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\}$, where the norm $\|f\|_{\mathcal{F}_+^b} := \sup_{\tau\geq 0} \|\Pi_{0,1} f(\tau + t)\|_{\mathcal{F}_{0,1}}$.

We remember that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. We need from our Banach space $\mathcal{F}_+^b$ only one fact that it should define bounded subsets in the trajectory space $\mathcal{K}^+$. For constructing a trajectory attractor in $\mathcal{K}^+$, instead of considering the corresponding uniform convergence topology of the Banach space $\mathcal{F}_+^b$, we use much weaker topology, i.e. the local convergence topology Θ_+^{loc}.

Assume that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, that is, every trajectory $u(t) \in \mathcal{K}^+$ of equation (1) has a finite norm. We define an attracting set and a trajectory attractor of the translation semigroup $\{S(\tau)\}$ acting on $\mathcal{K}^+$.

Definition 1 *A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an* attracting set *of the semigroup $\{S(\tau)\}$ acting on $\mathcal{K}^+$ in the topology Θ_+^{loc} if for any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{K}^+$ the set $\mathcal{P}$ attracts $S(\tau)\mathcal{B}$ as $\tau \to +\infty$ in the topology Θ_+^{loc}, i.e., for any ε-neighbourhood $O_\varepsilon(\mathcal{P})$ in Θ_+^{loc} there exists $\tau_1 \geq 0$ such that $S(\tau)\mathcal{B} \subseteq O_\varepsilon(\mathcal{P})$ for all $\tau \geq \tau_1$.*

It is easy to see that the attracting property of $\mathcal{P}$ can be formulated equivalently: we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{P}) \longrightarrow 0 \quad (\tau \to +\infty),$$

where $\mathrm{dist}_{\mathcal{M}}(X, Y) := \sup_{x\in X} \mathrm{dist}_{\mathcal{M}}(x, Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathcal{M}}(x, y)$ is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$. We remember that the Hausdorff semidistance is not symmetric, for any $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$.

Definition 2 ([17]) *A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the* trajectory attractor *of the translation semigroup $\{S(\tau)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if*

(i) $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc},

(ii) the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(\tau)\mathfrak{A} = \mathfrak{A}$ for all $\tau \geq 0$,

(iii) $\mathfrak{A}$ is an attracting set for $\{S(\tau)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$ we have $\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(\tau)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (\tau \to +\infty)$.

Let us formulate the main assertion on the trajectory attractor for equation (1).

Theorem 1 ([16, 17]) *Assume that the trajectory space $\mathcal{K}^+$ corresponding to equation (1) is contained in $\mathcal{F}_+^b$. Suppose that the translation semigroup $\{S(t)\}$ has an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$*

which is bounded in $\mathcal{F}_+^b$ *and compact in* Θ_+^{loc}. *Then the translation semigroup* $\{S(\tau), \tau \geq 0\}$ *acting on* $\mathcal{K}^+$ *has the trajectory attractor* $\mathfrak{A} \subseteq \mathcal{P}$. *The set* $\mathfrak{A}$ *is bounded in* $\mathcal{F}_+^b$ *and compact in* Θ_+^{loc}.

Let us describe in detail, i.e., in terms of complete trajectories of the equation, the structure of the trajectory attractor $\mathfrak{A}$ to equation (1). We study the equation (1) on the time axis,i.e. $t \in \mathbb{R}$.

Note that the trajectory space $\mathcal{K}^+$ of equation (1) on $\mathbb{R}_+$ have been defined. We need this notion on the entire $\mathbb{R}$. If a function $f(t)$, $s \in \mathbb{R}$, is defined on the entire time axis, then the translations $S(\tau)f(t) = f(\tau + t)$ are also defined for negative τ. A function $u(t), t \in \mathbb{R}$ is a *complete trajectory* of equation (1) if $\Pi_+ u(\tau + t) \in \mathcal{K}^+$ for all $\tau \in \mathbb{R}$. Here $\Pi_+ = \Pi_{0,\infty}$ denotes the restriction operator to $\mathbb{R}_+$.

We have $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$, and Θ_+^{loc}. Let us define spaces $\mathcal{F}^{loc}, \mathcal{F}^b$, and Θ^{loc} in the same way:

$$\mathcal{F}^{loc} := \{f(t), t \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2} \ \forall \ [t_1, t_2] \subseteq \mathbb{R}\}; \quad \mathcal{F}^b := \{f(t) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h \in \mathbb{R}} \|\Pi_{0,1} f(\tau + t)\|_{\mathcal{F}_{0,1}}. \tag{3}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$ and, by definition, $f_k(t) \to f(t)$ $(k \to \infty)$ in Θ^{loc} if $\Pi_{t_1,t_2} f_k(t) \to \Pi_{t_1,t_2} f(t)$ $(k \to \infty)$ in Θ_{t_1,t_2} for each $[t_1, t_2] \subseteq \mathbb{R}$. It is easy to see that Θ^{loc} is a metric space as well as Θ_+^{loc}.

Definition 3 *The* kernel $\mathcal{K}$ *in the space* $\mathcal{F}^b$ *of equation* (1) *is the union of all complete trajectories* $u(t), t \in \mathbb{R}$, *of equation* (1) *that are bounded in the space* $\mathcal{F}^b$ *with respect to the norm* (3), *i.e.* $\|\Pi_{0,1} u(\tau + t)\|_{\mathcal{F}_{0,1}} \leq C_u \quad \forall \tau \in \mathbb{R}$.

Theorem 2 *Assume that the hypotheses of Theorem* 1 *holds. Then* $\mathfrak{A} = \Pi_+ \mathcal{K}$, *the set* $\mathcal{K}$ *is compact in* Θ^{loc} *and bounded in* $\mathcal{F}^b$.

To prove this assertion one can use the approach from [17].

In this paper we investigate evolution equations and their trajectory attractors depending on a small parameter $\varepsilon > 0$.

Definition 4 *We say that the trajectory attractors* $\mathfrak{A}_\varepsilon$ *converge to the trajectory attractor* $\overline{\mathfrak{A}}$ *as* $\varepsilon \to 0$ *in the topological space* Θ_+^{loc} *if for any neighbourhood* $\mathcal{O}(\overline{\mathfrak{A}})$ *in* Θ_+^{loc} *there is an* $\varepsilon_1 \geq 0$ *such that* $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ *for any* $\varepsilon < \varepsilon_1$, *that is, for each* $M > 0$ *we have* $\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} \mathfrak{A}_\varepsilon, \Pi_{0,M} \overline{\mathfrak{A}}) \to 0$ $(\varepsilon \to 0)$.

2.2 The probabilistic framework and main assumptions

Throughout the paper, we assume that all the random fields and random variables are defined on a probability space $(\Omega, \mathcal{A}, \mu)$. The random fields considered in the paper are statistically homogeneous.

Definition 5 A family of measurable maps $T_x : \Omega \to \Omega$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, is called a *d-dynamical system* if the following properties hold true:

- Group property:

$$T_{x+y} = T_x T_y \ \ \forall x, y \in \mathbb{R}^d, \ \ T_0 = Id \ \ (Id \text{ is the identical mapping});$$

- Isometry property:

$$T_x \mathcal{U} \in \mathcal{A}, \quad \mu(T_x \mathcal{U}) = \mu(\mathcal{U}), \ \ \forall x \in \mathbb{R}^d, \ \ \forall \mathcal{U} \in \mathcal{A};$$

- Measurability: for any measurable functions $\phi(\omega)$ on Ω, the function $\phi(T_x\omega)$ is measurable on $\Omega \times \mathbb{R}^d$, where the space $\mathbb{R}^d$ is equipped with the Borel σ-algebra $\mathcal{B}$.

Definition 6 Let $\phi(\omega)$ be a measurable function (i.e. a random variable) on Ω. The function $\phi(T_x\omega)$ of $x \in \mathbb{R}^d$ and $\omega \in \Omega$ is called *statistically homogeneous random field*, and, for fixed $\omega \in \Omega$, $\phi(T_x\omega)$ is called the *realization* of the random field ϕ.

Let $L_q(\Omega)$ $(q \geq 1)$ be the space of measurable functions and integrable in the power q with respect to the measure μ. The following assertion holds, see [14] and [13] for the proof.

Proposition 1 *Assume that $\phi \in L_q(\Omega)$. Then almost all realizations $\phi(T_x\omega)$ belong to $L_q^{loc}(\mathbb{R}^d)$. If the sequence $\{\phi_k\} \subset L_q(\Omega)$ converges in $L_q(\Omega)$ to the function ϕ, then there exists a subsequence $\{\phi_{k'}\}$ such that almost all realizations $\phi_{k'}(T_x\omega)$ converge in $L_q^{loc}(\mathbb{R}^d)$ to the realization $\phi(T_x\omega)$.*

Definition 7 A measurable function $\phi(\omega)$ on Ω is called *invariant* if, for any $x \in \mathbb{R}^d$, $\phi(T_x\omega) = \phi(\omega)$ almost surely.

Definition 8 A d-dynamical system T_x is said to be *ergodic* if all its invariant functions are almost surely constant.

Definition 9 Let $\theta \in L_1^{loc}(\mathbb{R}^d)$. We say that the function θ has a spatial average if the limit

$$M(\theta) = \lim_{\varepsilon \to 0} \frac{1}{|B|} \int\limits_B \theta\left(\frac{x}{\varepsilon}\right) \ dx$$

exists for any bounded Borel set $B \in \mathcal{B}$ with $|B| > 0$, and moreover this limit does not depend on the choice of B. The quantity $M(\theta)$ is called the *spatial average* of the function θ.

The following results are proved in [14].

Proposition 2 *Let P be a measurable subset of $\mathbb{R}^d$ containing a neighbourhood of the origin. Let $q \geq 1$ or $q = \infty$. Suppose that a measurable function $\theta(x, \xi)$, $x \in P$, $\xi \in \mathbb{R}^d$, has a space mean value $M(\theta)(x)$ in $\mathbb{R}^d_\xi$ (that is, with respect to the variable ξ) for every $x \in P$ and the family $\{\theta\big(x, \frac{x}{\varepsilon}\big),\ 0 < \varepsilon \leqslant 1\}, x \in \mathcal{K}$, is bounded in $L_q(\mathcal{K})$, where $\mathcal{K}$ is an arbitrary bounded subset in P containing a neighbourhood of the origin.*

Then $M(\theta)(\cdot) \in L_q^{loc}(P)$ and, for $q \geq 1$, we have $\theta\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup M(\theta)(x)$ weakly in $L_q^{loc}(P)$ as $\varepsilon \to 0$,

while, for $q = \infty$, we have $\theta\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup M(\theta)(x)$ $*$-weakly in $L_\infty^{loc}(P)$ as $\varepsilon \to 0$.

From now on we use the following notation $\widehat{x} = (x_1, \ldots, x_{d-1})$. For a given group $T_x, x \in \mathbb{R}^d$, we also consider its subgroup $T_{\widehat{x}} : \Omega \to \Omega,\ \ \widehat{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1},\ T_{\widehat{x}} = T_{(\widehat{x},0)}$.

Let T_x be a d-dynamical system in Ω. Wa assume that $T_{\widehat{x}}$ is also a $(d-1)$-dynamical system in Ω (see Definition 5 with the number d replaced with $(d-1)$).

All the above Definitions and Propositions hold true for the $(d-1)$-dynamical system $T_{\widehat{x}}$ with evident modifications. In particular, we have the following

Definition 10 A random field $\zeta(\widehat{x}, \omega)$ ($\widehat{x} \in \mathbb{R}^{d-1}$, $\omega \in \Omega$) is called *statistically homogeneous* if the following representation holds $\zeta(\widehat{x}, \omega) = \zeta(T_{\widehat{x}}\omega)$, where ζ is a random variable on $(\Omega, \mathcal{A}, \mu)$ and $T_{\widehat{x}}$ is a $(d-1)$-dynamical system on Ω.

All along the article, we make use of the Birkhoff ergodic theorem in the following particular form (see, for instance, [14] and [13] for more details).

Theorem 3 (Birkhoff ergodic theorem) *Let T_x, $x \in \mathbb{R}^d$, be a d-dynamical system and let $\psi(\omega) \in L_1(\Omega, \mu)$. Then, for almost all $\omega \in \Omega$, the realization $\psi(T_x\omega)$ has the space mean value $M(\psi(T_x\omega))$ in $\mathbb{R}^d$. Moreover, $M(\psi(T_x\omega))$ is an invariant function and*

$$\mathbb{E}(\psi) \equiv \int_\Omega \psi(\omega)\ d\mu = \int_\Omega M(\psi(T_x\omega))\ d\mu,$$

where $\mathbb{E}(\phi)$ is the mathematical expectation of ψ. In particular, if T_x is ergodic then, for almost all $\omega \in \Omega$, we have the identity

$$\mathbb{E}(\psi) = M(\psi(T_x\omega)).$$

We shall also apply Birkhoff ergodic theorem to the ergodic $(d-1)$-dynamical system $T_{\widehat{x}},\ \widehat{x} \in \mathbb{R}^{d-1}$.

We are now ready to make assumptions on the random fields $F(\widehat{\xi}, \omega)$, $p(\widehat{\xi}, \omega)$ and $q(\widehat{\xi}, \omega)$ which we use in the definition of the stochastic geometry and coefficients in the Fourier boundary condition. First, we assume that these random fields are statistically homogeneous, that is

$$F(\widehat{\xi}, \omega) = \mathbf{F}(T_{\widehat{\xi}}\omega), \quad p(\widehat{\xi}, \omega) = \rho(T_{\widehat{\xi}}\omega), \quad q(\widehat{\xi}, \omega) = \varrho(T_{\widehat{\xi}}\omega), \quad \forall \widehat{\xi} \in \mathbb{R}^{d-1},$$

where $\mathbf{F}$, ρ and ϱ are random variables on $(\Omega, \mathcal{A}, \mu)$, and $T_{\widehat{x}}$ is an ergodic $(d-1)$-dynamical system on Ω.

Moreover, we assume that $\mathbf{F}$ has, almost surely, continuously differentiable or locally Lipschitz realizations. We denote

$$\partial^i_\omega \mathbf{F}(\omega) = \partial_{\xi_i} \mathbf{F}(T_{\widehat{\xi}}\omega)\big|_{\widehat{\xi}=0}, \quad \partial_\omega \mathbf{F}(\omega) = \nabla_{\widehat{\xi}} \mathbf{F}(T_{\widehat{\xi}}\omega)\big|_{\widehat{\xi}=0}.$$

We have $\nabla_{\widehat{\xi}} F(\widehat{\xi}, \omega) = \partial_\omega \mathbf{F}(T_{\widehat{\xi}}\omega)$ (see, for instance, [13]). Finally, we make the following assumptions on the functions $\mathbf{F}$, ρ and ϱ:

(h1) $\mathbf{F} \in L_\infty(\Omega),\ \ \mathbf{F}(\omega) \le 0$ a.s.;

(h2) $\partial_\omega \mathbf{F} \in (L_2(\Omega))^{d-1}$;

(h3) $\rho \in L_\infty(\Omega),\ \ \rho(\omega) \ge 0$ a.s., $\ \mu\{\omega\ :\ \rho(\omega) > 0\} > 0$.

(h4) $\varrho \in L_2(\Omega), \quad \varrho\partial_\omega \mathbf{F} \in (L_2(\Omega))^{d-1}$;

3 Homogenezation of attractors to the reaction-diffusion system in a domain with locally periodic oscillating boundary

3.1 Statement of the problem

Let D be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with smooth boundary $\partial D = \Gamma_1 \cup \Gamma_2$, where D lies in a half-space $x_d > 0$ and $\Gamma_1 \subset \{x : x_d = 0\}$. Given smooth nonpositive 1-periodic in the $\hat{\xi}$ function $F(\hat{x}, \hat{\xi})$, $\hat{x} = (x_1, ..., x_{d-1}), \hat{\xi} = (\xi_1, ..., \xi_{d-1})$, define the domain D_ε as follows: $\partial D_\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2$, where we set $\Gamma_1^\varepsilon = \{x = (\hat{x}, x_d) : (\hat{x}, 0) \in \Gamma_1, x_d = \varepsilon^\alpha F(\hat{x}, \hat{x}/\varepsilon)\}$, $0 < \alpha < 1$, i.e. we add thin oscillating layer $\Pi_\varepsilon = \{x = (\hat{x}, x_d) : (\hat{x}, 0) \in \Gamma_1, x_d \in [0, \varepsilon^\alpha F(\hat{x}, \hat{x}/\varepsilon))\}$ to the domain D. Usually, we assume $F(\hat{x}, \hat{\xi})$ to be compactly supported on Γ_1 uniformly in $\hat{\xi}$. Consider the following boundary-value problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \lambda \Delta u_\varepsilon - a\left(x, \frac{x}{\varepsilon}\right) f(u_\varepsilon) + h\left(x, \frac{x}{\varepsilon}\right), & x \in D_\varepsilon, t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^\beta p(\hat{x}, \frac{\hat{x}}{\varepsilon}) u_\varepsilon = \varepsilon^{1-\alpha} g(\hat{x}, \frac{\hat{x}}{\varepsilon}), & x = (\hat{x}, x_d) \in \Gamma_1^\varepsilon, t > 0, \\ u_\varepsilon = 0, & x \in \Gamma_2, t > 0, \\ u_\varepsilon = U(x), & x \in D_\varepsilon, t = 0, \end{cases} \tag{4}$$

where $u_\varepsilon = u_\varepsilon(x,t) = (u^1, \ldots, u^n)^\top$ is an unknown vector function, the nonlinear function $f = (f^1, \ldots, f^n)^\top$ is given, $h = (h^1, \ldots, h^n)^\top$ is the known right-hand side function, and λ is an $n \times n$-matrix with constant coefficients, having a positive symmetrical part: $\frac{1}{2}(\lambda + \lambda^\top) \geq \varpi I$ (where I is the unit matrix with dimension n). We assume that $\beta > 0, p\left(\hat{x}, \hat{\xi}\right) = \operatorname{diag}\{p^1, \ldots, p^n\}$, $g\left(\hat{x}, \hat{\xi}\right) = (g^1, \ldots, g^n)^\top$ are continuous, 1-periodic in $\hat{\xi}$ and $p^i\left(\hat{x}, \hat{\xi}\right)$, $i = 1, \ldots n$, are positive. Here $\dfrac{\partial u_\varepsilon}{\partial \nu} = \left(\dfrac{\partial u_\varepsilon^1}{\partial \nu}, \ldots, \dfrac{\partial u_\varepsilon^n}{\partial \nu}\right)^\top$ is the normal derivative of the vector function u_ε multiplied by the matrix λ, where $\dfrac{\partial u_\varepsilon^i}{\partial \nu} := \displaystyle\sum_{j=1}^n \sum_{k=1}^d \lambda_{ij} \frac{\partial u_\varepsilon^j}{\partial x_k} N_k$, $i = 1, \ldots, n$ and $N = (N_1, \ldots, N_d)$ is the unit outer normal to the boundary of the domain.

Function $a(x, \xi) \in C(\overline{D}_\varepsilon \times \mathbb{R}^d)$ such that $0 < a_0 \leq a(x, \xi) \leq A_0$ with some coefficient a_0, A_0. Assuming that function $a_\varepsilon(x) = a\left(x, \frac{x}{\varepsilon}\right)$ has average $\overline{a}(x)$ when $\varepsilon \to 0+$ in space $L_{\infty,*w}(D)$, that is

$$\int_D a\left(x, \frac{x}{\varepsilon}\right) \varphi(x) dx \to \int_D \overline{a}(x) \varphi(x) dx \quad (\varepsilon \to 0+) \tag{5}$$

for any function $\varphi \in L_1(D)$.

Denote by D^+ such a domain that $D_\varepsilon \subset D^+$ for any ε. For the vector function $h\left(x, \xi\right)$, assume that for any $\varepsilon > 0$ the function $h_\varepsilon^i(x) = h^i\left(x, \frac{x}{\varepsilon}\right) \in L_2(D^+)$ and has the average $\overline{h^i}(x)$ in the space $L_2(D^+)$ for $\varepsilon \to 0+$, that is

$$h^i\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup \overline{h^i}(x) \quad (\varepsilon \to 0+) \text{ weakly in } L_2(D^+),$$

or

$$\int_{D^+} h^i\left(x, \frac{x}{\varepsilon}\right) \varphi(x) dx \to \int_{D^+} \overline{h^i}(x) \varphi(x) dx \quad (\varepsilon \to 0+) \tag{6}$$

for any function $\varphi \in L_2(D^+)$ and for all $i = 1, \dots, n$.

From the condition (6) it follows that the norm of the function $h^i_\varepsilon(x)$ are bounded uniformly in ε, in the space $L_2(D_\varepsilon)$, i.e. $\|h^i_\varepsilon(x)\|_{L_2(D_\varepsilon)} \le M_0, \quad \forall \varepsilon \in (0,1]$.

It is assumed that the vector function $f(v) \in C(\mathbb{R}^n;\mathbb{R}^n)$ satisfies the following inequalities

$$\sum_{i=1}^{n} |f^i(v)|^{p_i/(p_i-1)} \le C_0 \left(\sum_{i=1}^{n} |v^i|^{p_i} + 1 \right), \quad 2 \le p_1 \le \ldots \le p_{n-1} \le p_n, \tag{7}$$

$$\sum_{i=1}^{n} \gamma_i |v^i|^{p_i} - C \le \sum_{i=1}^{n} f^i(v) v^i, \quad \forall v \in \mathbb{R}^n, \tag{8}$$

for $\gamma_i > 0$ for any $i = 1, \dots, n$. The inequality (7) is due to the fact that in real reaction-diffusion systems, the functions $f^i(u)$ are polynomials with possibly different degrees. Inequality (8) calls *dissipativity condition* for the reaction-diffusion system (4). In a simple model case $p_i \equiv p$ for any $i = 1, \dots, n$, condition (7) and (8) reduce to the following inequalities

$$|f(v)| \le C_0 \left(|v|^{p-1} + 1 \right), \quad \gamma |v|^p - C \le f(v)v, \quad \forall v \in \mathbb{R}^n.$$

Note that the fulfillment of the Lipschitz condition for the function $f(v)$ relative to the variable v *not expected.*

Remark 1 *Using the methods presented, it is also possible to study systems in which nonlinear terms have the form* $\sum_{j=1}^{m} a_j \left(x, \frac{x}{\varepsilon}\right) f_j(u)$, *where* a_j *are matrices whose elements allow averaging and* $f_j(u)$ *polynomial vectors of* u, *which satisfy conditions of the form* (7)–(8). *For brevity, we study the case* $m = 1$ *and* $a_1\left(x, \frac{x}{\varepsilon}\right) = a\left(x, \frac{x}{\varepsilon}\right) I$, *where* I *is the identity matrix.*

Denote

$$G(\hat{x}) = \int\limits_{[0,1)^{d-1}} \sqrt{|\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\, g(\hat{x}, \hat{\xi})\, d\hat{\xi}, \tag{9}$$

$$P(\hat{x}) = \int\limits_{[0,1)^{d-1}} \sqrt{|\nabla_{\hat{\xi}} F(\hat{x}, \hat{\xi})|^2}\, p(\hat{x}, \hat{\xi})\, d\hat{\xi}. \tag{10}$$

and we have the following convergences (see [3]):

$$\varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon} g^i \left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \upsilon \left(\hat{x}, \varepsilon^\alpha F\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right)\right) ds \to \int\limits_{\Gamma_1} G^i\left(\hat{x}\right) \upsilon\left(x\right)\, ds$$

and

$$\varepsilon^{\beta} \int\limits_{\Gamma_1^\varepsilon} p^i \left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) u \left(\hat{x}, \varepsilon F\left(\frac{\widehat{x}}{\varepsilon}, \omega\right)\right) \upsilon \left(\hat{x}, \varepsilon F\left(\frac{\widehat{x}}{\varepsilon}, \omega\right)\right) ds \to \int\limits_{\Gamma_1} P^i\left(\hat{x}\right)\, u(x)\, \upsilon\left(x\right)\, ds$$

for any $v \in H^1(D_\varepsilon)$ by $\varepsilon \to 0$, $i = 1, \dots, n$. Here ds is the element of $(d-1)$-dimensional measure on the hypersurface.

Let us introduce the following notation for the spaces $\mathbf{H} := [L_2(D)]^n$, $\mathbf{H}_\varepsilon := [L_2(D_\varepsilon)]^n$, $\mathbf{V} := [H^1(D, \Gamma_2)]^n$, $\mathbf{V}_\varepsilon := [H^1(D_\varepsilon; \Gamma_2)]^n$. Here, $H^1(D, \Gamma_2)$ (respectively $H^1(D_\varepsilon, \Gamma_2)$), denotes the space of functions from the Sobolev space $H^1(D)$ (respectively $H^1(D_\varepsilon)$) with zero trace on Γ_2. The norms in these spaces are determined as follows

$$\|v\|^2 := \int\limits_D \sum_{i=1}^n |v^i(x)|^2 dx, \ \|v\|_\varepsilon^2 := \int\limits_{D_\varepsilon} \sum_{i=1}^n |v^i(x)|^2 dx,$$

$$\|v\|_1^2 := \int\limits_D \sum_{i=1}^n |\nabla v^i(x)|^2 dx, \ \|v\|_{1,\varepsilon}^2 := \int\limits_{D_\varepsilon} \sum_{i=1}^n |\nabla v^i(x)|^2 dx.$$

We denote by $\mathbf{V}'$ the dual space to the space $\mathbf{V}$, and by $\mathbf{V}'_\varepsilon$ the dual space to the space $\mathbf{V}_\varepsilon$.

Let $q_i = p_i/(p_i - 1)$ for any $i = 1, \dots, n$. We will use the following vector notation $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$, and also define spaces

$$\mathbf{L_p} := L_{p_1}(D) \times \ldots \times L_{p_n}(D), \ \mathbf{L}_{\mathbf{p},\varepsilon} := L_{p_1}(D_\varepsilon) \times \ldots \times L_{p_n}(D_\varepsilon),$$

$$\mathbf{L_p}(\mathbb{R}_+; \mathbf{L_p}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(D)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(D)),$$
$$\mathbf{L_p}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(D_\varepsilon)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(D_\varepsilon)).$$

As in [17, 26] we study weak solutions of the initial boundary value problem (4), that is, functions

$$u_\varepsilon(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon})$$

which satisfy the equation (4) in the distributional sense (the sense of generalized functions), that is, the integral identity holds

$$- \int\limits_{D_\varepsilon \times \mathbb{R}_+} u_\varepsilon \cdot \frac{\partial \psi}{\partial t} \, dxdt + \int\limits_{D_\varepsilon \times \mathbb{R}_+} \lambda \nabla u_\varepsilon \cdot \nabla \psi \, dxdt + \int\limits_{D_\varepsilon \times \mathbb{R}_+} a_\varepsilon(x) f(u_\varepsilon) \cdot \psi \, dxdt +$$
$$+ \varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) u_\varepsilon \cdot \psi \, dsdt = \int\limits_{D_\varepsilon \times \mathbb{R}_+} h_\varepsilon(x) \cdot \psi \, dxdt + \varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \cdot \psi \, dsdt$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V}_\varepsilon \cap \mathbf{L}_{\mathbf{p},\varepsilon})$. Here $y_1 \cdot y_2$ means scalar product of vectors $y_1, y_2 \in \mathbb{R}^n$.

If $u_\varepsilon(x,t) \in \mathbf{L_p}(0, M; \mathbf{L}_{\mathbf{p},\varepsilon})$, then from the condition (7) it follows that $f(u(x,t)) \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon})$. At the same time, if $u_\varepsilon(x,t) \in \mathbf{L}_2(0, M; \mathbf{V}_\varepsilon)$, then $\lambda \Delta u_\varepsilon(x,t) + h_\varepsilon(x) \in \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon)$. Therefore, for an arbitrary weak solution $u_\varepsilon(x,s)$ to problem (4), satisfies

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon).$$

From the Sobolev embedding theorem follows that $\mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon) \subset \mathbf{L_q}\left(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}}\right)$, where space $\mathbf{H}_\varepsilon^{-\mathbf{r}} := H^{-r_1}(D_\varepsilon) \times \ldots \times H^{-r_n}(D_\varepsilon)$, $\mathbf{r} = (r_1, \dots, r_n)$ and indexes

$r_i = \max\{1, d(1/q_i - 1/2)\}$ by $i = 1, \ldots, n$. Here $H^{-r}(D_\varepsilon)$ denotes the space conjugate to the Sobolev space $\overset{\circ}{W}{}_2^r(D_\varepsilon)$ with index $r > 0$ in the domain D_ε.

Therefore, for any weak solution $u_\varepsilon(x,t)$ to problem (4) it's time derivative $\frac{\partial u_\varepsilon(x,t)}{\partial t}$ belongs to $\mathbf{L_q}\left(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}}\right)$.

Remark 2 *Existence of a weak solution $u(x,t)$ to problem (4) for any initial data $U \in \mathbf{H}_\varepsilon$ and fixed ε, can be proved in the standard way (see, for example, [16], [26]). This solution may not be unique, since the function $f(v)$ satisfies only the conditions (7), (8) and it is not assumed that the Lipschitz condition is satisfied with respect to v.*

The following Lemma is proved in a similar way to the proposition XV.3.1 from [17].

Lemma 1 *Let $u_\varepsilon(x,t) \in \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon})$ be a weak solution of problem (4). Then*

(i) $u_\varepsilon \in \mathbf{C}(\mathbb{R}_+; \mathbf{H}_\varepsilon)$;

(ii) *function $\|u_\varepsilon(\cdot,t)\|^2$ is absolutely continuous on $\mathbb{R}_+$, and moreover*

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(\cdot,t)\|^2 + \int\limits_{D_\varepsilon} \lambda \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t)dx + \int\limits_{D_\varepsilon} a_\varepsilon(x) f(u_\varepsilon(x,t)) \cdot u_\varepsilon(x,t)dx + \tag{11}$$

$$+\varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) u_\varepsilon(x,t) \cdot u_\varepsilon(x,t)\, ds = \int\limits_{D_\varepsilon} h_\varepsilon(x) \cdot u_\varepsilon(x,t)dx + \varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \cdot u_\varepsilon(x,t)\, ds,$$

for almost all $t \in \mathbb{R}_+$.

To define the trajectory space $\mathcal{K}_\varepsilon^+$ for (4), we use the general approaches of Section 2.1 and for every $[t_1, t_2] \in \mathbb{R}$ we have the Banach spaces

$$\mathcal{F}_{t_1,t_2} := \mathbf{L_p}(t_1, t_2; \mathbf{L_p}) \cap \mathbf{L}_2(t_1, t_2; \mathbf{V}) \cap \mathbf{L}_\infty(t_1, t_2; \mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L_q}\left(t_1, t_2; \mathbf{H}^{-r}\right) \right\}$$

(sometimes we omit the parameter ε for brevity) with the following norm:

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{\mathbf{L_p}(t_1,t_2;\mathbf{L_p})} + \|v\|_{\mathbf{L}_2(t_1,t_2;\mathbf{V})} + \|v\|_{\mathbf{L}_\infty(0,M;\mathbf{H})} + \left\|\frac{\partial v}{\partial t}\right\|_{\mathbf{L_q}(t_1,t_2;\mathbf{H}^{-r})}.$$

Setting $\mathcal{D}_{t_1,t_2} = \mathbf{L_q}\left(t_1, t_2; \mathbf{H}^{-r}\right)$ we obtain $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$ and for $u(t) \in \mathcal{F}_{t_1,t_2}$ we have $A(u(t)) \in \mathcal{D}_{t_1,t_2}$. One considers now weak solutions to (4) as solutions of an equation in the general scheme of Section 2.1.

Consider the spaces

$$\mathcal{F}_+^{loc} = \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L_p}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^{loc}(\mathbb{R}_+; \mathbf{H}^{-r}) \right\},$$

$$\mathcal{F}^{loc}_{\varepsilon,+} = \mathbf{L}^{loc}_{\mathbf{p}}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}^{loc}_2(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}^{loc}_\infty(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}^{loc}_{\mathbf{q}}(\mathbb{R}_+;\mathbf{H}_\varepsilon{}^{-r}) \right\}.$$

We introduce the following notation. Let $\mathcal{K}^+_\varepsilon$ be the set of all weak solutions to (4). For any $U \in \mathbf{H}$ there exists at least one trajectory $u(\cdot) \in \mathcal{K}^+_\varepsilon$ such that $u(0) = U(x)$. Consequently, the space $\mathcal{K}^+_\varepsilon$ to (4) is not empty and is sufficiently large.

We define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathcal{F}_{t_1,t_2}$ by means of the norms from $\mathbf{L}_2(t_1,t_2;\mathbf{H})$. We get

$$\rho_{t_1,t_2}(u,v) = \left(\int_{t_1}^{t_2} \|u(t) - v(t)\|^2_{\mathbf{H}} dt \right)^{1/2} \quad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{t_1,t_2}.$$

The topology Θ^{loc}_+ in $\mathcal{F}^{loc}_+$ is generated by these metrics. Let us recall that $\{v_k\} \subset \mathcal{F}^{loc}_+$ converges to $v \in \mathcal{F}^{loc}_+$ as $k \to \infty$ in Θ^{loc}_+ if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{L}_2(t_1,t_2;\mathbf{H})} \to 0$ $(k \to \infty)$ for all $[t_1,t_2] \subset \mathbb{R}_+$. The topology Θ^{loc}_+ is metrizable. We consider this topology in the trajectory space $\mathcal{K}^+_\varepsilon$ of (4). Similarly, we define the topology $\Theta^{loc}_{\varepsilon,+}$ in $\mathcal{F}^{loc}_{\varepsilon,+}$.

Denote by $S(\tau)$ the translation semigroup, i.e. $S(\tau)\,u(t) = u(t+\tau)$. The translation semigroup $S(\tau)$ acting on $\mathcal{K}^+_\varepsilon$, is continuous in the topology $\Theta^{loc}_{\varepsilon,+}$. It is easy to see that $\mathcal{K}^+_\varepsilon \subset \mathcal{F}^{loc}_{\varepsilon,+}$ and the space $\mathcal{K}^+_\varepsilon$ is translation invariant, i.e. $S(\tau)\mathcal{K}^+_\varepsilon \subseteq \mathcal{K}^+_\varepsilon$ for all $\tau \geq 0$.

Using the scheme of Section 2.1, one can define bounded sets in the space $\mathcal{K}^+_\varepsilon$ by means of the Banach space $\mathcal{F}^b_{\varepsilon,+}$. We naturally get

$$\mathcal{F}^b_{\varepsilon,+} = \mathbf{L}^b_{\mathbf{p}}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}^b_2(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}^b_{\mathbf{q}}(\mathbb{R}_+;\mathbf{H}^{-r}_\varepsilon) \right\}$$

and the space $\mathcal{F}^b_{\varepsilon,+}$ is a subspace of $\mathcal{F}^{loc}_{\varepsilon,+}$.

Suppose that $\mathcal{K}_\varepsilon$ is the kernel to (4), that consists of all weak complete solutions $u(t), t \in \mathbb{R}$, to our system, bounded in

$$\mathcal{F}^b_\varepsilon = \mathbf{L}^b_{\mathbf{p}}(\mathbb{R};\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}^b_2(\mathbb{R};\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R};\mathbf{H}_\varepsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}^b_{\mathbf{q}}(\mathbb{R};\mathbf{H}^{-r}_\varepsilon) \right\}.$$

In analogous way we define the topology Θ^{loc}_ε in $\mathcal{F}^b_\varepsilon$.

Proposition 3 *Problem (4) has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space $\Theta^{loc}_{\varepsilon,+}$. The set $\mathfrak{A}_\varepsilon$ is bounded in $\mathcal{F}^b_{\varepsilon,+}$ and compact in $\Theta^{loc}_{\varepsilon,+}$. Moreover, $\mathfrak{A}_\varepsilon = \Pi_+\mathcal{K}_\varepsilon$, the kernel $\mathcal{K}_\varepsilon$ is non-empty and bounded in $\mathcal{F}^b_\varepsilon$ and compact in Θ^{loc}_ε.*

To prove this proposition we use the approach of the proof from [17]. To prove the existence of an absorbing set (bounded in $\mathcal{F}^b_{\varepsilon,+}$ and compact in $\Theta^{loc}_{\varepsilon,+}$) one can use Lemma 1 similar to [17].

3.2 Homogenized reaction-diffusion system and convergence of attractors in the critical case ($\beta = 1 - \alpha$)

Now we study the behaviour of the problem (4) as $\varepsilon \to 0$ in the critical case $\beta = 1 - \alpha$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}\,(x)\, f(u_0) + \overline{h}\,(x)\,, & x \in D, t > 0, \\ \frac{\partial u_0}{\partial \nu} + P(\hat{x}) u_0 = G(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1, t > 0, \\ u_0 = 0, & x \in \Gamma_2, t > 0, \\ u_0 = U(x), & x \in D, t = 0, \end{cases} \tag{12}$$

Here $\overline{a}(x)$ and $\overline{h}(x)$ are defined in (5) and (6), respectively, $G(\hat{x})$ and $P(\hat{x})$ were defined in (9) and (10).

As before, we consider weak solutions of the problem (12), that is, functions

$$u_0(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{loc}\left(\mathbb{R}_+; \mathbf{L}_\mathbf{p}\right),$$

which satisfy the following integral identity:

$$\begin{gathered} -\int\limits_{D\times\mathbb{R}_+} u_0 \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{D\times\mathbb{R}_+} \lambda \nabla u_0 \cdot \nabla \psi \, dxdt + \int\limits_{D\times\mathbb{R}_+} \bar{a}(x) f(u_0) \cdot \psi \, dxdt + \\ + \int\limits_{\Gamma_1\times\mathbb{R}_+} P\left(\hat{x}\right) u_0 \cdot \psi \, dsdt = \int\limits_{D\times\mathbb{R}_+} \overline{h}(x) \cdot \psi \, dxdt + \int\limits_{\Gamma_1\times\mathbb{R}_+} G\left(\hat{x}\right) \cdot \psi \, dsdt \end{gathered} \tag{13}$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L_p})$. For any weak solution $u(x,t)$ to problem (12), we have that $\frac{\partial u_0(x,t)}{\partial t} \in \mathbf{L_q}\left(0, M; \mathbf{H^{-r}}\right)$ (see Section 3.1). Recall, that the "limit" domain D in (12) and (13) is independent of ε and it boundary contains the plain part Γ_1.

Similar to (4), for any initial data $U \in \mathbf{H}$, the problem (12) has at least one weak solution (see Remark 2). Lemma 1 also holds true for the problem (12) with replacing the ε-depending coefficients a, h, p and g by the corresponding averaged coefficients $\overline{a}(x), \overline{h}(x), P(\hat{x})$, and $G(\hat{x})$.

As usual, let $\overline{\mathcal{K}}^+$ be the the trajectory space for (12) (the set of all weak solutions), that belong to the corresponding spaces $\mathcal{F}_+^{loc}$ and $\mathcal{F}_+^b$ (see Section 2.1). Recall that $\overline{\mathcal{K}}^+ \subset \mathcal{F}_+^{loc}$ and the space $\overline{\mathcal{K}}^+$ is translation invariant with respect to translation semigroup $\{S(\tau)\}$, that is, $S(\tau)\overline{\mathcal{K}}^+ \subseteq \overline{\mathcal{K}}^+$ for all $\tau \geq 0$. We now construct the trajectory attractor in the topology Θ_+^{loc} for the problem (12) (see Sections 3.1 and 2.1).

Similar to Proposition 3 we have

Proposition 4 *Homogenized problem has the trajectory attractor $\overline{\mathfrak{A}}$ in the topological space Θ_+^{loc}. The set $\overline{\mathfrak{A}}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover, $\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}}$, the kernel $\overline{\mathcal{K}}$ of the homogenized problem is non-empty and bounded in $\mathcal{F}^b$.*

Here we formulate the main result concerning the limit behaviour of the trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (4) as $\varepsilon \to 0$ in the critical case $\beta = 1 - \alpha$.

Theorem 4 *The following limit holds in the topological space Θ_+^{loc}*

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \quad as\ \varepsilon \to 0+ .$$

Moreover,

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \text{ as } \varepsilon \to 0+ \text{ in } \Theta^{loc}.$$

Finally, we consider the reaction–diffusion systems for which the uniqueness theorem is true for the Cauchy problem. It suffices to assume that the nonlinear term $f(u)$ in (4) satisfies the condition

$$(f(v_1) - f(v_2), v_1 - v_2) \geq -C|v_1 - v_2|^2 \text{ for any } v_1, v_2 \in \mathbb{R}^n. \tag{14}$$

(see [17, 26]). In [26] it was proved that if (14) is true, then (4) and (12) generate dynamical semigroups in $\mathbf{H}$, possessing global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ are bounded in $\mathbf{V}$ (see also [16], [15]). Moreover

$$\mathcal{A}_\varepsilon = \{u(0) \mid u \in \mathfrak{A}_\varepsilon\}, \ \overline{\mathcal{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}.$$

Corollary 1 *Under the assumption of Theorem 4 the limit formula takes place*

$$\operatorname{dist}_{\mathbf{H}^{-\delta}}\left(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\right) \to 0 \ (\varepsilon \to 0+).$$

3.3 Homogenized reaction-diffusion system and convergence of attractors in the subcritical case ($\beta > 1 - \alpha$)

In the next sections, we study the behaviour of the problem (4) as $\varepsilon \to 0$ in the subcritical case $\beta > 1 - \alpha$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}\left(x\right) f\left(u_0\right) + \overline{h}\left(x\right), & x \in D, t > 0, \\ \frac{\partial u_0}{\partial \nu} = G(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1, t > 0, \\ u_0 = 0, & x \in \Gamma_2, t > 0, \\ u_0 = U(x), & x \in D, t = 0, \end{cases} \tag{15}$$

Here $\overline{a}(x)$ and $\overline{h}(x)$ are defined in (5) and (6), respectively, $G(\hat{x})$ was defined in (9).

As before, we consider weak solutions of the problem (15), that is, functions

$$u(x,t) \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\mathbf{p}^{loc}\left(\mathbb{R}_+; \mathbf{L}_\mathbf{p}\right),$$

which satisfy the following integral identity:

$$\begin{gathered} -\int\limits_{D\times\mathbb{R}_+} u \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{D\times\mathbb{R}_+} \lambda \nabla u \cdot \nabla \psi\, dxdt + \int\limits_{D\times\mathbb{R}_+} \bar{a}(x) f(u) \cdot \psi\, dxdt = \\ = \int\limits_{D\times\mathbb{R}_+} \bar{h}(x) \cdot \psi\, dxdt + \int\limits_{\Gamma_1\times\mathbb{R}_+} G\left(\hat{x}\right) \cdot \psi\, dsdt \end{gathered} \tag{16}$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L}_\mathbf{p})$. For any weak solution $u(x,t)$ to problem (15), we have that $\frac{\partial u(x,t)}{\partial t} \in \mathbf{L}_\mathbf{q}\left(0, M; \mathbf{H}^{-\mathbf{r}}\right)$ (see Section 3.1). Recall, that the "limit" domain D in (15) and (16) is independent of ε and it boundary contains the plain part Γ_1.

For homogenized problem (15) holds Proposition 4.

For trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (4) as $\varepsilon \to 0$ in the subcritical case $\beta > 1 - \alpha$ holds Theorem 4 and Corollary 1.

3.4 Homogenized reaction-diffusion system and convergence of attractors in the supercritical case ($\beta < 1-\alpha$)

In the next sections, we study the behaviour of the problem (4) as $\varepsilon \to 0$ in the supercritical case $\beta < 1-\alpha$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda\Delta u_0 - \overline{a}\,(x)\, f(u_0) + \overline{h}\,(x)\,, & x \in D, t > 0, \\ u_0 = 0, & x \in \partial D, t > 0, \\ u_0 = U(x), & x \in D, t = 0, \end{cases} \tag{17}$$

Here $\overline{a}(x)$ and $\overline{h}(x)$ are defined in (5) and (6), respectively.

We note that, in the supercritical case, the influence of the boundary layer on the part of the boundary Γ_1 completely disappears (compare with critical case [44] and subcritical case mentioned in Subsection 3.3).

As before, we consider weak solutions of the problem (17), that is, functions

$$u_0(x,t) \in \mathbf{L}_{\infty}^{loc}(\mathbb{R}_+;\mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}_{\mathbf{p}}^{loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p}})\,,$$

which satisfy the following integral identity:

$$-\int_{D\times\mathbb{R}_+} u_0 \cdot \frac{\partial\psi}{\partial t}\,dxdt + \int_{D\times\mathbb{R}_+} \lambda\nabla u_0\cdot\nabla\psi\,dxdt + \int_{D\times\mathbb{R}_+} \bar{a}(x) f(u_0)\cdot\psi\,dxdt = \int_{D\times\mathbb{R}_+} \bar{h}(x)\cdot\psi\,dxdt \tag{18}$$

for any function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+;\mathbf{V}\cap\mathbf{L}_\mathbf{p})$. For any weak solution $u(x,t)$ to problem (17), we have that $\frac{\partial u_0(x,t)}{\partial t} \in \mathbf{L}_\mathbf{q}(0,M;\mathbf{H}^{-\mathbf{r}})$ (see Section 3.1). Recall, that the "limit" domain D in (17) and (18) is independent of ε and it boundary contains the plain part Γ_1.

For homogenized problem (17) holds Proposition 4.

For trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (4) as $\varepsilon \to 0$ in the supercritical case $\beta < 1-\alpha$ holds Theorem 4 and Corollary 1.

4 Homogenezation of attractors to the reaction-diffusion system in a domain with randomly oscillating boundary

4.1 Statement of the problem

Let $D \subset \mathbb{R}^d \cap \{x \,|\, x_d > 0\}$, $d \ge 2$, be a smooth bounded domain whose boundary has a nontrivial flat part $\Gamma_1 = \partial D \cap \{x \,|\, x_d = 0\}$ with a nonempty $(d-1)$-dimensional interior $\overset{\circ}{\Gamma}_1$. We perturb the flat part of the boundary in such a way that the perturbed domain has an oscillating boundary. To this end, we define a smooth nonnegative function $g(\widehat{x})$, $\widehat{x} = (x_1,\dots,x_{d-1})$, such that $\operatorname{supp} g(\widehat{x}) \subset \Gamma_0 \Subset \overset{\circ}{\Gamma}_1$, and, given a statistically homogeneous non-positive random function $F(\widehat{\xi},\omega)$, $\widehat{\xi} = (\xi_1,\dots,\xi_{d-1})$, which has smooth realizations and is defined on a standard probability space $(\Omega,\mathcal{A},\mu)$, we set, for $\varepsilon > 0$,

$$\Pi_\varepsilon = \{x \in \mathbb{R}^d \;:\; \widehat{x} \in \Gamma_1,\; \varepsilon g(\widehat{x}) F\left(\frac{\widehat{x}}{\varepsilon},\omega\right) < x_d \le 0\}$$

and, finally, introduce the desired domain with random boundary as follows: $D_\varepsilon = D \cup \Pi_\varepsilon$. For more detailed definitions of randomness we refer to the next section. According to the above construction, the boundary ∂D_ε consists of the parts Γ_2 and $\Gamma_1^\varepsilon = \Big\{ x \in \partial D_\varepsilon \,:\, (\widehat{x}, 0) \in \Gamma_1,\, x_d = \varepsilon g(\widehat{x}) F\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) \Big\}$ forming together the domain boundary.

We consider the boundary-value problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \lambda \Delta u_\varepsilon - a\left(x, \frac{x}{\varepsilon}, \omega\right) f(u_\varepsilon) + r(x, \frac{x}{\varepsilon}, \omega), & x \in D_\varepsilon, t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} + g(\hat{x}) p\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) u_\varepsilon = g(\hat{x}) q\left(\frac{\widehat{x}}{\varepsilon}, \omega\right), & x = (\hat{x}, x_d) \in \Gamma_1^\varepsilon, t > 0, \\ u_\varepsilon = 0, & x \in \Gamma_2, t > 0, \\ u_\varepsilon = U(x), & x \in D_\varepsilon, t = 0, \end{cases} \tag{19}$$

where $u_\varepsilon = u_\varepsilon(x,t) = (u^1, \dots, u^n)^\top$ is an unknown vector function, the nonlinear function $f = (f^1, \dots, f^n)^\top$ is given, $r = (r^1, \dots, r^n)^\top$ is the known right-hand side function, and λ is an $n \times n$-matrix with constant coefficients, having a positive symmetrical part: $\frac{1}{2}(\lambda + \lambda^\top) \geq \varpi I$ (where I is the unit matrix with dimension n and $\varpi > 0$). We assume that $p\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) = \operatorname{diag}\{p^1, \dots, p^n\}$, $q\left(\frac{\widehat{x}}{\varepsilon}, \omega\right) = (q^1, \dots, q^n)^\top$ are random statistically homogeneous functions and $p^i\left(\frac{\widehat{x}}{\varepsilon}, \omega\right)$, $i = 1, \dots n$, are positive.

We assume that the random functions $a_\varepsilon(x,\omega) = a\Big(x, \frac{x}{\varepsilon}, \omega\Big)$ and $r_\varepsilon(x,\omega) = r\Big(x, \frac{x}{\varepsilon}, \omega\Big)$ are statistically homogeneous, that is $a(x,\xi,\omega) = \mathbf{A}(x, T_\xi \omega)$, $\quad r(x,\xi,\omega) = \mathbf{R}(x, T_\xi\omega)$, where $\mathbf{A} : D \times \Omega \to \mathbb{R}$ and $\mathbf{R} : D \times \Omega \to \mathbb{R}^n$ are measurable.

We also assume that $\mathbf{A}(x,\omega) \in C_b(\overline{D})$ for almost all $\omega \in \Omega$ and $0 < \alpha_0 \leq \mathbf{A}(x,\omega) \leq \alpha_1$, $\quad |\mathbf{R}(x,\omega)| \leq \phi(x)$, $\quad \forall x \in D$, where $\phi(x)$ is a positive function such that $\phi \in L_2(D)$.

Birkhoff ergodic theorem implies that the functions $a\,(x,\xi,\omega)$ and $r\,(x,\xi,\omega)$ have the space mean value

$$\overline{a}(x) := M(a)(x) = \mathbb{E}(\mathbf{A})(x), \qquad \overline{r}(x) := M(r)(x) = \mathbb{E}(\mathbf{R})(x)$$

for every $x \in D$. Note that the functions $\overline{a}(x)$ and $\overline{r}(x)$ also satisfy the inequality $\alpha_0 \leq \overline{a}(x) \leq \alpha_1$, $\quad |\overline{r}(x)| \leq \phi(x)$, $\quad \forall x \in D$. It follows from Proposition 2, that almost surely in $\omega \in \Omega$

$$\int\limits_D a_\varepsilon(x,\omega)\varphi(x)dx \to \int\limits_D \overline{a}\,(x)\,\varphi(x)dx \ (\varepsilon \to 0+) \quad \forall \varphi \in L_1(D), \tag{20}$$

$$\int\limits_{D^+} r_\varepsilon^i\,(x,\omega)\,\varphi(x)dx \to \int\limits_{D^+} \overline{r}^i\,(x)\,\varphi(x)dx \ (\varepsilon \to 0+) \quad \forall \varphi \in L_2(D^+),\ i = 1, \dots, n. \tag{21}$$

Here D^+ is such a domain that $D_\varepsilon \subset D^+$ for any ε.

We assume that the vector function $f(v) \in C(\mathbb{R}^n; \mathbb{R}^n)$ satisfies inequalities (7) and (8).

From (21) it follows that the norms of $r_\varepsilon^i(x,\omega)$ are almost surely uniformly bounded $\|r_\varepsilon^i\|_{L_2(D)} \leq M_0$, $\quad \forall \varepsilon \in (0,1]$ in the space $L_2(D)$.

Denote

$$P(\widehat{x}) = \mathbb{E}\left(\rho(\omega)\ \sqrt{1 + (g(\widehat{x})\,\partial_\omega \mathbf{F}(\omega))^2}\right), \qquad Q(\widehat{x}) = \mathbb{E}\left(\varrho(\omega)\ \sqrt{1 + (g(\widehat{x})\,\partial_\omega \mathbf{F}(\omega))^2}\right).$$

(22)

and, due to Birkhoff ergodic theorem and Proposition 2, we have almost surely the following convergence (see [40]):

$$\int\limits_{\Gamma_1^\varepsilon} g(\widehat{x})q^i\left(\frac{\hat{x}}{\varepsilon},\omega\right)\, \upsilon\left(\hat{x},\varepsilon g(\widehat{x})F\left(\frac{\widehat{x}}{\varepsilon},\omega\right)\right)\, ds \to \int\limits_{\Gamma_1} g(\widehat{x})Q^i\left(\hat{x}\right)\, \upsilon\left(x\right)\, ds$$

and

$$\int\limits_{\Gamma_1^\varepsilon} g(\widehat{x})p^i\left(\frac{\hat{x}}{\varepsilon},\omega\right)\, u\left(\hat{x},\varepsilon g(\widehat{x})F\left(\frac{\widehat{x}}{\varepsilon},\omega\right)\right)\, \upsilon\left(\hat{x},\varepsilon g(\widehat{x})F\left(\frac{\widehat{x}}{\varepsilon},\omega\right)\right)\, ds \to \int\limits_{\Gamma_1} g(\widehat{x})P^i\left(\hat{x}\right)\, u(x)\,\upsilon\left(x\right)\, ds$$

for any $u,\upsilon\in H^1(D_\varepsilon)$ as $\varepsilon\to 0$, $i=1,\dots,n$. Here ds is the element of $(d-1)$-dimensional measure on the hypersurface.

As in [17,26] we study weak solutions of the initial boundary value problem (19), that is, functions

$$u_\varepsilon(x,t)\in \mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon)\cap \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon)\cap \mathbf{L}_{\mathbf{p}}^{loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon})$$

which satisfy the equation (19) in the distributional sense (the sense of generalized functions), that is, the integral identity holds

$$-\int\limits_{D^\varepsilon\times\mathbb{R}_+} u_\varepsilon\cdot\frac{\partial\psi}{\partial t}\,dxdt+\int\limits_{D^\varepsilon\times\mathbb{R}_+}\lambda\nabla u_\varepsilon\cdot\nabla\psi\,dxdt+\int\limits_{D^\varepsilon\times\mathbb{R}_+} a_\varepsilon(x,\omega)f(u_\varepsilon)\cdot\psi\,dxdt+$$
$$+\int\limits_{\Gamma_1^\varepsilon\times\mathbb{R}_+} g(\hat{x})p\left(\frac{\hat{x}}{\varepsilon},\omega\right)u_\varepsilon\cdot\psi\,dsdt=\int\limits_{D^\varepsilon\times\mathbb{R}_+} r_\varepsilon(x,\omega)\cdot\psi\,dxdt+\int\limits_{\Gamma_1^\varepsilon\times\mathbb{R}_+} g(\hat{x})q\left(\frac{\hat{x}}{\varepsilon},\omega\right)\cdot\psi\,dsdt$$

for any function $\psi\in\mathbf{C}_0^\infty(\mathbb{R}_+;\mathbf{V}_\varepsilon\cap\mathbf{L}_{\mathbf{p},\varepsilon})$. Here $y_1\cdot y_2$ means scalar product of vectors $y_1,y_2\in\mathbb{R}^n$.

For any weak solution $u_\varepsilon(x,t)$ to problem (19) the time derivative $\frac{\partial u_\varepsilon(x,t)}{\partial t}\in\mathbf{L}_{\mathbf{q}}\left(0,M;\mathbf{H}_\varepsilon^{-\mathbf{r}}\right)$ (see Section 3.1).

Remark 3 *Existence of a weak solution $u(x,t)$ to problem (19) for any initial data $U\in\mathbf{H}_\varepsilon$ and fixed ε, can be proved in the standard way (see, for example, [16], [26]). This solution may not be unique, since the function $f(v)$ satisfies only the conditions (8) and it is not assumed that the Lipschitz condition is satisfied with respect to v.*

Proposition 5 *Under the hypotheses (7) and (8) the system (19) has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space $\Theta_{\varepsilon,+}^{loc}$. The set $\mathfrak{A}_\varepsilon$ is ω-almost surely bounded in $\mathcal{F}_{\varepsilon,+}^b$ and compact in $\Theta_{\varepsilon,+}^{loc}$. Moreover, $\mathfrak{A}_\varepsilon=\Pi_+\mathcal{K}_\varepsilon$, the kernel $\mathcal{K}_\varepsilon$ is non-empty, bounded in $\mathcal{F}_\varepsilon^b$ and compact in Θ_ε^{loc}.*

4.2 Homogenized reaction-diffusion system and convergence of attractors

In the next sections, we study the behaviour of the problem (19) as $\varepsilon \to 0$. We have the following "formal" limit problem with inhomogeneous Fourier boundary condition

$$\begin{cases} \frac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}\,(x)\, f(u_0) + \overline{r}\,(x)\,, & x \in D, t > 0, \\ \frac{\partial u_0}{\partial \nu} + g(\hat{x}) P(\hat{x}) u_0 = g(\hat{x}) Q(\hat{x}), & x = (\hat{x}, 0) \in \Gamma_1, t > 0, \\ u_0 = 0, & x \in \Gamma_2, t > 0, \\ u_0 = U(x), & x \in D, t = 0, \end{cases} \tag{23}$$

Here $\overline{a}(x)$ and $\overline{r}(x)$ were defined in (20) and (21), respectively, $Q(\hat{x})$ and $P(\hat{x})$ were defined in (22).

As before, we consider weak solutions of the problem (23), that is, functions

$$u_0(x,t) \in \mathbf{L}_{\infty}^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_{\mathbf{p}}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p}}),$$

which satisfy the following integral identity:

$$\begin{aligned} &- \int\limits_{D\times\mathbb{R}_+} u_0 \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{D\times\mathbb{R}_+} \lambda \nabla u_0 \cdot \nabla \psi \, dxdt + \int\limits_{D\times\mathbb{R}_+} \bar{a}(x) f(u_0) \cdot \psi \, dxdt + \\ &+ \int\limits_{\Gamma_1\times\mathbb{R}_+} g(\hat{x}) P\,(\hat{x})\, u_0 \cdot \psi \, dsdt = \int\limits_{D\times\mathbb{R}_+} \bar{r}(x) \cdot \psi \, dxdt + \int\limits_{\Gamma_1\times\mathbb{R}_+} g(\hat{x}) Q\,(\hat{x}) \cdot \psi \, dsdt \end{aligned} \tag{24}$$

for any function $\psi \in \mathbf{C}_0^{\infty}(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L}_{\mathbf{p}})$. For any weak solution $u(x,t)$ to problem (23), we have that $\frac{\partial u_0(x,t)}{\partial t} \in \mathbf{L}_{\mathbf{q}}(0, M; \mathbf{H}^{-\mathbf{r}})$ (see Section 4.1). Recall, that the "limit" domain D in (23) and (24) is independent of ε and it boundary contains the plain part Γ_1.

For homogenized problem (23) holds Proposition 4.

Under assumptions (**h1**)–(**h4**), for trajectory attractors $\mathfrak{A}_\varepsilon$ of the reaction-diffusion systems (19) as $\varepsilon \to 0$, ω-almost surely holds Theorem 4 and Corollary 1.

5 Conclusion

In the paper we consider reaction–diffusion systems with rapidly oscillating terms in equations and in boundary conditions in domains with locally periodic or randomly oscillating boundary (rough surface) depending on a small parameter. We define the trajectory attractors of these systems and express that they converge (almost surely) in a weak sense to the trajectory attractors of the limit (homogenized) reaction–diffusion systems in domain independent of the small parameter.

Acknowledgment

The first part of this research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No.AP26199535), the second part of the research is supported by the Russian Scientific Foundation (grant number 25-11-00133) .

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Received: April 16, 2025
Accepted: June 8, 2025

IRSTI 27.43.15 DOI: https://doi.org/10.26577/JMMCS2025126202

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THE MULTIPLICATIVE INTEGRAL AND THE EVOLUTION OF THE MAGNETIC FIELD IN THE MARKOV LINEAR MODEL

The paper is devoted to the probabilistic asymptotic analysis of the magnetic field and magnetic energy in a Markov linear model of an incompressible fluid. Firstly, the paper introduces a brief history of the problem under consideration and presents the main results of the previous studies, which ultimately lead to the study of the product of independent random matrices with an increasing number of multiplicands. After that, the description of the Markov linear model considered in the paper is given, the so-called Lyapunov (generally speaking, random) bases for the multiplicative (stochastic) integral contained in the integral representation of the magnetic field are constructed. In conclusion, by decomposing the multiplicative integral over the constructed Lyapunov basis and relying on the properties of the basis, the main results - theorems on the asymptotic behavior of the magnetic field and magnetic energy - have been proven.
Key words: Multiplicative integral, Markov linear model, magnetic field, Lyapunov exponent, Lyapunov basis.

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Мультипликативті интеграл және марковтық сызықтық моделдегі магнит өрісінің эволюциясы

Жұмыс сығылмайтын сұйықтық марковтың сызықтық моделіндегі магнит өрісі мен магниттік энергияның ықтималдықтық - асимптотикалық талдауына арналған. Жұмыста алдымен қарастырылып отырған есептің бұған дейін басқа авторлар қарастырған, ақыр соңында көбейткіштерінің саны өсе беретін тәуелсіз кездейсоқ матрицалардың көбейтінділерін зерттеуге келтірілетін, жұмыстардың қысқаша тарихы баяндалған. Сосын жұмыста қарастырылатын марковтың сызықтық модельдің сипаттамасы берілген, магниттік өрістің интегралдық жазылымында пайда болатын мультипликативтік (стохастикалық) интеграл үшін ляпуновтық деп аталатын (жалпы алғанда, кездейсоқ) базис құрастырылған. Ең соңында, мультипликативтік интегралды құрастырылған ляпуновтық базис арқылы жіктеп және бұл базистік қасиеттеріне сүйене отырып, негізгі нәтижелер - магнит өрісі мен магниттік энергияның асимптотикалық беталыстары туралы теоремалар дәлелденген.
Түйін сөздер: Мультипликативтік интеграл, марковтық сызықтық модель, магнит өрісі, Ляпунов көрсеткіші, ляпуновтық базистер.

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Мультипликативный интеграл и эволюция магнитного поля в марковской линейной модели

В работе сначала изложена краткая история рассматриваемой задачи, приведены основные результаты предыдущих,сводящихся в конечном итоге к изучению произведения независимых случайных матриц при возрастании числа сомножителей, работ. После дано описание рассматриваемой в работе марковский линейной модели, построены так называемые ляпуновские (вообще говоря, случайные) базисы для содержащегося в интегральном представлении магнитного поля мультипликативного (стохастического) интеграла.

В заключение, разлагая мультипликативный интеграл по построенному ляпуновскому базису и опираясь на свойства этих базисов, доказаны основные результаты - теоремы об асимптотических поведениях магнитного поля и магнитной энергии.
Ключевые слова: Мультипликативный интеграл, марковская линейная модель, магнитное поле, показатель Ляпунова, ляпуновские базисы.

1 Introduction

The problem of the evolution of a magnetic field in a random turbulent flow of a conducting fluid is one of the most important in many physical applications. First of all, astrophysical applications can be mentioned here: stars, planets, and galaxies have magnetic fields that can vary greatly in time and space. A huge number of works have been devoted to various physical and mathematical aspects of this problem (see, for example, the monographs [1], [2], and the recent work [3]). One of the central and actual issues in this area is the study of asymptotic properties (completely non-trivial from the mathematical point of view) of the solution of the Cauchy problem for the induction equation. In this paper, the problem of the evolution of the magnetic field is considered in a kinematic formulation: this means that the statistical characteristics of a given random velocity field do not change with time, although the statistical characteristics of the magnetic field, generally speaking, change. In other words, the reverse effect of the magnetic field on the velocity field is not taken into account. The kinematic formulation allows one to remain within the linear approximation (i.e., for a given fluid velocity field), while the problem of the joint evolution of the velocity field and magnetic field requires the study of a nonlinear system of equations in six dimensions. We note that the asymptotic behavior of solutions at very large Reynolds numbers is related to the famous (until now unsolved) problem of the hydromagnetic dynamo (see [4] and the bibliography cited there).

2 Literature review and problem statement.

While for a given fluid flow the process of magnetic field transfer is fundamentally clear, the very problem of describing a turbulent fluid flow is known to be extremely complex. Therefore, one or another method of modeling the motion of a fluid is usually resorted to. In [5]- [6], the question of the evolution of the magnetic field was studied in the so-called linear model with updating, and ultimately the problem under consideration was reduced to studying the product of independent random matrices with increasing number of factors. Our present work is a generalization of works [5]- [6] in the sense that we study the asymptotic behavior of the solution of the Cauchy problem for the magnetic induction equation in a more general (than the updated model) model - a given Markov linear model (for a description of the model, see below, in Section 5) at long times. We will also consider a similar question for the total magnetic energy. In this case, we will essentially use the main result of works [7]- [8] - the Ferstenberg- type theorem (the theorem that establishes the existence of a strictly positive Lyapunov exponent associated with the introduced Markov model of a multiplicative stochastic integral of a special form). It should be noted that this Ferstenberg- type theorem for the multiplicative stochastic integral is, in a certain sense, a generalization of similar results for the product of unimodular random matrices [9], in particular, the product of independent [10] or random matrices forming a Markov sequence [11] (see also the survey article [12]).

3 Purpose and objectives of the study

The purpose and objectives of our work are to study the asymptotic properties of the solution of the Cauchy problem for the equation of magnetohydrodynamics and to generalize and extend the main results of [5]- [6] to the case of a Markov linear model of a given velocity field. In this case, special attention will be paid to the problem of finding the asymptotic form of the magnetic field and total magnetic energy present in the integral representations and determined by the introduced Markov linear model of a multiplicative stochastic integral of a special kind.

4 Materials and methods.

In the work, some well-known results and methods of the theory of magnetic fields in random media, the theory of matrices and multiplicative integral, partial differential equations and stochastic analysis will be used and refined in cases necessary for our purposes.

5 Mean result.

5.1 Model of a Markov linear velocity field.

Let $b(t), t \geq 0$ is a Brownian motion on a compact Riemannian manifold K, $dimK = \nu \geq 3$, with metric form ds^2 having the form in local coordinates $x^1, ..., x^\nu$ on K $ds^2 = \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} g_{ij} dx^i dx^j, d\sigma = \sqrt{detg} dx-$Riemannian volume element. The infinitesimal operator of the process $b(s), s \geq 0$, is the Beltrami - Laplace operator $\frac{1}{2}\Delta$, where

$$\Delta = \frac{1}{\sqrt{detg}} \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \left(\frac{\partial}{\partial x^i} \left(g_{ij} \sqrt{detg} \right) \right).$$

Let $C(\cdot) : K \to SL(\nu, R)$, where $SL(\nu, R)$ is the linear space of square $\nu \times \nu$ matrices with zero trace ($TrC = 0$). Functions $g_{ij}(x)$, $c_{ij}(x)$ are functions of the class $C^\infty(K)$. Then the Markov linear model of the velocity field is the velocity field of the form

$$\vec{V}(t, x) = C(b_t)x, \tag{1}$$

where the process $b_t = b(t)$, the manifold and the matrix $C(\cdot)$ are defined and described by the above conditions.

5.2 Evolution of the magnetic field in a Markov linear model.

It is well known that the evolution of the initial distribution

$$\vec{H}(x) = (H_{01}(x), H_{02}(x), ..., H_{0\nu}(x))$$

ν-dimensional $\nu \geq 3$ magnetic field

$$\vec{H}(t,x) = (H_1(t,x), H_2(t,x), ..., H_\nu(t,x))$$

in a given speed field

$$\vec{V}(t,x) = (V_1(t,x), V_2(t,x), ..., V_\nu(t,x))$$

with constant magnetic diffusion ν_m ($\nu_m > 0$) is described by the induction equation

$$\frac{\partial}{\partial t}\vec{H} = \nu_m \Delta \vec{H} + \text{rot}\left[\vec{V} \times \vec{H}\right], \tag{2}$$

$$\vec{H}(0,x) = \vec{H}(x), \tag{3}$$

where $t \geq 0, x \in \mathbb{R}^\nu$.

If we assume that the velocity field $\vec{V}$ and the initial field $\vec{H}_0$ are incompressible, i.e. (divergences in x)

$$\text{div}\,\vec{V} = 0, \;\; \text{div}\,\vec{H}_0 = 0, \tag{4}$$

then problem (2)-(3), under the condition

$$\text{div}\,\vec{H} = 0, \tag{5}$$

reduces to solving the Cauchy problem (herein after, the bracket $(...,...)$ means the scalar product)

$$\frac{\partial}{\partial t}\vec{H} = \nu_m \Delta \vec{H} - \left(\vec{V}, \nabla\right)\vec{H} + \left(\vec{H}, \nabla\right)\vec{V}, \quad \vec{H}(0,x) = \vec{H}_0(x). \tag{6}$$

Note that condition (5) is a consequence of condition (4): from $\text{div}\,\vec{H}_0(x) = 0$ it follows that $\text{div}\,\vec{H}(t,x) = 0$ for all $t \geq 0$.

Indeed, taking the divergence from both parts of (2) and taking into account the relation $\text{div}\,(rot) = 0$, we obtain

$$\frac{\partial}{\partial t}\text{div}\,\vec{H} = \nu_m \Delta\left(\text{div}\,\vec{H}\right). \tag{7}$$

Therefore, by the uniqueness theorem, condition (5) will be satisfied for all $t > 0$ if it is satisfied for $t = 0$, i.e. for the initial condition $\vec{H}_0(x)$.

The initial magnetic field $\vec{H}_0(x)$ is given by the distribution of currents, and these currents are concentrated in a limited region of space. It is known that then $\vec{H}_0(x) = O\left(|x|^{-\nu}\right), x \to \infty$ and this condition ensures the solvability of system (6). Let us now write out the solution of equation (6) in the Markov linear model (1).

To do this, we first look for a particular solution in the form

$$\vec{H}(t,x) = \vec{h}(t,k) exp\left\{i(\vec{\varkappa}(t,\vec{k}), x)\right\}. \tag{8}$$

where $t \geq 0, \vec{k} \in \mathbb{R}^\nu$, i is imaginary unit, and $\vec{\varkappa}(\vec{h}_0, \vec{k}) = \vec{k}$. Substituting (8) into (6) and equating the real and imaginary parts of the received relation, we obtain

$$\frac{d}{dt}\vec{\varkappa}(t,\vec{k}) = -C^*(b_t)\vec{\varkappa}(t,\vec{k}), \quad \vec{\varkappa}(0,\vec{k}) = \vec{k}, \tag{9}$$

$$\frac{d}{dt}\vec{h}(t,\vec{k}) + \nu_m \vec{\varkappa}^2 \vec{h}(t,\vec{k}) = C(b_t)\vec{h}(t,\vec{k}), \quad \vec{h}(0,\vec{k}) = h_0(\vec{k}), \tag{10}$$

where $*$ is the transposition operation, the scalar square $\vec{\varkappa}^2 = \left(\vec{\varkappa}(t,\vec{k}), \vec{\varkappa}(t,\vec{k})\right)$ and the condition $\operatorname{div} \vec{H}_0(x) = 0$ is equivalent to the condition $\left(\vec{h}_0, \vec{k}\right)$. In addition, the condition $\operatorname{div} \vec{V} = 0$ means that the matrices $C(b_t), C^*(b_t)$ have zero traces:

$$trC(b_t) = trC^*(b_t) = 0,$$

those $C(b_t) \in SL(\nu, \mathbb{R})$. As is known ([13] Ch. XV, §5, §6), the unimodular $\nu \times \nu$ matrix $X(t)$, is a solution to the equation

$$\frac{d}{dt}X(t) = C(t)X(t), \quad X(0) = E,$$

where the E is identity matrix is called the multiplicative integral (in terms of [13]-matrix) and is denoted by the symbol

$$X(t) = \Omega_0^t(D) = \int_0^t (E + D(s)ds)\,.$$

Then, introducing into consideration the matrix (multiplicative integral, more precisely, multiplicative stochastic integral)

$$G_t = \int_0^t (E + C(b_s)ds)\,. \tag{11}$$

as a solution to the equation

$$\frac{d}{dt}G_t = -C(b_t)G_t, \quad G_0 = E,$$

and noting that the matrix of system (9) is $(G_t^*)^{-1}$ and the property 2^0 ([13], p. 431) we get that the solutions of systems (9) - (10) can be written, respectively, in the form

$$\vec{\varkappa}(t,\vec{k}) = (G_t^*)^{-1}\vec{k},$$

$$\vec{h}(t,\vec{k}) = G_t \vec{h}_0 exp\left\{-\nu_m \int_0^t \vec{\varkappa}^2(s,\vec{k})ds\right\},$$

Now, to find a general solution of the problem (6) in model (7), we need to expand the initial condition $\vec{H}_0(x)$ into a Fourier integral and force each component of this expansion to evolve according to systems (9), (10). In other words, if by $\widehat{\vec{H}}_0(\vec{k})$ we denote the Fourier image of the initial field $\vec{H}_0(x)$:

$$\vec{H}_0(x) = \frac{1}{(2\pi)^{\nu/2}} \int\limits_{\mathbb{R}^\nu} e^{i(\vec{k},x)} \widehat{\vec{H}}_0(\vec{k}) d\vec{k}. \tag{12}$$

then

$$\vec{H}(t,x) = \frac{1}{(2\pi)^{\nu/2}} \int\limits_{\mathbb{R}^\nu} G_t \widehat{\vec{H}}_0(\vec{k}) exp\left\{ i((G_t^*)^{-1}\vec{k},x)\right\} \cdot exp\left\{ -\nu_m \int\limits_0^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds \right\} d\vec{k}. \tag{13}$$

It is easy to see that for the total magnetic energy $E(t)$ we obtain the integral representation

$$\mathcal{E}(t) = \int\limits_{\mathbb{R}^\nu} \vec{H}^2(t,x)dx = \int\limits_{\mathbb{R}^\nu} \left(G_t \widehat{\vec{H}}_0(\vec{k})\right)^2 exp\left\{ -2\nu_m \int\limits_0^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds \right\} d\vec{k} \tag{14}$$

Thus, the solution $\vec{H}(t,x)$ of the magnetic induction equation (6) in the Markov linear model (1) by formula (13), and its magnetic energy $\mathcal{E}(t)$ by formula (14) are expressed as some functionals of the multiplicative integral (matrix) of the form (11). And this means that in order to study questions about the asymptotic behaviors of the magnetic field $\vec{H}(t,x)$ and magnetic energy $\mathcal{E}(t)$ as $t \to \infty$, it is important to know the asymptotic behavior as the multiplicative integral G_t itself and integrals (13) and (14) depending on it as $t \to \infty$.

In connection with the above, the following tasks arise:

A) Find out the asymptotic behavior of G_t as $t \to \infty$;

B) Carry out an asymptotic analysis of the magnetic field $\vec{H}(t,x)$ and magnetic energy $\mathcal{E}(t)$ at $t \to \infty$.

Other equally interesting problems are also possible (for example, those related to various moments of the magnetic field in the Markov linear model). But in this paper we will not consider such problems.

The solution of problem A) was announced in [7], and the complete solution of problem A) was given in [8] in the following setting.

Let θ_0 is an arbitrary (non-random) column vector of unit length: $\theta_0 \in S^{\nu-1}$ is the unit sphere in $\mathbb{R}^\nu$, $\nu \geq 3$. Let us act on θ_0 by the multiplicative integral G_t with the matrix $C(\cdot)$: $K \to SL(\nu,\mathbb{R})$ and denote by r_t the Euclidean length of the resulting vector: $r_t = \|G_t\theta_0\|$.

The Lyapunov exponent of the matrix G_t is the almost-probably (a.s.) limit

$$\gamma = \lim_{t\to\infty} \frac{1}{t} ln\, r_t.$$

Then the following is true.

Theorem 1 *(Ferstenberg type theorem) For any fixed initial phase $\theta_0 \in S^{\nu-1}$ there exists a.s.and a strictly positive limit*

$$\gamma = \lim_{t\to\infty} \frac{1}{t} ln \|G_t\theta_0\| > 0. \tag{15}$$

In other words, it was proved in [8] that for$t \to \infty$; the asymptotic $r_t = \|G_t\theta_0\| \sim exp\{\gamma t\}, \gamma > 0$, i.e., for sufficiently large$t$, the action of the matrix G_t on a vector of unit length leads to its exponential expansion. The proposed paper is devoted to solving problem C). In this case, the above Theorem 1 from [8] will play an essential role.

5.3 Construction of Lyapunov bases for G_t

This section is devoted to obtaining such results on the multiplicative integral G_t of a Markov random matrix that would be convenient for studying the magnetic field $\vec{H}(t,x)$ and its energy $\mathcal{E}(t)$ expressed by formulas (13) and (14).

Let us now proceed to the construction of Lyapunov (generally speaking, random) bases and show that, with the appropriate use of these bases, the multiplicative integral G_t defined in model (1) by formula (11) as $t \to \infty$ almost does not differ from the degree of some constant matrix.

To do this, first of the multiplicative integral (matrix) G_t, like any matrix, we represent as a product of orthogonal (U_t) and upper triangular (K_t) matrices. Technically, this can be done as follows. We orthonormalize the columns of the matrix G_t, starting from the first one, and form a new basis from them. As U_t, we take the transition matrix from the original basis to the new one. Then the diagonal elements of the matrix K_t will have the following form: K_{11} is the length of the first column of the matrix G_t, K_{22} are the length of the component of the second column orthogonal to the first column, and so on. Let us now substitute this representation $G_t = U_tK_t$ into the equation for G_t, if the matrix $U_t^{-1}C(b_t)U_t$ is represented as the sum of antisymmetric (F_t) and upper triangular (B_t) matrices, then for U_t and K_t we obtain the equations

$$\frac{dU_t}{dt} = U_tF_t, \quad \frac{dK_t}{dt} = B_tK_t \tag{16}$$

The first of these equations is a non-linear equation closed with respect to the orthogonal matrix U_t, which determines the orientation of the matrix G_t. After solving this equation, we can find the diagonal elements of K_t:

$$K_{ii} = K_{ii}(0)exp\left\{t\gamma_i + \sqrt{t}\xi_i(t)\right\},$$

where

$$\gamma_i = \lim_{t\to\infty} \frac{1}{t}\int_0^t b_{ii}(s)ds, \quad \xi_i(t) = \frac{1}{\sqrt{t}}\int_0^t (b_{ii}(s) - \gamma_i)\, ds. \tag{17}$$

According to the central limit theorem, as $t \to \infty$, the process $\xi_i(t)$ has a normal distribution (note that we can always put $K_{ii}(0) = 1$).

Moreover, the matrix U_t is Markov and the group of orthogonal matrices is compact in $SL(\nu, \mathbb{R})$. Whence follows the existence of a stationary distribution μ of the matrices U, over which the averaging is performed in the first of formulas (17). Further, $detK_t = 1$, so $\gamma_1 + \gamma_2 + ... + \gamma_\nu = 0$. From the decomposition method of G_t into the product $U_t K_t$ it follows that $\gamma_1 \geq \gamma_2 \geq ...\gamma_{\nu-1} \geq \gamma_\nu$. Take $\theta_0 = (1, 0, ..., 0) \in S^{\nu-1}$ and make sure that $\gamma_1 = \gamma > 0$, where γ_ν is the largest Lyapunov exponent appearing in the Ferstenberg-type theorem (see (15)). After that, we act to the unit vector $\theta_0 \in S^{\nu-1}$ by the inverse matrix G_t^{-1}. Then the resulting vector, for the same reasons as above, changes as $exp|\gamma_\nu|t$, where γ_ν is the "highest exponent in the reverse course of time i.e. the lowest index and $\gamma_\nu < 0$. However, the negativity of γ_ν also follows from the fact that $\gamma_1 > 0, \gamma_1 + \gamma_2 + ... + \gamma_\nu = 0$. The signs of other $\gamma_j (j = 2, 3..., \nu - 1)$ can be arbitrary. For example, if $\nu = 3$ and the distribution of matrices also has symmetry under the change $G_t \to G_t^{-1}$, then $\gamma_2 = 0,\ \gamma_1 = -\gamma_3$. In addition, some of γ_j can be the same. However, it turns out that in our assumptions, they are all different, or rather true

Theorem 2 *(simplicity theorem for the spectrum of characteristic Lyapunov exponents). The exponents of the Lyapunov matrix G_t are different, i.e. there are strict inequalities*

$$\gamma = \gamma_1 \geq \gamma_2 \geq ...\gamma_{\nu-1} \geq \gamma_\nu. \tag{18}$$

Proof 1 *We divide the interval $[0, t]$ into n parts by points $0 = t_0 < t_1 < t_2 < ... < t_n = t$ and represent the matrix G_t as ([13], p. 433, formula (46))*

$$G_t = g_n \cdot g_{n-1} \cdot ... \cdot g_1, \tag{19}$$

where

$$g_k = \int\limits_{t_{k-1}}^{t_k} (E + C(b_s)ds), \quad k = 1, 2, ..., n. \tag{20}$$

It is clear that $g_1, g_2...$ are stationary Markov processes with values in the group $SL(\nu, \mathbb{R})$ and mean value $Mln\|g_1\| < \infty$. Studying this sequence $g_1, g_2...$ from the point of view of [11]. Given that, according to the results of [8], the pair (b_t, G_t) has a smooth transition density, we see that this sequence satisfies all the requirements of the main theorem of [11] ([11], §2, p.122). Thus, according to the main theorem [11], the characteristic Lyapunov exponents are simple, i.e.

$$\gamma_1 > \gamma_2 > ... > \gamma_\nu.$$

The simplicity of the spectrum of characteristic exponents is one of the central properties that determine the asymptotic behavior of the multiplicative integral G_t. Below, using this property, we will prove a theorem that, when using an appropriate Lyapunov basis, for large t $(t \to \infty)$ G_t almost does not differ from the degree of some fixed matrix (see Theorem 3 below). For these purposes, we first prove an auxiliary lemma.

Lemma 1 *a) For large t, matrix K_t can be represented as $K_t = D_t\overline{K}_t$, where K_t is an upper triangular matrix such that there are limits $\lim_{t\to\infty} \overline{K}_{ij}(t) = \overline{K}_{ij}(\infty)$, and D_t is diagonal matrix:*

$$D_t = diag\left\{exp\left\{(\gamma_1 + \alpha_{11}(t))t\right\}, ..., exp\left\{(\gamma_\nu + \alpha_{\nu\nu}(t))t\right\}\right\} \tag{21}$$

$$\alpha_{ij} = \frac{1}{t}\int_0^t (b_{ij}(s) - \gamma_j)\, ds.$$

b) Let $\overline{K}(\infty)$ is a matrix with entries $\overline{K}_{ij}(\infty)$. Let us set $\overline{K}_t = \overline{\overline{K}}_t\overline{K}(\infty)$. Then for any $\alpha > 0$ there are numbers $\beta_{ij} > 0$ such that, for sufficiently large t, the inequalities hold for $j < i$ with probability 1

$$-\beta_{ij}exp\left\{(\gamma_j - \gamma_i)t - \alpha t\right\} \leq \overline{\overline{K}}_{ij}(t) \leq \beta_{ij}exp\left\{(\gamma_j - \gamma_i)t + \alpha t\right\} \tag{22}$$

where $\overline{\overline{K}}_{ij}(t)$ are the elements of the matrix $\overline{\overline{K}}_t$.

Proof 2 *For simplicity, we will carry out the proof for matrices K_t of order 3×3.*

a) We know the form of the diagonal elements of K_t (formula (16). Using (17) for off-diagonal elements, we obtain:

$$K_{12}(t) = K_{11}(t)\overline{K}_{12}(t),$$

$$K_{13}(t) = K_{11}(t)\overline{K}_{13}(t),$$

$$K_{22}(t) = K_{22}(t)\overline{K}_{23}(t),$$

where

$$\overline{K}_{12}(t) = 1 + \int_0^t b_{12}(s)K_{22}(s)K_{11}^{-1}ds,$$

$$\overline{K}_{13}(t) = 1 + \int_0^t \left(b_{12}(s)K_{23}(s) + b_{13}(s)K_{33}(s)\right)K_{11}^{-1}ds, \tag{23}$$

$$\overline{K}_{23}(t) = 1 + \int_0^t b_{23}(s)K_{33}(s)K_{22}^{-1}ds,$$

whence $K_t = D_t\overline{K}_t$, where the matrix D_t is defined by proportions (21).

Further, due to the boundedness of the norm matrix $C(x)$ on the compact set K, the elements of the matrix B_t with probability 1 are bounded functions of t. Therefore, from formula (21), from the fact that inequalities $j > i$ are found for $\gamma_j - \gamma_i < 0$, means of limitation follow for $\overline{K}_{ij}(t)$ at $t \to \infty$.

b) by the definition of a matrix, $\overline{\overline{K}}_t$ we have:

$$\overline{\overline{K}}_{11}(t) = \overline{\overline{K}}_{22}(t) = \overline{\overline{K}}_{33}(t) = 1, \overline{\overline{K}}_{ij}(t) = 0,\ (j < i), \overline{\overline{K}}_{12}(t) = \overline{K}_{12}(G) - \overline{K}_{12}(\infty),$$

$$\overline{\overline{K}}_{23}(t) = \overline{K}_{23}(t) - \overline{K}_{23}(\infty),\ \overline{\overline{K}}_{13}(t) = \overline{K}_{13}(t) - \overline{K}_{13}(\infty) + \overline{\overline{K}}_{12}(t)\overline{K}_{23}(\infty).$$

Substituting now the values $\overline{K}_{ij}(t)$ from (23) into the last relations, and evaluating the integrals obtained, we obtain the inequalities we need (22).

Definition 1 *Let column vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_\nu$ form a matrix e, satisfying the condition*

$$\overline{K}(\infty)e = E,$$

where E is the identity matrix. Then the basis $\vec{e}_1, \vec{e}_2, ..., \vec{e}_\nu$ called corresponding indicators $\gamma_1, \gamma_2, ..., \gamma_\nu$ Lyapunov basis.

The name "Lyapunov bases"is justified to some extent by the following theorem.

Theorem 3 *On Lyapunov bases $\vec{e}_1, \vec{e}_2, ..., \vec{e}_\nu$ the relations are fulfilled*

$$\gamma_j = \lim_{t\to\infty} \frac{1}{t} ln \left\| G_t \vec{e}_j \right\|, \quad j = 1, 2, ..., \nu. \tag{24}$$

Proof 3 *By definition of the Lyapunov basis*

$$\overline{K}(\infty)\vec{e} = (0, ..., 0, 1, 0, ..., 0)^*,$$

where 1(one) is in the $j-th$ place, and the $*$ is sign is the transposition operation.

Therefore, $\left\| G_t \vec{e}_j \right\|^2$ the length is simply the square of the length of the $j-th$ column of the matrix $D_t \overline{\overline{K}}_t$, i.e.,

$$\left\| G_t \vec{e}_j \right\|^2 = \sum_{l=1}^{\nu} \overline{\overline{K}}_{ij}^2 exp\left\{2(\gamma_l + \alpha_{ll}(t))t\right\}$$

Using inequalities (22), we obtain that for any $\alpha > 0$ there are numbers $\beta_1, \beta_2 > 0$ such that

$$\beta_1 exp\left\{2\gamma_j t - \alpha t\right\} \leq \left\| G_t \vec{e}_j \right\|^2 \leq \beta_2 exp\left\{2\gamma_j t + \alpha t\right\}$$

Now the relations (21) we need follow from the last inequality (due to the arbitrariness of α).

Note that the Lyapunov exponents are, as averages, non-random numbers. Mean while, the Lyapunov basis corresponding to them is random; it is different for different implementations of the process b_t. However, for this implementation, the basis is the same and does not depend on time.

5.4 Asymptotic analysis of the magnetic field and total magnetic energy

Now let the vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_\nu$ constitute the Lyapunov basis, and $\vec{e} = \lambda_1\vec{e}_1 + \lambda_2\vec{e}_2 + ... + \lambda_\nu\vec{e}_\nu$. Denote by $[|G_t\vec{e}|]^2$ approximate value of the square of the Euclidean norm, calculated under the assumption of the replacement $\overline{K}(t)$ with $\overline{K}(\infty)$ and D_t on $\overline{D}_t = diag exp(\gamma_1 t), ..., exp(\gamma_\nu t)$. In other words

$$[|G_t\vec{e}|]^2 = \sum_{j=1}^{\nu} \lambda_j^2 exp(2\gamma_j t). \tag{25}$$

On the other hand, due to the infinite smallness of $\alpha_{jj}(t)(j=1,...,\nu)ast \to \infty$, for any $a>0$ there exists $T_1 = T_1(a) > 0$, such that for $t > T_1$ the inequalities hold

$$e^{-at}\left[|G_t\vec{e}|\right]^2 \le \left\|D_t\overline{K}(\infty)\vec{e}\right\|^2 = \sum_{j=1}^{\nu}\lambda_j^2 e^{2(\gamma_j+\alpha_{jj}(t))t} \le e^{at}\left[|G_t\vec{e}|\right]^2. \tag{26}$$

Using (6), we can write similar inequalities for the norms $\left\|D_t\overline{K}(\infty)\vec{e}\right\|^2$ and $\|G_t\vec{e}\|^2$. As a result, we come to the conclusion that the theorem is true.

Theorem 4 *For any $a>0$ there exists $T=T(a)>0$ such that for $t>T$ uniformly over all vectors $\vec{e}\in\mathbb{R}^\nu$ the inequalities hold*

$$e^{-at} \le \frac{\left[|G_t\vec{e}|\right]^2}{\|G_t\vec{e}\|^2} \le e^{at}. \tag{27}$$

Inequalities (27) will later play an essential role in the study of the asymptotic behavior of the magnetic field strength and its total energy. Now, before proceeding to these studies, we note the following useful information.

If $\gamma_1 > \gamma_2 > ... > \gamma_\nu$ are the Lyapunov exponents for the matrix G_t, then the Lyapunov exponents for $(G_t^*)^{-1}$ will be $(-\gamma_1) < (-\gamma_2 < ... < (-\gamma_\nu)$. The corresponding Lyapunov basis $\vec{e}_1', \vec{e}_2', ..., \vec{e}_\nu'$ satisfies the condition

$$\left(\overline{K}(\infty)^*\right)^{-1} e' = E$$

(e' composed of column vectors $\vec{e}_1', \vec{e}_2', ..., \vec{e}_\nu'$ matrix). In other words, $e^*e' = E$, i.e. the Lyapunov bases for G_t and $(G_t^*)^{-1}$ are biorthogonal.

In addition, for the elements of the matrix $(G_t^*)^{-1}$ inequalities similar to (22) hold, so inequalities (27) are also valid for $(G_t^*)^{-1}$.

Note also that in formulas (12) and (13) the quantities $(G_t^*)^{-1}\vec{k}$ and $G_t\widehat{\vec{H}}_0(\vec{k})$ increase exponentially with probability 1. Therefore, the multiplier $exp\left\{-\nu_m\int\limits_0^t\left((G_t^*)^{-1}\vec{k}\right)^2 ds\right\}$ decreases as $t\to\infty$ (and any $\nu_m > 0$) as a double exponent. But in the integral sense (due to the influence of values of $\vec{k}$ close to zero) the double exponent decreases only at the rate of the usual exponent. This circumstance determines the nontriviality of the analysis of integral expressions (12) and (13).

Further, for simplicity, we will assume that $\nu=3$ and proceed from formulas (12) and (13). In addition, below we will additionally assume that $\gamma_2 \ne 0$ herefore, two qualitatively different cases are possible:

$$a)\gamma_1 > 0 > \gamma_2 > \gamma_3,$$

$$b)\gamma_1 > \gamma_2 > 0 > \gamma_3.$$

Let us now formulate the main results - the theorem on the exponential decrease in the magnetic field and the theorem on the exponential growth of the total magnetic energy. (Below, $|\vec{a}|$ denotes the length of the vector $\vec{a}$, and we use the sign $\|\cdot\|$ to denote the norm).

Theorem 5 *Let $|\vec{H}_0(x)| \in L^{1+\beta}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ where $0 < \beta \leq 1$. Then there is $\alpha > 0$ (which does not depend on ν_m, provided that $\nu_m > 0$) such that with probability 1 as $t \to \infty$*

$$\sup_x |\vec{H}(t,x)| = o(exp(-\alpha t)). \tag{28}$$

Theorem 6 *Let $\widehat{\vec{H}}_0(0) \neq 0$, $|\vec{H}_0(x)| \in L^1(\mathbb{R}^3)$ and the initial field $\vec{H}_0(x)$ either be nonrandom or random, but does not depend on the fluid flow, i.e. from process b_t. Then there exists a positive with probability 1 function $B(\omega)$ of the elementary ω and a constant $\alpha > 0$ (the same for all $\nu_m > 0$) such that for sufficiently large t the inequality*

$$\varepsilon(t) > B(\omega)(exp(\alpha t)). \tag{29}$$

The proofs of these theorems will essentially be based on the inequalities (27), and the methodologies of the proofs from the technical point of view are in many respects similar and rather lengthy. In this connection, here we omit the detailed proofs of these assertions and give only the proof of Theorem 5 in case b).

We will proceed from the integral representation (12) for the magnetic field strength H $\vec{H}(t,x)$. Omitting the constant before the integral, we can write

$$|\vec{H}(t,x)| \leq \int\limits_{\mathbb{R}^3} exp\left\{-\nu_m \int\limits_0^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds\right\} \left|G_t \widehat{\vec{H}}_0(\vec{k})\right| d\vec{k}. \tag{30}$$

But according to Theorem 2, for any $\alpha > 0$ there exists $T = T(a) > 0$ such that for $t > T$ we have

$$\left|G_t \widehat{\vec{H}}_0(\vec{k})\right| \leq exp\left\{(\gamma_1 + \alpha)t\right\} \left|\widehat{\vec{H}}_0(\vec{k})\right| \tag{31}$$

On the other hand, $\vec{H}_0(x) \in L^{1+\beta}(\mathbb{R}^3)$, therefore, by the Hausdorff-Young inequality, $\widehat{\vec{H}}_0(\vec{k}) \in L^q(\mathbb{R}^3)$, $\frac{1}{q} + \frac{1}{1+\beta} = 1$. Furthermore,

$$\left\|\widehat{\vec{H}}_0(\vec{k})\right\|_{L^q} \leq c_\beta \left\|\vec{H}_0(x)\right\|_{L^{1+\beta}},$$

where C_β is a constant depending on β. Therefore, applying the Holder inequality to the right-hand side of (21), we obtain

$$\left|\vec{H}(t,x)\right| \leq c_\beta \int\limits_{\mathbb{R}^3} \left(exp\left\{-(1+\beta)\nu_m \int\limits_0^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds\right\}\right)^{\frac{1}{1+\beta}} \left\|\widehat{\vec{H}}_0(\vec{k})\right\|_{L^q} exp\left\{(\gamma_1+\alpha)t\right\} d\vec{k}.$$

Let us estimate the triple integral on the right side of the last inequality. Let k_1, k_2, k_3 are the coordinates of the vector $\vec{k}$ in the Lyapunov basis for $(G_t^*)^{-1}$. Then, according to Theorem 5, for any $\alpha > 0$ and sufficiently large t, the inequalities are fulfilled

$$\int\limits_0^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds \geq \int\limits_0^\tau \left((G_s^*)^{-1}\vec{k}\right)^2 ds + \int\limits_{t-\tau}^t \left((G_s^*)^{-1}\vec{k}\right)^2 ds \geq$$

$$\delta(\omega)\left(k_1^2 + k_2^2 + k_3^2\right) + \sum_{j=1}^3 f_j(t) k_j^2,$$

where $\delta(\omega)=\delta$ is a finite, positive value with probability 1 (ω is an elementary event), τ is a sufficiently small number,

$$f_j(t)=\int\limits_{t-\tau}^{t} exp\left\{-2\left(\gamma_j+\alpha\right)s\right\}ds.$$

Since $\gamma_1>\gamma_2>0>\gamma_3$ and an α is an arbitrary small number, then for large t

$$\int\limits_0^t \left((G_t^*)^{-1}\vec{k}\right)^2 ds \geq \delta(\omega)k^2+C_1(\tau)exp\left(-2\left(\gamma_1+\alpha\right)t\right)k_1^2+$$
$$C_2(\tau)exp\left(-2\left(\gamma_2+\alpha\right)t\right)k_2^2+exp\left(-2\left(\gamma_3+\alpha\right)t\right)k_3^2\geq$$
$$\delta(\omega)k^2+exp\left(-2\left(\gamma_3+\alpha\right)t\right)k_3^2,$$

where C_j are some positive constants depending on τ.
Hence

$$\int\limits_{\mathbb{R}^3} exp\left\{-(1+\beta)\nu_m\int\limits_0^t\left((G_s^*)^{-1}\vec{k}\right)^2 ds\right\}d\vec{k}\leq\left(\frac{\pi}{(1+\beta)\nu_m}\right)^{\frac{3}{2}} exp\left(\left(\gamma_3+\alpha\right)t\right).$$

Substituting this into (31) and taking into account the fact that $\gamma_1+\gamma_3=-\gamma_2<0$, we verify the assertion of Theorem 5. in case b).

Remark 1 *Usually in applications the initial field* $\vec{H}_0(x)$ *is decreasing at infinity as* $|x|^{-(3+\alpha)},\alpha>0$*, so that in the most important practical cases the condition of Theorem 5 is satisfied.*

If the initial function is not random, then Theorem 5 can be strengthened by assuming only the finiteness of the magnetic energy, i.e. by assuming that $|\vec{H}_0(x)|\in L^2(\mathbb{R}^3)$. Namely, the following is true.

Theorem 7 *If the initial field is nonrandom or independent of the fluid flow, from the finiteness of its magnetic energy follows an exponential, with probability 1, decrease of the magnetic field as* $t\to\infty$*: for some* $\alpha>0$ *with probability 1*

$$\sup_x|\vec{H}(t,x)|=o(exp(-\alpha t)),\quad t\to\infty. \tag{32}$$

The proof of this theorem is similar to the proof of Theorem 5.

6 Discussion of the results

The main results of the work are presented in section 5. At the same time, at the beginning, in Section 5.4, a description of the Markov linear model considered in this paper is introduced and given. Note that such a model of the velocity field in the form (1) was first defined in the previous works of the first of the authors of this article. Such a representation of the velocity field in [7]- [8] made it possible to apply the theory of degenerate elliptic-parabolic Hermander operators to prove the existence of non-degenerate joint transition densities constructed according to the multiplicative integral (matrix) G_t of diffusion processes defined by formula (11), and, ultimately, to prove theorems of Furstenberg type (Theorem 1). This theorem plays the main role later in paragraph (5.) in the construction of Lyapunov bases for G_t.

In paragraph 5.2. the magnetic induction equation is considered within the introduced Markov linear model. Explicit integral representations (in the form of some functionals of the multiplicative integral G_t) of the desired solution $\vec{H}(t,x)$ of the magnetic induction equation (6) and its total magnetic energy ε_t (formulas (13) and (14), respectively) are obtained. the main problem was reduced to studying the asymptotic behavior of the multiplicative integral G_t and integrals (14) and (15) depending on it as $t \to \infty$.

Section 5.3 is devoted to the construction of Lyapunov (generally speaking, random) bases for G_t. At the same time, using the results known from the theory of matrices (decomposition of a matrix into a product of orthogonal and upper triangular matrices, etc.) and the central limit theorem, we first find the characteristic Lyapunov exponents (formulas (17)), prove a theorem on their simplicity (Theorem 3), and the Lyapunov bases corresponding to these $t(t \to \infty)$ exponents are defined (Definition1).

In the last subsection 5.2, we present and prove the main results of this paper, Theorems 5 and 6 In proving these theorems, the results of Theorem 4 (i.e., inequalities (27)) were essentially used. Theorems 5 and 6 actually mean that in the Markov linear model for large $t(t \to \infty)$ the exponential decrease of the magnetic field occurs with probability 1, however, its total magnetic energy grows exponentially throughout space. This property of the field, which at first glance seems paradoxical, can be explained simply: in the linear model, due to the increase in velocity, the volume of space occupied by the field rapidly increases, which entails an increase in the total magnetic energy.

The results obtained in this work are similar to the results of [5]- [6], where the problem of the evolution of a magnetic field in a random linear velocity field was also studied in a kinematic setting (i.e., for a given velocity field), but the authors of these papers had to deal with with the product of independent random matrices as the number of factors increases. Our paper covers a more general situation (than papers [5]- [6]), because we are studying the magnetic induction equation in a more general (Markovian) linear model, and we had to investigate the asymptotic behavior of a more general product - a multiplicative stochastic integral.

7 Conclusion

The work was devoted to the asymptotic analysis of the solution of the Cauchy problem for the induction equation in a given Markov linear velocity model. Explicit, containing (associated with a given velocity field) some multiplicative stochastic integral G_t of a Markov random matrix, integral representations for the magnetic field and its total energy were obtained. So-called Lyapunov bases are constructed and it is shown that, with an appropriate choice of the corresponding Lyapunov bases, G_t for large $t(t \to \infty)$ almost does not differ from the degree of some constant matrix. As a result, theorems were proved on the exponential decrease in the magnetic field at each point in space and the exponential growth of the total magnetic energy (over the entire space) in a given Markov linear model.

Acknowledgment

The authors are grateful to Professor S.A. Molchanov for statement of the problems and useful discussions.

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Received: March 2, 2025
Accepted: June 7, 2025

IRSTI 27.17.29 DOI: https://doi.org/10.26577/JMMCS2025126203

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A REGULARIZED TRACE OF A TWO-FOLD DIFFERENTIATION OPERATOR WITH NON-LOCAL MATCHING CONDITIONS ON A STAR GRAPH WITH ARCS OF THE SAME LENGTH

In this paper, we study the regularized trace of a two-fold differentiation operator with non-local matching conditions on a star graph consisting of arcs of the same length. We consider both the integrable case, when the potentials belong to the space L_1, and the singular case, in which the potentials admit more general features, including distributions. The main attention is paid to the derivation of the asymptotic decomposition of the characteristic function corresponding to the boundary value problem on a graph and the calculation of regularized traces using spectral theory methods. The main goal is to calculate the first regularized trace of an operator, which is defined as the limit of the sum of the differences of the eigenvalues of the operator and its modification. It is shown that in the integrable case, the regularized trace is a linear functional of the potential coefficients, whereas in the singular case (when the potentials are represented as generalized functions), it acquires a nonlinear dependence. Explicit formulas for the regularized trace using characteristic determinants and integral representation methods are derived. The results of this work generalize the well-known formulas of regularized traces applied to operators on a segment to the case of more complex structures such as graphs. The work is of interest to specialists in the field of spectral theory of operators and differential equations on graphs.
Key words: Regularized trace, star graph, differential operator, Sturm-Liouville operator.

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Ұзындығы бірдей доғалары бар граф жұлдызындағы локальды емес сәйкестік шарттары бар екі еселенген дифференциалдау операторының реттелген ізі

Бұл жұмыста бірдей ұзындықтағы доғалардан тұратын граф-жұлдыздағы локальды емес сәйкестендіру шарттары бар екі реттік дифференциалдау операторының реттелген ізі зерттеледі. Потенциалдар L_1 кеңістігіне жататын интегралданатын жағдай да, потенциалдар үлестіруді қоса алғанда, жалпы ерекшеліктерге мүмкіндік беретін сингулярлық жағдай да қарастырылады. Графтағы шеткі есептерге сәйкес келетін сипаттамалық функцияның асимптотикалық ыдырауын анықтауға және спектрлік теория әдістерін қолдана отырып, реттелген іздерді есептеуге баса назар аударылады. Негізгі мақсат-оператордың меншікті мәндерінің айырмашылықтарының қосындысының шегі және оның модификациясы ретінде анықталатын оператордың бірінші реттелген ізін есептеу. Интегралданған жағдайда реттелген із потенциал коэффициенттерінің сызықтық функционалы болып табылады, ал сингулярлық жағдайда (потенциалдар жалпыланған функциялар түрінде үсынылған кезде) ол сызықтық емес тәуелділікке ие болады. Сипаттамалық анықтауыштар мен интегралды бейнелеу әдістерін қолдана отырып, реттелген ізге арналған нақты формулалар шығарылды. Бұл жұмыстың нәтижелері графтар сияқты күрделі құрылымдар жағдайында кесіндідегі операторларға қолданылатын реттелген іздердің белгілі формулаларын жинақтайды. Жұмыс операторлардың спектрлік теориясы және графтардағы дифференциалдық теңдеулер саласындағы мамандарды қызықтырады.
Түйін сөздер: Реттелген із, граф жұлдызы, дифференциалды оператор, Штурм-Лиувилль операторы.

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Регуляризованный след оператора двух кратного дифференцирования с нелокальными условиями согласования на граф-звезде с дугами одинаковой длины

В данной работе исследуется регуляризованный след оператора двукратного дифференцирования с нелокальными условиями согласования на графе-звезде, состоящем из дуг одинаковой длины. Рассматриваются как интегрируемый случай, когда потенциалы принадлежат пространству L_1, так и сингулярный случай, при котором потенциалы допускают более общие особенности, включая распределения. Основное внимание уделяется выводу асимптотического разложения характеристической функции, соответствующей краевой задаче на графе, и вычислению регуляризованных следов с использованием методов спектральной теории. Основной целью является вычисление первого регуляризованного следа оператора, который определяется как предел суммы разностей собственных значений оператора и его модификации. Показано, что в интегрируемом случае регуляризованный след является линейным функционалом от коэффициентов потенциала, тогда как в сингулярном случае (когда потенциалы представлены в виде обобщённых функций) он приобретает нелинейную зависимость. Выведены явные формулы для регуляризованного следа, использующие характеристические определители и методы интегрального представления. Результаты данной работы обобщают известные формулы регуляризованных следов, применяемые к операторам на отрезке, на случай более сложных структур, таких как графы. Работа представляет интерес для специалистов в области спектральной теории операторов и дифференциальных уравнений на графах.

Ключевые слова: регуляризованный след, граф-звезда, дифференциальный оператор, оператор Штурма-Лиувилля.

1 Formulation of the Problem

In [1], the first regularized trace of the Sturm-Liouville operator B was calculated, generated by the differential expression

$$l(y) = -y^{(2)}(x) + \left(h\delta\left(x - \frac{\pi}{2}\right) - \frac{h}{\pi}\right) y(x)$$

on the segment $[0, \pi]$ with Dirichlet boundary conditions. The eigenvalues of operator B are denoted by λ_n for $n > 0$. Then the formula is valid

$$\sum_{n=1}^{\infty}(\lambda_n - n^2 - \frac{1}{\pi} + (-1)^n \frac{1}{\pi}) = -\frac{h^2}{8}. \tag{1}$$

Further generalizations of A.M.Savchuk's formula can be found in [2,3]. In this paper, the formula (1) is generalized to the case of a differential operator on a star graph.

N.P.Bondarenko [4] considers a star graph with more than two arcs. The lengths of the arcs are considered equal to π. In the article [4], the eigenvalue problem B for a twofold differentiation operator on a graph is investigated

$$-y_j^{(2)}(x) = \lambda y_j(x), x \in (0, \pi), \qquad j = \overline{1, m},$$

with Robin conditions in the boundary vertices

$$y_j^{(1)}(0) - h_j y_j(0) = 0, \qquad j = \overline{1, m},$$

with continuity conditions in the inner vertex

$$y_j(\pi) = y_1(\pi), \qquad j = \overline{2, m},$$

and the matching conditions in the inner vertex

$$\sum_{j=1}^{m} \left(y_j^{(1)}(\pi) + \int_0^{\pi} p_j(x) y_j(x)\, dx \right) = 0.$$

Here λ is a spectral parameter, and nonzero numbers h_j are complex numbers. In the first part of the article, the functions $p_j(x)$ belong to the space $L_2(0, \pi)$. The eigenvalues of operator B are denoted by $[\lambda_n, n \geq 1]$. Along with operator B, we also consider operator B_0, which is obtained from operator B when $p_j(x) \equiv 0$, $h_j = 0$, $j = \overline{1, m}$. The eigenvalues of the operator B_0 are denoted by $\{\lambda_n^0, n \geq 1\}$.

The purpose of this article is to calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} \left(\lambda_k - \lambda_k^0 - a_k \right),$$

where are the sequences $\{a_k\}$ and $\{m_n\}$ they are selected in a special way. Thus, the calculated sum is called the first regularized trace of the operator B defined on the star graph. Formulas of regularized traces for different classes of differential operators can be found in the works of V.A. Sadovnichy and his students [5].

The work consists of two parts. First, we study the integrable case when the functions $p_j(x)$ belong to the space $L_2(0, \pi)$. In the second part, we study the singular case when the functions $p_j(x)$ represent distributions. In this case, it is assumed that the generalized primitive $q_j(x) = \int p_j(x)\, dx$ are functions of limited variation. It is proved that in the integrable case, the regularized trace is a linear functional of the functions $p_j(x)$. At the same time, the regularized trace in the singular case is a nonlinear functional of the functions $p_j(x)$.

2 The main result in the integrable case

It is convenient to introduce notation to formulate the results. Let

$$P(x) = \sum_{j=1}^{m} p_j(x), \qquad P_1(x) = \sum_{j=1}^{m} h_j p_j(x), \qquad P_2(x) = \sum_{j=1}^{m} \frac{p_j(x)}{h_j},$$

$$H_1 = \sum_{j=1}^{m} h_j, \qquad H_2 = \sum_{j=1}^{m} \frac{1}{h_j}, \qquad \lambda = z^2.$$

Theorem 1 (integrable case) . *Let the nonzero numbers h_j be complex, and the functionsp$_j$ (x) belong to the space $L_2(0,\pi)$. Also assume that the function $P(x)=\sum_{j=1}^{m} p_j(x)$ satisfies the Dini condition at the point $x=\pi$. The, for $m>2$, the formula for the regularized trace is valid*

$$\lim_{n\to\infty}\sum_{k=1}^{m_n}\left(\lambda_k-\lambda_k^0\right)=P(\pi).$$

Here m_n is the number of eigenvalues of the original problem in the circle λ plane of radius $\left(n+\frac{1}{4}\right)^2$ centered at zero.

It was shown in [4] that the characteristic determinants of the operators B_0 and B are given by the equalities

$$\varDelta_0\left(z^2\right)=-mz\sin(z\pi)\left(\cos(z\pi)\right)^{m-1},$$

$$\varDelta(\lambda)=\varDelta_0(\lambda)\left(1+F(\lambda)\right), \tag{2}$$

where

$$F\left(z^2\right)=-\int_0^\pi P(x)\frac{\cos(zx)}{z\sin(z\pi)}dx-\int_0^\pi P_1(x)\frac{\sin(zx)}{z^2\sin(z\pi)}dx-H_2\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1}$$

$$-H_1\left(\frac{tg(z\pi)}{z}\right)^{m-1}\int_0^\pi P_2(x)\frac{\cos(zx)}{z\sin(z\pi)}dx-H_1\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1}\int_0^\pi P(x)\frac{\sin(zx)}{z^2\sin(z\pi)}dx.$$

In fact, the characteristic determinants $\Delta_0(\lambda)$ and $\Delta(\lambda)$ are integer functions of λ. The zeros of the integer functions $\Delta_0(\lambda)$ and$\Delta(\lambda)$ represent the eigenvalues of operators B_0 and B. Thus, the sequence $\{\lambda_n^0, n\geq 1\}$ represents a sequence of zeros taking into account their multiplicities of the whole function $\Delta_0(\lambda)$. Similarly, the sequence $\{\lambda_n, n\geq 1\}$ it is associated with the zeros of the whole function $\Delta(\lambda)$. Their asymptotic behavior was clarified in [4]. Since the zeros of the whole function $\Delta_0(\lambda)$ break up into series, the zeros of the whole function $\Delta(\lambda)$ also have an asymptotically serial structure (see Theorem 1.2 from [4]). Their asymptotic behavior is clarified in the work [4].

Let us use γ_n to denote a circle in the z-plane of radius $n+\frac{1}{4}$ centered at zero. It is easy to understand that the function $\operatorname{tg}(z\pi)$ on the circles γ_n is bounded by a constant independent of n. If x is fixed between zero and π, then the functions $\frac{\cos(zx)}{\sin(z\pi)}$ and $\frac{\sin(zx)}{\sin(z\pi)}$ on the circles γ_n are bounded by a constant independent of n. For sufficiently large n, the function $\ln(1+F(\lambda))$ is holomorphic on the circle γ_n. Now let's try to calculate the integral $\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\varDelta\left(z^2\right)$. According to the principle of the argument, we have

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\varDelta\left(z^2\right)=2\sum_{k=1}^{m_n}\lambda_k. \tag{3}$$

Here $m_n = \frac{1}{2\pi i}\oint_{\gamma_n} d\ln\Delta\left(z^2\right) = \frac{1}{2\pi i}\oint_{\gamma_n} d\ln\Delta_0\left(z^2\right)$ for sufficiently large n. On the other hand, the ratio (2) implies

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\Delta\left(z^2\right) = \frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\Delta_0\left(z^2\right) + \frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right).$$

It is clear that

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\Delta_0\left(z^2\right) = 2\sum_{k=1}^{m_n}\lambda_k^0. \tag{4}$$

Applying the piecemeal integration formula allows us to write the ratio

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right) = -2\frac{1}{2\pi i}\oint_{\gamma_n} z\ln\left(1+F\left(z^2\right)\right)dz.$$

Let $m > 3$. For sufficiently large n, we rewrite the last equality as

$$\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right) = -2\frac{1}{2\pi i}\oint_{\gamma_n} zF\left(z^2\right)dz + o(1). \tag{5}$$

It is taken into account here that for $n\to\infty$ on the circles γ_n, the function $F\left(z^2\right) = o\left(\frac{1}{n}\right)$, since the functions $p_j(x)$ belong to the space $L_2(0,\pi)$. It remains to calculate the integral $\frac{1}{2\pi i}\oint_{\gamma_n} zF\left(z^2\right)dz$ using the deduction theorem. When $n\to\infty$ we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\gamma_n} zF\left(z^2\right)dz &= \frac{1}{2\pi i}\oint_{\gamma_n} z\left(-\int_0^\pi P(x)\frac{\cos(zx)}{z\sin(z\pi)}dx\right)dz + o(1)\\ &= -\int_0^\pi P(x)\left(\frac{1}{2\pi i}\oint_{\gamma_n}\frac{\cos(zx)}{z\sin(z\pi)}dz\right)dx + o(1)\\ &= -\frac{1}{\pi}\int_0^\pi P(x)\left(1+2\sum_{k=1}^{n}(-1)^k\cos(kx)\right)dx + o(1).\end{aligned} \tag{6}$$

Thus, it follows from the relations (3)–(6) that for $m > 3$ the formula of the regularized trace has the form

$$\lim_{n\to\infty}\sum_{k=1}^{m_n}\left(\lambda_k - \lambda_k^0\right) = P(\pi). \tag{7}$$

Now consider the case of $m = 3$. For sufficiently large n, equality (5) can be written as

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right)\\ &= -2\frac{1}{2\pi i}\oint_{\gamma_n} z\left(-\int_0^\pi P(x)\frac{\cos(zx)}{z\sin(z\pi)}dx\right)dz + 2H_2\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\operatorname{tg}(z\pi)}{z}\right)^{m-1}dz + o(1).\end{aligned}$$

We need to calculate the integral $\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\text{tg}(z\pi)}{z}\right)^2 dz$ using the deduction theorem:

$$\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\text{tg}\,(z\pi)}{z}\right)^2 dz = 0$$

Thus, the contribution from the integral $\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\text{tg}(z\pi)}{z}\right)^2 dz$ in the sum (7) is missing. In the case of $m = 3$, the formula (7) is preserved.

3 The singular case

In this paragraph, the generalized primitive $q_j\left(x\right) = \int p_j\left(x\right)dx$ are functions of limited variation. In particular, the case when the generalized primordial $q_j\left(x\right)$ represent the jump functions is studied in detail. In this case, the Stieltjes integral is calculated using the formula

$$\int_0^{\pi} y\left(x\right)dQ\left(x\right) = \sum_{s=1}^{r} t_s y\left(x_s\right),$$

where $dQ\left(x\right) = P\left(x\right)dx$.

Then

$$F\left(z^2\right) = -\sum_{s=1}^{r} t_s \frac{\cos\left(zx_s\right)}{z\sin\left(z\pi\right)} - \sum_{s=1}^{r_1} t_{s1}\frac{\sin\left(zx_{s1}\right)}{z^2\sin\left(z\pi\right)} - H_2\left(\frac{\text{tg}\,(z\pi)}{z}\right)^{m-1}$$

$$-H_1\left(\frac{\text{tg}\,(z\pi)}{z}\right)^{m-1}\sum_{s=1}^{r_2} t_{s2}\frac{\cos\left(zx_{s2}\right)}{z\sin\left(z\pi\right)} - H_1\left(\frac{\text{tg}\,(z\pi)}{z}\right)^{m-1}\sum_{s=1}^{r} t_s\frac{\sin\left(zx_s\right)}{z^2\sin\left(z\pi\right)}.$$

In this case, the ratio (5) will be written as

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\gamma_n} z^2 d\ln\left(1+F\left(z^2\right)\right) &= -2\frac{1}{2\pi i}\oint_{\gamma_n} z\ln\left(1+F\left(z^2\right)\right)dz\\ &= 2\frac{1}{2\pi i}\oint_{\gamma_n} z\sum_{s=1}^{r} t_s\frac{\cos\left(zx_s\right)}{z\sin\left(z\pi\right)}dz + 2\frac{1}{2\pi i}\oint_{\gamma_n} z\sum_{s=1}^{r_1} t_{s1}\frac{\sin\left(zx_{s1}\right)}{z^2\sin\left(z\pi\right)}dz\\ &+ 2H_2\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\frac{\text{tg}\,(z\pi)}{z}\right)^{m-1}dz-\\ &- \frac{1}{2\pi i}\oint_{\gamma_n} z\left(\sum_{s=1}^{r} t_s\frac{\cos\left(zx_s\right)}{z\sin\left(z\pi\right)}\right)^2 dz + o(1), \qquad n\to\infty.\end{aligned}$$

It remains to calculate the required integrals using the deduction theorem:

$$2\frac{1}{2\pi i}\oint_{\gamma_n}\sum_{s=1}^{r} t_s\frac{\cos\left(zx_s\right)}{\sin\left(z\pi\right)}dz = 2\sum_{s=1}^{r}\left(2\sum_{k=1}^{n}\frac{\cos\left(kx_s\right)}{\pi\left(-1\right)^k} + \frac{1}{\pi}\right)t_s,$$

$$2\frac{1}{2\pi i}\oint_{\gamma_n} z\sum_{s=1}^{r_1} t_{s1}\frac{\sin(zx_{s1})}{z^2\sin(z\pi)}dz = 2\sum_{s=1}^{r_1} t_{s1}\left(\frac{x_{s1}}{\pi} + 2\sum_{k=1}^{n}(-1)^k\frac{\sin(kx_{s1})}{k\pi}\right),$$

$$\frac{1}{2\pi i}\oint_{\gamma_n} z\left(\sum_{s=1}^{r} t_s\frac{\cos(zx_s)}{z\sin(z\pi)}\right)^2 dz = 2\sum_{k=1}^{r}\frac{1}{k}\left(\sum_{s=1}^{r} t_s\cos(kx_s)\right)^2.$$

Thus, in the singular case, the regularized trace formula has the form

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{m_n}\left(\lambda_k-\lambda_k^0\right)-\sum_{s=1}^{r}\left(2\sum_{k=1}^{n}\frac{\cos(kx_s)}{\pi(-1)^k}+\frac{1}{\pi}\right)t_s\right\}$$
$$= -\sum_{w=1}^{r}\sum_{s=1}^{r} t_w t_s\left(C(x_w+x_s)+C(x_w-x_s)\right)+\sum_{s=1}^{r_1} t_{s1}A_s, \qquad (8)$$

where $A_s = \dfrac{x_{s1}}{\pi} + 2\sum_{k=1}^{\infty}(-1)^k\dfrac{\sin(kx_{s1})}{k\pi}$, $C(x_s) = \dfrac{1}{2}\sum_{k=1}^{\infty}\dfrac{\cos(kx_s)}{k}$. Since $A_s = 0$. Then (8) will take the form

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{m_n}\left(\lambda_k-\lambda_k^0\right)-\sum_{s=1}^{r}\left(2\sum_{k=1}^{n}\frac{\cos(kx_s)}{\pi(-1)^k}+\frac{1}{\pi}\right)t_s\right\}$$
$$= -\sum_{w=1}^{r}\sum_{s=1}^{r} t_w t_s\left(C(x_w+x_s)+C(x_w-x_s)\right) \qquad (9)$$

Let requirement 1 be fulfilled: for an arbitrary continuous function $y(x)$, the integrals satisfy the equalities

$$\int_0^\pi y(x)\,dQ(x) = \sum_{s=1}^{r} t_s y(x_s),$$

$$\int_0^\pi y(x)\,dQ_1(x) = \sum_{s=1}^{r_1} t_{s1} y(x_{s1}),$$

where $dQ(x) = P(x)\,dx$ and $dQ_1(x) = P_1(x)\,dx$.

Theorem 2 (the singular case) *Let the nonzero numbers h_j be complex, the set of functions $\{p_j(x)\}$ is subject to requirement 1. Then, for $m > 2$, the formula (9) is valid for a regularized trace.*

Thus, from formula (9) we see that the regularized trace in the singular case is a nonlinear functional from jumps $\{t_s\}$. At the same time, from the theorem 1 we see that the regularized trace in the integrable case is a linear functional of the functions $\{p_j(x)\}$. A similar effect in the case of differential operators on a segment was noted in [1, 2, 3]. In this paper, it is shown that the A. M. Savchuk effect is also preserved for second-order differential operators on a star graph.

4 Acknowledgements

The work has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP19678089).

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Received: May 27, 2025
Accepted: June 9, 2025

IRSTI 27.31.15 DOI: https://doi.org/10.26577/JMMCS2025126204

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ON THE SOLVABILITY OF BOUNDARY VALUE PROBLEMS WITH GENERAL CONDITIONS FOR THE TRIHARMONIC EQUATION IN A BALL

The need to study boundary value problems for elliptic and parabolic equations is dictated by numerous practical applications in the theoretical study of processes in hydrodynamics, electrostatics, mechanics, heat conduction, elasticity theory, and quantum physics. This paper investigates the solvability of a boundary value problem with general conditions for the triharmonic equation in a unit ball.The validity of the analogue of the Almansi representation is proved. For completeness of presentation, a representation of the Green's functions of the Dirichlet-2 problem is given. This article indicates the difference between the Green's function of the real Dirichlet problem and the Green's function of the Dirichlet-2 problem. It is known that the results of differential equations with partial derivatives in the entire space or differential equations without boundary conditions are in a sense final. The theory of boundary value problems for general differential operators is currently a relevant and rapidly developing part of the theory of differential equations. However, there is a shortage of explicitly solvable problems on the path of further development of the theory of boundary value problems of differential equations. Over the past decades, sufficient material has been accumulated on the constructive construction of solutions to boundary value problems for model equations with partial derivatives. This article relates to this topical issue.

Key words: Green's function, triharmonic equation, Dirichlet-2 problem, boundary value problem with general conditions, integral representation of the solution.

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Көп өлшемді шарда үш гармоникалық теңдеу үшін жалпы шарттары бар шеттік есептердің шешімділігі туралы

Эллиптикалық және параболалық теңдеулер үшін шекаралық есептерді зерттеу қажеттілігі гидродинамика, электростатика, механика, жылу өткізгіштік, серпімділік теориясы және кванттық физика процестерін теориялық зерттеуде көптеген практикалық қолданулардан туындайды.Бұл жұмыста көп өлшемді бірлік шарда үш гармоникалық теңдеу үшін жалпы шарттары бар шеттік есептердің шешімділігі зерттеледі. Альманси өрнегінің аналогы дұрыстығы дәлелденеді. Материалдың толықтығы үшін Дирихле-2 есебінің Грин функциясының өрнегі келтірілген. Бұл мақалада нағыз Дирихле есебі мен Дирихле-2 есебінің Грин функцияларының айырмашылығы көрсетілген. Дербес туындылы дифференциалдық теңдеулерге бүкіл кеңістіктегі немесе шекаралық шарттары жоқ дифференциалдық теңдеулерге қатысты нәтижелер белгілі бір мағынада түпкілікті дәрежеде зерттелген болып табылады. Жалпы дифференциалдық операторлар үшін шекаралық есептер теориясы қазіргі уақытта дифференциалдық теңдеулер теориясының өзекті және қарқынды дамып келе жатқан саласы болып табылады. Алайда дифференциалдық теңдеулердің шекаралық есептерінің теориясын одан әрі дамыту жолында анық шешілетін есептердің тапшылығы байқалады.

Соңғы онжылдықтарда жеке туындылары бар модельдік теңдеулер үшін шекаралық есептердің шешімдерін конструктивті құру бойынша жеткілікті материал жинақталды. Бұл мақала осы өзекті тақырыпқа арналады.
Түйін сөздер: Грин функциясы, үш гармоникалық теңдеу, Дирихле-2 есебі, жалпы шарттары бар шеттік есептер, шешімнің интегралдық өрнегі.

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О разрешимости краевых задач с общими условиями для тригармонического уравнения в шаре

Необходимость исследования краевых задач для эллиптических и параболических уравнений диктуется многочисленными практическими приложениями при теоретическом исследовании процессов гидродинамики, электростатики, механики, теплопроводности, теории упругости, квантовой физики. В данной работе исследуется разрешимость краевой задачи с общими условиями для тригармонического уравнения в единичном шаре. Доказана справедливости аналога представление Альманси. Для полноты изложения приведен представление функций Грина задачи Дирихле-2. В данной статье указана разница между Функцией Грина настоящей задачи Дирихле с функцией Грина задачи Дирихле-2. Известно, что результаты дифференциальных уравнений с частными производными во всем пространстве или дифференциальных уравнений без краевых условий являются в некотором смысле окончательными. Теория краевых задач для общих дифференциальных операторов в настоящее время является актуальной и бурно развивающейся частью теории дифференциальных уравнений. Однако ощущается дефицит явнорешаемых задач на пути дальнейшего развития теории краевых задач дифференциальных уравнений. За последние десятилетия накоплен достаточный материал по конструктивному построению решений краевых задач для модельных уравнений с частными производными. К этой актуальной теме относится данная статья.
Ключевые слова: Функция Грина, тригармоническое уравнение, задача Дирихле-2, краевая задача с общими условиями, интегральное представление решения.

1 Introduction

One of the effective methods of representing solutions to boundary value problems for elliptic equations is a method based on constructing the Green's function of the problem. Many works are devoted to constructing the Green's function in explicit form for various classical boundary value problems. The explicit form of the Green's function of the Dirichlet problem for the polyharmonic equation in the unit ball is constructed in various ways in the works [1–6]. In [7, 8] the solvability of some local and nonlocal boundary value problems with involution for the biharmonic equation is investigated and Green's functions are constructed. Solvability conditions for some versions of boundary value problems for the biharmonic equation in a ball are also obtained in [9]. In [10], solutions to the Dirichlet and Neumann problems for a homogeneous polyharmonic equation were found without using the Green's function. In [11], the Green's functions of the Navier [12] and Riquier-Neumann problems for a biharmonic equation in a ball are given, and in [13], the Green's functions of such problems for a polyharmonic equation are constructed. In [14, 15] the conditions for solvability of some boundary value problems for the polyharmonic equation are found and examples are given for the biharmonic and triharmonic equations. In [16, 17] the Fredholm solvability was investigated and the index formulas for the generalized Neumann problem for high-order elliptic equations containing powers of normal derivatives in the boundary conditions were calculated.

2 Statement of the problem and the main result

In this paper, we study the following boundary value problem with general conditions for the triharmonic equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\Delta^3 u(x) = 0, \quad x \in S, \tag{1}$$

$$\begin{cases} a_{00}u + a_{01}\frac{\partial}{\partial\nu}u + a_{02}\Delta u + a_{03}\frac{\partial}{\partial\nu}\Delta u + a_{04}\Delta^2 u = \varphi_1(x),\ x \in \partial S, \\ a_{11}\frac{\partial}{\partial\nu}u + a_{12}\Delta u + a_{13}\frac{\partial}{\partial\nu}\Delta u + a_{14}\Delta^2 u + a_{15}\frac{\partial}{\partial\nu}\Delta^2 u = \varphi_2(x),\ x \in \partial S, \\ a_{21}\frac{\partial}{\partial\nu}u + a_{22}\Delta u + a_{23}\frac{\partial}{\partial\nu}\Delta u + a_{24}\Delta^2 u + a_{25}\frac{\partial}{\partial\nu}\Delta^2 u = \varphi_3(x),\ x \in \partial S, \end{cases} \tag{2}$$

where $\frac{\partial}{\partial\nu}$ is the outer normal derivative to ∂S, a_{ij}, $(i = 0,\ j = \overline{0,4},\ i = 1, 2,\ j = \overline{1,5})$and are some constants.

This problem generalizes the Dirichlet problem ($a_{00} \neq 0$, $a_{11} \neq 0$, $a_{22} \neq 0$, $a_{ij} = 0$ for the remaining i, j), the Riquier problem ($a_{00} \neq 0$, $a_{12} \neq 0$, $a_{23} \neq 0$, $a_{ij} = 0$ for the remaining i, j), but does not generalize the Neumann problem.

Theorem 1. *a) The solution of problem (1)-(2) from the class $C^5(\overline{S}) \cap C^6(S)$ for arbitrary functions $\varphi_1(x) \in C^4(\partial S)$, $\varphi_2(x) \in C^3(\partial S)$, $\varphi_3(x) \in C^3(\partial S)$ exists and is unique if and only if the polynomial*

$$\det P(\lambda) =$$

$$\begin{vmatrix} a_{00} + \lambda a_{01} & 2[a_{01} + 2(2\lambda + n)(a_{02} + \lambda a_{03})] & a^*_{02} \\ \lambda a_{11} & 2[a_{11} + 2(2\lambda + n)(a_{12} + \lambda a_{13})] & a^*_{12} \\ \lambda a_{21} & 2[a_{21} + 2(2\lambda + n)(a_{22} + \lambda a_{23})] & a^*_{22} \end{vmatrix} \tag{3}$$

does not have integer roots in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where

$$a^*_{02} = 8[a_{02} + a_{03} + (2 + 2\lambda + n)(2\lambda + n)a_{04}],$$

$$a^*_{i2} = 8\left[a_{i2} + a_{i3} + (2 + 2\lambda + n)(2\lambda + n)(a_{i4} + \lambda a_{i5})\right],\ i = 1, 2.$$

b) If $\det P(m) = 0$, then the homogeneous problem (1)-(2) has a solution

$$u(x) = \left[C_1 - C_2 + (C_2 - C_3)|x|^2 + (C_3 - C_2)|x|^4\right] H_m(x),$$

where $H_m(x)$ is a homogeneous harmonic polynomial of degree m [18], and the constants C_1, C_2, C_3 are found from the system of equations

$$P(m)\overrightarrow{C} = 0, \quad \overrightarrow{C} = (C_1,\ C_2,\ C_3)^\top. \tag{4}$$

Proof. Let us prove that the homogeneous problem (1)-(2) has only a zero solution. Any triharmonic function in S $u(x) \in C^5(\overline{S})$ can be expanded in a power series [18] and therefore the solution of problem (1)-(2) can be represented in the form

$$u(x) = u_0(x) + |x|^2 u_1(x) + |x|^4 u_2(x) = \sum_{m=0}^{\infty} \sum_{i=1}^{h_m} \left[u^{(i)}_{0m} + |x|^2 u^{(i)}_{1m} + |x|^4 u^{(i)}_{2m}\right] H^{(i)}_m(x), \tag{5}$$

where $h_m = \frac{2m+n-2}{n-2}(m+n-3...n-3)$, a $H_m^{(i)}(x)$, $m \in \mathbb{N}_0$, $i = \overline{1, h_m}$– is a complete orthogonal system of harmonic polynomials on ∂S [18].
Series (5) converges uniformly at $|x| \leq \varepsilon < 1$.

Let's look at the operators

$$L_0 = a_{00} + a_{01}\Lambda + a_{02}\Delta + a_{03}\Lambda\Delta + a_{04}\Delta^2,$$

$$L_j = a_{j1}\Lambda + a_{j2}\Delta + a_{j3}\Lambda\Delta + a_{j4}\Delta^2 + a_{j5}\Lambda\Delta^2,\ j = 1, 2,$$

where $\Lambda = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$.
Since $u(x) \in C^5(\overline{S})$, it follows from the properties of the operator Λ that

$$L_0 u(x) \to a_{00}u + a_{01}\frac{\partial}{\partial \nu}u + a_{02}\Delta u + a_{03}\frac{\partial}{\partial \nu}\Delta u + a_{04}\Delta^2 u, \quad x \to s \in \partial S,$$

$$L_i u(x) \to a_{i1}\frac{\partial}{\partial \nu}u + a_{i2}\Delta u + a_{i3}\frac{\partial}{\partial \nu}\Delta u + a_{i4}\Delta^2 u + a_{i5}\frac{\partial}{\partial \nu}\Delta^2 u,\ i = 1, 2, \quad x \to s \in \partial S, \quad (6)$$

and the limit is uniform in $s \in \partial S$.
It is easy to see that the polynomials $L_j(u_{0m}^{(i)} + |x|^2\, u_{1m}^{(i)} + |x|^4\, u_{2m}^{(i)})\, H_m^{(i)}(x)|_{x=s}$ are orthogonal on ∂S for fixed $j = 0, 1, 2$ and for all $m \in \mathbb{N}_0$, $i = \overline{1, h_m}$.

Let for some $m \in \mathbb{N}_0$, $i = \overline{1, h_m}$ either $u_{0m}^{(i)} \neq 0$, or $u_{1m}^{(i)} \neq 0$, or $u_{2m}^{(i)} \neq 0$ in expansion (5). Then, due to the uniform convergence of the series from (5) for $|x| \leq \varepsilon < 1$, we have

$$\int_{|x|=\varepsilon} H_m^{(i)}(x)\, L_j u(x)\, ds_x = \int_{|x|=\varepsilon} H_m^{(i)}(x)\, L_j \sum_{p=0}^{\infty}\sum_{k=1}^{h_p} \left[u_{0p}^{(k)} + |x|^2\, u_{1p}^{(k)} + |x|^4\, u_{2p}^{(k)}\right] H_p^{(k)}(x)\, ds_x =$$

$$\int_{|x|=\varepsilon} H_m^{(i)}(x)\, L_j \left[u_{0m}^{(i)} + |x|^2\, u_{1m}^{(i)} + |x|^4\, u_{2m}^{(i)}\right] H_m^{(i)}(x)\, ds_x.$$

Directing $\varepsilon \to 1$ in the resulting equality and using (6) we obtain

$$\int_{|x|=1} H_m^{(i)}(x)\, L_j \left[u_{0m}^{(i)} + |x|^2\, u_{1m}^{(i)} + |x|^4\, u_{2m}^{(i)}\right] H_m^{(i)}(x)\, ds_x = 0,\ j = 0, 1, 2. \quad (7)$$

Let's calculate the integrands. Let's use the following properties of operators Λ, Δ:

$$\Lambda(u\, v) = u\Lambda v + v\Lambda u, \quad \Delta\big(|x|^{2k+2}\, Q_s(x)\big) = (2k+2)(2k+2s+n)|x|^{2k}\, Q_s(x).$$

Then on ∂S we have

$$L_0\big(u_{0m}^{(i)} + |x|^2\, u_{1m}^{(i)} + |x|^4\, u_{2m}^{(i)}\big)\, H_m^{(i)}(x) =$$

$$\big(a_{00} + a_{01}\Lambda + a_{02}\Delta + a_{03}\Lambda\Delta + a_{04}\Delta^2\big)[u_{0m}^{(i)} + |x|^2\, u_{1m}^{(i)} + |x|^4\, u_{2m}^{(i)}]\, H_m^{(i)}(x) =$$

$$[u_{0m}^{(i)}(a_{00} + ma_{01}) + u_{1m}^{(i)}\big(a_{00}|x|^2 + (2+m)a_{01}|x|^2 + 2(2m+n)a_{02} + 2(2m+n)ma_{03}\big) +$$

$$u_{2m}^{(i)}\big(a_{00}|x|^4 + (4+m)|x|^4 a_{01} + 4(2+2m+n)|x|^2 a_{02} + 4(2+2m+n)m|x|^2 a_{03} +$$

$$4(2+2m+n)2(2m+n)a_{04}\big)]\, H_m^{(i)}(x) = |x \in \partial S| =$$

$$[u_{0m}^{(i)}(a_{00} + ma_{01}) + u_{1m}^{(i)}\big(a_{00} + (2+m)a_{01} + 2(2m+n)a_{02} + 2(2m+n)ma_{03}\big) +$$

$$u_{2m}^{(i)}\big(a_{00}+(4+m)a_{01}+4(2+2m+n)a_{02}+4(2+2m+n)ma_{03}+4(2+2m+n)2(2m+n)a_{04}\big)\big]\, H_m^{(i)}(x);$$

$$L_j\big(u_{0m}^{(i)}+|x|^2\,u_{1m}^{(i)}+|x|^4\,u_{2m}^{(i)}\big)\,H_m^{(i)}(x)=$$

$$\big(a_{j1}\Lambda+a_{j2}\Delta+a_{j3}\Lambda\Delta+a_{j4}\Delta^2\big)[u_{0m}^{(i)}+|x|^2\,u_{1m}^{(i)}+|x|^4\,u_{2m}^{(i)}]\,H_m^{(i)}(x)=$$

$$\big[u_{0m}^{(i)}\,ma_{j1}+u_{1m}^{(i)}\big((2+m)a_{j1}+2(2m+n)a_{j2}+2(2m+n)ma_{j3}\big)+$$

$$u_{2m}^{(i)}\big((4+m)a_{j1}+4(2+2m+n)a_{j2}+4(2+2m+n)ma_{j3}+4(2+2m+n)2(2m+n)a_{j4}+$$

$$+4(2+2m+n)2(2m+n)ma_{j5}\big)\big]\,H_m^{(i)}(x),\; j=1,2;$$

Therefore (7) can be rewritten as follows

$$\begin{cases} \big(u_{0m}^{(i)}\,(a_{00}+ma_{01})+u_{1m}^{(i)}a_{01}^*+u_{2m}^{(i)}a_{02}^*\big)\,\|H_m^{(i)}\|^2_{L_2(\partial S)}=0,\\ \big(u_{0m}^{(i)}ma_{11}+a_{11}^*u_{1m}^{(i)}+a_{12}^*u_{2m}^{(i)}\big)\,\|H_m^{(i)}\|^2_{L_2(\partial S)}=0,\\ \big(u_{0m}^{(i)}ma_{21}+u_{1m}^{(i)}a_{21}^*+u_{2m}^{(i)}a_{22}^*\big)\,\|H_m^{(i)}\|^2_{L_2(\partial S)}=0,\end{cases}$$

where

$$a_{01}^*=a_{00}+(2+m)a_{01}+2(2m+n)a_{02}+2(2m+n)ma_{03},$$

$$a_{j1}^*=(2+m)a_{j1}+2(2m+n)a_{j2}+2(2m+n)ma_{j3},$$

$$a_{02}^*=a_{00}+(4+m)a_{01}+4(2+2m+n)a_{02}+4(2+2m+n)ma_{03}+4(2+2m+n)2(2m+n)a_{04},$$

$$a_{j2}^*=(4+m)a_{j1}+4(2+2m+n)a_{j2}+4(2+2m+n)ma_{j3}+$$

$$4(2+2m+n)2(2m+n)a_{j4}+4(2+2m+n)2(2m+n)ma_{j5},\; j=1,2.$$

Since $\|H_m^{(i)}\|^2_{L_2(\partial S)}\neq 0$, we get

$$\begin{pmatrix} a_{00}+ma_{01} & a_{01}^* & a_{02}^*\\ ma_{11} & a_{11}^* & a_{12}^*\\ ma_{21} & a_{21}^* & a_{22}^*\end{pmatrix}\overrightarrow{U}=0. \qquad (8)$$

Let us calculate the determinant of this system. It is equal to $\det P(m)$.
If $\det P(m)\neq 0$, then system (8) has only zero $\overrightarrow{U}=(u_{0m}^{(i)},u_{1m}^{(i)},u_{2m}^{(i)})^\top=0$. This contradicts the assumption that either $u_{0m}^{(i)}\neq 0$, or $u_{1m}^{(i)}\neq 0$, or $u_{2m}^{(i)}\neq 0$ in the expansion (5). Thus, problem (1)-(2) has only the zero solution.

If $\det P(m)=0$, then system (4) has a non-zero solution $\overrightarrow{C}=(C_1,C_2,C_3)^\top$, which means

$$P(m)\begin{pmatrix} C_1-C_2\\ C_2-C_3\\ C_3-C_2\end{pmatrix}=0. \qquad (9)$$

Therefore, on ∂S the equalities

$$L_j\big[C_1-C_2+(C_2-C_3)\,|x|^2+(C_3-C_2)\,|x|^4\big]\,H_m(x)=0,\; j=0,1,2,$$

are true and therefore

$$u(x) = \left[C_1 + C_2(-1 + |x|^2 - |x|^4) + C_3(-|x|^2 + |x|^4)\right] H_m(x)$$

is a solution to the homogeneous problem (1)-(2). Theorem 1 is proven.

Theorem 2. *If $u(x)$ is a triharmonic function in S, then for harmonic functions in S*

$$u_0(x) = \Delta^2 u(x) - \frac{|x|^2}{2}\int_0^1 t^{n-1}\Delta^2 u(t^2x)\,dt,$$

$$u_1(x) = \frac{1}{2}\int_0^1 t^{n-1}\Delta^2 u(t^2x)\,dt - \frac{|x|^2}{2}\int_0^1 t^{n-1}\Delta^2 u(t^2x)\,dt,$$

$$u_2(x) = \frac{1}{2}\int_0^1 t^{n-1}\Delta^2 u(t^2x)\,dt \tag{10}$$

Almansi's representation is fair

$$u(x) = u_0(x) + |x|^2\, u_1(x) + |x|^4\, u_2(x). \tag{11}$$

Proof. If the functions $u_0(x)$, $u_1(x)$, $u_2(x)$ are defined by equalities (10), then the representation (11) is true. Let us prove that the functions $u_0(x)$, $u_1(x)$, $u_2(x)$ from (10) are harmonic in S. Obviously, the functions $u_1(x), u_2(x)$ are harmonic in S if the function $u(x)$ is triharmonic in S.
Further, since the equality $\Delta(|x|^2\, v(x)) = (2 + 4\Lambda)\, v(x)$ is true for the harmonic function $v(x)$, and the chain of equalities

$$\Delta(|x|^2\, u_2) = (2m + 4\Lambda)\frac{1}{2}\int_0^1 t^{n-1}w(t^2x)\,dt =$$

$$n\int_0^1 t^{n-1}w(t^2x)\,dt + 2\int_0^1 t^{n+1}\sum_{i=1}^{n} x_i w_{x_i}(t^2x)\,dt = n\int_0^1 t^{n-1}w(t^2x)\,dt + \int_0^1 t^n w_t(t^2x)\,dt =$$

$$n\int_0^1 t^{n-1}w(t^2x)\,dt + t^n w(t^2x)|_0^1 - n\int_0^1 t^{n-1}w(t^2x)\,dt = w(x),$$

then we have

$$\Delta u_0 = \Delta^3 u - \frac{1}{2}\Delta\Big(|x|^2\int_0^1 t^{n-1}\Delta^2 u(t^2x)\,dt\Big) = \Delta^3 u - (2m + 4\Lambda)\frac{1}{2}\int_0^1 t^{n-1}\Delta^2 u(t^2x)\,dt =$$

$$\Delta^3 u(x) - \Delta^3 u(x) = 0.$$

This means that the function $u_0(x)$ is harmonic in S. Theorem 2 is proven.

For the sake of completeness, we present the results of V.V. Karachik [20] on the representation of the solution of the following boundary value problem for the triharmonic equation in the unit ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$

$$\Delta^3 u(x) = 0, \quad x \in S, \tag{12}$$

$$u|_{\partial S} = \varphi_0, \quad , \frac{\partial u}{\partial \nu}|_{\partial S} = \varphi_1, \quad , \Delta u|_{\partial S} = \varphi_2, \tag{13}$$

which can be called the Dirichlet-2 problem. This problem turned out to be close to the Dirichlet problem, they have the same Green's function. The solution to the problem is sought in the class $u \in C^5(\overline{S}) \cap C^6(S)$.

The Green's function of the Dirichlet problem for the Poisson equation in the ball S for $n \geq 2$ has the form

$$G_2(x,\xi) = E_2(x,\xi) - E_2(\frac{x}{|x|}, |x|\xi), \tag{14}$$

where $E_2(x,\xi)$ is an elementary solution of the Laplace equation, as A.V. Bitsadze called it [19]. In the work [20], an elementary solution of the biharmonic equation was determined

$$E_4(x,\xi) = \begin{cases} \frac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, & n > 4,\ n = 3 \\ -\frac{1}{4}\ln|x-\xi|, & n = 4 \\ \frac{|x-\xi|^2}{4}(\ln|x-\xi| - 1) & n = 2, \end{cases} \tag{15}$$

was determined, and in the paper [21] for the 3-harmonic equation

$$E_6(x,\xi) = \begin{cases} \frac{|x-\xi|^{6-n}}{2\cdot 4(n-2)(n-4)(n-6)}, & n \geq 3,\ n \neq 4,6 \\ -\frac{1}{64}\ln|x-\xi|, & n = 6 \\ \frac{|x-\xi|^2}{32}(\ln|x-\xi| - \frac{3}{4}), & n = 4 \\ -\frac{|x-\xi|^4}{64}(\ln|x-\xi| - \frac{3}{2})\ln|x-\xi|, & n = 2. \end{cases} \tag{16}$$

In addition, the Green functions $G_4(x,\xi)$ and $G_6(x,\xi)$ corresponding to the Dirichlet problems in S were found.

If we denote $E_k^*(x,\xi) = E_k(\frac{x}{|x|}, |x|\xi)$, then $G_6(x,\xi)$ has the form

$$G_6(x,\xi) = E_6(x,\xi) - E_6^*(x,\xi) - \frac{1}{2}\frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}E_4^*(x,\xi) - \frac{1}{4}\frac{(|x|^2-1)^2}{4}\frac{(|\xi|^2-1)^2}{4}E_2^*(x,\xi). \tag{17}$$

Based on the functions $E_4(x,\xi)$ and $E_6(x,\xi)$, an elementary solution of the m-harmonic equation $\Delta^m u = 0$ was introduced in [23]. If $m \in \mathbb{N}$, then $\mathbb{N}\backslash\{1\}$ can be partitioned into two disjoint sets $\mathbb{N}_m = \{n \in \mathbb{N} : n > 2m > 1\} \cup (2\mathbb{N}+1)$ and its complement $\mathbb{N}_m^c = \{2, 4, ..., 2m\}$. Since the set $\mathbb{N}_m^c$ is finite, $\mathbb{N}_m$ is infinite. It is clear that $\mathbb{N}_{m-1}^c \subset \mathbb{N}_m^c$, and therefore $\mathbb{N}_m \subset \mathbb{N}_{m-1}$. We define the elementary solution $E_{2m}(x,\xi)$ as

$$E_{2m}(x,\xi) = \begin{cases} \frac{(-1)^m\,|x-\xi|^{2m-n}}{(2-n,2)_m(2,2)_{m-1}}, & n \in \mathbb{N}_m, \\ \frac{(-1)^m\,|x-\xi|^{2m-n}}{(2-n,2)_m^*(2,2)_{m-1}}(\ln|x-\xi| - \sum_{k=1}^{m-n/2}\frac{1}{2k} - \sum_{k=n/2}^{m-1}\frac{1}{2k}), & n \in \mathbb{N}_m^c, \end{cases} \tag{18}$$

where $(a,b)_k = a(a+b)...(a+kb-b)$ is a generalized Pochhammer symbol with the convention $(a,b)_0 = 1$, and the symbol $(a,b)_k^*$ means that if among the factors $a, (a+b), ..., (a+kb-b)$, included in $(a,b)_k$, there is 0, then it should be replaced by 1, for example, $(-2,2)_3^* = (-2)\cdot 1\cdot 2 = -4$. In addition, if in the sums included in (18) the upper index becomes less than the lower index, then the sum is considered to be equal to zero. Note that $(2-n,2)_m = (2-n)(4-n)...(2m-n) \neq 0$ for $n \in \mathbb{N}_m$ and therefore the right-hand side of formula (18) is defined correctly.

In [23] for $n \in \mathbb{N}_{m-1}^c$ the Green's function was constructed

$$G_{2m}(x,\xi) = E_{2m}(x,\xi) - \sum_{k=0}^{m-1}\frac{(|x|^2-1)^k(|\xi|^2-1)^k}{(2m-2,-2)_k(2,2)_k}E_{2m-2k}^*(x,\xi). \tag{19}$$

In [22] the elementary solution $E_{2m}(x,\xi)$ was slightly corrected and another function $\mathcal{E}_{2m}(x,\xi)$ was introduced, which is related to the function $E_{2m}(x,\xi)$ by the formula

$$\mathcal{E}_{2m}(x,\xi)=\begin{cases} E_{2m}(x,\xi), \quad n\in\mathbb{N}_{m-1}, \\ E_{2m}(x,\xi)+\frac{(-1)^m\,|x-\xi|^{2m-n}}{(2-n,2)^*_m(2,2)_{m-1}}\sum_{k=n/2}^{m-1}\frac{1}{2k}, \quad n\in\mathbb{N}^c_{m-1}. \end{cases} \tag{20}$$

It is clear that $\mathcal{E}_{2m}(x,\xi)=E_{2m}(x,\xi)$ for all $n\geq 2$. For $m=2$ we have $\mathbb{N}^c_1=\{2\}$ and, therefore, in formula (15) only the last line will change

$$\mathcal{E}_4(x,\xi)=\begin{cases} \frac{1}{2(n-2)(n-4)}|x-\xi|^{4-n}, \quad n>4,\, n=3 \\ -\frac{1}{4}\ln|x-\xi|, \quad n=4 \\ \frac{|x-\xi|^2}{4}(\ln|x-\xi|-1/2) \quad n=2, \end{cases} \tag{21}$$

If $m=3$, then $\mathbb{N}^c_2=\{2,4\}$ and therefore the last two lines will change

$$\mathcal{E}_6(x,\xi)=\begin{cases} \frac{|x-\xi|^{6-n}}{2\cdot 4(n-2)(n-4)(n-6)}, \quad n\geq 3,\, n\neq 4,6 \\ -\frac{1}{64}\ln|x-\xi|, \quad n=6 \\ \frac{|x-\xi|^2}{32}(\ln|x-\xi|-\frac{1}{2}), \quad n=4 \\ -\frac{|x-\xi|^4}{64}(\ln|x-\xi|-\frac{3}{4})\ln|x-\xi|, \quad n=2. \end{cases} \tag{22}$$

Replacing in (17) $E_{2m}(x,\xi)$ with $\mathcal{E}_{2m}(x,\xi)$ we obtain a new function

$$\mathcal{G}_6(x,\xi)=\mathcal{E}_6(x,\xi)-\mathcal{E}^*_6(x,\xi)-\frac{1}{2}\frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}\mathcal{E}^*_4(x,\xi)-\frac{1}{4}\frac{(|x|^2-1)^2}{4}\frac{(|\xi|^2-1)^2}{4}\mathcal{E}^*_2(x,\xi). \tag{23}$$

If we put $m=3$ and $n=4$ in it, then in relation to (20)

$$\mathcal{E}_6(x,\xi)=E_6(x,\xi)+\frac{1}{4}\frac{|x-\xi|^2}{32},\quad \mathcal{E}_4(x,\xi)=E_4(x,\xi),$$

and therefore

$$\begin{aligned}\mathcal{G}_6(x,\xi)&=\mathcal{E}_6(x,\xi)-\mathcal{E}^*_6(x,\xi)-\frac{1}{2}\frac{|x|^2-1}{2}\frac{|\xi|^2-1}{2}\mathcal{E}^*_4(x,\xi)-\frac{1}{4}\frac{(|x|^2-1)^2}{4}\frac{(|\xi|^2-1)^2}{4}\mathcal{E}^*_2(x,\xi)\\ &=G_6(x,\xi)+\frac{1}{4}(\frac{|x-\xi|^2}{32}-\frac{|x/|\xi|-\xi|x||^2}{32})\\ &=G_6(x,\xi)-\frac{1}{4}\frac{(|\xi|^2-1)(|x|^2-1)}{32}.\end{aligned}$$

It turns out that the function $\mathcal{G}_6(x,\xi)$ obtained for $m=3$ and $n=4$ coincides with the Green's function for the 3-harmonic Dirichlet problem (12)-(13) from the works [21].

Theorem 3. **[20]** *If a solution to problem (12)-(13) exists, then it can be written in the form*

$$u(x)=\frac{1}{\omega_n}\int_{\partial S}\big(-\frac{\partial\Delta^2\mathcal{G}_6(x,\xi)}{\partial\nu}\varphi_0(\xi)+\Delta^2\mathcal{G}_6(x,\xi)\varphi_1(\xi)-\frac{\partial\Delta\mathcal{G}_6(x,\xi)}{\partial\nu}\varphi_0(\xi)\big)ds_\xi$$
$$-\frac{1}{\omega_n}\int_S\mathcal{G}_6(x,\xi)\,f(\xi)\,d\xi, \tag{24}$$

where $\omega_n=|\partial S|$ is the area of an unit sphere in $\mathbb{R}$ and ν is the outward unit normal to ∂S, the Green's function $\mathcal{G}_6(x,\xi)$ is defined in (23).

Acknowledgment

This research work has been funded by Grant number AP19678182 the Ministry of Science and Higher Education of the Republic of Kazakhstan.

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Received: April 15, 2025
Accepted: June 16, 2025

IRSTI 27.03.66 DOI: https://doi.org/10.26577/JMMCS2025126205

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ON O-MINIMALITY FOR EXPANSIONS OF A DENSE MEET-TREE

This paper aims to define the notion of o-minimality for partially ordered sets. Originally, the notion of o-minimality was introduced for linearly ordered sets in the following way: A linearly ordered structure is said to be o-minimal if any definable subset is a finite union of intervals and points. For partially ordered sets, this definition does not work. One of the main reasons for this is that the complement of an interval need not be a finite union of intervals, as happens in linearly ordered sets. Here we suggest a notion of a generalized interval which makes possible defining o-minimality for such a partial case of partially ordered sets as a dense meet-tree in a classical way: an expansion of a dense meet-tree is said to be o-minimal if any definable subset is a finite union of generalized interval and points. We think that this approach allows us to transfer the machinery for investigating o-minimality for linearly ordered structures to partially ordered structures.
Key words: Ehrenfeucht's theory, small theory, linearly ordered set, partially ordered set, o-minimality.

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Тығыз кездесу ағашының кеңейтілген құрылымдарындағы о-минималдылығы туралы

Бұл жұмыстың мақсаты жартылай реттелген жиындар үшін о-минималдылық түсінігін анықтау болып табылады. О-минималдылық түсінігі бастапқыда сызықты реттелген жиындар үшін келесідей енгізілген болатын: егер сызықты реттелген құрылымның әрбір формульді ішкі жиыны интервалдар мен нүктелердің ақырлы бірігуі болса, онда осы сызықты реттелген құрылым о-минималды деп аталады. Бұл анықтама жартылай реттелген жиындар үшін орындалмайды. Мұның басты себептерінің бірі интервалдың толықтауышы сызықты реттелген жиындардағы сияқты интервалдардың ақырлы бірігуі ретінде әрдайым бола бермейді. Мұнда біз классикалық жолмен жартылай реттелген жиындардың мысалы ретінде тығыз кездесу ағашы үшін о-минималдылығын анықтауға мүмкіндік беретін жалпыланған интервал түсінігін ұсынамыз: егер әрбір формульді ішкі жиын жалпыланған интервал мен нүктелердің ақырлы бірігуі болса, онда тығыз кездесу ағашының кеңеюі о-минималды деп аталады. Бұл тәсіл сызықты реттелген құрылымдар үшін о-минималдылықты зерттеу аппаратын жартылай реттелген құрылымдарға ауыстыруға мүмкіндік береді деп есептейміз.
Түйін сөздер: Эренфойхт теориясы, шағын теория, сызықтық реттелген жиын, жартылай реттелген жиын, о-минималдылық.

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Об о-минимальности для обогащений плотного дерева встреч

Целью данной статьи является определение понятия о-минимальности для частично упорядоченных множеств. Первоначально понятие о-минимальности было введено для линейно упорядоченных множеств следующим образом: линейно упорядоченная структура называется о-минимальной, если любое формульное подмножество является конечным объединением интервалов и точек. Для частично упорядоченных множеств это определение не работает. Одной из главных причин этого является то, что дополнение интервала не обязательно должно быть конечным объединением интервалов, как это происходит в линейно упорядоченных множествах.

Здесь мы предлагаем понятие обобщенного интервала, которое позволяет определить о-минимальность для такого частичного случая частично упорядоченных множеств как плотное дерево встреч классическим способом: обогащение плотного дерева встреч называется о-минимальным, если любое формульное подмножество является конечным объединением обобщенного интервала и точек. Мы считаем, что этот подход позволяет нам перенести аппарат исследования о-минимальности для линейно упорядоченных структур на частично упорядоченные структуры.
Ключевые слова: эренфойхтова теория, малая теория, линейно упорядоченное множество, частично упорядоченное множество, о-минимальность.

1 Introduction

This article aims to apply the notion of o-minimality to partially ordered structures. We start with the dense meet tree [2,4] as a sufficiently tame partial order to examine our approach, where a dense meet-tree means a lower semilinear order $<$ in which each pair of elements a, b has a greatest common lower bound, their meet $a \sqcap b$ without the least and greatest elements such that:

(a) for each pair of incomparable elements, their join does not exist;

(b) for each pair of distinct comparable elements, there is an element between them;

(c) for each element a there exist infinitely many pairwise incomparable elements greater than a, whose infimum is equal to a.

The first paper on o-minimality for partially ordered sets was by Carlo Toffalori [6], who gave the following definition. A partially ordered structure is *o-minimal* if each its definable over some set X subset is a finite Boolean combination of sets defined by formulae $x \leq a$ or $x \geq b$, where these a and b are in the algebraic closure of X. As we know, all other notions of o-minimality and weak o-minimality of partially ordered structures are based on this definition, for instance, [3]. We suggest another approach, which was first discovered by S. Sudoplatov and V. Verbovskiy in [5] for weak o-minimality of partially ordered structures.

The standard notion of o-minimality for totally ordered structures is not convenient for partially ordered structures because of the following reasons. In a totally ordered set the complement of an interval is just a union of at most two intervals, while in partially ordered sets this is no more true. That is why Toffalori suggested using a Boolean combination of intervals in place of a finite union of intervals. Here we suggest another approach: we do not change "a finite union", but we change the notion of an interval, and we introduce the notion of a generalized interval. So, our definition of an o-minimal partially ordered set is the following.

Definition 1 *A partially ordered structure is said to be o-minimal if each of its definable subsets is a finite union of generalized intervals and points.*

In the rest of the paper, we discuss the notion of a generalized interval.

The aim of this paper is to find a way of extending the notion of o-minimality to partially ordered structure, because the notion of o-minimality and its generalizations, as o-stability [1,8,9] already proved its own fruitfulness. Perhaps, it will be complicated to extend the notion of stability in a direct way to partially ordered structures, but we can also use a more general notion of relative stability [7], where the scheme of creating different classes of theories was suggested.

2 Definable subsets of DMT

This section introduces the concept of generalized intervals, which extends the classical representation of intervals. This concept allows us to work with our structure, namely the dense meet-tree.

To begin with, we give a definition below.

Definition 2 (V. V. Verbovskiy) *Let $(M, <, \sqcap)$ be a model of the DMT theory. A subset of M is said to be a generalized interval if it is either an interval or is equal to*

$$\bigcup_{a \in A} (a, +\infty)$$

for some definable with parameters in the signature $\{<, \sqcap\}$ subset A of M on one of the following forms:

1. *$A = (c, +\infty) \setminus ([a_1]_c \cup \cdots \cup a_{nc})$, for positive integer n and some elements $a_1, \ldots, a_n \in (c, +\infty)$;*
2. *$A = (c, a)$;*
3. *$A = \{(b, c) : b \in (-\infty, a),\ c \in (b, +\infty)\}$*

For our reasoning, we need the following definition.

Definition 3 An element b is said to be a *partial infimum* of a set A if there exists a partition of A into sets A^+ and A^0 such that $A^+ \neq \emptyset$, $b = \inf A^+$ and $b \parallel a_0$ whenever $a_0 \in A^0$.

Observe that the elementary theory of a dense meet tree admits quantifier elimination, so in the above definition we can use just subsets which are definable by a quantifier-free formula with parameters.

Let $\mathcal{M} = (M; <, \sqcap)$ be a dense meet-tree. For every $c \in M$ it is possible to define an equivalence relation above c, that is, on the set $(c, +\infty)$:

$$a \sim_c b \Leftrightarrow a \sqcap b > c.$$

We call an equivalence class of this equivalence relation an *open cone above c*.

Lemma 1 *An equivalence class for the equivalence relation $\sim_c$ is expressible as written below:*

$$[a]_c = \bigcup_{d \in (c,a)} (d, +\infty).$$

Proof. Assume that $b \in [a]_c$. By definition, $a \sqcap b > c$. Since the order is dense, there exists $d_0 \in (c, a \sqcap b)$. Then $b \geq a \sqcap b > d_0$ and $b \in (d_0, +\infty)$. Hence, $[a]_c \subseteq \bigcup_{d \in (c,a)} (d, +\infty)$.

Let $b \in \bigcup_{d \in (c,a)} (d, +\infty)$. Then $b \in (d, +\infty)$ for some element $d \in (c, a)$. It means that $d < b$ and $d < a$, so $c < d \leq a \sqcap b$, that is, $a \sim_c b$. We proved the inverse inclusion and, thus, the equality of two sets. □

Note that for each class $[a]_c$ there exists its infimum.

The difference between a total ordering and a partial one is the existence of incomparable elements. So, we express the set of incomparable to a elements as a union of intervals.

Lemma 2 *For any element a the following holds:*

$$a \parallel x \Leftrightarrow x \in \bigcup_{b\in(-\infty,a)} \bigcup_{c\in(b,+\infty)\setminus[a]_b} (c,+\infty)$$

Proof. First, we show the necessary condition. Let $b = x \sqcap a$. It means that $x \notin [a]_b$, namely x is outside the class a over b. Then there is an element c between b and x such that $b < c < x$. Note that $c \notin [a]_b$. So, $x \in (c, +\infty)$ and this interval is one of the intervals of the union.

Now we show the sufficient condition. Let $x \in (c, +\infty)$ for some $c \in (b, +\infty) \setminus [a]_b$ where $b < a$. Assume the contrary, that x and a are comparable. If $a \leq x$, then a and c are comparable, because both are less than x. Then $a \sqcap c = \min(a, c) > b$. It means that $c \in [a]_b$, for a contradiction. Let $x < a$. Since $c < x$ we obtain that $c < a$ and $a \sqcap c = c > b$, for a contradiction. □

Note that the set of all incomparable elements to some element does not have infimum.

Thus, using the notations about the class of equivalence relations and incomparable elements, we can express the complements of the equivalence class above c.

Below we use the following notation. Let $\psi(x)$ be a formula. Then

$$(\psi(x))^{\mathcal{M}} = \{a \in M : \mathcal{M} \models \psi(a)\}.$$

Lemma 3 *For any elements a and c with $c < a$ the following holds:*

$$\overline{[a]}_c = (-\infty, c] \cup (x \parallel c)^{\mathcal{M}} \cup \bigcup_{d\in(c,+\infty)\setminus[a]_c} (d,+\infty).$$

In particular,

$$\overline{[a]}_c \cap (c,+\infty) = \bigcup_{d\in(c,+\infty)\setminus[a]_c} (d,+\infty).$$

In particular, the set $\overline{[a]}_c$ does not have infimum and $c = \overline{[a]}_c \cap (c, +\infty)$.

Proof. Let $b \notin \overline{[a]}_c$. Then we have the following possibilities:

$$\neg(b \sqcap a > c) \Leftrightarrow (b \sqcap a = c) \vee (b \sqcap a < c) \vee (b \sqcap a \parallel c)$$

We consider each disjunct separately:

1. If $b \sqcap a = c$ then $b \geq c$ and $b \notin [a]_c$. Now we write the set of all such b's as follows:

$$[c, +\infty) \setminus [a]_c = \{c\} \cup \bigcup_{d\in(c,+\infty)\setminus[a]_c} (d,+\infty)$$

Let $b \in (c, +\infty) \setminus [a]_c$. Since the order is dense, there exists $d \in (c, b)$. Note that $d \in [b]_c \neq [a]_c$, then $d \notin [a]_c$ and $b \in (d, +\infty)$.

Conversely, let

$$b \in \{c\} \cup \bigcup_{d\in(c,+\infty)\setminus[a]_c} (d,+\infty)$$

If $b = c$, then $b \in [c, +\infty] \setminus [a]_c$. Assume that $b \neq c$. We choose $d \in (c, +\infty) \setminus [a]_c$ so that $b \in (d, +\infty)$. Then $b > d > c$ implies $b > c$ and $b \in [d]_c \neq [a]_c$. That is why $b \notin [a]_c$.

2. Since $b \sqcap a < c$ and $a > c$, this implies $b \sqcap a = b \sqcap c$ and $b \sqcap c < c$, so $b < c$ or $b \parallel c$. The set of all b's such that $b < c$ is an interval $(-\infty, c)$. The set of all b's such that $b \parallel c$ is a generalized interval. So, the set of all b's such that $b \sqcap a < c$ is a union of an interval and a generalized interval.

3. Since $a > c$ and $a \geq b \sqcap a$, then c and $b \sqcap a$ are comparable, so $b \sqcap a \parallel c$ is inconsistent. $\square$

Now, we describe definable subsets of DMT. It is well-known that the theory of DMT admits quantifier elimination [4], so any formula in one free variable x is a Boolean combination of formulae of the following kinds:

(1) $t(x, \overline{u}) = t(x, \overline{v})$ (5) $t(x, \overline{u}) \neq t(x, \overline{v})$

(2) $t(x, \overline{u}) < t(x, \overline{v})$ (6) $t(x, \overline{u}) \not< t(x, \overline{v})$

(3) $t(x, \overline{u}) > t(x, \overline{v})$ (7) $t(x, \overline{u}) \not> t(x, \overline{v})$

(4) $t(x, \overline{u}) \parallel t(x, \overline{v})$ (8) $t(x, \overline{u}) \not\parallel t(x, \overline{v})$

The formulae with negation can be transformed by the following equivalences:

$$\begin{aligned}
t(x,\overline{u}) \neq t(x,\overline{v}) &\Leftrightarrow [t(x,\overline{u}) < t(x,\overline{v})] \vee [t(x,\overline{u}) > t(x,\overline{v})] \vee [t(x,\overline{u}) \parallel t(x,\overline{v})];\\
t(x,\overline{u}) \not< t(x,\overline{v}) &\Leftrightarrow [t(x,\overline{u}) > t(x,\overline{v})] \vee [t(x,\overline{u}) = t(x,\overline{v})] \vee [t(x,\overline{u}) \parallel t(x,\overline{v})];\\
t(x,\overline{u}) \not> t(x,\overline{v}) &\Leftrightarrow [t(x,\overline{u}) < t(x,\overline{v})] \vee [t(x,\overline{u}) = t(x,\overline{v})] \vee [t(x,\overline{u}) \parallel t(x,\overline{v})];\\
t(x,\overline{u}) \not\parallel t(x,\overline{v}) &\Leftrightarrow [t(x,\overline{u}) < t(x,\overline{v})] \vee [t(x,\overline{u}) > t(x,\overline{v})] \vee [t(x,\overline{u}) = t(x,\overline{v})].
\end{aligned}$$

So, we can assume that any formula is a disjunction of conjunctions of formulae of the kinds (1)–(4).

The operation $\sqcap$ is idempotent, commutative, and associative. That is why any term $t(x, \bar{u})$ is equal to $x \sqcap \tilde{t}(\bar{u})$ for some term $\tilde{t}$. So, we obtain the following types of atomic formulae:

(1.1) $x = v$

(1.2) $x \sqcap u = x$

(1.3) $x \sqcap u = v$

(1.4) $x \sqcap u = x \sqcap v$

(2.1) $x < v$

(2.2) $x \sqcap u < x$

(2.3) $x \sqcap u < v$

(2.4) $x \sqcap u < x \sqcap v$

(3.1) $x > v$

(3.2) $x \sqcap u > x$

(3.3) $x \sqcap u > v$

(3.4) $x \sqcap u > x \sqcap v$

(4.1) $x \parallel v$

(4.2) $x \sqcap u \parallel x$

(4.3) $x \sqcap u \parallel v$

(4.4) $x \sqcap u \parallel x \sqcap v$

We consider each case separately and show that each formula can be described as a point, an interval, or a generalized interval.

Cases (2.2), (3.2), (4.2), and (4.4) are false. For Case (1.1) we have a point, Cases (2.1) and (3.1) give the intervals. Also, Case (1.2) is equivalent to $x \leq u$, and Case (4.1) is the generalized interval.

Now we look at the remaining cases in more detail.

Case (1.3). We obtain the following

$$x \sqcap u = v \Leftrightarrow [u = v \wedge x \geq v] \vee [u > v \wedge x = v] \vee [u > v \wedge x > v \wedge \neg(x \sim_v u)]$$

Here, the first disjunct $u = v \wedge x \geq v$ defines an interval and the second disjunct defines a single point. From Lemma 3 it follows that the third disjunct define a generalized interval.

Case (2.3). We see that x belongs either to the interval or to the generalized interval

$$\begin{aligned} x \sqcap u < v \quad \Leftrightarrow \quad & [u < v] \vee [u \geq v \wedge (x < v \vee x \parallel v)] \vee \\ & \vee [u \parallel v \wedge (x \leq u \sqcap v \vee x \parallel u \sqcap v \vee (x > u \sqcap v \wedge \neg(x \sim_{u \sqcap v} u)))] \end{aligned}$$

Obviously, each disjunct defines an interval or a generalized interval.

Case (3.3). Since $u \leq v$ is impossible in this case, we can see that

$$x \sqcap u > v \quad \Leftrightarrow \quad [u > v \wedge x \sim_v u]$$

Case (4.3). This case is possible only under the condition $u \parallel v$:

$$x \sqcap u \parallel v \quad \Leftrightarrow \quad [u \parallel v \wedge x \sim_{u \sqcap v} u]$$

Case (1.4). Here we have two possibilities: $x \sqcap u < u \sqcap v$ and $x \sqcap u = u \sqcap v$. The first case is similar to *Case* (2.3). So, we consider $x \sqcap u = x \sqcap v = u \sqcap v$.

This case is written as follows

$$x \sqcap u = x \sqcap v = v \sqcap u \Leftrightarrow x = u \sqcap v \vee \\ \vee (x > u \sqcap v \wedge \neg(x \sim_{u \sqcap v} u) \wedge \neg(x \sim_{u \sqcap v} v))$$

Case (2.4). It is certain that $u \geq v$ is false, then we get the following

$$x \sqcap u < x \sqcap v \Leftrightarrow [u < v \wedge x \sim_u v] \vee [u \parallel v \wedge x \sim_{u \sqcap v} v]$$

Case (3.4). Similar to *Case* (2.4).

Theorem 1 *Notion of o-minimality for an expansion of the DMT theory is correct, that is, any Boolean combination of sets which are either a point or a generalized interval can be expressed as a finite union of points and generalized intervals.*

For any definable set there exist at most finitely many partials infima.

Proof. We prove both statements of this theorem by the simultaneous induction in the complexity of the construction of a Boolean combination.

Note that any set defined by an atomic formula has at most one infimum because it is either a point, or an interval, or a generalized interval. Because of quantifier elimination, it is sufficient to consider just an arbitrary Boolean combination of atomic formulas.

1. It is obvious that the union of generalized intervals and points is a finite union of generalized intervals and points. Obviously, any finite union of sets that have at most finitely many partial infima also has at most finitely many infima.

2. We consider the intersection of two finite unions of generalized intervals and points. Since

$$\bigcup_i A_i \cap \bigcup_j B_j = \bigcup_{i,j} (A_i \cap B_j)$$

it is sufficient to consider the intersection of generalized intervals and points.

The intersection of generalized intervals with a point either is empty or a point. The intersection of generalized intervals with an interval of the form (a, b), where $a \in M \cup \{-\infty\}, b \in M$ is a subset of a linearly ordered set (a, b), and in linearly ordered sets the intersection of intervals either is empty or an interval.

We consider an intersection of the form: $(a, +\infty) \cap (b, +\infty)$. If a and b are comparable, then this is $(\max(a, b), +\infty)$, otherwise it is empty. We can see that in these two cases the intersection of two sets has at most one infimum.

We consider the intersection of generalized intervals:

$$\left(\bigcup_{a\in A} (a, +\infty)\right) \cap \left(\bigcup_{b\in B} (b, +\infty)\right) = \bigcup_{a\in A} \bigcup_{b\in B} ((a, +\infty) \cap (b, +\infty)) = \bigcup_{c\in C} (c, +\infty)$$

where $C = \{c \in A \cup B \mid c = \max(a, b)$ for some $(a, b) \in A \times B$ such that a and b are comparable$\}$. Therefore, the intersection of two generalized intervals is itself a generalized interval.

Note that if both A and B have at most finitely many partial infima, then C also has at most finitely many partial infima, then $\bigcup_{c\in C} (c, +\infty)$ has at most finitely many infima.

We consider the intersection of a generalized interval with an interval of the form $(b, +\infty)$, but the last interval can be written as $\bigcup_{b\in\{b\}}(b, +\infty)$, thus we get into the previous case.

3. Now we consider the complements of the finite union of generalized intervals and points. $\overline{\bigcup_i A_i} = \bigcap_i \overline{A}_i$, therefore, it is sufficient to consider only the complement of a point, an interval, and a generalized interval.

We should also consider the complement of intervals since we know that the complement of intervals is a finite union of intervals, which is what the below states.

$$\begin{aligned}\overline{(-\infty, a)} &= [a, +\infty) \cup (x \parallel a)^{\mathcal{M}} \\ \overline{(a, +\infty)} &= (-\infty, a] \cup (x \parallel a)^{\mathcal{M}} \\ \overline{(a, b)} &= (-\infty, a] \cup [b, +\infty) \cup (x \parallel a)^{\mathcal{M}} \cup [(a, +\infty) \cup (b \parallel x)^{\mathcal{M}}]\end{aligned}$$

Note that the complements of a point a is $x < a \cup x > a \cup x \parallel a$.

As one can see, any of the above sets has at most one infimum.

So, we consider the complement of a generalized interval:

$$\overline{\bigcup_{a\in A}(a, +\infty)} = \bigcap_{a\in A} \overline{(a, +\infty)} = \bigcap_{a\in A}((-\infty, a] \cup (x \parallel a)^{\mathcal{M}}) =$$
$$= \bigcap_{a\in A}(-\infty, a] \cup \bigcap_{a\in A}(x \parallel a)^{\mathcal{M}})$$

Note that $(-\infty, a] \cap (-\infty, b] = (-\infty, a \sqcap b]$ for any a and b. Since A is definable in $(M, \leq, \sqcap)$ and $Th(\mathcal{M})$ admits quantifier elimination, there exists $c = \inf\{a \sqcap b : a, b \in A\}$ or $\bigcap_{a\in A}(-\infty, a] = \emptyset$. So, $\bigcap_{a\in A}(-\infty, a]$ being non-empty is equal to $(-\infty, c)$, provided that $c = \min\{a \sqcap b : a, b \in A\}$, and to $(-\infty, c)$ otherwise.

Now we consider $\bigcap_{a\in A}(x \parallel a)^{\mathcal{M}})$. Observe that

$$(x \parallel a)^{\mathcal{M}} \cap (x \parallel b)^{\mathcal{M}} = (x \parallel a)^{\mathcal{M}}$$

for any pair $a < b$. So, if A contains the least element, say, c, we obtain $\bigcap_{a\in A}(x \parallel a)^{\mathcal{M}} = (x \parallel c)^{\mathcal{M}}$. If A is not bounded below, we obtain $\bigcap_{a\in A}(x \parallel a)^{\mathcal{M}} = \emptyset$. So, we assume that A is bounded below. By induction hypothesis A has at most finitely many partial infima, say $c_1, \ldots, c_n$. Let $A_1, \ldots, A_n$ be a partition of A such that $c_i = \inf A_i$ for each i. As we have noticed, if $c_i \in A_i$, then $\bigcap_{a\in A_i}(x \parallel a)^{\mathcal{M}} = (x \parallel c_i)^{\mathcal{M}}$.

Assume that $c_i \notin A_i$. Then obviously, $\bigcap_{a\in A_i}(x \parallel a)^{\mathcal{M}} \supseteq (x \parallel c_i)^{\mathcal{M}}$. Indeed, if an element is comparable with some a, then it is comparable to c. Now we consider an element d that is comparable to c. If $d \leq c$, then by transitivity $d < a$ for each $a \in A_i$. So, we consider only those d, that $d > c$. We denote $D = \{d > c : d \notin A_i\}$.

By the quantifier elimination result it holds that either D is contained in finitely many $\sim_c$-classes or D contains cofinitely many $\sim_c$-classes. Also, at most finitely many $\sim_c$-classes intersect D but not subsets of D. Let D_1 consist of those $\sim_c$-classes, that are subsets of D and $D_2 = D \setminus D_1$. Then $\bigcap_{a\in A_i}(x \parallel a)^{\mathcal{M}} \supseteq (x \parallel c_i)^{\mathcal{M}} \cup D_1$.

Let $d' \in D_2$ and $d = \inf(-\infty, d') \cap D_2$. If $d > c_i$ then we obtain a similar situation as before, we consider $\sim_d$-classes. So, we assume that $d = c_i$. We obtain $(c_i, d'] \subseteq D_2$, this means that $(c_i, d'] \cap A_i = \emptyset$. By the definition of D_2, we have $[d']_{c_i} \not\subseteq D_2$.

In order to obtain A_i from $(c_i, +\infty)$ we remove finitely many subsets definable by a conjunction of atomic formulas. We can remove

1. an equivalence class $[u]_v$ for some $v \geq c_i$ and u,
2. an interval of the form $(v, +\infty)$ for some $v > c_i$,
3. an interval of the form (v, u) for some $u > v \geq c_i$,
4. the set of all elements that are not comparable with v for some $v > c_i$.

We have already described the way to deal with Case (1).

Assume we have removed an interval of the form $(v, +\infty)$ for some $v > c_i$. Since the order is dense, there exists $u \in (c_i, v)$. Then $(x \parallel t)^{\mathcal{M}} \supseteq (x \parallel u)^{\mathcal{M}}$ whenever $t > u$. Then

$$\bigcap_{a \in A_i} (x \parallel a)^{\mathcal{M}} = \bigcap_{a \in A_i \cup (v,\infty)} (x \parallel a)^{\mathcal{M}}$$

Assume that we have removed an interval of the form (v, u) for some $u > v \geq c_i$. If $v > c$ we proceed as above just replacing (v, ∞) with (v, u). So, we assume that $v = c_i$, that is, we have removed an interval of the form (c_i, u). Let b be the supreme of all $t > u$ such that $(c_i, t) \subseteq D_2$ and $(w, +\infty) \cap A_i \neq \emptyset$ for each $w \in (c_i, t)$. So, we have removed an interval of the form (c_i, b) or $(c_i, b]$ and this set is a maximal connected set that contains u, is a subset of D_2 and any element of D_2 that is comparable with some element of (c_i, b) (or $(c_i, b]$) then this element is comparable with all elements of (c_i, b) (or $(c_i, b]$). In this case we obtain

$$\bigcap_{a \in A_i} (x \parallel a)^{\mathcal{M}} \supseteq (x \parallel c_i)^{\mathcal{M}} \cup \big([b]_c \setminus [b, +\infty)\big)$$

or $\bigcap_{a \in A_i} (x \parallel a)^{\mathcal{M}} \supseteq (x \parallel c_i)^{\mathcal{M}} \cup \big([b]_c \setminus (b, +\infty)\big)$ depending which kind of an interval do we have: (c_i, b) or $(c_i, b]$.

Now we consider the last case: we have removed from $(c, +\infty)$ the set of all elements that are not comparable with v for some $v > c_i$. Note that

$$(c, +\infty) \setminus (x \parallel v)^{\mathcal{M}} = (c, v) \cup \{v\} \cup (v, +\infty)$$

Also we observe that

$$\bigcap_{a \in (c,v) \cup [v,+\infty)} (x \parallel a)^{\mathcal{M}} = \bigcap_{a \in (c,v)} (x \parallel a)^{\mathcal{M}} = (x \parallel c_i)^{\mathcal{M}} \cup \big((c, +\infty) \setminus [v]_c\big)$$

Since we can make only finitely many removals from $(c, +\infty)$, we end with finitely many steps describing $\bigcap_{a \in A_i} (x \parallel a)^{\mathcal{M}}$.

Also we can see that this operation cannot give a definable set with infinitely many partial infima. □

So, the next is clear.

Theorem 2 $(M, <, \sqcap)$ *is o-minimal, that is, any of its definable subsets is a finite union of generalized intervals and points.*

Acknowledgment

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23484665).

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Received: February 13, 2025
Accepted: June 9, 2025

IRSTI 27.31.44 DOI: https://doi.org/10.26577/JMMCS2025126206

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SYMMETRY EQUIVALENCES OF BOUNDARY VALUE PROBLEMS FOR THE NON-UNIFORM BEAMS

In this paper, the models of Euler–Bernoulli non-uniform beams with the axial loads on the Winkler foundations are considered. The non-uniform beam in the model is described by three variable parameters/coefficients: bending stiffness, foundation and beam mass per unit length. The key finding of this study is the clear demonstration of how the agreed symmetry of variable parameters affects the spectral properties of a problem. The qualitative results for the symmetric equivalence (factorisation of sets of eigenvalues and eigenfunctions) of eigenvalues of non-uniform beams for two types of fixing at the ends (clamped-clamped and hinged-hinged) have been obtained. In order to demonstrate equivalence, a hybrid algorithm has been devised, based on the qualitative spectral properties of fourth-order ordinary differential equations and axial load calculations. The results have been validated using examples on the Maple computer package and compared with the experimental measurements.
Key words: Euler–Bernoulli beam, non-uniform beam, eigenvalue, symmetry, equivalence.

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Біркелкі емес бөренелер үшін шекаралық есептердің симметриялық эквиваленттігі

Бұл жұмыста іргесі Винклер бойынша осьтік жүктемелері бар Эйлер-Бернулли біркелкі емес бөренелердің модельдері қарастырылған. Модельдегі біркелкі емес бөрене үш айнымалы параметрмен/коэффициенттермен сипатталады: иілу қаттылығы, ұзындық бірлігіне қатысты бөрененің іргесі мен массасы. Бұл зерттеудің негізгі тұжырымы айнымалы параметрлердің келісілген симметриясының есептің спектрлік қасиеттеріне қалай әсер ететінін айқын көрсету болып табылады. Біркелкі емес бөренелердің меншікті мәндерінің симметриялық эквиваленттілігінің (меншікті мәндерінің және меншікті функциялар жиынның көбейткіштерге жіктелінуі) екі ұштарында бекітудің (қатты-қатты және топсалы-топсалы бекітілген) түрлері үшін сапалы нәтижелер алынды. Эквиваленттілігін көрсету үшін төртінші ретті қарапайым дифференциалдық теңдеулердің сапалы спектрлік қасиеттеріне және осьтік жүктемені есептеуге негізделген гибридті алгоритм жасалды. Нәтижелер Maple компьютер пакетіндегі мысалдар арқылы расталды және эксперименттік өлшемдермен салыстырылды.
Түйін сөздер: Эйлер–Бернулли бөренесі, біркелкі емес бөренелер, меншікті мән, симметрия, эквиваленттік.

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Симметричная эквивалентность краевых задач для неоднородных балок

В данной работе рассматриваются модели неоднородных балок Эйлера–Бернулли с осевыми нагрузками на основание Винклера. Неоднородная балка в модели описывается тремя переменными параметрами/коэффициентами: жесткостью изгиба, основанием и массой балки на единицу длины. Ключевым выводом данного исследования является наглядная демонстрация того, как согласованная симметрия переменных параметров влияет на спектральные свойства задачи. Получены качественные результаты для симметричной эквивалентности (факторизации наборов собственных значений и собственных функций) собственных значений неоднородных балок для двух типов закрепления на концах (защемленно-защемленное и шарнирно-шарнирное). Для демонстрации эквивалентности разработан гибридный алгоритм, основанный на качественных спектральных свойствах обыкновенных дифференциальных уравнений четвертого порядка и расчетах осевой нагрузки. Результаты были проверены с использованием примеров в компьютерном пакете Maple и сравнены с экспериментальными измерениями.
Ключевые слова: балка Эйлера–Бернулли, неоднородная балка, собственное значение, симметрия, эквивалентность.

1 Introduction

The majority of mechanical systems comprising beam construction, as employed in technology and engineering, are defined by their geometric and physical variable parameters. Such structures include parabolic tapering and functionally graded beams [1–3], which can be adopted for a light-weight design or specific wave propagation effects [4, 5], as well as piezoelectric energy harvesting [6,7]. In [3], a closed-form dynamic stiffness formulation for the analysis of transverse free vibration in non-uniform symmetric Euler–Bernoulli beams was proposed, and effects of boundary conditions were investigated. A beam with a heterogeneous temperature distribution exhibits variable physical properties. The presence of variable parameters introduces a significant degree of complexity into the dynamic analysis. The modelling of mechanical systems comprising non-uniform beam construction gives rise to the formation of fourth-order linear equations with variable coefficients. Consequently, both approximate analytical [8–10] and numerical methods [12–16] for solving differential equations with variable coefficients under different conditions are being actively developed. A thorough literature review on the solution methods for transverse vibration of non-uniform beams with variable cross-sections can be found in [9]. In [10], approximate analytical expressions for the natural frequencies of non-uniform beams were obtained in terms of asymptotic theory. The isospectral problems for non-uniform beams were studied in [11,12]. The isospectral problems between non-uniform and uniform beams were presented in [12]. The natural frequencies of free boundary value problems for beams with symmetric coefficient without an axial load were studied in [9]. In [15], a regular variation approach to finding natural frequencies and modes of vibration of non-homogeneous beams were studied.

In the modelling of mechanical systems, it is essential to have a closed analytical formula for natural frequencies [17, 18, 20, 23]. In [17], a closed-form solution for non-uniform beams was proposed using special functions. In [18], an asymptotic formula of natural frequencies for the non-uniform beams with different boundary conditions was derived based on perturbation method. Eigenvalue asymptotics of an even order differential ordinary operator with square integrable potential were obtained in [19]. In [20], a solution for the free vibrations of non-uniform beams on a non-uniform Winkler foundation was presented, employing the Laguerre collocation method. The influence of axial loads on the natural frequencies of uniform beams with various boundary conditions were investigated in [21, 22]. Additionally, the papers revealed critical values of axial loads. In [23], several results pertaining to the closed-form expression for the natural frequencies of uniform beams were modified. Additionally, the concept of symmetrical equivalence was demonstrated for a uniform Euler-Bernoulli beam subjected to an axial load. The spectral properties of hinged-hinged beams, both with and without axial loads on an elastic foundation, were investigated based on the characteristic determinant in [24, 25] and [26], respectively. The effects of a foundation coefficient for calculating of the critical load were presented in [27]. The symmetric equivalence of boundary value problems for the uniform beams without and with axial loads lying on a Winkler's type foundation were studied in [28] and [29], respectively. Nevertheless, the symmetric equivalence of boundary value problems for non-uniform beams remains an area of incomplete investigation. One of the methods for studying non-uniform beams is to represent them as stepped beams. The symmetric equivalence of stepped beams can be employed for identification problems pertaining to the physical properties of beams, as evidenced by the findings set forth in the paper by [30].

The goal of this research is to identify conditions for the variable coefficients of bending stiffness, foundation and mass of the beam per unit length and fixing types of a non-uniform beam under which it is possible to establish the equivalence of the eigenvalues and eigenfunctions. The novelty of the paper is the agreed symmetry of the variable coefficients (see Theorems 1). The results presented here extend several known results from the cited sources, namely [23, 28, 29].

The problem of transverse vibrations of a non-uniform beam of unit length

$$\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}+k(x)w(x,t)+T\frac{\partial^2 w(x,t)}{\partial x^2}+\frac{\partial^2}{\partial x^2}\left(EJ(x)\frac{\partial w(x,t)}{\partial x^2}\right)=0,$$

after replacement $w(x,t)=v(\lambda,x)\sin(\omega t)$ reduces to the following spectral problem:

$$(EJ(x)v''(\lambda,x))''+Tv''(\lambda,x)+k(x)v(\lambda,x)=\lambda\rho A(x)v(\lambda,x),\quad x\in I_p, p=1,2,\tag{1}$$

where $v(\lambda,x)$ are the eigenfunctions of the transverse static deflection of the beam; $EJ(x)$ is the bending stiffness; $\rho\, A(x)$ is mass of the beam per unit length; T is corresponding to a constant compressive force if $T>0$ or a constant tensile force if $T<0$; $\lambda=\rho\,\omega^2$ are the eigenvalues; ω is the circular frequency; ρ is the material density; $k(x)$ is the variable coefficient of foundation, $I_1=(0,\,1)$, $I_2=\left(\frac{1}{2},\,1\right)$. Notice that $J(x)$ and $A(x)$ are assumed twice continuously differentiable and strictly positive, $k(x)$ is the real-valued summable function.

In this study, two types of beams are considered. The first is the hinged-hinged beam on the interval I_1 with the boundary conditions (see Figure 1)

$$v(\lambda,0)=0,\ v''(\lambda,0)=0, v(\lambda,1)=0,\ v''(\lambda,1)=0,\tag{2}$$

and the second is the clamped-clamped beam on the interval I_1 with the boundary conditions (see Figure 2)

$$v(\lambda,0)=0,\ v'(\lambda,0)=0, v(\lambda,1)=0,\ v'(\lambda,1)=0.\tag{3}$$

In addition, we introduce the sliding-hinged boundary conditions

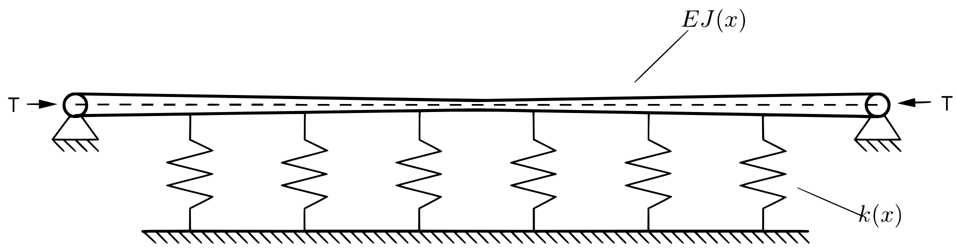

Figure 1: Hinged-hinged Euler–Bernoulli non-uniform beam.

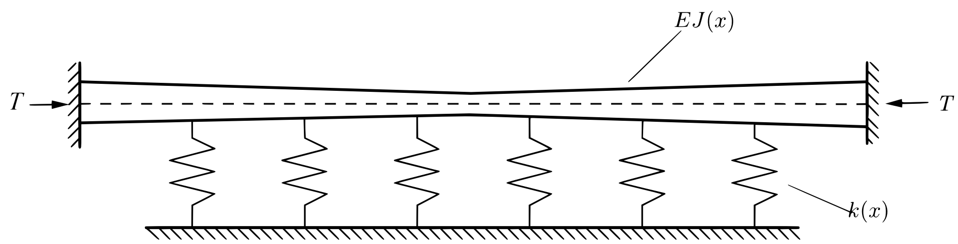

Figure 2: Clamped-clamped Euler–Bernoulli non-uniform beam.

$$v'\left(\lambda, \frac{1}{2}\right) = 0,\ v'''\left(\lambda, \frac{1}{2}\right) = 0,\ v(\lambda, 1) = 0,\ v''(\lambda, 1) = 0, \tag{4}$$

and hinged-hinged boundary conditions

$$v\left(\lambda, \frac{1}{2}\right) = 0,\ v''\left(\lambda, \frac{1}{2}\right) = 0, v(\lambda, 1) = 0,\ v''(\lambda, 1) = 0 \tag{5}$$

which are connected with hinged-hinged fixing on the interval I_2. Furthermore, we introduce the sliding-clamped boundary conditions

$$v'\left(\lambda, \frac{1}{2}\right) = 0,\ v'''\left(\lambda, \frac{1}{2}\right) = 0,\ v(\lambda, 1) = 0,\ v'(\lambda, 1) = 0, \tag{6}$$

and hinged-clamped boundary conditions

$$v\left(\lambda, \frac{1}{2}\right) = 0,\ v''\left(\lambda, \frac{1}{2}\right) = 0, v(\lambda, 1) = 0,\ v'(\lambda, 1) = 0 \tag{7}$$

which are connected with clamped-clamped fixing on the interval I_2.

2 Main results

Let $\sigma(A_1), \sigma(B_1), \sigma(C_1)$ be a set of eigenvalues of problems $A_1 - \lambda I,\ B_1 - \lambda I,\ C_1 - \lambda I$ generated by Equation (1) on finite intervals by boundary conditions (2), (4), (5), respectively.

Theorem 1 *Let $J(x)$, $k(x)$ and $A(x)$ be the symmetric functions with respect to the point $x = \frac{1}{2}$*

$$J(x) = J(1-x), \quad k(x) = k(1-x), \quad A(x) = A(1-x), \quad x \in \left[0; \frac{1}{2}\right] \tag{8}$$

and $T < T_{cr}$. The following statements are true:
1. $\sigma(A_1) \equiv \sigma(B_1) \cup \sigma(C_1)$
2. If $\lambda \in \sigma(B_1)$ or $\lambda \in \sigma(C_1)$, then the eigenfunctions of problems $A_1 - \lambda I$ corresponding to the eigenvalues λ are symmetric or asymmetric with respect to the middle of the beam at the point $x = \frac{1}{2}$ on the interval $(0, 1)$, respectively.

Let $\sigma(A_2)$, $\sigma(B_2)$, $\sigma(C_2)$ be a set of eigenvalues of problems $A_2 - \lambda I$, $B_2 - \lambda I$, $C_2 - \lambda I$ generated by Equation (1) on finite intervals by boundary conditions (3), (6), (7), respectively.

Theorem 2 *Let $J(x)$, $k(x)$ and $A(x)$ be the symmetric functions with respect to the point $x = \frac{1}{2}$, i.e. the condition in Equation (8) holds and $T < T_{cr}$. The following statements are true:*
1. $\sigma(A_2) \equiv \sigma(B_2) \cup \sigma(C_2)$
2. If $\lambda \in \sigma(B_2)$ or $\lambda \in \sigma(C_2)$, then the eigenfunctions of problems $A_2 - \lambda I$ corresponding to the eigenvalues λ are symmetric or asymmetric with respect to the middle of the beam at the point $x = \frac{1}{2}$ on the interval $(0, 1)$, respectively.

The proof of Theorems 1 and 2 is ideologically similar to that presented in work [28]. Nevertheless, there is a single discrepancy, which require is calculating of the critical value T_{cr}. Further will be described the scheme for proving Theorems 1 and 2.

First step. The following functions $J(x)$, $k(x)$ and $A(x)$ will be selected to satisfy condition (8).

Second step. The critical value of T_{cr} will be calculated that corresponding to the first step and the value of T will be selected such that $T < T_{cr}$. The calculation of T_{cr} will be conducted using well-known numerical method (see, [22]).

Third step. The final step will employ the same technique used to prove the result presented in [28].

Upon completion of the aforementioned three steps, the proofs of Theorems 1 and 2 will be obtained. In the third step, the analytical or numerical method may be employed. It should be noted that if the functions $J(x)$, $k(x)$ and $A(x)$ satisfy condition (8) and the additional condition from [12], then the non-uniform beam can be transformed into a uniform one.

Remark 1 *Results from Theorem 1 and Theorem 2 are preserved for stepped beams. Experimental and numerical simulations for the clamped-clamped stepped beam were carried out in [30,31]. The symmetric equivalence of the clamped-clamped stepped beam was used for solving the inverse coefficients problems in [30].*

3 Examples and discussion

In this section, we calculate approximately the four or five eigenvalues of boundary value problems $A_n - \lambda I$, $B_n - \lambda I$, $C_n - \lambda I$ $(n = 1, 2, 3)$ generated by the Euler–Bernoulli equation for the various coefficients $J(x)$, $k(x)$, $A(x)$ and $p(x)$. The results of calculation of the eigenvalues are shown in the corresponding columns of Tables 1–4.

Example 1 *In this analysis, we examine three steps.*

First step. Let $J(x) = 1 + x(1-x)$, $k(x) = 4x(1-x)$, $A(x) = x(1-x)$ *and* $E = 1$.

Second step. In this example $T_{cr} \approx 12.09$ *and take* $T = 5$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 1 are shown in Table 1.*

Table 1: Numerical calculations of the first five eigenvalues from the Example 1

Hinged-hinged at the points $x = 0$, $x = 1$	Sliding at the point $x = \frac{1}{2}$, hinged at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, hinged at the point $x = 1$
(2)	(4)	(5)
17.95	17.95	95.65
95.65	229.93	420.31
229.93	666.79	969.34
420.31	1327.96	1742.63
666.79	2213.36	2740.14

The calculations presented in Example 1 provide corroboration for the validity of Statement 1 of Theorem 1 pertaining to the factorization of the set of eigenvalues.

Example 2 *In this analysis, we consider three steps.*

First step. Let $J(x) = x(1-x)$, $k(x) = 5(1+x)^3$, $A(x) = x(1-x)$ *and* $E = 1$.

Second step. In this example $T_{cr} \approx 3.73$ *and take* $T = 1$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 2 are shown in Table 2.*

Table 2: Numerical calculations of the first five eigenvalues from the example 2.

Hinged-hinged at the points $x = 0$, $x = 1$	Sliding at the point $x = \frac{1}{2}$, hinged at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, hinged at the point $x = 1$
(2)	(4)	(5)
11.31	12.37	37.09
36.38	84.70	152.81
84.31	240.98	349.07
152.55	476.99	624.72
240.81	792.23	979.52

The violation of the regularity of factorization of eigenvalues in Example 2 is due to the failure to satisfy the symmetry condition for the function $k(x)$. The aforementioned calculations in Example 2 confirm the validity of Statement 1 of Theorem 1.

Example 3 *We consider three steps.*

First step. Let $J(x) = 1 + x(1-x)$, $k(x) = 4$, $A(x) = x^2(1-x)^2$ *and* $E = 1$.

Second step. In this example $T_{cr} \approx 45.71$ *and take* $T = 30$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 3 are shown in Table 3.*

Table 3: Numerical calculations of the first five eigenvalues from the Example 3

Clamped-clamped at the points $x = 0$, $x = 1$	Sliding at the point $x = \frac{1}{2}$, clamped at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, clamped at the point $x = 1$
(3)	(6)	(7)
27.4	27.4	121.79
121.79	285.1	511.35
285.1	800.52	1152.37
511.35	1566.79	2043.71
800.52	2583.07	3184.87

The calculations which represent in Example 3 confirm the validity of Statement 1 of Theorem 2 on the factorization of the set of eigenvalues.

Example 4 *We consider three steps.*

First step. Let $J(x) = 1 + x(1+x)$, $k(x) = 4$, $A(x) = x^2(1-x)^2$ *and* $E = 1$.

Second step. In this example $T_{cr} \approx 67.4$ *and take* $T = 30$.

Third step. The numerical results of the first five eigenvalues' square root $\sqrt{\lambda}$ *for Example 4 are shown in Table 4.*

Table 4: Numerical calculations of the first four eigenvalues from the Example 4

Clamped-clamped at the points $x = 0$, $x = 1$	Sliding at the point $x = \frac{1}{2}$, clamped at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, clamped at the point $x = 1$
(3)	(6)	(7)
43.21	62.46	191.41
159.65	420.72	725.72
357.61	1126.92	1602.73
631.47	3562.85	2820.71

The violation of the regularity of factorization of eigenvalues in Example 4 is is due to the failure to satisfy the symmetry condition for the function $J(x)$. The aforementioned calculations in Example 4 confirm the validity of Statement 1 of Theorem 2.

Example 5 *In this study, we consider a two-stepped beam with clamped-clamped boundary conditions. The geometric dimensions of the composite beam are as follows:* $L_1 = L_3 =$

247.6mm $L_2 = 508mm$, $h_1 = h_2 = h_3 = 3.6mm$, $b_1 = b_3 = 38.2mm$ and $b_2 = 25.7mm$ and these are illustrated in in Fig.3. Young's modulus and the density are $28.3GPa$ and $\rho = 1800kg/m^3$, respectively. In this example, $T = 0$, $k(x) = 0$. The natural frequencies

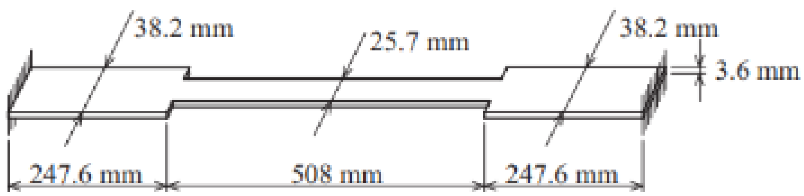

Figure 3: Two-stepped composite beam with clamped-clamped boundary conditions [30,31].

are computed by transcendental eigenvalue problem (TEP) and the results compared with the experimental measurements [30] in Table 5.

Table 5: The first five natural frequencies (Hz) from the Example 5

Clamped-clamped at the points $x = 0$, $x = 1$. Experiment [30]	Sliding at the point $x = \frac{1}{2}$, clamped at the point $x = 1$	Hinged at the point $x = \frac{1}{2}$, clamped at the point $x = 1$
(3)	(6)	(7)
16.1 ± 0.16	16.12	41.01
41.3 ± 0.16	78.67	130.55
79.3 ± 0.16	195.01	270.97
134.0 ± 0.16	360.78	465.47
196.5 ± 0.16	581.72	708.17

The calculations presented in Example 5 confirm the validity of Statement 1 of Theorem 2 on the factorization of the set of eigenvalues.

A numerical method was employed for the calculation of the eigenvalues at the variable coefficients of $J(x)$, $A(x)$ and $k(x)$ with the polynomial expansion and integral techniques as outlined in [14]. The degree of the polynomial was selected as $N = 25$, which ensures accuracy of calculations. The numerical calculations were carried out using the Maple computer mathematics system [32].

The results obtained in this work permits to study the qualitative spectral properties of a non-uniform beam. Symmetrical equivalence permits to calculate the natural frequencies of a full beam using the natural frequencies of two short beams with different lengths and fixing methods. This paper presents examples of partial factorization of the eigenvalues of a full-length beam in the case of an asymmetric foundation coefficient. Furthermore, the length of short beams is also contingent upon the agreed symmetry. To illustrate, when the parameters are symmetric about the $x = 1/2$, the length of the short beams is equal to half the length of the full beam. The aforementioned capabilities are play a significant role in computer calculations and the modeling of mechanical systems with complex structures. It is therefore anticipated that future research will focus on the behavior of mechanical systems of complex structure, with the star graph serving as a case in point [33] .

4 Conclusions

In this paper, the problems for determining the eigenvalues of the non-uniform Euler–Bernoulli beam with the axial load lying on the Winkler's type foundation at two types of fixings at the ends have been solved: clamped-clamped and hinged-hinged. A sufficient condition has been found for the variable coefficients of the differential equation of the beam, which a symmetrical equivalence of the eigenvalues and eigenfunctions is satisfied.

5 Acknowledgments

The research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan, grant number AP19579114

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Received: April 28, 2025
Accepted: June 8, 2025

IRSTI 27.39.21 DOI: https://doi.org/10.26577/JMMCS2025126207

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SYMMETRIC BANACH-KANTOROVICH SPACES

Let B be a complete Boolean algebra, let $Q(B)$ be the Stone compact of B, let $C_\infty(Q(B))$ be the commutative unital algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, possibly assuming the values $\pm\infty$ on nowhere-dense subsets of $Q(B)$. We consider Maharam measure m defined on B, which takes values in the algebra L^0 of all real measurable functions. With the help of the property of equimeasurability of elements from $C_\infty(Q(B))$, associated with such a measure m, the notion of a symmetric Banach-Kantorovich space $(E, \|\cdot|_E)$ over L^0 is introduced and studied in detail. Here $E \subset C_\infty(Q(B))$, and $\|\cdot|_E$ – L^0-valued norm in E, endowing it with the structure of a Banach-Kantorovich space. Examples of symmetric Banach-Kantorovich spaces are given, which are vector-valued analogues of classical L^p-spaces for $1 \le p \le \infty$, associated with a numerical σ-finite measure.
Key words:: the Banach-Kantorovich space, order complete vector lattice, vector-valued measure, vector integration, symmetric space.

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Симметриялық Банах-Канторович кеңістіктері

B толық бульдік алгебра, $Q(B)$ B бульдік алгебраға сәйкес келуші стоундік компакт болсын және $L^0(B) := C_\infty(Q(B))$ $Q(B)$- да анықталған және $\pm$ мәндерді $Q(B)$-дағы еш жерде тығыз болмаған жиындарда ғана қабылдайтын $x : Q(B) \to [-\infty, +\infty]$ барлық өзіліссіз функциялардың алгебрасы болсын. Магарамның вектор мәнді $m : B \to L^0(\Omega)$ өлшемдері қарастырылады, олардың мәндері өлшемі σ-ақырлы болған (Ω, Σ, μ) өлшемді кеңістіктегі нақты өлшемді функцияларға дерлік барлық жерде тең барлық кластардың $L^0(\Omega)$ алгебрасында болады. m өлшеммен байланысқан $L^0(B)$ - дағы элементтердің теңөлшемділік қасиеті көмегімен $(E, \|\cdot|_E)$ симметриялық Банах-Канторович кеңістігі түсінігі енгізіледі, мұнда $E \subset L^0(B)$, $\|\cdot|_E$ $E-$дегі $L^0(\Omega)$-мәнді норма, бұл норма оған Банах-Канторович кеңістігінің құрылымын береді. Симметриялық Банах-Канторович кеңістігіне мысалдар келтіріледі, олар сандық σ-ақырлы өлшеммен байланысқан классикалық L^p, $1 \le p \le \infty$ кеңістіктердің вектор мәнді аналогтары болады.
Түйін сөздер: векторлық интегралдау, Магарам өлшемі, тең өлшем, Банах – Канторович кеңістігі.

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Симметричные пространства Банаха-Канторовича

Пусть B произвольная полная булева алгебра, $Q(B)$ стоуновский компакт, соответствующий булевой алгебре B и $L^0(B) := C_\infty(Q(B))$ алгебра всех непрерывных функций $x : Q(B) \to [-\infty, +\infty]$, определенных на $Q(B)$ и принимающих значения $\pm\infty$ лишь на нигде не плотных множествах из $Q(B)$. Рассматриваются векторнозначные меры Магарам $m : B \to L^0(\Omega)$ со значениями в алгебре $L^0(\Omega)$ всех классов равных почти всюду действительных измеримых функций на измеримом пространстве (Ω, Σ, μ) с σ-конечной мерой. С помощью свойства равноизмеримости элементов из $L^0(B)$, ассоциированного с такой мерой m, вводится понятие симметричного пространства Банаха-Канторовича $(E, \|\cdot|_E)$ над $L^0(\Omega)$, где $E \subset L^0(B)$, и $\|\cdot|_E$ – $L^0(\Omega)$-значная норма в E, наделяющая его структурой пространства Банаха-Канторовича.

Приводятся примеры симметричных пространств Банаха-Канторовича, являющихся векторнозначными аналогами классических L^p-пространств, $1 \le p \le \infty$, ассоциированных с числовой σ-конечной мерой.
Ключевые слова: векторное интегрирование, мера Магарам, равноизмеримость, пространство Банаха - Канторовича.

Introduction

The development of the theory of Banach-Kantorovich space theory began with the construction of integration for measures with the values in order complete vector lattice (K-spaces), in particular, in the algebra $L^0(\Omega)$ of all classes of almost everywhere equal real measurable functions on the measurable space (Ω, Σ, μ) with a σ-finite numerical measure μ. Important examples of the Banach-Kantorovich spaces include the "vector-valued" analogues of the L_p-spaces, $1 \le p < \infty$ [1], [2], and the Orlicz spaces [3], [4], [5]. If Ω is a singleton, then the class of Banach-Kantorovich spaces coincides with the class of real Banach spaces, important examples of which are functional symmetric spaces. The theory of symmetric spaces contains many profound results and has important applications in a wide variety of fields of function theory and functional analysis, in particular, in the theory interpolation of linear operators, ergodic theory and harmonic analysis (see for example, [6], [7], [8]). The development of the theory of Banach-Kantorovich spaces naturally involves the introduction and study of symmetric Banach-Kantorovich spaces. In this paper, we consider a measure m defined on a complete Boolean algebra B, which takes on value in the algebra $L^0(\Omega)$. With the help of this measure, the associated distribution function for elements of the algebra $L^0(B) := C_\infty(Q(B))$ of all continuous functions $x : Q(B) \to \overline{\mathbb{R}} = [-\infty, +\infty]$, defined on the Stone compact $Q(B)$ of a Boolean algebra B, such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(B)$, is determined. Then the notion of a symmetric Banach-Kantorovich space $(E, \|\cdot|_E)$ over $L^0(\Omega)$ is introduced, where $E \subset L^0(B)$ and $\|\cdot\|_E - L^0(\Omega)$-valued norm in E, endowing it with the structure of the Banach-Kantorovich space. Examples of symmetric Banach-Kantorovich spaces are given, which are vector-valued analogues of classical L^p-spaces, $1 \le p \le \infty$, associated with a numerical σ-finite measure. Throughout the paper, we use the terminology and notation of the theory of Boolean algebras [9], an order complete vector lattice [10], the theory of vector integration and the theory of Banach-Kantorovich spaces [1], as well as the terminology of the general theory of symmetric spaces [6].

1 Preliminaries

Let (Ω, Σ, μ) be a measurable space with σ-finite measure μ, and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ be the algebra of all real measurable functions on (Ω, Σ, μ) (functions coinciding almost everywhere are identified). $L^0(\Omega)$ is an order complete vector lattice with respect to the natural partial order $(f \le g \Leftrightarrow g - f \ge 0$ almost everywhere). The weak unit is $\mathbf{1}(\omega) \equiv 1$, and the set $B(\Omega)$ of all idempotents in $L^0(\Omega)$ is a complete Boolean algebra. Denote $L^0(\Omega)_+ = \{f \in L^0(\Omega) : f \ge 0\}$.

Let X be a vector space over the field $\mathbb{R}$ of real numbers. A mapping $\|\cdot\| : X \to L^0(\Omega)$ is called an $L^0(\Omega)$-valued norm on X if the following relations hold for any $x, y \in X$ and $\lambda \in \mathbb{R}$:

(1) $\|x\| \ge 0$, $\|x\| = 0 \Leftrightarrow x = 0$;

(2) $\|\lambda x\| = |\lambda|\ \|x\|$;

(3) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a lattice-normed space over $L^0(\Omega)$. A lattice-normed space X is said to be d-decomposable if for any $x \in X$ and any decomposition $\|x\| = f_1 + f_2$ into a sum of nonnegative disjoint elements $f_1, f_2 \in L^0(\Omega)$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$, $\|x_1\| = f_1$ and $\|x_2\| = f_2$.

A net $\{x_\alpha\}_{\alpha\in A}$ of elements of $(X, \|\cdot\|)$ is said to (bo)-converge to $x \in X$ if the net $\{\|x - x_\alpha\|\}_{\alpha\in A}$ (o)-converges to zero in $L^0(\Omega)$, that is, there exists a decreasing net $\{f_\gamma\}_{\gamma\in\Gamma}$ in $L^0(\Omega)$ such that $f_\gamma \downarrow 0$ and for each $\gamma \in \Gamma$ there is $\alpha(\gamma) \in A$ with $\|x - x_\alpha\| \leq f_\gamma$ $(\alpha \geq \alpha(\gamma))$ [1, 1.3.4] (note, that the o-convergence of a net in $L^0(\Omega)$ is equivalent to its convergence almost everywhere). A net $\{x_\alpha\}_{\alpha\in A} \subset X$ is called (bo)-fundamental if the net $\{x_\alpha - x_\beta\}_{(\alpha,\beta)\in A\times A}$ (bo)-converges to zero.

The Banach-Kantorovich space over $L^0(\Omega)$ is defined as a (bo)-complete d-decomposable lattice-normed space over $L^0(\Omega)$. If a Banach Kantorovich space $(X, \|\cdot\|)$ is in addition a vector lattice and the norm $\|\cdot\|$ is monotone (i.e. the conditions $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in X$) then it is called a Banach-Kantorovich lattice over $L^0(\Omega)$ (see [1], [2]). Useful examples of Banach-Kantorovich lattices are constructed using vector integration theory. Let us recall some basic notions of the theory of vector integration (see [1], [2]).

Let B be a arbitrary complete Boolean algebra with zero $\mathbf{0}$ and unit $\mathbf{1}$. A mapping $m : B \to L^0(\Omega)$ is called a $L^0(\Omega)$-*valued measure* if it satisfies the following conditions:

1) $m(e) \geq 0$ for all $e \in B$;

2) $m(e \vee g) = m(e) + m(g)$ for any $e, g \in B$ with $e \wedge g = \mathbf{0}$;

3) $m(e_\alpha) \downarrow 0$ for any net $e_\alpha \downarrow \mathbf{0}$, $\{e_\alpha\} \subset B$.

A measure m is said to be *strictly positive*, if $m(e) = 0$ implies $e = \mathbf{0}$. In this case, B is a Boolean algebra of countable type, thus, in condition 3) above, instead of the net $e_\alpha \downarrow \mathbf{0}$, one can take a sequence $e_n \downarrow \mathbf{0}$, $\{e_n\}_{n\in\mathbb{N}} \subset B$.

A strictly positive $L^0(\Omega)$-valued measure m is said to be *decomposable*, if for any $e \in B$ and a decomposition $m(e) = f_1 + f_2$, $f_1, f_2 \in L^0(\Omega)_+$ there exist $e_1, e_2 \in B$, such that $e = e_1 \vee e_2$, $m(e_1) = f_1$ and $m(e_2) = f_2$. A measure m is decomposable if and only if it is a Maharam measure, that is, the measure m is strictly positive and for any $e \in B$, $0 \leq f \leq m(e)$, $f \in L^0(\Omega)$, there exist $q \in B$, $q \leq e$, such that $m(q) = f$ [11].
The following statement shows that, in the case of the Maharam measure m, there is a natural embedding of the Boolean algebra $B(\Omega)$ into the Boolean algebra B.

Proposition 1 ([12], Proposition 2.3) . *For each $L^0(\Omega)$-valued Maharam measure $m : B \to L^0(\Omega)$ there exists a unique injective completely additive Boolean homomorphism $\varphi : B(\Omega) \to B$ such that $\varphi(B(\Omega))$ is a regular Boolean subalgebra of B, and $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$.*

Let $Q(B)$ be the Stone compact of a complete Boolean algebra B, and let $L^0(B) := C_\infty(Q(B))$ be the algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(B)$. Denotes by $C(Q(B))$ the Banach algebra of all continuous real functions on $Q(B)$ with the uniform norm $\|x\|_\infty = \sup\limits_{t\in Q(B)} |x(t)|$.

We denote by $s(x) := \sup_{n\geq 1}\{|x| > n^{-1}\}$, the support of an element $x \in L^0(B)$, where $\{|x| > \lambda\} \in B$ is the characteristic function χ_{E_λ} of the set E_λ which is the closure in $Q(B)$ of the set $\{t \in Q(B) : |x(t)| > \lambda\},\ \lambda \in \mathbb{R}$.

Let $m : B \to L^0(\Omega)$ be a Maharam measure. We identify B with the complete Boolean algebra of all idempotents in $L^0(B)$, i.e., we assume $B \subset L^0(B)$. By Proposition 1, there exists a regular Boolean subalgebra $\nabla(m)$ in B and a Boolean isomorphism φ from $B(\Omega)$ onto $\nabla(m)$ such that $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$. In this case, the algebra $L^0(\Omega)$ is identified with the algebra $L^0(\nabla(m)) = C_\infty(Q(\nabla(m)))$ (the corresponding isomorphism will also be denoted by φ), and the algebra $C_\infty(Q(\nabla(m)))$ itself can be considered as a subalgebra and as a regular vector sublattice in $L^0(B) = C_\infty(Q(B))$ (this means that the exact upper and lower bounds for bounded subsets of $L^0(\nabla(m))$ are the same in $L^0(B)$ and in $L^0(\nabla(m))$. In addition, $L^0(B)$ is an $L^0(\nabla(m))$-module.

Denote by $\mathcal{S}(B)$ the vector sublattice of $L^0(B)$ comprising all B-simple (finite-valued) elements, i.e. $x \in \mathcal{S}(B)$ means, that there is a representation $x = \sum_{i=1}^{n} \alpha_i e_i$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $e_1, \ldots, e_n \in B$ are pairwise disjoint. Let $m : B \to L^0(\Omega)$ be a strongly positive measure on a complete Boolean algebra B. If $x \in \mathcal{S}(B)$ then we put by definition

$$I_m(x) := \int x\, dm := \sum_{k=1}^{n} \alpha_k m(e_k)\ (x \in \mathcal{S}(B)).$$

This formula correctly defines a linear order continuous operator $I_m : \mathcal{S}(B) \to L^0(\Omega)$ [1, 6.1.1, 6.1.2].

We say that a positive element $x \in L^0(B)$ is *integrable* by m, or m*-integrable* if there is an increasing sequence $(x_n)_{n\in\mathbb{N}}$ of positive elements in $\mathcal{S}(B)$ (o)-converging in $L^0(B)$ to x and the supremum $\sup_{n\in\mathbb{N}} \int x_n\, dm$ existing in L^0. In this case, the sequence of integrals $(I_m(x_n))_{n\in\mathbb{N}}$ is (o)-fundamental sequence (see [1, 6.1.3]). Therefore, we may define the integral of x by putting

$$I_m(x) := \int x\, dm := \text{(o)-}\lim_{n\to\infty} \int x_n\, dm.$$

An element $x \in L^0(B)$ is *integrable* (= m*-integrable*), if its positive part x_+ and the negative part x_- are integrable. Denote by $L^1(B, m)$ the set of all integrable elements and, given $x \in L^1(B, m)$, put

$$I_m(x) := \int x\, dm := \int x_+\, dm - \int x_-\, dm.$$

It is known, that $L^1(B, m)$ is an order-dense ideal in $L^0(B)$ and $I_m : L^1(B, m) \to L^0(\Omega)$ is a linear operator. For each $x \in L^1(B, m)$, let $\|x\|_{1,m} := \int |x|\, dm$. Then $(L^1(B, m), \|x\|_{1,m})$ is a lattice-normed space over $L^0(\Omega)$ (see [1, 6.1.3]).

Let $p > 1$, and let

$$L^p(B, m) = \{x \in L^0(B) : |x|^p \in L^1(B, m)\},$$

$$\|x\|_{p,m} := \Big[\int |x|^p dm\Big]^{\frac{1}{p}}, \quad x \in L^p(B, m).$$

The following is known

Theorem 1 *([1], [2]). Let $m : B \to L^0(\Omega)$ be a Maharam measure. Then*
(i). $(L^1(B,m), \|x\|_{1,m})$ is a Banach-Kantorovich space over $L^0(\Omega)$, moreover,

$$L^0(\nabla(m)) \cdot L^1(B,m) \subset L^1(B,m), \int \varphi(\alpha) x dm = \alpha \int x dm,$$

for every $x \in L^1(B,m)$, $\alpha \in L^0(\Omega)$;
(ii). $(L^p(B,m), \|x\|_{p,m})$ is a Banach-Kantorovich space over $L^0(\Omega)$.

In what follows we identify $\varphi(L^0(\Omega))$ and $L^0(\nabla(m))$, and instead of $\varphi(f)$ we will write $f \in L^0(\Omega)$.

The element $x \in L^0(B)$ is called $L^0(\Omega)$-*bounded,* if there exists an element $f \in L^0(\Omega)_+$ such that $|x| \le f$. Denote by $L^\infty(B, L^0(\Omega))$ the set of all $L^0(\Omega)$-bounded elements from $L^0(B)$. It is clear that $L^\infty(B, L^0(\Omega))$ is a subalgebra in $L^0(B)$, as well as order complete vector sublattice in $L^0(B)$, moreover, $L^0(\Omega) \subset L^\infty(B, L^0(\Omega))$, $C(Q(B)) \subset L^\infty(B, L^0(\Omega))$.

For each $x \in L^\infty(B, L^0(\Omega)$ put

$$\|x\|_{\infty, L^0(\Omega)} = \inf\{f \in L^0(\Omega)_+ : |x| \le f\}.$$

It follows directly from the definition of element $\|x\|_{\infty, L^0(\Omega)} \in L^0(\Omega)_+$ that $|x| \le \|x\|_{\infty, L^0(\Omega)}$.

Proposition 2 ([13], Propositions 4 and 5) . *The map*

$$\| \cdot \|_{\infty, L^0(\Omega)} : L^\infty(B, L^0(\Omega)) \to L^0(\Omega)$$

is a $L^0(\Omega)$-valued d-decomposable norm on $L^\infty(B, L^0(\Omega))$. Moreover, if $|y| \le |x|$, $y, x \in L^\infty(B, L^0(\Omega))$, then $\|y\|_{\infty, L^0(\Omega)} \le \|x\|_{\infty, L^0(\Omega)}$.

Theorem 2 . $(L^\infty(B, L^0(\Omega)), \| \cdot \|_{\infty, L^0(\Omega)})$ *is the Banach-Kantorovich lattice over $L^0(\Omega)$*

Proof. According to Proposition 2, $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty, L^0(\Omega)})$ is a d-decomposable lattice-normed space over $L^0(\Omega)$.

It remains to show that this lattice-normed space is (bo)-complete. Take an (bo)-fundamental net $\{x_\alpha\}_{\alpha \in A} \subset L^\infty(B, L^0(\Omega))$. Then by the definition of fundamentality, net $(x_\alpha - x_\beta)_{(\alpha,\beta) \in A \times A}$ (bo)-converges to zero. Hence, there exists a net $\{h_\gamma\}_{\gamma \in \Gamma} \downarrow 0$, $h_\gamma \in L^\infty(B, L^0(\Omega))$ such that for any h_γ there is $\alpha(\gamma) \in A$, that

$$|x_\alpha - x_\beta| \le \|x_\alpha - x_\beta\|_{\infty, L^0(\Omega)} \le h_\gamma \text{ for all } \alpha \ge \alpha(\gamma), \ \beta \ge \alpha(\gamma). \quad (1)$$

This means that the net $\{x_\alpha\}_{\alpha \in A}$ is a (o)-fundamental net from order complete vector lattice $L^\infty(B, L^0(\Omega))$. Consequently, this net (o)-converges in $L^\infty(B, L^0(\Omega))$, i.e., there exists an element $x \in L^\infty(B, L^0(\Omega))$, for which $x_\alpha \xrightarrow{(o)} x$. In particular, for each fixed $\beta \ge \alpha(\gamma)$ the net $\{x_\alpha - x_\beta\}_{\alpha \in A, \ \alpha \ge \alpha(\gamma)}$ (o)-converges in $L^\infty(B, L^0(\Omega))$ to element $x - x_{\alpha(\gamma)}$. Hence by (1) we have $|x - x_\beta| \le h_\gamma \in L^\infty(B, L^0(\Omega))$ for all $\beta \ge \alpha(\gamma)$, which implies the inequality

$$\|x - x_\beta\|_{\infty, L^0(\Omega)}) \le h_\gamma \text{ for all } \beta \ge \alpha(\gamma).$$

This means that the net $\{x_\alpha\}_{\alpha \in A}$ (bo)-converges to the element $x \in L^\infty(B, L^0(\Omega))$. $\square$

2 Symmetric spaces of Banach-Kantorovich

Denote by $L^0(\Omega)_{++}$ the set of all positive elements $\lambda \in L^0_+(\Omega)$ such that $s(\lambda) = \mathbf{1}$. It is clear that for any $\lambda \in L^0(\Omega)_{++}$ there exists $\lambda^{-1} \in L^0(\Omega)_{++}$ such that $\lambda \cdot \lambda^{-1} = \mathbf{1}$.

Let m be $L^0(\Omega)$-valued Maharam measure on a complete Boolean algebra B. In the rest of this section we assume that $m(\mathbf{1}) = \mathbf{1}$.

Definition 1 . *Let $0 \leq x \in L^0(B)$ and $h \in L^0_{++}(\Omega)$. The $L^0(\Omega)$-valued distribution function $\eta_x : L^0_{++}(\Omega) \to L^0(\Omega)_+$ is defined by setting*

$$\eta_x(h) := m(\{x > h\}),$$

where $\{x > h\} \in B$ is the idempotent in the algebra $L^0(B)$, which is the characteristic function $\chi_{E_h(x)}$ of the closure $E_h(x)$ of the set $\{s \in Q(B) : x(s) > h(s)\}$.

Proposition 3 . *A mapping η_x is decreasing, and right-continuous, that is, if $h_n \in L^0_{++}(\Omega), n = 0, 1, ...,$ and $h_n \downarrow h_0$, then $\eta_x(h_0) = \sup\limits_{n\geq 1} \eta_x(h_n)$*

Proof. The first statement follows from the following implications

$$h_1 \leq h_2 \Rightarrow E_{h_1}(x) \supseteq E_{h_2}(x) \Rightarrow \{x > h_1\} \geq \{x > h_2\} \Rightarrow \eta_x(h_1) \geq \eta_x(h_2).$$

To establish right-continuity, let $q_h = \{x > h\}$, $(h \in L^0_{++}(\Omega))$, and fix $h_0 \in L^0_{++}(\Omega)$. The sets $E_{h_n}(x)$ increase as h_n decreases, and $E_{h_0}(x) = \bigcup\limits_{n=1}^{\infty} E_{h_n}(x)$. Hence, by the monotone convergence property of measure m,

$$\eta_x(h_n) = m(q_{h_n}) \uparrow m(q_{h_0}) = \eta_x(h_0). \qquad \square$$

Proposition 4 . *Suppose $x, y, x_n\,(n = 1, 2, \ldots)$ belong to $L^0(B)$, and let $h, g \in L^0_{++}(\Omega)$. Then*

(i) if $|x| \leq |y|$, then $\eta_{|x|}(h) \leq \eta_{|y|}(h)$;

(ii) $\eta_{g|x|}(h) = \eta_{|x|}(\frac{h}{g})$;

(iii) if $x \geq 0, y \geq 0$, $h_1, h_2 \in L^0_{++}(\Omega)$, then $\eta_{x+y}(h_1 + h_2) \leq \eta_x(h_1) + \eta_y(h_2)$;

(iv) if $|x_n| \uparrow |x|$, then $\eta_{|x_n|}(h) \uparrow \eta_{|x|}(h)$ for every $h \in L^0_{++}(\Omega)$.

Proof. (i). If $|x| \leq |y|$, then $\{|x| > h\} \leq \{|y| > h\}$. Consequently, $\eta_{|x|}(h) = m(\{|x| > h\}) \leq m(\{|y| > h\}) = \eta_{|y|}(h)$.

(ii). $\eta_{g|x|}(h) = m(\{g|x| > h\}) = m(\{|x| > \frac{h}{g}\}) = \eta_{|x|}(\frac{h}{g})$.

(iii). If $s \in Q(B)$ and $x(s) + y(s) > h_1(s) + h_2(s)$, then either $x(s) > h_1(s)$ or $y(s) > h_2(s)$. Therefore $\{x + y > h_1 + h_2\} \leq \{x > h_1\} \vee \{y > h_2\}$. Consequently,

$$\eta_{x+y}(h_1 + h_2) = m(\{x + y > h_1 + h_2\}) \leq m(\{x > h_1\}) + m(\{y > h_2\}) = \eta_x(h_1) + \eta_y(h_2).$$

(iv). We fix $h \in L^0_{++}(\Omega)$ and put $G_h(x) = \{s \in Q(B) : |x(s)| > h(s)\}$, $G_h(x_n) = \{s \in Q(B) : |x_n(s)| > h(s)\}$, $(n = 1, 2, ...)$. Since $|x_n| \leq |x_{n+1}|$, then $G_h(x_n) \subset G_h(x_{n+1})$.

Furthermore, the condition $|x_n| \uparrow |x|$ imply that $G_h(x) = \bigcup_{n=1}^{\infty} G_h(x_n)$. Hence, by the monotone convergence property of measure m,

$$\eta_{|x_n|}(h) = m(\{|x_n| > h\}) \uparrow m(\{|x| > h\}) = \eta_{|x|}(h). \qquad \square$$

Examples 1 . *1. Let $x = e \in B$ and $h \in L^0_{++}(\Omega)$. Then* $\eta_e(h) = m(\{e > h\}) = m(e)$, *if* $h < \mathbf{1}$, *and* $\eta_e(h) = \mathbf{0}$, *if* $h \geq \mathbf{1}$.

2. It will be worthwhile to formally compute the $L^0(\Omega)$-valued distribution function η_x of a positive B-simple element $x \in \mathcal{S}(B)$. Suppose

$$x = \sum_{k=1}^{n} \alpha_k e_k,$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+ = (0, \infty)$, and $e_1, \ldots, e_n$ are pairwise disjoint elements of B. Without loss of generality we may assume that $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$. If $h \in L^0_{++}(\Omega)$ and $h \geq \alpha_1 \cdot \mathbf{1}$, then clearly $\eta_x(h) = 0$. However, if $\alpha_2 \cdot \mathbf{1} \leq h < \alpha_1 \cdot \mathbf{1}$, then $\{x > h\} = e_1$, and so $\eta_x(h) = m(e_1)$. Similarly, if $\alpha_3 \cdot \mathbf{1} \leq h < \alpha_2 \cdot \mathbf{1}$, then $\{x > h\} = e_1 \vee e_2$, and so $\eta_x(h) = m(e_1 \vee e_2) = m(e_1) + m(e_2)$. In general, we have

$$\eta_x(h) = \sum_{i=1}^{k} m(e_i) \quad \text{if} \ \ \alpha_{k+1} \cdot \mathbf{1} \leq h < \alpha_k \cdot \mathbf{1} \ (h \in L^0_{++}(\Omega)),$$

where $k = 1, 2, \ldots, n$, *and* $\alpha_{n+1} = 0$.

Definition 2 . *Positive elements $x, y \in L^0(B)$ are called m-equimeasurable, if $\eta_x = \eta_y$, i.e.,*

$$m\{x > h\} = m\{y > h\}$$

for all $h \in L^0_{++}(\Omega)$.

Examples 2 . *1. Two idempotents $e_1, e_2 \in B$ are m-equimeasurable if and only if $m(e_1) = m(e_2)$ (see Example 1.1.)*

2. Let $x, y \in \mathcal{S}(B)_+$, $x = \sum_{k=1}^{n} \alpha_k e_k$ and $y = \sum_{k=1}^{n} \beta_k g_k$, where $\alpha_k, \beta_k \in \mathbb{R}_+$, $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$, $\beta_1 > \beta_2 > \ldots > \beta_n > 0$, and $\{e_k\}$, respectively, $\{g_k\}$ are pairwise disjoint elements of B. By Example 1.2, we have

$$\eta_x(h) = \sum_{i=1}^{k} m(e_i) \quad \text{if}\ \alpha_{k+1} \cdot \mathbf{1} \leq h < \alpha_k \cdot \mathbf{1},$$

$$\eta_y(h) = \sum_{i=1}^{k} m(g_i) \quad \text{if}\ \beta_{k+1} \cdot \mathbf{1} \leq h < \beta_k \cdot \mathbf{1} \ (h \in L^0_{++}(\Omega)),$$

where $k = 1, 2, \ldots, n$, *and* $\alpha_{n+1} = \beta_{n+1} = 0$.

From equality $\eta_x(t) = \eta_y(t)$ we get $\alpha_k = \beta_k$ and $\sum_{i=1}^{k} m(e_i) = \sum_{i=1}^{k} m(g_i)$ for all $k = 1, \ldots, n$. Of the last equalities by $k = 1$ we have $m(e_1) = m(g_1)$. Further, if $k = 2$ the equality

$m(e_1) + m(e_2) = m(g_1) + m(g_2)$ *is true, thus* $m(e_2) = m(g_2)$, *etc., when* $k = n$ *we get* $m(e_n) = m(g_n)$.

Thus the elements x *and* y m*-equimeasurable if and only if* $\alpha_k = \beta_k$ *and* $m(e_k) = m(g_k)$ *for all* $k = 1, \dots, n$.

3. If x *and* y *are positive elements of* $L^0(B)$ *and* $m\{x > t \cdot \mathbf{1}\} = m\{y > t \cdot \mathbf{1}\}$ *for all* $t > 0$, *then*

$$m\{x \le t \cdot \mathbf{1}\} = m\{y \le t \cdot \mathbf{1}\} \quad \text{and} \quad m\{s \cdot \mathbf{1} < x \le t \cdot \mathbf{1}\} = m\{s \cdot \mathbf{1} < y \le t \cdot \mathbf{1}\}$$

for any $0 < s < t$.

Indeed, $m\{x \le t \cdot \mathbf{1}\} = m(\mathbf{1}) - m\{x > t \cdot \mathbf{1}\} = m(\mathbf{1}) - m\{y > t \cdot \mathbf{1}\} = m\{y \le t \cdot \mathbf{1}\}$.

Further, using the equalities $\{s \cdot \mathbf{1} < x \le t \cdot \mathbf{1}\} = \{x > s \cdot \mathbf{1}\} - \{x > t \cdot \mathbf{1}\}, \{s \cdot \mathbf{1} < y \le t \cdot \mathbf{1}\} = \{y > s \cdot \mathbf{1}\} - \{y > t \cdot \mathbf{1}\}$, *we get*

$$m\{s \cdot \mathbf{1} < x \le t \cdot \mathbf{1}\} = m\{x > s \cdot \mathbf{1}\} - m\{x > t \cdot \mathbf{1}\} =$$

$$m\{y > s \cdot \mathbf{1}\} - m\{y > t \cdot \mathbf{1}\} = m\{s \cdot \mathbf{1} < y \le t \cdot \mathbf{1}\}.$$

The following theorem establishes the equality of integrals for m-equimeasurable elements.

Theorem 3 . *If* x, y *are* m*-equimeasurable, where* $y \in L^1(B, m)$, *then* $x \in L^1(B, m)$ *and* $\int x dm = \int y dm$.

Proof. Let $x, y \in S(B)_+$, $x = \sum\limits_{k=1}^{n} \alpha_k e_k$ and $y = \sum\limits_{k=1}^{n} \beta_k g_k$. Then by m-equimeasurable x and y (see Example 2.2.),

$$\int x dm = \sum_{k=1}^{n} \alpha_k m(e_k) = \sum_{k=1}^{n} \beta_k m(g_k) = \int y dm.$$

Let now $x \in L^0(B)_+$, $0 \le y \in L^1(B, m)$ and $\eta_x = \eta_y$. Let us, first assume that $y \in C(Q(B))$. Recall that by assumption $m(\mathbf{1}) = \mathbf{1}$, and therefore $C(Q(B)) \subset L^1(B, m))$, in this case, $\|y\|_{1,m} \le \|y\|_\infty \mathbf{1}$. Without loss of generality we may assume that $\|y\|_\infty \le 1$. Since $\eta_x = \eta_y$, then $m\{x > \mathbf{1}\} = m\{y > \mathbf{1}\} = \mathbf{0}$, that is $\|x\|_\infty \le 1$.

Consider the following two increasing sequences of positive simple elements

$$x_n = (\sum_{k=1}^{2^n} \frac{k-1}{2^n} e_k) \uparrow x, \quad y_n = (\sum_{k=1}^{2^n} \frac{k-1}{2^n} g_k) \uparrow y,$$

where $e_k = \{\frac{k-1}{2^n} \cdot \mathbf{1} < x \le \frac{k}{2^n} \cdot \mathbf{1}\}$, $g_k = \{\frac{k-1}{2^n} \cdot \mathbf{1} < y \le \frac{k}{2^n} \cdot \mathbf{1}\}$. Since $\eta_x = \eta_y$, then $m(e_k) = m(g_k)$ (see example 2.3.), and therefore

$$\int x_n dm = \sum_{k=1}^{2^n} \frac{k-1}{2^n} m(e_k) = \sum_{k=1}^{2^n} \frac{k-1}{2^n} m(g_k) = \int y_n dm \uparrow \int y dm.$$

Hence,

$$\int x dm = (o)\text{-} \lim_{n \to \infty} \int x_n dm = \int y dm.$$

Now let y be an arbitrary positive element $L^1(B,m)$. Consider two increasing sequences of positive elements of $C(Q(B))$

$$x_n = xp_n \uparrow x, \quad y_n = yq_n \uparrow y,$$

where $p_n = \{x \leq n \cdot \mathbf{1}\}$, $q_n = \{y \leq n \cdot \mathbf{1}\}$. Using example 2.3., we obtain

$$m\{x_n > t \cdot \mathbf{1}\} = m\{xp_n > t \cdot \mathbf{1}\} = m\{t \cdot \mathbf{1} < x \leq n \cdot \mathbf{1}\} = m\{t \cdot \mathbf{1} < y \leq n \cdot \mathbf{1}\} =$$

$$= m\{yq_n > t \cdot \mathbf{1}\} = m\{y_n > t \cdot \mathbf{1}\}$$

for any $t \in \mathbb{R}^+$. Since y_n is an integrable element of $C(Q(B))$, it follows from the above, that $\int x_n dm = \int y_n dm$. At the same time, there is a limit

$$(o)\text{-}\lim_{n\to\infty} \int x_n dm = (o)\text{-}\lim_{n\to\infty} \int y_n dm = \int y dm.$$

Hence $x \in L^1(B,m)$ and $\int x dm = \int y dm$. □

Corollary 1 . *Let $0 \leq x \in L^0(B)$ and $0 \leq y \in L^p(B,m)$, $p > 1$. If x and y m-equimeasurable, then $x \in L^p(B,m)$ and $\|x\|_{p,m} = \|y\|_{p,m}$.*

Proof. Since $y^p \in L^1(B,m)$ and

$$m\{x^p > t \cdot \mathbf{1}\} = m\{x > t^{\frac{1}{p}} \cdot \mathbf{1}\} = m\{y > t^{\frac{1}{p}} \cdot \mathbf{1}\} = m\{y^p > t \cdot \mathbf{1}\}$$

for any $t \in \mathbb{R}^+$, $p > 1$, then for the elements x^p and y^p the proof of Theorem 3 is preserved, by virtue of which we obtain

$$x^p \in L^1(B,m) \text{ and } \int x^p dm = \int y^p dm,$$

i.e. $x \in L^p(B,m)$ and $\|x\|_{p,m} = \|y\|_{p,m}$. □

Definition 3 . *Let E – be a nonzero linear subspace in $L^0(B)$ with the property of ideality, i.e. for $x \in L^0(B)$ and $y \in E$, from $|x| \leq |y|$ it follows that $x \in E$. Consider the $L^0(\Omega)$-valued norm $\|\cdot\|_E$ on E, which endows E with the structure of a Banach-Kantorovich lattice. We say that E is a symmetric Banach-Kantorovich space over $L^0(\Omega)$, if m-equimeasurablity of the elements x and y, where $x \in L^0(B)_+$, $0 \leq y \in E$, implies that $x \in E$ and $\|x\|_E = \|y\|_E$.*

The main and most important examples of symmetric Banach-Kantorovich spaces are the spaces $L^p(B,m)$, $1 \leq p < \infty$, and $L^\infty(B, L^0(\Omega))$.

Theorem 4 . *(i). $(L^p(B,m), \|\cdot\|_{p,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$ for every $1 \leq p < \infty$.*

(ii). $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

Proof. (i). According to [1, Section 6.1], linear subspace $L^1(B,m) \subset L^0(B)$ has the ideality property, moreover, the norm $\|\cdot\|_{1,m}$ is monotone, and the space $L^1(B,m)$, equipped with this norm, is a Banach-Kantorovich lattice. It remains to apply theorem 3, by virtue of which the pair $(L^1(B,m), \|\cdot\|_{1,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

Now let $|x| \le |y|$, $x \in L^0(B)$, $y \in L^p(B,m)$, where $1 < p < \infty$. Since $|x|^p \le |y|^p \in L^1(B,m)$, then $|x|^p \in L^1(B,m)$ and

$$\|x\|_{p,m}^p = \||x|^p\|_{1,m} \le \||y|^p\|_{1,m} = \|y\|_{p,m}^p,$$

and therefore $\|x\|_{p,m} \le \|y\|_{p,m}$, i.e. $\|\cdot\|_{p,m}$ is $L^0(\Omega)$-valued monotone norm on $L^p(B,m)$, which endows $L^p(B,m)$ with the structure of a Banach-Kantorovich lattice over $L^0(\Omega)$. It remains to apply Corollary 1, by virtue of which the pair
$(L^p(B,m), \|\cdot\|_{p,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

(ii). By Theorem 2, the pair $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ is a Banach-Kantorovich lattice, moreover, it is clear that $L^\infty(B, L^0(\Omega))$ has the ideality property and the norm $\|\cdot\|_{\infty,L^0(\Omega)}$ is monotone on $L^\infty(B, L^0(\Omega))$.
Let $x \in L^0(B)$, $y \in L^\infty(B, L^0(\Omega))$, and let x and y be m-equimeasurable. Assign $h(\varepsilon) = \|y\|_{\infty,L^0(\Omega)} + \varepsilon \cdot \mathbf{1}$, $\varepsilon > 0$. Since $h(\varepsilon) \in L^0_{++}(\Omega)$, then

$$m\{|x| > h(\varepsilon)\} = m\{|y| > h(\varepsilon)\} = \mathbf{0}.$$

Hence, $|x| \le h(\varepsilon)$, and therefore $x \in L^\infty(B, L^0(\Omega)$, moreover, $\|x\|_{\infty,L^0(\Omega)} \le h(\varepsilon)$ for every $\varepsilon > 0$. From this it follows that $\|x\|_{\infty,L^0(\Omega)} \le \|y\|_{\infty,L^0(\Omega)}$.

Let's put now $h_1(\varepsilon) = \|x\|_{\infty,L^0(\Omega)} + \varepsilon \cdot \mathbf{1} \in L^0_{++}(\Omega)$, $\varepsilon > 0$. Using equalities

$$m\{|y| > h_1(\varepsilon)\} = m\{|x| > h_1(\varepsilon)\} = \mathbf{0},$$

we get that $\|y\|_{\infty,L^0(\Omega)} \le h_1(\varepsilon)$ for every $\varepsilon > 0$. This means that $\|y\|_{\infty,L^0(\Omega)} \le \|x\|_{\infty,L^0(\Omega)}$. Thus, $\|x\|_{\infty,L^0(\Omega)} = \|y\|_{\infty,L^0(\Omega)}$.

Consequently, $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$. □

Following the general theory of functional symmetric spaces, consider a space $L^1(B,m) \cap L^\infty(B, L^0(\Omega)$ with a norm

$$\|x\|_{L^1 \cap L^\infty} = \|x\|_{1,m} \vee \|x\|_{\infty,L^0(\Omega)}, x \in L^1(B,m) \cap L^\infty(B, L^0(\Omega)).$$

Proposition 5 . *$(L^1(B,m) \cap L^\infty(B, L^0(\Omega)), \|\cdot\|_{L^1 \cap L^\infty})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.*

Proof. Since $m(\mathbf{1}) = \mathbf{1}$, and for every $x \in L^\infty(B, L^0(\Omega))$ the inequality $|x| \le \|x\|_{\infty,L^0(\Omega)}$ is true, then $L^\infty(B, L^0(\Omega)) \subset L^1(B,m)$, moreover, $\|x\|_{1,m} \le \|x\|_{\infty,L^0(\Omega)}$. Hence, $L^1(B,m) \cap L^\infty(B, L^0(\Omega) = L^\infty(B, L^0(\Omega))$ and $\|x\|_{1,m} \vee \|x\|_{\infty,L^0(\Omega)} = \|x\|_{\infty,L^0(\Omega)}$. Thus, the pair

$$(L^1(B,m) \cap L^\infty(B, L^0(\Omega)), \|\cdot\|_{L^1 \cap L^\infty}(B, L^0(\Omega)) = (L^\infty(B, L^0(\Omega), \|\cdot\|_{\infty,L^0(\Omega)})$$

is a symmetric Banach-Kantorovich space over $L^0(\Omega)$ (see Theorem 4 (ii)). □

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Received: December 29, 2024
Accepted: June 5, 2025

IRSTI 27.41.19 DOI: https://doi.org/10.26577/JMMCS2025126208

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MATHEMATICAL MODEL WITH NON-LOCAL BOUNDARY CONDITION OF INCOMPRESSIBLE FLUID FILTRATION

This work is devoted to an actual problem today – the creation of cost-effective technology of combined development of several reservoirs. Joint development of oil reservoirs combining two or more oil reservoirs into one production facility by simultaneous extraction of reservoir fluids from them by a single network of wells [1, 2]. Oil fields, as a rule, are multilayer, and the productive formations are heterogeneous, first of all, by reservoir properties: first of all, they have different permeability and thickness. It is economically unprofitable to drill a different production grid for each of the productive formations. One of the primary tasks of putting an oil field into commercial development is to combine productive formations into single production facilities and to carry out joint development of these formations. After the reservoirs are combined into a single production facility, they are drilled using a single grid of production and injection wells [3, 4]. This paper considers a two-layer reservoir with different permeability and thickness. Numerical solution of the model is proposed to determine the pressure field of incompressible fluid at known total flow rate. The technology of combined development of several reservoirs isolated from each other is used. We construct special difference equations in the neighborhood of internal boundaries that allow us to apply the integro-interpolation method in a two-connected domain. Special differences equations in the vicinity of internal borders, allowing overcome the difficulties arising from the borders of the domain are constructed. The necessity of combined development in order to reduce the economic costs is revealed and justified. Based on the numerical investigations of the problem, obtained numerical results in programming language Fortran, and graphics in Tecplot for double-layer reservoirs. The article also found an analytical solution of this problem for the two reservoirs and made a comparative analysis of the results. Given conclusions about the quality and accuracy of used iterative method. The scientific novelty of this work is to research several layers by simultaneous selection of the reservoir fluids single well.The method of solving and analysis of the results will be of interest to those skilled in the development the oil fields.

Key words: combined development, doubly connected domain, reservoir thickness, mesh of wells, numerical solution, finite difference method, analytical solution.

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Шекаралық шарты локалды емес сығылмайтын сұйықтың фильтрленуінің математикалық моделі

Зерттеу жұмысы қазіргі кездегі өзекті – бірнеше пласты бірге өндірудің экономикалық тиімді технологиясын құру мәселесіне арналған. Екі немесе көпқатпарша пластарды бірмезгілде ұңғыма жүйесін енгізіп өндіру жолдарына [1, 2] байланысты математикалық модельдердің шешімін қарастыру бірмезгілде мұнайды өндірудің қажеттілігін талдау мүмкіндіктерін береді. Мұнай қоры жерасты қабаттарының әртүрлі коллекторлық қасиеттеріне және пластқабатшалардың қалыңдығына байланысты орналасқан. Соған байланысты мұнай қоры бар қабатшаларға жеке ұңғыма жүйесін пайдаланудың тиімділігі шамалы. Сондықтан мұнай қоры бар пластқабатшаларды бірге өндіру жолдарын қарастрып зерттеу мұнай механикасында жиі қолданылады [3, 4]. Өзара байланыспаған екі пластағы сұйық қорын өндіру есебі зерттелген. Мұнда ұңғыманың ықпалдық радиусымен шектелген аймақта арнайы ақырлы-айырымдық схема алынып, барлық аймақ үшін интегро-интерполяциялық әдісті қолдану жолы қойылған. Осымен бірге екі пласт үін шекаралық шарты локальды емес сығылмайтын сұйықтық фильтрлену есебінің аналитикалық шешімі алынып, жуық шешіміне талдау жасалады. Экономикалық шығындарды азайтуда бірге өндірудің қажеттігі көрсетіліп, дәлелденеді. Жүргізілген сандық зерттеулердің нәтижесінде Fortran бағдарламалау тілінде екі пласт үшін есептің сандық мәндері және Tecplot бағдарламалық пакетінде сызбалары алынған. Қолданылған итерациалық әдістің сапасы дәлелденген. Жұмыстың басқа қарастырылған жұмыстармен салыстырғандағы ғылыми жаңалығы – өзара байланыспаған пластардағы сұйық қорын бір мезгілде бір ұңғымалар торымен өндіру есебін зерттеу болып табылады. Модельді зерттеу әдістері мен алынған қорытынды нәтижелері мұнай кен орындары мамандарының қызығушылығын тудыруы мүмкін.

Түйін сөздер: бірге өндіру, екі байланысқан аймақ, пластың қуаты, ұңғымалар торы, сандық шешім, ақырлы-айырымдық әдіс, аналитикалық шешім.

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Математическая модель с нелокальным граничным условием фильтрации несжимаемой жидкости

Данная работа посвящена актуальной на сегодняшний день проблеме – созданию экономически эффективной технологии совместной разработки нескольких пластов. Совместная разработка нефтяных пластов – объединение двух и более нефтяных пластов в один эксплуатационный объект путём одновременного отбора из них пластовой жидкости единой сеткой скважин рассматривались многими авторами [1,2]. Нефтяные месторождения, как правило, являются многопластовыми, причём продуктивные пласты неоднородны, прежде всего по коллекторским свойствам имеют первую очередь различную проницаемость и толщину. На каждый из продуктивных пластов бурить свою сетку добывающих экономически убыточно. Одной из первоочередных задач ввода нефтяного месторождения в промышленную разработку является объединение продуктивных пластов в единые эксплуатационные объекты и проведение совместной разработки этих пластов. После объединения пластов в единый эксплуатационный объект их разбуривают по единой сетке добывающих скважин [3, 4]. В работе рассматривается двухслойный пласт различной проницаемости и толщины. Предложено численное решение модели определения давления несжимаемой жидкости, когда известен суммарный дебит при одновременной совместной разработке нескольких изолированных между собой пластов методом конечных разностей. Построены специальные разностные уравнения в окрестности внутренних границ, позволяющие применить интегро-интерполяционный метод в двухсвязной области. На основе проведенного исследования поставленной задачи получены численные результаты на языке программирования Fortran и графики модели на Tecplot для двухслойного пласта. Также найдено аналитическое решение данной задачи для двух пластов и сделан сравнительный анализ полученных результатов. Получено аналитическое и численное решение задачи с нелокальным граничным условием при совместной разработке двухслойных пластов с заданным суммарным расходом. Научная новизна работы заключается в исследовании нескольких пластов путем одновременного отбора пластовой жидкости единой сеткой скважины. Метод решения и анализ полученных результатов будут интересны специалистам в области разработки нефтяных месторождений.

Ключевые слова: совместная разработка, двухсвязная область, мощность пласта, сетка скважин, численное решение, метод конечных разностей, аналитическое решение

1 Introduction

The isothermal filtration of a homogeneous liquid in two formations isolated from each other, but penetrated by one well, is considered. Thus, the problem of planned filtration of fluid into a well comes down to finding a solution to Laplace's equation in a doubly connected region, the outer boundary of which is the contour of the filtration area, and the inner boundary is the contour of the well.

Due to the fact that the size of the filtration area, as a rule, is much larger than the size of the well contour, when solving the problem using the grid method, approximating the filtration area by the grid area so as to take into account the size and shape of the well presents certain difficulties [5]– [7].

When the well is replaced by a material point – the well in which the source (sink) is located, the function p at point O_0 becomes unlimited, and the flow rate q is defined as the limit

$$\lim_{l\to 0}\oint_l \sigma\frac{\partial p}{\partial n}dl = q, \tag{1}$$

where l is some closed contour covering the well, n is the external normal lose to l.

At filtration of homogeneous fluid the condition (1) is quite justified.If we keep the boundary $\partial\omega_\varepsilon$, associated with the control well, then specifying only the well flow rate for them is not sufficient; additional conditions are needed on the well contour, i.e.

$$\int_{\partial\omega_\varepsilon} \sigma\frac{\partial p}{\partial n}d\gamma = q, \tag{2}$$

$$p(x,y) = C, \text{ at } (x,y)\in\partial\omega_\varepsilon. \tag{3}$$

where C is some unknown constants. In this case it follows from relations (2), (3) that in ε neighborhood of the well the function $p(x,y)$ is represented as

$$p = u + \alpha ln\frac{r}{r_c}, \tag{4}$$

where $\frac{q}{2\pi\sigma_c}$, r_c is the radius of the well.

Then from (2) we have

$$\sigma_c = \frac{1}{2\pi}\int_0^{2\pi}\sigma(r_c,\varphi)d\varphi \tag{5}$$

Apparently, using relation (5) and hydraulic conductivity, it is necessary to continue to the inside of the well so that at the well point it takes σ_c, and require the fulfillment of condition (1) instead of condition (2). In this case, we can apply the substitution method [7]. However, as a result of such a transition, condition (3) will be fulfilled only approximately. Following the work [4, 8] given conditions (2), (3) on the well and taking into account the logarithmic dependence (4), the pressure function in ε neighborhoods of the well, a finite-difference method of solving the problem is constructed.

When studying the issue of fluid flow to production wells in a multi-layer system or in layers with a permeable roof and bottom, it is necessary to take into account its possible flows from one horizon to another, which greatly complicates theoretical studies and mathematical solutions of practical problems. We will investigate the plane-radial motion of liquid in two-layer formations isolated from each other, but opened by a single well. Considering that the thickness of the formation H is small compared to its dimensions in the horizontal plane, that the roof and the sole of the layers are impermeable, it is possible to pre-carry out all the necessary averaging of the parameters by power and thus move from spatial tasks to flat ones.Let's direct the OZ axis against gravity and introduce the reduced pressure function $p^* = p + \rho gh$. Then we write the filtration rate in the form $\vec{v} = -\frac{kH}{\mu}\text{grad}p^*$. In the following we will omit the asterisk at p^* and by the function p we will understand the reduced pressure.

2 Mathematical model of fluid filtration in two-layer formations

Let us introduce the notations: ω_ε-area, enclosed within the contour $\partial\omega_\varepsilon$, Ω-area enclosed within $\partial\Omega$, Q_0 is total flow rate of the well, selection from two layers, $k = 1, 2$ is layer number. Then Ω_k is a flat doubly connected region $\partial\Omega_k$, and $\omega_\varepsilon \in \Omega_k$ is a circle of radius $r_c = \varepsilon \ll \text{diam}\Omega_k$. Assume that the center of the circle coincides with the origin. Let us pose the problem of finding pressures in the region $\Omega_{\varepsilon,k} = \Omega_k/\bar{\omega}_\varepsilon$ that satisfy the equation

$$\text{div}\,\sigma_k\,\text{grad}p_k = 0, \quad k = 1, 2 \quad (x, y) \in \Omega_{\varepsilon,k}, \tag{6}$$

where ε is the radius of the well hereafter for convenience we will assume $r_c = \varepsilon$. On the contour $\partial\Omega_k$ takes the given values

$$p_k(x, y) = \varphi_0(x, y), \quad (x, y) \in \partial\Omega_k, \tag{7}$$

on $\partial\omega_\varepsilon$ satisfy the following conditions

$$\sum_{k=1}^{2} \oint_{\partial\omega_\varepsilon} \sigma_k \frac{\partial p_k}{\partial n} d\gamma = Q_0 \quad (x, y) \in \partial\omega_\varepsilon \tag{8}$$

$$p_2(x, y) = p_1(x, y) + \rho g z_c \quad \text{at}(x, y) \in \partial\omega_\varepsilon \tag{9}$$

where $\sigma_k = \frac{k_k H_k}{\mu} > 0$ is coefficient of hydraulic conductivity, $p_1(x, y) = C$ is some unknown constant value, $z_c = \text{const}$ is trance between the center surfaces (horizontal planes) of the two layers, ρ is the density of liquids, g is the acceleration of free fall.

3 Analytical solution of the problem (6)–(9)

Let's pass to polar coordinates in two-dimensional space

$$r^2 = x^2 + y^2, \quad x = r\cos\varphi, \quad y = r\sin\varphi, \quad \tan\varphi = y/x.$$

In polar coordinates, the desired function $p(r,\varphi)$ must be periodic with period $2\pi:\ p(r,\varphi+2\pi)=p(r,\varphi)$. Let us write the isobaric pressure field for a circular reservoir with constant hydraulic conductivity coefficient ($\sigma_k=\text{const}$):

$$\frac{d^2p_i}{dr^2}+\frac{1}{r}\frac{dp_i}{dr}=0,\quad r_c<r<R. \tag{10}$$

On the contour of two-layer strata

$$p_i(R)=p_0. \tag{11}$$

We set the total flow rate Q_0 on the well

$$\sum_{i=1}^{2}\oint_{\partial\omega_\varepsilon}\sigma_i\frac{\partial p_i}{\partial r}d\gamma=Q_0, \tag{12}$$

and unknown pressure on the well contour

$$p_2(r_c)=p_1(r_c)+\rho gH. \tag{13}$$

Let us represent the general solution in layers in the following form

$$p_i(r)=A_i\ln r+B_i,\quad i=1,2, \tag{14}$$

Let us take a sector of a circular layer (Figure 1), where $\angle AOB=d\varphi$, $d\gamma=AB$. Then for small $\left(\frac{d\varphi}{2}\right)$ we have $\frac{d\gamma}{2}=r_c\sin\left(\frac{d\varphi}{2}\right)$ or write $d\gamma\approx r_c d\varphi$.

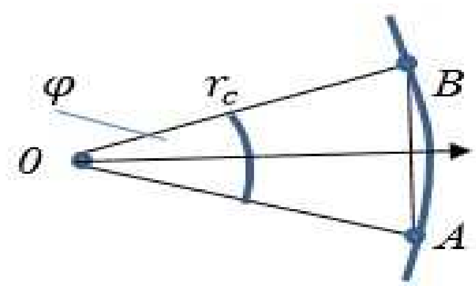

Figure 1. Sector of the circle $d\varphi=\angle AOB$.

Now let us write the condition (14) with period 2π in the form $\sum\limits_{i=1}^{2}\int\limits_{0}^{2\pi}\sigma_i\frac{\partial p_i}{\partial r}r_c d\varphi=Q_0$. Then after integration on the basis of the general solution we obtain

$$2\pi\sigma_1A_1+2\pi\sigma_2A_2=Q_0. \tag{15}$$

Taking into account the boundary conditions on the well contour, we determine the integral constants for the first and second reservoirs:

$$A_1=\left(P_0-p_1(r_c)\right)/\ln\left(\frac{R}{r_c}\right), \tag{16}$$

$$A_2 = (p_0 - p_1(r_c) - \rho g H) / \ln \left(\frac{R}{r_c} \right). \tag{17}$$

From equations (15)–(17) we find the pressure on the well contour of the first and second layers:

$$p_1(r_c) = p_0 - \frac{Q_0}{2\pi} \ln \left(\frac{R}{r_c} \right) / (\sigma_1 + \sigma_2) - \sigma_2 \cdot \rho g H / (\sigma_1 + \sigma_2). \tag{18}$$

$$p_2(r_c) = p_0 - \frac{Q_0}{2\pi} \ln \left(\frac{R}{r_c} \right) / (\sigma_1 + \sigma_2) + (1 - \sigma_2)/(\sigma_1 + \sigma_2) \cdot \rho g H. \tag{19}$$

Knowing the boundary conditions on the contour of two layers $p_i(R) = p_0$ and conditions on the contour of the well (18) and (19), we plot the field of pressure changes in the first and second layers:

$$p_1(r) = p_0 + \left[\frac{\frac{Q_0}{2\pi}}{\sigma_1 + \sigma_2} + \frac{\sigma_2 \cdot \rho g H / (\sigma_1 + \sigma_2)}{\ln \left(\frac{R}{r_c} \right)} \right] \cdot \ln \left(\frac{r}{R} \right), \tag{20}$$

$$p_2(r) = p_0 + \left[\frac{\frac{Q_0}{2\pi}}{\sigma_1 + \sigma_2} - \frac{(1 - \sigma_2/(\sigma_1 + \sigma_2)) \cdot \rho g H}{\ln \left(\frac{R}{r_c} \right)} \right] \cdot \ln \frac{r}{R}. \tag{21}$$

If $\rho g H = 0$, we get a pressure field where the downhiller pressure at the well's same for the two layers, $P_1(r_c) = P_2(r_c) = \text{const}$. It should be noted that problems with nonlocal boundary condition for elliptic equation are considered by many authors. For example, in [9, 10] the asymptotes of solutions of nonlocal elliptic equations are considered in flat bounded domains.

4 Numerical solution of the problem by finite difference method

In the case of joint reservoir development, the finite element method is proposed in [11]. It is shown here that when using the finite difference method, the influence of the well radius on the filtration process presents certain difficulties. Following the work [8], we construct the solution by the finite difference method. First, we will construct a solution method for one layer (for the prostate we will take $p = p_1 = p_2$, $\sigma = \sigma_1 = \sigma_2$, $\Omega = \Omega_1 = \Omega_2$, $q_1 = \text{const}$). Let the area Ω is covered by a grid $\Omega_h (h \gg r_c)$. We will place the well point O_0 at node (i_0, j_0). The point O_0 does not belong to the area Ω, so the node (i_0, j_0) does not belong to Ω_h either. We include all points formed by the intersection of grid lines with the boundary $\partial\omega$. We denote this set of nodes by $\partial\Omega_h$, and we denote an area (cell) by $\Omega_{i,j}$ and its boundary by $\partial\omega_{i,j}$. Then the cell $\partial\Omega_{i_0,j_0}$ is doubly connected, has an internal boundary ω_0 and, unlike other elementary areas, contains not one grid point, but a set of nodes $\partial\omega_{0,h}$ (see Fig. 1). From the generalized Green's formula and applying the condition (8), we obtain

$$\iint_{\partial\omega_{i_0,j_0}} \text{div}\sigma\text{grad}p dV = \oint_{\partial\omega_{i_0,j_0}} \sigma \frac{\partial p}{\partial n} d\gamma - \oint_{\partial\omega_0} \sigma \frac{\partial p}{\partial n} d\gamma = \oint_{\partial\omega_{i_0,j_0}} \sigma \frac{\partial p}{\partial n} d\gamma - q_1.$$

The grid cell Ω_{i_0,j_0} is bi-connected because the cell contains a well with radius r_c (Figure 2).

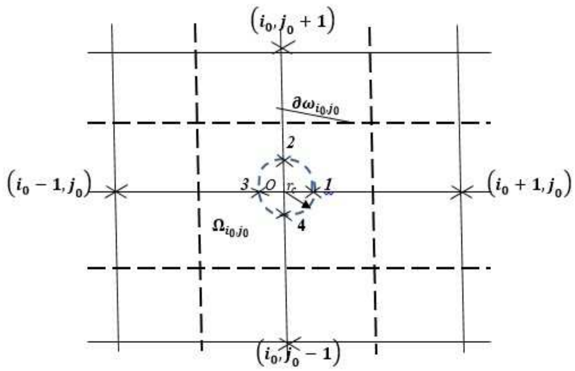

Figure 2. The well is located at a grid node.

Let the grid be square $x = ih$, $y = jh$, $i, j = \pm 1, \pm 2, \dots$ then

$$\overline{\Omega}_{i,j} = \left\{ x_{i-1/2} \le x \le x_{i+1/2},\ y_{j-1/2} \le y \le y_{j+1/2} \right\}.$$

Thus, as a result of integration of equation (6) over the cells $\Omega_{i,j}$ we obtain

$$\oint_{\partial\omega_\varepsilon} \sigma \frac{\partial p}{\partial n} d\gamma = \Phi_{i,j}, \tag{22}$$

where $\Phi_{i,j} = \begin{cases} 0, \text{ at } (i,j) = (i_0, j_0), \\ q_1, \text{ at } (i,j) = (i_0, j_0). \end{cases}$

The radius (ρ) of influence of the well is comparable to the grid spacing (h). Then in the neighborhoods of ρ point the function p has a logarithmic dependence. Therefore, in ρ neighborhoods we introduce an auxiliary function

$$u = p - \alpha_0 \ln r, \quad r^2 = (x - x_0)^2 + (y - y_0)^2 \tag{23}$$

where α is an undetermined constant.

Then we write (22) in the following form

$$\oint_{\partial\omega_\varepsilon} \sigma \frac{\partial p}{\partial n} d\gamma = \Phi_{i,j} - \alpha_0 \oint_{\partial\omega_\varepsilon} \sigma \frac{\partial lnr}{\partial n} d\gamma, \tag{24}$$

where $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\cos(\widehat{n,x}) + \frac{\partial u}{\partial y}\cos(\widehat{n,y})$ and taking into account the difference derivative along the normal $\nabla n, m$ to the boundary γ_m (Figure 2). Thus γ_m is orthogonal to the m-th grid line leaving node (i,j) of the grid. As a result of numerical differentiation and integration of the left side of equation (24), we obtain

$$\sum_m \sigma_m \nabla_{n,m} u \nabla\gamma_m = \Phi_{i,j} - \alpha_0 \sum_m \sigma_m \int_{\gamma_m} \sigma \frac{\partial \ln r}{\partial n} d\gamma. \tag{25}$$

Here $\nabla\gamma_m$ is the length of the boundary γ_m.

Consider the cell Ω_{i_0,j_0} (Figure 2). Its outer boundary is formed by line segments $x = x_0 \pm \frac{1}{2}(h + r_c)$, $y = y_0 \pm \frac{1}{2}(h + r_c)$. In this cell, four points belong to the grid Ω_h. Obviously, the distance from all points to the well point O is equal to r_c. When $m = 1, 3$ and $m = 2, 4$, we write down the difference approximation $\nabla_{x,m} u\,(h + r_c)$ and $\nabla_{y,m} u\,(h + r_c)$. Then we make the reverse transition on the grid, i.e. we exclude the grid values of the auxiliary function $u(x, y)$ using equality (23) connecting the values of the functions $p(x, y)$ and $u(x, y)$. Since in our case $h \gg r_c$, we can get:

$$\nabla_{x,m} u = \left(p_{i_0 \pm 1, j_0} - p_{i_0,j_0} - \alpha ln \frac{h}{r_c} \right), \quad \nabla_{y,m} u = \left(p_{i_0, j_0 \pm 1} - p_{i_0,j_0} - \alpha ln \frac{h}{r_c} \right),$$

It is not difficult to make sure that for all m we have

$$\int_{\gamma_m} \frac{\partial ln r}{\partial n} d\gamma = \int_{-1/2}^{1/2} \frac{1/2 h dy}{1/4 h^2 + (x - i_0 h)} = \frac{\pi}{2}.$$

Here $\alpha_0 = \frac{q_1}{2\pi \overline{\sigma}_{cp}}$, where $\overline{\sigma}_{cp} = \frac{1}{4}\left(\sigma_{i_0+1/2,j_0} + \sigma_{i_0-1/2,j_0} + \sigma_{i_0,j_0+1/2} + \sigma_{i_0,j_0-1/2} \right)$.

Finally, for the cell Ω_{i_0,j_0}, the difference equations with a given flow rate q_1 for one layer will be written in the form

$$h^2 L(\sigma, p)_{i_0,j_0} = \sigma_{i_0+1/2,j_0} \left(p_{i_0+1,j_0} - p_{i_0,j_0} \right) + \sigma_{i_0-1/2,j_0} \left(p_{i_0-1,j_0} - p_{i_0,j_0} \right)$$

$$+ \sigma_{i_0,j_0+1/2} \left(p_{i_0,j_0+1} - p_{i_0,j_0} \right) + \sigma_{i_0,j_0-1/2} \left(p_{i_0,j_0-1} - p_{i_0,j_0} \right) = q_1 \ln \frac{h}{r_c}.$$

Thus, when the radius of influence of the well (ρ) is equal to the grid step (h), for one reservoir the condition (22) has the following difference approximation

$$h^2 L(\sigma, p)_{i_0,j_0} = \frac{2}{\pi} q_1 \ln \frac{h}{r_c}. \tag{26}$$

Accordingly, expression (26) gives a difference approximation at the point-well and for the second layer. Let us assume that the grid spacing for the two layers is the same. In this case, the following expression can be written for the two layers

$$h^2 L_1(\sigma_1, p)_{i_0,j_0} + h^2 L_2(\sigma_2, p)_{i_0,j_0} = \frac{2}{\pi} q_1 \ln \frac{h}{r_c} + \frac{2}{\pi} q_2 \ln \frac{h}{r_c}. \tag{27}$$

Then, taking into account the total flow rate ($Q_0 = q_1 + q_2$) from two layers, we obtain

$$h^2 L_1(\sigma_1, p)_{i_0,j_0} + h^2 L_2(\sigma_2, p)_{i_0,j_0} = \frac{2}{\pi} Q_0 \ln \frac{h}{r_c}, \tag{28}$$

We construct numerical solutions of the problem (6)–(9) by the longitudinal-transverse scheme proposed by Peaceman–Rachford.

5 The Peaceman–Rachford method for solving the problem (6)–(9)

Let the grid be square $x = ih$, $y = jh$, $i, j = \pm 1, \pm 2, \ldots$ Then the left side of the difference equation inside the domain on a five-point template with an error of $O(h^2)$ will have the form [12]:

$$h^2 L_k(\sigma_m, p)_{i,j} = \sigma_{k,i+\frac{1}{2},j} (p_{i+1,j} - p_{i,j}) + \sigma_{k,i-\frac{1}{2},j} (p_{i-1,j} - p_{i,j})$$

$$+\sigma_{k,i,j+\frac{1}{2}} + (p_{i,j+1} - p_{i,j}) + \sigma_{k,i,j-\frac{1}{2}} (p_{i,j-1} - p_{i,j}), \tag{29}$$

here $k = 1, 2$ layer numbers.

Let's write down the boundary conditions on the well

$$h^2 L_1(\sigma_1, p)_{i_0,j_0} + h^2 L_2(\sigma_2, p)_{i_0,j_0} = \frac{2}{\pi} Q_0 \ln \frac{h}{r_c},$$

$$p_{2,i_0,j_0} = p_{1,i_0,j_0} + \rho h H. \tag{30}$$

On the contour $r = R$ of two-layered interlayers

$$p_k(r) = \varphi_0, \quad k = 1, 2. \tag{31}$$

Let us write equation (29) with constant hydraulic conductivity coefficient $\sigma = \text{const}$. Let us introduce the difference operators:

$$p_{\pm 1} p \equiv p_{\pm 2} p \equiv p_{i,j\pm 1}, \quad Ep \equiv p_{i,j}.$$

Let us represent the operator L_k as a sum, e.g. for $k = 1$.

$$L_1 = A_1 + A_2, \quad A_1 = \frac{\sigma_1}{h^2}(p_{+1} - 2E + p_{-1}), \quad A_2 = \frac{\sigma_1}{h^2}(p_{+2} - 2E + p_{-2}).$$

If $p^k = \left\{ p^k_{i,j} \right\}$, is known, it is done in two steps through finding the intermediate value $p^{k+1/2} = \left\{ p^{k+1/2}_{i,j} \right\}$:

$$\frac{p^{k+1/2}_{i,j} - p^k_{i,j}}{\omega} = h^2 A_1 p^{k+1/2} + h^2 A_2 p^k, \quad 1 \le i \le N-1, \ 1 \le j \le N-1, \tag{32}$$

the corresponding entry is the same at $k = 2$.

Condition (33) at the intermediate stage

$$A_{10} p^{k+1/2} + A_{20} p^k = L_2(\sigma, p)_{i_0,j_0} + \frac{2}{\pi} Q_0 \ln \frac{h}{r_c} / h^2. \tag{33}$$

On the contour of the layers, $p^{k+1/2}_{i,j} = \varphi_0$.

Similarly, the Peaceman–Rachford method is constructed for the second layer in the case of the operator $L_2(\sigma_2, p)$, [15]. The computational algorithm of the problem (6)–(9)

consists of two stages – internal and external iteration.In the case of internal iteration, the pressure function in individual formations is determined. And the outer iteration ends when the nonlocal boundary condition (30) is satisfied.

During the computational process, for example, for the first layer at a well point, from expression (23) we determine $L_1(\sigma_1, p)_{i_0,j_0}$ with fixed $L_2(\sigma_2, p)_{i_0,j_0}$ and we solve $L_1(\sigma_1, p)_{i,j}$ i.e. we perform an internal iteration for the first layer. Then we carry out the internal iteration for the second layer with fixed $L_1(\sigma_1, p)_{i_0,j_0}$ and solve $L_2(\sigma_2, p)_{i,j}$. This procedure continues until condition (30) is satisfied, at which point the outer iteration ends.

Let us briefly dwell on the issue of convergence of the Peaceman–Rachford method [13]. Matrices A_1 and A_2 are symmetric and negative – definite and have a common complete orthonormal system of eigenvectors

$$2\sin(i\pi_k h_1)\sin(i\pi l h);\quad k=\overline{1,N_1-1};\ l=\overline{1,N_2-1}.$$

The eigenvalues of the operator A_1 and A_2 are $(\lambda_i)_{A_1} = -\frac{4}{h_1^2}\sin^2\left(\frac{i\pi h_1}{2}\right)$, $(\lambda_j)_{A_2} = -\frac{4}{h_2^2}\sin^2\left(\frac{j\pi h_2}{2}\right)$, $1\le i\le N_1-1$, $1\le j\le N_2-1$.

In the case of a tridiagonal matrix, excluding $p^{k+1/2}$, we can write

$$(E-\omega A_1)(E-\omega A_2)p^{k+1} = (E+\omega A_1)(E+\omega A_2)p^k,$$

Hence $p^{k+1} = (E-\omega A_2)^{-1}(E-\omega A_1)^{-1}(E+\omega A_1)(E+\omega A_2)p^k$. It is known that the eigenvalues of the transition operator are taken

$$B = (E-\omega A_1)(E-\omega A_2)(E+\omega A_1)(E+\omega A_2)$$

equal

$$(\lambda_{i,j})_B = \frac{1+\omega(\lambda_i)_{A_1}}{1-\omega(\lambda)_{A_1}}\cdot\frac{1+\omega(\lambda_j)_{A_2}}{1-\omega(\lambda_j)_{A_1}}. \tag{34}$$

Since $(\lambda_i)_{A_1} < 0$, $(\lambda_j)_{A_2} < 0$, then $(\lambda_{i,j})_B < 1$ and $\omega > 0$ for any $\omega > 0$. Therefore, the Peaceman–Rachford method converges.

If the internal iteration associated with the Peaceman-Reckford method converges, then obviously the external iteration also converges.

In this method, the question of the optimal choice of ω is a complex issue that is not resolved in all cases. You can proceed as follows: for the first N-1 iterations put [14,15]

$$\omega_{k+1} = -\frac{1}{(\lambda_i)_{A_1}},\quad k=\overline{0,N_1-2}.$$

Then $(\lambda_{i,j})_B$, $1\le i\le N_1-1$ from (34) will vanish. If at the same time the inequality

$$\max_{\substack{1\le i\le N_1-1\\ 1\le j\le N_2-1}} \left|p_{i,j}^{k+1}-p_{i,j}^k\right| < \varepsilon,$$

then ω is then chosen to be equal to

$$\omega_k = -\frac{1}{(\lambda_{k-N_1+2})_{A_2}},\quad k = N_1-1, N_1,\ldots,N_1+N_2-3.$$

R^k will be a value of the order of $e^{-\gamma_N k}$. The value γ_N characterizes the quality of the iterative method. As is known, for the simplest iteration process $P^{k+1} = P^k + \alpha(Au^k - h^2 f)$ under certain restrictions on α we have $\gamma_N = 1/N^2$.

Conclusion

The solution of the problem (6)–(9) was carried out with the following parameters. On the contour of the layers, the same pressure was maintained at 15 MPa, the fluid flow rate from the two layers was 80 t/day, the thickness of the layers was 10 m, 15 m, and the thickness of the impermeable interlayer bridge was 0.5 m, the fluid viscosity was 4 Sp, the permeability's were different 0.3 D, 0.5 D. The results obtained by finite difference method were compared with the exact solution. In the filtration area of the radius of influence of the well, the error averaged 0.1 per cent. However, this error did not exceed 0.05 per cent when approaching the reservoir contour, i.e. it decreased. For $\frac{1}{4}$ area they obtained solutions by formulas (18)–(19) are shown in Figures 1 and 2. The same pressure of 14 MPa was maintained at the contour of the two layers, and the well radius was 12 cm. Here, the isobaric surfaces of the pressure field $P(x,y)$ differ by 0.7 MPa. In Figure 4, the isobaric surfaces are obtained with a contour pressure of 13 MPa. Naturally, when the contour pressure decreases, the concentric surfaces occupy less area.

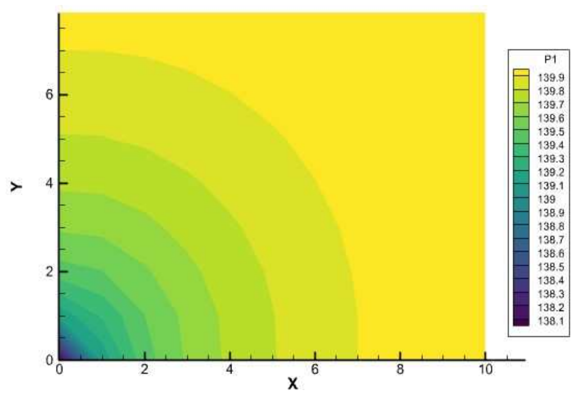

Figure 3. Pressure field for the first.

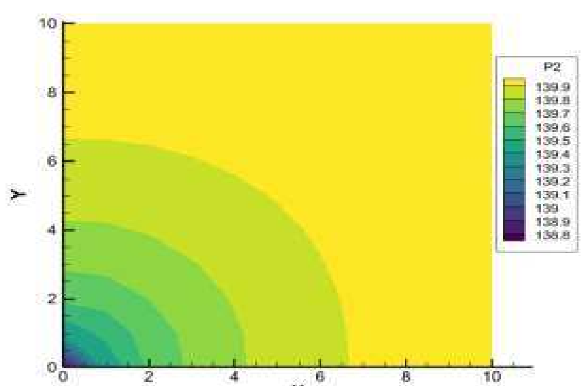

Figure 4. Pressure field for the second layer with permeability of 0.5 D.
Layer with permeability of 0.3 D. The contour pressure is 14 MPa.
The contour pressure is 14 MPa.

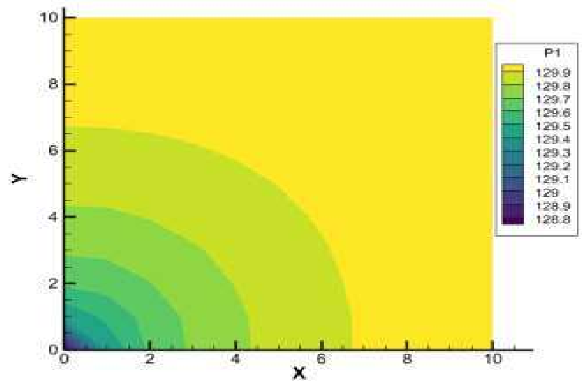

Figure 5. Pressure field for the first layer with a permeability of 0.5 D.
Pressure on the contour is 13 MPa.

Joint development of several formations with one well can be cost-effective, especially for low-productivity formations that cannot be exploited separately because they are not economically feasible. The results obtained with the analytical method show the correctness and high accuracy of the numerical finite difference method.

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Received: May 23, 2025
Accepted: June 5, 2025

IRSTI 27.39.21 DOI: https://doi.org/10.26577/JMMCS2025126209

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RECOVERING A SURFACE IN ISOTROPIC SPACE USING DUAL MAPPING ACCORDING TO CURVATURE INVARIANTS

The problem of recovering a surface according to its curvature is one of the fundamental problems of differential geometry. Problems of recovering surfaces in various spaces by their total or mean curvature have been widely studied in many works.Recovering of a surface by its total curvature is equivalent to solving the Monge-Ampere equation of elliptic type; such problems are solved in special cases. When the right part is given concretely.The Monge-Ampere equation is solved using a dual mapping of isotropic space, in which the dual surface is a transfer surface. Also, some special cases are used to find the surface equation.The connection between dual mean curvature and amalgamatic curvature is studied.The equivalence of the problem of recovering by dual mean and amalgamatic curvature is shown. In particular, the problem of recovering surfaces with total negative constant curvature, the mean curvature of which is a function of one variable, is solved. Furthermore, the problems of the recovering surfaces are solved according to their dual mean curvature, amalgamatic and Casorati curvatures.
Key words: isotropic space, Monge-Ampere equation, dual mapping, amalgamatic curvature, Casorati curvature.

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Қисықтық инварианттары бойынша дуаль бейнелеуді қолдану аркылы изотроптық кеңістікте бетті қалпына келтіру

Беткейді оның қисығына қарап қалпына келтіру мәселесі – дифференциалдық геометриядағы негізги міндеттердің бірі болып табылады.Әртүрлі кеңістіктерде беткейлерді толық немесе орташа қисығына қарап қалпына келтіру мәселелері көптеген еңбектерде кеңінен зерттелген. Беткейді оның толық қисығына қарап қалпына келтіру Монж-Ампердің эллиптикалық типтегі теңдеуін шешуге тең. Мұндай есептер кейбір жекелеген жағдайларда, оң жақ бөлігі нақты берілген кезде шешілген. Монж–Ампер теңдеуі изотропты кеңістіктегі дуальды бейнелеу арқылы шешіледі, мұнда дуальды беткей бұл көшу беткейі болып табылады. Сондайақ беткей теңдеуін табу үшін кейбір жеке жағдайлар қолданылған.Дуальды орташа қисығымен және амальгамалық қисығымен байланыс зерттелген. Дуальды орташа қисық пен амальгамалық қисық бойынша қалпына келтіру мәселесінің эквиваленттілігі көрсетілген. Атап айтқанда, толық теріс тұрақты қисыққа ие беткейлерді, олардың орташа қисығы бір айнымалыға тәуелді болған жағдайда, қалпына келтіру есебі шешілген. Сонымен қатар, беткейлерді дуальды орташа қисығына, амальгамалық және Касорати қисығына сәйкес қалпына келтіру есептері де қарастырылған.

Түйін сөздер: Изотропты кеңістік, Монж–Ампер теңдеуі, дуаль бейнелеу, амальгаматик қисықтық, Касорати қисықтығы.

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Восстановление поверхности в изотропном пространстве с использованием двойственного отображения по инвариантам кривизны

Задача восстановления поверхности по ее кривизне является одной из основных задач дифференциальной геометрии. Задачи восстановления поверхностей в различных пространствах по их полной или средней кривизне широко изучались во многих работах. Восстановление поверхности по ее полной кривизне эквивалентно решению уравнения Монжа-Ампера эллиптического типа, такие задачи решена в частных случаях, когда правая част дана конкретно. Уравнение Монжа-Ампера решается с помощью двойственного отображения изотропного пространства, в котором двойственная поверхность является поверхностью переноса. Также для нахождение уравнению поверхности использована некоторые частные случае. Изучается связь между дуальной средней кривизной и амальгаматической кривизной. Показана эквивалентность задачи восстановления по двойственному среднему и амальгаматической кривизне. В частности, решена задача восстановления поверхностей с полной отрицательной постоянной кривизной, средняя кривизна которых является функцией одной переменной. Кроме того, задачи восстановления поверхностей решаются в соответствии с их двойной средней кривизной, амальгаматической и кривизной Касорати.

Ключевые слова: Изотропное пространство, уравнение Монжа-Ампера, дуальное отображение, амальгаматическая кривизна, кривизна Касорати.

1 Introduction

K.Strubecker studied the basic concepts related to isotropic geometry [1, 2]. Currently, many mathematicians are conducting scientific research on isotropic space. M.E. Aydin studied the types of transfer surfaces by a given constant curvature in isotropic space [3, 4]. Z.M. Sipus found equations of transfer surfaces by a given constant Gaussian and mean curvature in 3-dimensional isotropic space.Also she studied transfer Wiengarten surfaces in this space [5]. M.Karacan ,B.Bukcu, D.Yoon and N.Yuksel investigated transfer and ruled surfaces satisfying [6, 7]:

$$\Delta^J x_i = \lambda_i x_i$$

A.Cakmak, S.Kiziltug, M.Karacan found dual surface for the surface $z = f(u)+g(v)$ satisfying the condition

$$\Delta^J x_i^* = \lambda_i x_i^*$$

in 3-dimensional isotropic space. Besides that they solved the recovering problem of the transfer dual surface by given non-zero total and mean curvatures [8]. Several mathematicians solved the Monge-Ampere equation for transfer surfaces in some special cases. In the article [9], M.S.Lone,M.K.Karacan solved the problem of recovering a given dual transfer surface with total curvature being constant. Sh. Ismoilov solved this problem by given total curvature being the product of two functions with separate variables [10]. Moreover, in the article of A.Artykbaev and Sh. Ismoilov [11, 12], the connection of total curvatures between the given surface and dual surface is proved. In Euclidean space, A.D.Alexandrov solved the problem of existence and uniqueness of a surface by a given external curvature [13]. I.Y.Bakelman presented a solution to the Dirichlet problem for the elliptic Monge-Ampere equation related to this geometric problem [14].

In addition to the problem of recovering a surface from its total curvature, one of the important problems of differential geometry is also the problem of recovering it from its mean curvature. In many works, the problem of recovering a surface by its total or mean curvature was solved in different special cases. However, in addition to these geometric characteristics, the problem of surface recovering can be considered by other curvature invariants. In surface theory, there are the amalgamatic and Casorati curvatures, which are associated with principal normal curvatures that differ from the total and mean curvatures. Amalgamatic curvature in Euclidean space was studied by Suceava and a calculation formula was found [15]. Decu and Verstraelen investigated isotropic Casorati curvature [16]. The problem of recovering a surface in isotropic space by amalgamatic and Casorati curvatures was solved, where these curvatures are equal to zero and constant for surfaces with a total curvature of -1 [17]. In this work, we find the surface equation by solving the Monge-Ampere equation, in the case of that the dual surface is a transfer surface. Also, by studying the connection between the amalgamatic curvature of a surface and the mean curvature of its dual surface, we find the equation of the surface for surfaces with total curvature of -1 in isotropic space, where the mean curvature is a differentiable function of one variable.

2 Preliminaries

2.1 Geometry of isotropic space and duality

Let there be given an affine space A_3 with the coordinate system $Oxyz$. We consider $\vec{X}\{x_1, y_1, z_1\}$ and $\vec{Y}\{x_2, y_2, z_2\}$ vectors in A_3.

Definition 1 *If the scalar product of the two vectors $\vec{X}\{x_1, y_1, z_1\}$ and $\vec{Y}\{x_2, y_2, z_2\}$ is defined by the following formula:*

$$\begin{cases} (X,Y)_1 = x_1x_2 + y_1y_2 & if \quad (X,Y)_1 \neq 0 \\ (X,Y)_2 = z_1z_2 & if \quad (X,Y)_1 = 0 \end{cases}$$

then, the affine space A_3 is the isotropic space and denoted by R_3^2.

Two types of spheres are defined in isotropic space [18]. The first is the metric sphere, which is given by the following formula:

$$x^2 + y^2 = r^2 \tag{1}$$

where $(0, 0, z)$ is the center, r is radius.

The second sphere in isotropic space is defined as follows [18]:

$$x^2 + y^2 = 2z \tag{2}$$

It is called the isotropic sphere. Consider a plane Π in this space. Let this plane not be parallel to the axis Oz. The section of this sphere by the plane Π, forms a closed curve .This curve is an ellipse and denote it by γ [11]. Pass tangent planes to the isotropic sphere (2) through the points $P \in \gamma$. We denote the set of these planes to points Φ by $\{\Pi\}$. We get the following:

Theorem 1 *All planes belonging to the set $\{\Pi\}$ intersect at one point [11].*

If the plane Π_0 is as follows:

$$z = A_0x + B_0y + C_0 \tag{3}$$

then the intersection point of these planes belonging to the set $\{\Pi\}$ is $(A_0, B_0, -C_0)$.

Definition 2 *The point* $(A_0, B_0, -C_0)$ *is a dual point to the plane* (3) *with respect to the isotropic sphere* (2) *in the isotropic space [11].*

Let there be given a plane $z = T$. And γ be the section of this plane on the isotropic sphere. Consider a surface Φ that is given by the following:

$$\Phi : \{ z = f(x, y) | \, (x, y) \in D\} \tag{4}$$

And the curve γ be the boundary of the surface (4). The surface (4) is convex and it is located inside the part of the isotropic sphere bounded by the plane.

Let us pass a tangent plane Π_P to the given surface Φ at a point $P\,(x_0, y_0, z_0)$. Let us denote by P^* the dual image of the tangent plane Π_P with respect to the isotropic sphere (2). If the given point $P \in \Phi$ changes on the surface Φ, the dual image of this point forms a surface Φ^*.

Definition 3 *The surface* Φ^* *is called the dual surface to the given surface* Φ *in the isotropic space. If* Φ *has the following form, i.e.:*

$$z = f(x, y)$$

then the parametric equations for the dual surface Φ^* *are:*

$$\begin{cases} x^*\,(u, v) = {f_u}'\,(u, v) \\ y^*\,(u, v) = {f_v}'\,(u, v) \\ z^*\,(u, v) = u \cdot {f_u}'\,(u, v) + v \cdot {f_v}'\,(u, v) - f\,(u, v) \end{cases} \tag{5}$$

The above equation (5)is the dual mapping in isotropic space [10]. Following connection is valid between the total curvatures for the given surface Φ and its dual surface Φ^*

Theorem 2 *For the product of total curvatures* K *and* K^**, the following holds [11]:*

$$K \cdot K^* = 1 \tag{6}$$

From this, the total curvature K^* is equal to the following:

$$K^* = \frac{1}{K} \tag{7}$$

The following equality holds for the mean curvature of a given surface and the mean curvature of its dual surface:

$$H^* = \frac{H}{K} \tag{8}$$

The question "Can the result obtained from the problem solved for the dual surface be applied to the Φ surface?"is considered important. If we apply a dual mapping to the dual surface again, then we have the following theorem that the dual image of a dual surface is equal to the given surface, that is:

Theorem 3 *The dual image of the dual surface* Φ^* *coincides with the given surface* Φ *[19]:*

$$\Phi^{**} = \Phi \tag{9}$$

2.2 Transfer surfaces

M.E.Aydin classified transfer surfaces and found equations for these surfaces in the case that their total and mean curvatures are constant [3]. M.S.Lone, M.K.Karacan found a dual surface by a given constant total and mean curvatures of this surface [9]. Sh.Sh.Ismoilov solved in the case that the total curvature for the transfer surface is the product of two functions with separate variables for this class [12]. In general, the vector equation of the transfer surface can be expressed as the sum of two isotropic planar curves in isotropic space:

$$\overline{r}\,(u,v)=\overline{\rho}\,(u)+\overline{\sigma}\,(v)$$

where, $\overline{\rho}\,(u)$ and $\overline{\sigma}\,(v)$ are the vector forms of these curves. The surface is one-valued projected onto the Oxy plane. Let this surface not be parallel to the axis Oz in the isotropic space, then we obtain the following:

$$\overline{r}\,(u,v)=u\overline{i}+v\overline{j}+(f(u)+g(v))\,\overline{k}$$

where, $\overline{\rho}\,(u)=(u,0,f(u))$ and $\overline{\sigma}\,(v)=(0,v,g(v))$.

3 Solving the Monge-Ampere equation by using duality

The Monge-Ampere equation is generally as follows:

$$z_{xx}z_{yy}-z_{xy}^2=\varphi(x,y,z,z_x,z_y) \tag{10}$$

where the function $\varphi(x,y,z,z_x,z_y)-$ is the given function. In this paper, for

$$z_{xx}z_{yy}-z_{xy}^2=\varphi(z_x,z_y) \tag{11}$$

we will find the solution. If a regular surface is given by the following form

$$z=z(x,y),\quad (x,y)\in D\subset R_2$$

in the isotropic space R_3^2, then the total curvature of this surface is expressed by the following formula:

$$z_{xx}z_{yy}-z_{xy}^2=K \tag{12}$$

Where, K is the total curvature for the surface, the left side of the formula (12) is the Monge-Ampere operator. The problem of recovering the surface is equivalent to solve the Monge-Ampere equation in isotropic space [3]. Equation (11) can be solved for transfer surfaces if the dual mapping of isotropic space is used.

Theorem 4 *In the isotropic space, the Monge-Ampere equation is in the form (11), and the function on the right side can be written in the form* $\varphi(z_x,z_y)=\frac{1}{\psi_1(z_x)\psi_2(z_y)}$*, then the general solution of the transfer surface is equal to:*

$$z(x,y)=x\cdot\psi_1^{-1}\left(\frac{dx}{\tau}\right)+y\cdot\psi_2^{-1}\left(\tau dy\right)-\int xd\left(\psi_1^{-1}\left(\frac{dx}{\tau}\right)\right)-\int yd\left(\psi_2^{-1}\left(\tau dy\right)\right) \tag{13}$$

where, $\tau-$ is const. z_x,z_y are first-order derivatives of $z(x,y)$.

Proof of the Theorem 4. Let us assume that the regular surface Φ be given by the

$$z = z(x, y), \quad (x, y) \in D \subset R_2$$

in the space R_3^2. The Monge-Ampere equation for this surface is as follows:

$$z_{xx} z_{yy} - z_{xy}^2 = \varphi(z_x, z_y)$$

Let the function on the right side be given in the form $\varphi(z_x, z_y) = \frac{1}{\psi_1(z_x)\psi_2(z_y)}$. We write the Monge-Ampere equation of the dual surface respect to the given surface using a dual mapping (5) of the isotropic space, that is:

$$z^*_{x^*x^*} z^*_{y^*y^*} - \left(z^*_{x^*y^*}\right)^2 = \frac{1}{\varphi\left(x^*, y^*\right)} \tag{14}$$

where, the dual mapping is as follows:

$$\begin{cases} x^* = z_x \\ y^* = z_y \\ z^* = x \cdot z_x + y \cdot z_y - z \end{cases} \tag{15}$$

We solve the Monge-Ampere equation by the given formula (14)for transfer surfaces for the case where the total curvature of these surfaces is a product of two separate variable functions. The vector form of the transfer surface is as follows:

$$\overline{r}\left(x^*, y^*\right) = x^* \overline{i} + y^* \overline{j} + \left(f\left(x^*\right) + g\left(y^*\right)\right) \overline{k}$$

If we put it in the formula (14), we obtain the following:

$$f_{x^*x^*} \cdot g_{y^*y^*} = \psi_1\left(x^*\right) \psi_2\left(y^*\right)$$

From this,

$$\frac{f_{x^*x^*}}{\psi_1\left(x^*\right)} = \frac{\psi_2\left(y^*\right)}{g_{y^*y^*}} = \tau$$

$\tau-$ const.

The above equations are second-order differential equations. By solving these differential equations, we recover the given dual transfer surface according to its given total curvature.

1) $\frac{f_{x^*x^*}}{\psi_1(x^*)} = \tau$

$$\frac{f_{x^*x^*}}{\psi_1\left(x^*\right)} = \tau \quad \Rightarrow f_{x^*x^*} = \tau \cdot \psi_1\left(x^*\right) \quad \Rightarrow f_{x^*} = \tau \int \psi_1\left(x^*\right) dx^*$$

Integrating again, we get the following:

$$f\left(x^*\right) = \int \left[\tau \int \psi_1\left(x^*\right) dx^*\right] dx^*$$

2) $\frac{\psi_2(y^*)}{g_{y^*y^*}} = \tau$

$$\frac{\psi_2\left(y^*\right)}{g_{y^*y^*}} = \tau \quad \Rightarrow g_{y^*y^*} = \frac{\psi_2\left(y^*\right)}{\tau} \quad \Rightarrow g_{y^*} = \frac{1}{\tau} \int \psi_2\left(y^*\right) dy^*$$

From this,

$$g\left(y^{*}\right)=\int\left[\frac{1}{\tau}\int\psi_{2}\left(y^{*}\right)dy^{*}\right]dy^{*}$$

From this, we obtain the following equation for the transfer surface:

$$\Phi^{*}:\ z^{*}\left(x^{*},y^{*}\right)=f\left(x^{*}\right)+g\left(y^{*}\right)=\int\left[\tau\int\psi_{1}\left(x^{*}\right)dx^{*}\right]dx^{*}+\int\left[\frac{1}{\tau}\int\psi_{2}\left(y^{*}\right)dy^{*}\right]dy^{*} \quad (16)$$

If we apply the dual mapping (5) for the points of the surface Φ^{*}, we get the following:

$$\begin{cases} x^{**}=z^{*}_{x^{*}} \\ y^{**}=z^{*}_{y^{*}} \\ z^{**}=x^{*}\cdot z^{*}_{x^{*}}+y^{*}\cdot z^{*}_{y^{*}}-z^{*} \end{cases}$$

The parametric equations of the surface Φ^{**} are as follows:

$$\begin{array}{c} x^{**}=\tau\int\psi_{1}\left(x^{*}\right)dx^{*} \\ y^{**}=\frac{1}{\tau}\int\psi_{2}\left(y^{*}\right)dy^{*} \\ z^{**}=x^{*}\cdot\tau\int\psi_{1}\left(x^{*}\right)dx^{*}+y^{*}\cdot\frac{1}{\tau}\int\psi_{2}\left(y^{*}\right)dy^{*}-\int\left[\tau\int\psi_{1}\left(x^{*}\right)dx^{*}\right]dx^{*}-\int\left[\frac{1}{\tau}\int\psi_{2}\left(y^{*}\right)dy^{*}\right]dy^{*} \end{array} \quad (17)$$

Finding the following expressions from the first and second equalities of the system (17) above,

$$\begin{array}{c} x^{*}=\psi_{1}^{-1}\left(\frac{dx^{**}}{\tau}\right) \\ y^{*}=\psi_{2}^{-1}\left(\tau dy^{**}\right) \end{array}$$

if we put it in the third equation, we get the following equation of the surface Φ^{**}, i.e:

$$z^{**}=x^{**}\cdot\psi_{1}^{-1}\left(\frac{dx^{**}}{\tau}\right)+y^{**}\cdot\psi_{2}^{-1}\left(\tau dy^{**}\right)-\int x^{**}d\left(\psi_{1}^{-1}\left(\frac{dx^{**}}{\tau}\right)\right)-\int y^{**}d\left(\psi_{2}^{-1}\left(\tau dy^{**}\right)\right) \quad (18)$$

From the Theorem 3 above, the following holds for this surface:

$$\Phi^{**}=\Phi \quad (19)$$

From this, the surface equation Φ is also calculated according to the formula (18) and we get the following for this surface:

$$z(x,y)=x\cdot\psi_{1}^{-1}\left(\frac{dx}{\tau}\right)+y\cdot\psi_{2}^{-1}\left(\tau dy\right)-\int xd\left(\psi_{1}^{-1}\left(\frac{dx}{\tau}\right)\right)-\int yd\left(\psi_{2}^{-1}\left(\tau dy\right)\right)$$

The theorem is completely proved.

Theorem 5 *For the special case of the Monge-Ampere equation*

$$z_{xx}z_{yy}-z_{xy}^{2}=z_{x} \quad (20)$$

there is a solution in the family of transfer surfaces and it is as follows:

$$z\left(x,y\right)=\frac{\mu}{2}y^{2}-C_{2}\cdot\mu y+\mu e^{\frac{x-C_{1}}{\mu}}+C$$

Where, $\mu, C_1, C_2, C-$ *const.*

Proof of the Theorem 5. Let there be given a regular surface Φ and its equation is $z = z(x,y), \quad (x,y) \in D \subset R_2$. Assume that this surface satisfies the special case of the Monge-Ampere equation (20). We find the dual surface for the given surface by dual mapping (15):

$$z^*_{x^*x^*} z^*_{y^*y^*} - \left(z^*_{x^*y^*}\right)^2 = \frac{1}{x^*} \tag{21}$$

The vector form of the transfer surface Φ^* is as follows:

$$\overline{r}\left(x^*, y^*\right) = x^*\overline{i} + y^*\overline{j} + \left(f\left(x^*\right) + g\left(y^*\right)\right)\overline{k}$$

From this, $f_{x^*x^*} \cdot g_{y^*y^*} = \frac{1}{x^*} \quad \Rightarrow f_{x^*x^*} \cdot x^* = \frac{1}{g_{y^*y^*}} = \mu$, $\mu-$const.

1) $f_{x^*x^*} \cdot x^* = \mu$

$$f_{x^*x^*} = \frac{\mu}{x^*} \quad \Rightarrow f_{x^*} = \mu \ln\left|x^*\right| + C_1$$

We obtain the following:

$$f\left(x^*\right) = \mu \cdot x^*\left(\ln\left|x^*\right| - 1\right) + C_1 x^* + C_3$$

2) $\frac{1}{g_{y^*y^*}} = \mu$

$$g_{y^*y^*} = \frac{1}{\mu} \quad \Rightarrow g_{y^*} = \frac{y^*}{\mu} + C_2$$

From this:

$$g\left(y^*\right) = \frac{\left(y^*\right)^2}{2\mu} + C_2 y^* + C_4$$

$$F^*: \; z^*\left(x^*, y^*\right) = f\left(x^*\right) + g\left(y^*\right) = \mu \cdot x^*\left(\ln\left|x^*\right| - 1\right) + C_1 x^* + C_3 + \frac{\left(y^*\right)^2}{2\mu} + C_2 y^* + C_4 =$$

$$= \mu \cdot x^*\left(\ln\left|x^*\right| - 1\right) + C_1 x^* + \frac{\left(y^*\right)^2}{2\mu} + C_2 y^* + C$$

where, $C_3 + C_4 = C-$ const. If we also apply dual mapping for the transfer surface Φ^*, the following is valid:

$$x^{**} = \mu \cdot \ln\left|x^*\right| + C_1 \quad \Rightarrow x^* = e^{\frac{x^{**} - C_1}{\mu}}$$
$$y^{**} = \frac{y^*}{\mu} + C_2 \quad \Rightarrow y^* = \mu\left(y^{**} - C_2\right)$$
$$z^{**} = x^* \cdot x^{**} + y^* \cdot y^{**} - z^*$$

$$z^{**}\left(x^{**}, y^{**}\right) = \mu \cdot e^{\frac{x^{**} - C_1}{\mu}} + \frac{\mu}{2}\left(y^{**}\right)^2 - C_2 \mu y^{**} + \frac{\mu}{2} C_2^2 - C \tag{22}$$

According to the Theorem 3 above:

$$\Phi^{**} = \Phi \quad \Rightarrow z^{**}\left(x^{**}, y^{**}\right) = z\left(x, y\right)$$

From this, there exists a solution to the equation (20) and we get the following by simplifying the constant numbers:

$$z\left(x, y\right) = \frac{\mu}{2} y^2 - C_2 \cdot \mu y + \mu e^{\frac{x - C_1}{\mu}} + C \tag{23}$$

Theorem is proved.

4 Amalgamatic and Casorati curvatures in the isotropic space

For investigating of the theory of surfaces, studying the connection between their total and mean curvatures is important in solving many geometric problems. We know that in surface theory, the problems of recovering surfaces with respect to their total curvature K and mean curvature H were studied in many works [3–5, 9, 10, 12]. In addition these characteristics, studying the $\frac{K}{H}$, $\frac{K}{H^2}$ ratios also reveals some new features of the geometry of surfaces. The original idea can be found in the works of Weingarten [20, 21]. This ratio $\frac{K}{H}$ was later called amalgamatic curvature. The amalgamatic curvature of the surface and the information about it are given by B. Suceava [22]. The aim of studying amalgamatic curvature is to study surfaces by analogue the ratio $\frac{\tau}{k}$ of the torsion to the curvature of curves in higher-dimensional geometric objects.

Now let us define amalgamatic curvature:

Definition 4 *Let $\xi : G \subset R^2 \to R_3^2$ be a surface given by the smooth mapping ξ. Then the amalgamatic curvature at point p is:*

$$A = \frac{2k_1k_2}{k_1 + k_2}$$

To study surfaces through a certain connection between the total curvature K and the mean curvature H, the concept of Casorati curvature is presented in the following works [15, 16, 23, 24]. This curvature was introduced by Feliz Casorati in 1890 and is defined as follows [23]:

$$C = \frac{k_1^2 + k_2^2}{2}$$

In isotropic space, the amalgamatic and Casorati curvatures of a surface are respectively as follows [17]:

$$A = \frac{2k_1k_2}{k_1 + k_2} = \frac{K}{H} \quad C = \frac{k_1^2 + k_2^2}{2} = 2H^2 - K$$

Where, k_1, k_2- are principal curvatures. We know that the mean curvature of the dual surface to the given surface is determined by (8) [10]. The problems of surface recovering using the dual mean curvature are discussed in detail by the authors in the following works [8–10]. As can be seen from the formula for finding the amalgamatic curvature, it is inversely proportional to the dual mean curvature. So, from this, we can conclude that the problem of recovering a surface according to its amalgamatic curvature is equivalent to the problem of recovering the surface according to its dual mean curvature. The solutions to all problems when $H^* = \chi(u, v)$ are also the solutions to the problem of recovering a surface according to amalgamatic curvature. From this, the following theorem holds:

Theorem 6 *The following connection holds for the amalgamatic curvature of a given surface and the mean curvature of the its dual surface:*

$$A = \frac{1}{H^*} \tag{24}$$

Proof of the Theorem 6. We are given a surface F and its dual surface F^*. The amalgamatic curvature of the surface F is as follows:

$$A = \frac{K}{H}$$

For the mean curvature of the surface F^*:

$$H^* = \frac{H}{K}$$

From this, equality (24) follows. The isotropic Casorati curvature of a dual surface to the given surface is equal to:

$$C^* = 2(H^*)^2 - K^* = 2\left(\frac{H}{K}\right)^2 - \frac{1}{K} = \frac{C}{K^2}$$

From this,

$$C^* = \frac{C}{K^2} \tag{25}$$

Thus, this equality shows the connection between the Casorati curvatures of a surface and its dual surface.

5 Recovering surfaces with constant negative curvature according to their curvature invariants

Now we will present some properties of the curvatures of surfaces with a given negative curvature in isotropic space: Let the surface F be given as:

$$r\left(u,v\right) = \left(r_1\left(u,v\right), r_2\left(u,v\right), r_3\left(u,v\right)\right)$$

In this,

$$r_1\left(u,v\right) = f_1\left(u\right) + g_1\left(v\right)$$

$$r_2\left(u,v\right) = f_2\left(u\right) + g_2\left(v\right)$$

Let the condition

$$f_1{}'g_2{}' - f_2{}'g_1{}' \neq 0 \tag{26}$$

be fulfilled and $f_i, g_i \in C^2, \quad i = 1, 2$, also

$$\begin{vmatrix} f_1{}' & f_2{}' & r_{3u} \\ g_1{}' & g_2{}' & r_{3v} \\ f_1{}'' & f_2{}'' & r_{3uu} \end{vmatrix} = 0 \quad \begin{vmatrix} f_1{}' & f_2{}' & r_{3u} \\ g_1{}' & g_2{}' & r_{3v} \\ g_1{}'' & g_2{}'' & r_{3vv} \end{vmatrix} = 0 \tag{27}$$

If conditions (27) are valid, then the parametric curves will be asymptotic. From conditions (26) and (27), the functions $a\left(u,v\right)$ and $b\left(u,v\right)$ are found one-valued through the $r_3\left(u,v\right)$

$$r_{3u} = af_1{}' + bf_2{}' \quad r_{3v} = ag_1{}' + bg_2{}' \quad r_{3uu} = af_1{}'' + bf_2{}'' \quad r_{3vv} = ag_1{}'' + bg_2{}'' \tag{28}$$

From solving equations (28), $a_{uv} = b_{uv} = 0$ is valid. From this, $a = \alpha_1(u) + \beta_1(v)$, $b = \alpha_2(u) + \beta_2(v)$ are found. From the previous equation, for arbitrary functions $\alpha_i, \beta_i, \quad i = 1, 2$, we get:

$$\alpha_1{}' f_1{}' + \alpha_2{}' f_2{}' = 0 \quad \beta_1{}' g_1{}' + \beta_2{}' g_2{}' = 0$$

From this,

$$\alpha_1{}' = \lambda(u) f_2{}' \quad \alpha_2{}' = -\lambda(u) f_1{}' \quad \beta_1{}' = \delta(v) g_2{}' \quad \beta_2{}' = -\delta(v) g_1{}'$$

Then,

$$r_{3uv} = \delta(v)(g_2{}' f_1{}' - g_1{}' f_2{}') \qquad r_{3vu} = \lambda(u)(g_1{}' f_2{}' - g_2{}' f_1{}')$$

If, we get $\lambda(u) = -\delta(v) = const. = \kappa_1$

$$a = \kappa_1(f_2 - g_2) + \kappa_2, \quad b = -\kappa_1(f_1 - g_1) + \kappa_3$$

As a result, from these expressions

$$r_{3u} = \kappa_1(f_1{}' f_2 - f_1 f_2{}') + \kappa_1(f_2{}' g_1 - f_1{}' g_2) + \kappa_2 f_1{}' + \kappa_3 f_2{}'$$

$$r_{3v} = \kappa_1(g_2{}' g_1 - g_2 g_1{}') + \kappa_1(g_1{}' f_2 - g_2{}' f_1) + \kappa_2 g_1{}' + \kappa_3 g_2{}'$$

$r_{3uv} = r_{3vu}$ is valid. By integrating, we obtain:

$$r_3(u,v) = \kappa_1 \left\{ (f_2{}' g_1 - f_1{}' g_2) + \int (f_1{}' f_2 - f_1 f_2{}')\, du + \int (g_2{}' g_1 - g_2 g_1{}')\, dv \right\} + \kappa_2(f_1 + g_1) + \kappa_3(f_2 + g_2) + \kappa_4$$

So, if $\kappa_1 = 1, \ \kappa_2 = \kappa_3 = \kappa_4 = 0$, the surface equation is:

$$r(u,v) = \left(f_1 + g_1, f_2 + g_2, (f_2 g_1 - f_1 g_2) + \int (f_1{}' f_2 - f_1 f_2{}')\, du + \int (g_2{}' g_1 - g_2 g_1{}')\, dv \right) \quad (29)$$

In equation (29),isotropic total and mean curvatures of the surface are as follows [17]:

$$K = -1 \quad H = \frac{f_1{}' g_1{}' + f_2{}' g_2{}'}{f_2{}' g_1{}' - f_1{}' g_2{}'} \quad (30)$$

Let the following conditions be satisfied in equality (29), that is:

$$f_1 = u \quad f_2 = f' \quad g_1 = v \quad g_2 = g' \quad (31)$$

In this case, the equation of the surface is:

$$r(u,v) = (u + v, f' + g', 2(f - g) + (v - u)(f' + g')) \quad (32)$$

By calculating the fundamental forms of this surface, the isotropic total, mean, Casorati, and amalgamatic curvatures are as follows:

$$K = -1 \quad H = \frac{1 + f'' g''}{f'' - g''} \quad C = 1 + \frac{2(1 + f'' g'')^2}{(f'' - g'')^2} \quad A = \frac{g'' - f''}{1 + f'' g''}$$

From here, for the total and mean curvatures of the dual surface:

$$K^* = -1 \quad H^* = \frac{1 + f'' g''}{g'' - f''}$$

Now, let us consider the problem of recovering surfaces with total curvature -1 according to their different curvature invariants:

5.1 The problem of recovering the surface by the mean curvature

If $H = \phi_1(u)$ or $H = \phi_2(v)$ is an arbitrary continuously differentiable function, we consider the problem of recovering of the surface given by formula (32) by the mean curvature. In this case, by simplifying the equation $\frac{1+f''g''}{f''-g''} = \phi_1(u)$, we get:

$$\frac{f''\phi_1(u) - 1}{f'' + \phi_1(u)} = g'' = \eta \quad \eta = const$$

We get two ordinary differential equations with separate variables. Solving these equations, we find the following:

$$\begin{cases} f(u) = \int \left[\int \frac{1+\eta\phi_1(u)}{\phi_1(u)-\eta} du\right] du + c_0 u + c_1 \\ g(v) = \frac{\eta v^2}{2} + d_0 v + d_1 \end{cases} \tag{33}$$

Even if $H = \phi_2(v)$, by using the same method we obtain the following expressions:

$$\begin{cases} f(u) = \frac{\eta u^2}{2} + c_0 u + c_1 \\ g(v) = \int \left[\int \frac{\eta\phi_2(v)-1}{\phi_2(v)+\eta} dv\right] dv + d_0 v + d_1 \end{cases} \tag{34}$$

Theorem 7 *If the parameterization of the surface F is defined by formula (32) and the mean curvature is given by $H = \phi_1(u)$ or $H = \phi_2(v)$ arbitrary continuous differentiable functions, then the functions $f(u)$ and $g(v)$ are found by expressions (33) and (34), respectively.*

5.2 Problems of surface recovering from amalgamatic curvature and dual mean curvature

The amalgamatic curvature of a surface F with total curvature -1 is:

$$A = -\frac{1}{H} \tag{35}$$

and for the mean curvature of the dual surface F^* is:

$$H^* = \frac{1}{A} = -H \tag{36}$$

Therefore, it can be concluded that if the mean curvature of the surface F is given by the arbitrary continuous differentiable functions in Theorem 7, then from formulas (35) and (36) we will have solved the problems of recovering surface by the amalgamatic curvature and by the mean curvature of the its dual surface.

5.3 The problem of recovering the surface according to the Casorati curvature

For the Casorati curvature of the surface given by (32),

$$C = 1 + \frac{2(1 + f''g'')^2}{(f'' - g'')^2} \tag{37}$$

is valid. If the Casorati curvature is given by positive continuous differentiable functions $C = \theta_1(u)$ or $C = \theta_2(v)$, then for the functions $f(u)$ and $g(v)$ of the surface given by equation (32), we find the following equalities: In the case $C = \theta_1(u)$, $\theta_1(u) > 1$

$$\begin{cases} f(u) = \int \left[\int \frac{1+\eta\sqrt{\frac{\theta_1(u)-1}{2}}}{\sqrt{\frac{\theta_1(u)-1}{2}}-\eta} du \right] du + c_0 u + c_1 \\ g(v) = \frac{\eta v^2}{2} + d_0 v + d_1 \end{cases}$$

or $C = \theta_2(v)$, $\theta_2(v) > 1$:

$$\begin{cases} f(u) = \frac{\eta u^2}{2} + c_0 u + c_1 \\ g(v) = \int \left[\int \frac{\eta\sqrt{\frac{\theta_2(v)-1}{2}}-1}{\sqrt{\frac{\theta_2(v)-1}{2}}+\eta} dv \right] dv + d_0 v + d_1 \end{cases}$$

Corollary 1 *The following equality holds for the Casorati curvatures of a surface with total curvature -1 and its dual surface:*

$$C^* = C$$

Because, from equality (25), the problem of reconstructing surfaces with total curvature -1 according to the Casorati curvature is equivalent to the problem of recovering its dual surface by this curvature.

6 Conclusion

In this paper, in the first part of the main results, the application of the dual mapping of isotropic space to the theory of surfaces makes it possible to solve the Monge-Ampere equation in a special case. We know that this equation has applications in various fields. Namely, in the theory of surfaces in differential geometry, the recovering of a surface according to its total curvature coincides with the solution of this equation. Moreover, the problems of recovering surfaces according to other curvature invariants are also important in the study of surfaces. Therefore, in the second part of the results obtained in this work, problems of surface recovering from curvature invariants are considered. In addition to mathematical problems, in physics the connection between the Hamiltonian and Lagrange functions is studied using the dual mapping mentioned above. Putting the energy to the Lagrangian function can be used to solve extremal problems [25].

The problems solved in the article are a generalization of the problems considered in the works of M.E.Aydin [3], M. S. Lone and M. K. Karasen [9], A. Artikbaev and Sh. Ismoilov [10].

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Received: May 2, 2025
Accepted: June 8, 2025

IRSTI 27.39.21 DOI: https://doi.org/10.26577/JMMCS20251262010

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ON A SUBSET OF BAZILEVIČ FUNCTIONS IDENTIFIED BY THE THREE-LEAF FUNCTION, MILLER-ROSS FUNCTION, AND MULTIPLIER OPERATORS

A significant portion of the collection of analytic-univalent functions of the type

$$h(\zeta) = \zeta + \sum_{n=rm+1}^{\infty} a_n \zeta^n$$

whose definition is found in the unit disk

$$\Omega := \{z : |z| < 1\},$$

is investigated in this work. Several subsets of the well-known set of Bazilevič functions are included in this new set. The new set and its findings are developed using the Miller-Ross function, the Schwarz function, some multiplier operators, and some mathematical ideas such as subordination, set theory, infinite series generation, and convolution of some geometric expressions. Among the main achievements are the estimates for the coefficient bounds and the Fekete-Szegö functional. Generally speaking, the new set reduces to a number of known subsets with some supposedly unique results when some parameters are altered inside their declaration intervals.

Key words: analytic function, Miller-Ross function, Schwarz function, Bazilevič function, multiplier operator, three-leaf-type function.

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Үш деңгейлі функция, Миллер-Росс функциясы және көбейткіш операторлары арқылы анықталған Базилевич функцияларының ішкі жиынында

Типтің аналитикалық бір мәнді функцияларын жинаудың маңызды бөлігі

$$h(\zeta) = \zeta + \sum_{n=rm+1}^{\infty} a_n \zeta^n$$

оның анықтамасы

$$\Omega := \{z : |z| < 1\},$$

бірлік дискісінде осы жұмыста зерттеледі. Базилевич функцияларының белгілі жиынының бірнеше ішкі жиындары осы жаңа жиынға енгізілген. Жаңа жиын және оның нәтижелері Миллер-Росс функциясын, Шварц функциясын, кейбір көбейткіш операторларды және бағыну, жиын теориясы, шексіз қатарларды генерациялау және кейбір геометриялық өрнектерді айналдыру сияқты кейбір математикалық идеяларды пайдалана отырып әзірленді. Негізгі жетістіктердің қатарында коэффициенттер мен Фекете-Сего функционалдың байланыстырылған бағалаулар бар.

Жалпы айтқанда, жаңа жиын кейбір параметрлер жариялау аралықтарында әзгерген кезде кейбір болжамды бірегей нәтижелері бар белгілі ішкі жиындар санына дейін азаяды.
Түйін сөздер: аналитикалық функция, Миллер-Росс функциясы, Шварц функциясы, Басилевич функциясы, көбейту операторы, үш валентті функция, қосылу моделі.

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О подмножестве функций Базилевича, идентифицируемых трехлестной функцией, функцией Миллера-Росса и операторами множителей

Значительная часть коллекции аналитически-однозначных функций типа

$$h(\zeta) = \zeta + \sum_{n=rm+1}^{\infty} a_n \zeta^n$$

определение которых находится в единичном круге

$$\Omega := \{z : |z| < 1\},$$

исследуется в этой работе. Несколько подмножеств известного набора функций Базилевича включены в этот новый набор. Новый набор и его результаты разрабатываются с использованием функции Миллера-Росса, функции Шварца, некоторых операторов множителей и некоторых математических идей, таких как подчинение, теория множеств, генерация бесконечных рядов и свертка некоторых геометрических выражений. Среди основных достижений — оценки границ коэффициентов и функционал Фекете-Сего. Вообще говоря, новый набор сводится к ряду известных подмножеств с некоторыми предположительно уникальными результатами, когда некоторые параметры изменяются внутри их интервалов объявления.
Ключевые слова: аналитическая функция, функция Миллера-Росса, функция Шварца, функция Базилевича, оператор умножения, функция трехлистного типа.

1 Preliminary

In this study, the set of analytic functions of the series type

$$h(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n \quad (\zeta \in \Omega := \{\zeta \in \mathbb{C} : |\zeta| < 1\}). \tag{1}$$

is represented by $\mathcal{A}$. The nature of this function agrees with the fact that $h(0) = 0 = h'(0) - 1$. One of the fundamental principles in geometric function theory is the subordination principle. The principle states that if we have two analytic functions $h(\zeta)$ in (1) and

$$H(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n \quad (\zeta \in \Omega), \tag{2}$$

then h is subordinate to H (usually expressed in notational $h \prec H$) if there is another analytic function

$$d(\zeta) = \sum_{n=1}^{\infty} d_n \zeta^n \in \varDelta \quad (\zeta \in \Omega) \tag{3}$$

such that $|d(\zeta)| < 1$, $d(0) = 0$, and

$$h(\zeta) = H(d(\zeta)) \quad (\zeta \in \Omega).$$

Suppose H is also univalent in Ω, then the definition improves to say that

$$h \prec H \iff h(0) = H(0) \quad \text{and} \quad h(\Omega) \subset H(\Omega).$$

The Hadamard product (or convolution) of two functions h in (1) and H in (2) is the third analytic function declared as

$$(h \star H)(\zeta) = (H \star h)(\zeta) = z + \sum_{n=2}^{\infty} (a_n \times b_n)\zeta^n \quad (\zeta \in \Omega).$$

1.1 Bazilevič Functions

An analytic functions of the integral type

$$b(z) = \left\{ (\eta + i\gamma) \int_0^z \rho(\tau) s(\tau)^{\gamma} \tau^{-(1-i\gamma)} d\tau \right\}^{\frac{1}{\eta+i\gamma}} \tag{4}$$

such that $\eta > 0$, γ have real value, s is a starlike function, and $\rho \in \wp$ are called Bazilevič [2] functions. This set was shown to be the 'largest' subset of the set of univalent functions that is currently known. Numerous scholars have examined different properties of the subsets of the set of Bazilevič functions by varying the parameters in (4); for instance, see [8, 9, 17, 18].

Indeed, an important subset of analytic functions are Bazilevič functions. These functions have been thoroughly examined in a large body of research and are distinguished by their geometric features. Olukoya and Oyekan's work in [12] is noteworthy since it offers some polynomial bounds for functions in the set of modified hyperbolic tangent functions. The behavior of the various analytic functions in certain subsets of the Bazilevič functions was better understood as a result of these findings.

Another related study involved some results on Chebyshev polynomial bounds for sets of analytic-univalent functions, presented in [14]. The work extends the understanding of the geometric properties of Bazilevič functions, shedding more light on their analytical characteristics. Additionally, Oyekan and Awolere [15] explored the polynomial bounds for bi-univalent functions associated with the probability of generalized distribution defined by generalized polylogarithms via Chebyshev polynomial.

Gandhi [3] presented the analytic function

$$3\ell(\zeta) = 1 + \frac{4}{5}\zeta + \frac{1}{5}\zeta^4 \tag{5}$$

in 2020 called a '*three-leaf-type function*' and studied the analytical properties of a certain set of starlike functions defined by the conditions

$$z\frac{h'(z)}{h(z)} \prec 3\ell(z) \quad (\zeta \in \Omega)$$

so that by using (3) in (5), we get

$$3\ell(d(\zeta)) = 1 + \frac{4}{5}d(\zeta) + \frac{1}{5}(d(\zeta))^4$$

to give

$$3\ell(d(\zeta)) = 1 + \frac{2}{5}d_1\zeta + \left(\frac{2}{5}d_2 - \frac{1}{5}d_1^2\right)\zeta^2 + \cdots . \tag{6}$$

1.2 Some Analytic Functions and Operators

In 1993, Miller and Ross [11, p. 88] introduced the special function

$$E_{c,v}(\zeta) = \sum_{n=0}^{\infty} \frac{c^n}{\Gamma(n+v+1)}\zeta^{n+v} \quad (c, v, \zeta \in \mathbb{C}).$$

This special function is famously called Miller-Ross function. The Miller-Rose function is a generalization of many special functions, see [6]. The work of Eker and Ece [5, Eq. 2] introduced the normalized form of $E_{c,v}$ defined by

$$\widetilde{E}_{c,v}(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{c^{n-1}\Gamma(v+1)}{\Gamma(v+n)}\zeta^n \quad (v > -1;\ c, \zeta \in \Omega). \tag{7}$$

For $h \in \mathcal{A}$ of type (1), the multiplier operator $I^{\delta}_{\eta_1,\eta_2}$ that maps $\mathcal{A}$ to $\mathcal{A}$ and introduced by Hameed *et al.* [7] was defined on h as

$$I^{\delta}_{\varsigma_1,\varsigma_2}h(\zeta) = \zeta + \sum_{n=2}^{\infty} \left(\frac{1 + (\varsigma_1 + \varsigma_2)(l-1)}{1 + \varsigma_2(l-1)}\right)^{\delta} a_n\zeta^n \quad (\zeta \in \Omega) \tag{8}$$

for the parameters: $\delta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $l \geqq 1$, and $0 \leqq \varsigma_1 \leqq \varsigma_2$. More so, Oyekan [13] investigated the operator $\mathcal{D}^{\delta}$ that maps $\mathcal{A}$ to $\mathcal{A}$ by

$$\mathcal{D}^{\delta}h(\zeta) = \zeta + \sum_{n=2}^{\infty} \mathscr{L}_{\delta}^{\delta+n-1} a_n\zeta^n \quad (\zeta \in \Omega) \tag{9}$$

where

$$\mathscr{L}_{\delta}^{\delta+n-1} = \binom{\delta+n-1}{\delta} = \frac{(\delta+1)(\delta+2)\cdots(\delta+n-1)}{(n-1)!} = \frac{(\delta+1)_{n-1}}{(1)_{n-1}}. \tag{10}$$

In 2023, Oyekan [13] modified function $h \in \mathcal{A}$ and showcased the analytic function $\mathfrak{h} \in \mathcal{A}(r, m)$ as

$$\mathfrak{h}(\zeta) = \zeta + \sum_{n=rm+1}^{\infty} a_n\zeta^n \quad (\zeta \in \Omega) \tag{11}$$

where r is fixed and $r, m \in \mathbb{N}$. Observe that if $r = 1 = m$ in (11), then $\mathfrak{h} = h$ in (1) and if $r = 1$ (or $m = 1$), then we will have the function studied by Hameed *et al.* [7]. Now in the likes of (11), we have (7) expressed in the form

$$\mathcal{E}_{c,v}(\zeta) = \zeta + \sum_{n=rm+1}^{\infty} \frac{c^{n-1}\Gamma(v+1)}{\Gamma(v+n)} \zeta^n,$$

the multiplier operator (8) defined as

$$\mathcal{I}^{\delta}_{\varsigma_1,\varsigma_2}\mathfrak{h}(\zeta) = \zeta + \sum_{n=rm+1}^{\infty} \left(\frac{1 + (\varsigma_1 + \varsigma_2)(l-1)}{1 + \varsigma_2(l-1)} \right)^{\delta} a_n \zeta^n \quad (\zeta \in \Omega).$$

and the multiplier operator (9) modified as

$$\mathcal{D}^{\delta}\mathfrak{h}(\zeta) = \zeta + \sum_{n=rm+1}^{\infty} \mathscr{L}_{\delta}^{\delta+n-1} a_n \zeta^n \quad (\zeta \in \Omega)$$

where $\mathscr{L}_{\delta}^{\delta+n-1}$ is as defined by (10). Likewise, for $\mathfrak{h}$ of the series type (11), Hameed *et al.* [7] studied the operator $\mathcal{R}^{\delta}_{\varsigma_1,\varsigma_2}$ that maps $\mathcal{A}(r,m)$ to $\mathcal{A}(r,m)$ and defined by

$$\mathcal{R}^{\delta}_{\varsigma_1,\varsigma_2}\mathfrak{h}(\zeta) = \mathcal{I}^{\delta}_{\varsigma_1,\varsigma_2}\mathfrak{h}(\zeta) \star \mathcal{D}^{\delta}\mathfrak{h}(\zeta) = \zeta + \sum_{n=rm+1}^{\infty} \mathscr{L}_{\delta}^{\delta+n-1} \left(\frac{1 + (\varsigma_1 + \varsigma_2)(l-1)}{1 + \varsigma_2(l-1)} \right)^{\delta} a_n \zeta^n \quad (\zeta \in \Omega).$$

Now, we declare the analytic function

$$\begin{aligned} \mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta) &= \mathcal{E}_{c,v}(\zeta) \star \mathcal{R}^{\delta}_{\varsigma_1,\varsigma_2}\mathfrak{h}(\zeta) \\ &= \zeta + \sum_{n=rm+1}^{\infty} \frac{c^{n-1}\Gamma(v+1)}{\Gamma(v+n)} \mathscr{L}_{\delta}^{\delta+n-1} \left(\frac{1 + (\varsigma_1 + \varsigma_2)(l-1)}{1 + \varsigma_2(l-1)} \right)^{\delta} a_n \zeta^n \quad (\zeta \in \Omega) \end{aligned} \tag{12}$$

where all parameters are as aforementioned.

2 A Set of Lemmas

Let the function d be as defined in (3), then the following lemmas hold for the main results.

Lemma 2.1 ([4]) *Let $d \in \Delta$, then $|d_n| \leqq 1, \ \forall n \in \mathbb{N}$. Equality occurs for functions $d(\zeta) = e^{i\vartheta}\zeta^n$ $(\vartheta \in [0, 2\pi))$.*

Lemma 2.2 ([1]) *Let $d \in \Delta$, then for complex number ξ,*

$$|d_2 - \xi d_1^2| \leqq \max\{1; \ |\xi|\}.$$

Equality holds for functions $d(z) = \zeta$ or $d(z) = \zeta^2$.

3 Main Results

3.1 A Novel Class of Analytic Functions

Definition 3.1 *A function $\mathfrak{h} \in \mathcal{A}(r,m)$ of the form (11) belongs to the set $\nabla^{\delta,c,v}_{\sigma,\varsigma_1,\varsigma_2}(3\ell)$, if it satisfies the subordination condition*

$$\frac{\left(\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)\right)' \left(\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)\right)^{\sigma-1}}{\zeta^{\sigma-1}} \prec 3\ell(\zeta) \tag{13}$$

where we declare the parameters: $c,\zeta \in \Omega$, $\delta \in \mathbb{N}_0$, $v > -1$, $0 \leqq \varsigma_1 \leqq \varsigma_2$, and $0 \leqq \sigma \leqq 1$, for functions $\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)$ and $3\ell(\zeta)$ defined in (12) and (5), respectively.

This study aims to explore and analyze a subset of Bazilevič functions characterized by the Miller-Ross function and a certain multiplier operator in the space of a three-leaf function, a nuanced area in geometric function theory. Some achieved results are the upper estimates for $|a_{m+1}|$, $|a_{2m+1}|$, and $|a_{2m+1} - \xi a^2_{m+1}|$ functionals; see [9, 10, 12–14, 16] for some details on bounds. Several (presumably) new results are reported as corollaries and remarks.

3.2 Coefficient Estimates

Theorem 3.2 *Let $\mathfrak{h} \in \mathcal{A}(r,m)$ belong to the set $\nabla^{\delta,c,v}_{\sigma,\varsigma_1,\varsigma_2}(3\ell)$. Then*

$$|a_{m+1}| \leqq \frac{2}{5(\sigma+m)\frac{c^m\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}^{\delta+m}_{\delta}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}} \tag{14}$$

and

$$\begin{aligned}|a_{2m+1}| \leqq &\frac{1}{5(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}^{\delta+2m}_{\delta}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}\\ &+\frac{4}{25(\sigma+m)(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}^{\delta+2m}_{\delta}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}\\ &+\frac{4\sigma(m+1)}{25(\sigma+m)^2(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}^{\delta+2m}_{\delta}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}\\ &+\frac{2\sigma(\sigma-1)}{25(\sigma+m)^2(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}^{\delta+m}_{\delta}\mathscr{L}^{\delta+2m}_{\delta}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}.\end{aligned} \tag{15}$$

Proof. Since $\mathfrak{h} \in \nabla^{\delta,c,v}_{\sigma,\varsigma_1,\varsigma_2}(3\ell)$, then by the principle of subordination, we can express (13) such that

$$\frac{\left(\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)\right)' \left(\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)\right)^{\sigma-1}}{\zeta^{\sigma-1}} = 1 + \frac{4}{5}\omega(\zeta) + \frac{1}{5}(\omega(\zeta))^4$$

or in a simplified form

$$\frac{\zeta\left(\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)\right)'\left(\frac{\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)}{\zeta}\right)^{\sigma}}{\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)}=1+\frac{4}{5}\omega(\zeta)+\frac{1}{5}(\omega(\zeta))^4$$

and

$$\zeta\left(\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)\right)'\left(\frac{\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)}{\zeta}\right)^{\sigma}=\left(1+\frac{4}{5}\omega(\zeta)+\frac{1}{5}(\omega(\zeta))^4\right)\left(\mathcal{J}^{\delta,c,v}_{\varsigma_1,\varsigma_2}(\zeta)\right). \tag{16}$$

Putting (3) and (12) into (16) with some simplifications yields

$$\begin{aligned}&\left(\zeta+(rm+1)\frac{c^{rm}\Gamma(v+1)}{\Gamma(v+rm+1)}\mathscr{L}_{\delta}^{\delta+rm}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{rm+1}\zeta^{rm+1}\right)\\&\quad\times\left\{1+\sigma\frac{c^{rm}\Gamma(v+1)}{\Gamma(v+rm+1)}\mathscr{L}_{\delta}^{\delta+rm}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{rm+1}\zeta^{rm}\right.\\&\quad\left.+\frac{\sigma(\sigma-1)}{2}\frac{c^{2rm}\left[\Gamma(v+1)\right]^2}{\left[\Gamma(v+rm+1)\right]^2}\mathscr{L}_{2\delta}^{\delta+rm}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{2\delta}b_{rm+1}^2\zeta^{2rm}+\cdots\right\}\\&=\left(1+\frac{2}{5}d_1\zeta+[\frac{2}{5}d_2-\frac{1}{5}d_1^2]\zeta^2+\cdots\right)\\&\quad\times\left(\zeta+\frac{c^{rm}\Gamma(v+1)}{\Gamma(v+rm+1)}\mathscr{L}_{\delta}^{\delta+rm}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{rm+1}\zeta^{rm+1}\right).\end{aligned}$$

Now, for $r\in\{1,2,3,\ldots\}$ we have a simplified series

$$\begin{aligned}&\zeta+(\sigma+m+1)\frac{c^{m}\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}_{\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{m+1}\zeta^{m+1}\\&\quad+\left[\sigma(m+1)\frac{c^{2m}\left[\Gamma(v+1)\right]^2}{\left[\Gamma(v+m+1)\right]^2}\mathscr{L}_{2\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{2\delta}b_{m+1}^2\right.\\&\quad+\frac{\sigma(\sigma-1)}{2}\frac{c^{2m}\left[\Gamma(v+1)\right]^2}{\left[\Gamma(v+m+1)\right]^2}\mathscr{L}_{2\phi}^{\phi+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{2\delta}b_{m+1}^2\\&\quad\left.+(\sigma+2m+1)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{2m+1}\right]\zeta^{2m+1}+\cdots\\&=\zeta+\frac{2}{5}d_1\zeta^2+\frac{c^{m}\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}_{\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{m+1}\zeta^{m+1}\\&\quad+\left(\frac{2}{5}d_2-\frac{1}{5}d_1^2\right)\zeta^3+\frac{2}{5}d_1\frac{c^{m}\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}_{\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{m+1}\zeta^{m+2}\\&\quad+\cdots+\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{2m+1}\zeta^{2m+1}+\cdots.\end{aligned} \tag{17}$$

Clearly, taking the first corresponding coefficients in (17) shows that

$$(\sigma+m+1)\frac{c^m\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}_{\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)\,(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{m+1}$$
$$=\frac{2}{5}d_1+\frac{c^m\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}_{\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)\,(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{m+1}$$

so that

$$a_{m+1}=\frac{2d_1}{5(\sigma+m)\frac{c^m\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}_{\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}} \tag{18}$$

and

$$|a_{m+1}|\leqq\frac{2|d_1|}{5(\sigma+m)\frac{c^m\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}_{\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$

so that applying Lemma 2.1 gives the result in (14). Next, taking the second corresponding coefficients in (17) shows that

$$\sigma(m+1)\frac{c^{2m}\left[\Gamma(v+1)\right]^2}{\left[\Gamma(v+m+1)\right]^2}\mathscr{L}_{2\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)\,(l-1)}{1+\varsigma_2(l-1)}\right)^{2\delta}b_{m+1}^2$$
$$+\frac{\sigma(\sigma-1)}{2}\frac{c^{2m}\left[\Gamma(v+1)\right]^2}{\left[\Gamma(v+m+1)\right]^2}\mathscr{L}_{2\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)\,(l-1)}{1+\varsigma_2(l-1)}\right)^{2\delta}b_{m+1}^2$$
$$+(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)\,(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{2m+1}+\cdots$$
$$=\left(\frac{2}{5}d_2-\frac{1}{5}d_1^2\right)+\frac{2}{5}d_1\frac{c^m\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}_{\phi}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)\,(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}a_{m+1}+\cdots$$

so that

$$a_{2m+1}=\frac{\frac{2}{5}d_2-\frac{1}{5}d_1^2}{(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\phi+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$+\frac{4d_1^2}{25(\sigma+m)(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$+\frac{4\sigma(m+1)d_1^2}{25(\sigma+m)^2(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$-\frac{2\sigma(\sigma-1)d_1^2}{25(\sigma+m)^2(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+m}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}} \tag{19}$$

and

$$|a_{2m+1}| \leqq \frac{\frac{2}{5}|d_2 - \frac{1}{2}d_1^2|}{(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\phi+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$+\frac{4|d_1|^2}{25(\sigma+m)(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$+\frac{4\sigma(m+1)|d_1|^2}{25(\sigma+m)^2(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$-\frac{2\sigma(\sigma-1)|d_1|^2}{25(\sigma+m)^2(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\phi+m}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$

so that applying Lemmas 2.1 and 2.2 gives the result in (15).

Theorem 3.3 *If $\mathfrak{h} \in \nabla_{\sigma,\varsigma_1,\varsigma_2}^{\delta,c,v}(3\ell)$, then for a complex value ξ,*

$$|a_{2m+1} - \xi a_{m+1}^2| \leqq \frac{2}{5(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}} \times \max\{1,\chi\}$$

where

$$\chi = \left|\frac{2}{5(\sigma+m)} - \frac{2\sigma(m+1)}{5(\sigma+m)^2}\frac{\sigma(\sigma-1)}{5(\sigma+m)^2\mathscr{L}^{\phi+m}(\delta)} - \frac{1}{2}\right.$$
$$\left.-\xi\frac{2(\sigma+2m)\Gamma(v+2m+1)\mathscr{L}_{\delta}^{\delta+2m}}{5(\sigma+m)^2\frac{\Gamma(v+1)}{[\Gamma(v+m+1)]^2}\mathscr{L}_{2\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}\right|.$$

Proof. Using (18) and (19) means

$$a_{2m+1} - \xi a_{m+1}^2 = \frac{2d_2 - d_1^2}{5(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$+\frac{4d_1^2}{25(\sigma+m)(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$-\frac{4\sigma(m+1)d_1^2}{25(\sigma+m)^2(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$-\frac{2\sigma(\sigma-1)d_1^2}{25(\sigma+m)^2(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_{\delta}^{\delta+m}\mathscr{L}_{\delta}^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}$$
$$-\xi\left(\frac{2d_1}{5(\sigma+m)\frac{c^{m}\Gamma(v+1)}{\Gamma(v+m+1)}\mathscr{L}_{\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^{\delta}}\right)^2$$

where further simplification gives

$$a_{2m+1}-\xi a_{m+1}^2=\frac{2}{5(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_\delta^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta}$$
$$\times\left\{d_2+\left[\frac{2}{5(\sigma+m)}-\frac{2\sigma(m+1)}{5(\sigma+m)^2}-\frac{\sigma(\sigma-1)}{5(\sigma+m)^2\mathscr{L}_\delta^{\delta+m}}-\frac{1}{2}\right.\right.$$
$$\left.\left.-\xi\frac{2(\sigma+2m)\Gamma(v+2m+1)\mathscr{L}_\delta^{\delta+2m}}{5(\sigma+m)^2\frac{\Gamma(v+1)}{[\Gamma(v+m+1)]^2}\mathscr{L}_{2\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta}\right]d_1^2\right\}.$$

Therefore,

$$\left|a_{2m+1}-\xi a_{m+1}^2\right|\leqq\frac{2}{5(\sigma+2m)\frac{c^{2m}\Gamma(v+1)}{\Gamma(v+2m+1)}\mathscr{L}_\delta^{\delta+2m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta}$$
$$\times\left|d_2+\left[\frac{2}{5(\sigma+m)}-\frac{2\sigma(m+1)}{5(\sigma+m)^2}-\frac{\sigma(\sigma-1)}{5(\sigma+m)^2\mathscr{L}_\delta^{\delta+m}}-\frac{1}{2}\right.\right.$$
$$\left.\left.-\xi\frac{2(\sigma+2m)\Gamma(v+2m+1)\mathscr{L}_\delta^{\delta+2m}}{5(\sigma+m)^2\frac{\Gamma(v+1)}{[\Gamma(v+m+1)]^2}\mathscr{L}_{2\delta}^{\delta+m}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta}\right]d_1^2\right|.$$

so that applying Lemma 2.1 gives the result in the theorem.

Putting $m=1$ in Theorems 3.2 and 3.3 gives the following results.

Corollary 3.4 *If h given by (1) belongs to the set $\nabla_{\sigma,\varsigma_1,\varsigma_2}^{\delta,c,v}(3\ell)$, then*

$$|a_2|\leqq\frac{2}{5(\sigma+1)\frac{\Gamma(v+1)}{\Gamma(v+2)}\mathscr{L}_\delta^{\delta+1}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta},$$
$$|a_3|\leqq\frac{1}{5(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\delta+2}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta}$$
$$+\frac{4}{25(\sigma+1)(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\delta+2}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta}$$
$$+\frac{8\sigma}{25(\sigma+1)^2(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\delta+2}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta}$$
$$+\frac{2\sigma(\sigma-1)}{25(\sigma+1)^2(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\delta+1}\mathscr{L}_\delta^{\delta+2}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta},$$

and

$$|a_3-\xi a_2^2|\leqq\frac{2}{5(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\delta+2}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta}\times\max\{1,\chi\}$$

where

$$\chi = \left| \frac{2}{5(\sigma+1)} - \frac{4\sigma}{5(\sigma+1)^2} - \frac{\sigma(\sigma-1)}{5(\sigma+m)^2\mathscr{L}_\delta^{\delta+1}} - \frac{1}{2} \right.$$
$$\left. - \xi \frac{2(\sigma+2)\Gamma(v+3)\mathscr{L}_\delta^{\delta+2}}{5(\sigma+1)^2\frac{\Gamma(v+1)}{[\Gamma(v+2)]^2}\mathscr{L}_{2\delta}^{\delta+1}\left(\frac{1+(\varsigma_1+\varsigma_2)(l-1)}{1+\varsigma_2(l-1)}\right)^\delta} \right|.$$

Remark 3.5 *Corollary 3.4 presumably holds new results.*

Putting $\varsigma_1 = 0 = \varsigma_2$ in Corollary 3.4 gives the following results.

Corollary 3.6 *If h given by (1) belongs to the set $\nabla_{\sigma,0,0}^{\delta,c,v}(3\ell)$, then*

$$|a_2| \leqq \frac{2}{5(\sigma+1)\frac{\Gamma(v+1)}{\Gamma(v+2)}\mathscr{L}_\delta^{\delta+1}},$$
$$|a_3| \leqq \frac{1}{5(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\phi+2}} + \frac{4}{25(\sigma+1)(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\phi+2}}$$
$$+ \frac{8\sigma}{25(\sigma+1)^2(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\delta+2}} + \frac{2\sigma(\sigma-1)}{25(\sigma+1)^2(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\delta+1}\mathscr{L}_\delta^{\delta+2}},$$

and

$$\left|a_3 - \xi a_2^2\right| \leqq \frac{2}{5(\sigma+2)\frac{\Gamma(v+1)}{\Gamma(v+3)}\mathscr{L}_\delta^{\delta+2}} \times \max\{1, |\chi|\}$$

where

$$\chi = \frac{2}{5(\sigma+1)} - \frac{4\sigma}{5(\sigma+1)^2} - \frac{\sigma(\sigma-1)}{5(\sigma+1)^2\mathscr{L}_\delta^{\delta+1}} - \frac{1}{2} - \xi\frac{2(\sigma+2)\Gamma(v+3)\mathscr{L}_\delta^{\delta+2}}{5(\sigma+1)^2\frac{\Gamma(v+1)}{[\Gamma(v+2)]^2}\mathscr{L}_{2\delta}^{\delta+1}}.$$

Remark 3.7 *Corollary 3.6 presumably holds new results.*

Putting $v = 0$ in Corollary 3.6 gives the following results.

Corollary 3.8 *If h given by (1) belongs to the set $\nabla_{\sigma,0,0}^{1,1,\delta}(3\ell)$, then*

$$|a_2| \leqq \frac{2}{5(\sigma+1)\mathscr{L}_\delta^{\delta+1}},$$
$$|a_3| \leqq \frac{2}{5(\sigma+2)\mathscr{L}_\delta^{\delta+2}} + \frac{8}{25(\sigma+1)(\sigma+2)\mathscr{L}_\delta^{\delta+2}}$$
$$+ \frac{16\sigma}{25(\sigma+1)^2(\sigma+2)\mathscr{L}_\delta^{\delta+2}} + \frac{4\sigma(\sigma-1)}{25(\sigma+1)^2(\sigma+2)\mathscr{L}_\delta^{\delta+1}\mathscr{L}_\delta^{\delta+2}},$$

and

$$|a_3 - \xi a_2^2| \leqq \frac{4}{5(\sigma+2)\mathscr{L}_\delta^{\delta+2}} \times \max\{1, |\chi|\}$$

where

$$\chi = \frac{2}{5(\sigma+1)} - \frac{4\sigma}{5(\sigma+1)^2} - \frac{\sigma(\sigma-1)}{5(\sigma+1)^2\mathscr{L}_\delta^{\delta+1}} - \frac{1}{2} - \xi\frac{4(\sigma+2)\mathscr{L}_\delta^{\delta+2}}{5(\sigma+1)^2\mathscr{L}_{2\delta}^{\delta+1}},$$

$$\frac{\Gamma(1)}{\Gamma(2)} = 1 \qquad \text{and} \qquad \frac{\Gamma(1)}{\Gamma(3)} = \frac{1}{2}.$$

Remark 3.9 *Corollary 3.8 presumably holds new results.*

Putting $\sigma = 0$ in Corollary 3.8 gives the following results.

Corollary 3.10 *If h given by (1) belongs to the set $\nabla_{0,0,0}^{0,0,\delta}(3\ell)$, then*

$$|a_2| \leqq \frac{2}{5\mathscr{L}_\delta^{\delta+1}}, \ |a_3| \leqq \frac{18}{50\mathscr{L}_\delta^{\delta+2}}, \text{ and } |a_3 - \xi a_2^2| \leqq \frac{2}{5\mathscr{L}_\delta^{\delta+2}} \max\left\{1, \left|\frac{1}{10} + \xi\frac{8\mathscr{L}_\delta^{\delta+2}}{5\mathscr{L}_{2\delta}^{\delta+1}}\right|\right\}.$$

Remark 3.11 *Corollary 3.10 presumably holds new results.*

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Received: March 2, 2025
Accepted: June 7, 2025

IRSTI 27.31.21 DOI: https://doi.org/10.26577/JMMCS20251262011

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AN INVERSE PROBLEM FOR PSEUDOPARABOLIC EQUATION WITH MEMORY TERM AND DAMPING

In this paper, we study the inverse problem of determining, along with solution $u(x,t)$ of a pseudo-parabolic equation with memory (convolution term) and a damping term, also an unknown coefficient $f(t)$ determining the external effect (the free term). In the investigating inverse problem, the overdetermination condition is given in integral form, which represents the average value of a solution tested with some given function over all the domain. By reducing the considering inverse problem to an equivalent nonlocal direct problem. The applicability of the Faedo-Galerkin method to the inverse problem is analyzed. The damping term $\gamma\,|u|^{q-2}\,u$ affects as nonlinear source in the case $\gamma > 0$, and an absorption, if $\gamma < 0$. In all these cases, we establish the conditions on the range of exponent q, the dimension d, and the data of the problem for the global and local in time existence and uniqueness of a weak solution of the studing problem.
Key words: inverse problem, nonlinear pseudoparabolic equation, memory term, solvability.

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Жады бар және сызықты емес мүшелі псевдопараболалық теңдеу үшін кері есеп

Бұл жұмыста жады мүшесі бар (үйірткі түріндегі) және сызықты емес мүшелі псевдопараболалық теңдеудің $u(x,t)$ шешімімен қатар сыртқы әсерді сипаттайтын (бос мүше) қосылғыштың $f(t)$ коэффициентін анықтау кері есебі зерттелінген. Қарастырылып отырған кері есепте қайта анықтау қосымша шарты интегралдық түрде берілген, ал ол өз кезегінде шешімнің орташа мәні туралы ақпарат береді. Берілген кері есепті эквивалентті локалды емес тура есепке келтіру арқылы шешімнің бар болуы Фаэдо-Галеркин әдісімен дәлелденді. Теңдеудегі сызықты емес $\gamma\,|u|^{q-2}\,u$ мүшесі $\gamma > 0$ жағдайда жылу көзі, $\gamma < 0$ жағдайда абсорбция қосылғыш ретінде қатысады. Сондай-ақ, q көрсеткішіне, d кеңістік өлшеміне және бастапқы берілген функцияларға жеткілікті шарттары негізінде кері есептің әлсіз шешімінің локалды және глобалды бар болуы, сонымен қатар әлсіз шешімінің жалғыздығы дәлелденді.
Түйін сөздер: кері есеп, сызықты емес псевдопараболалық теңдеу, интегралдық мүше, шешімділік.

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Обратная задача для псевдопараболического уравнения с памятью и затуханием

В данной работе изучается обратная задача определения наряду с решением $u(x,t)$ псевдопараболического уравнения с памятью (членом свертки) и затухающим членом также неизвестного коэффициента $f(t)$, определяющего внешнее воздействие (свободный член). В исследуемой обратной задаче условие переопределения задается в интегральной форме, представляющей собой среднее значение решения, проверенного с некоторой заданной функцией по всей области. Путем сведения рассматриваемой обратной задачи к эквивалентной нелокальной прямой задаче анализируется применимость метода Фаэдо-Галеркина к обратной задаче. Затухающий член $\gamma\,|u|^{q-2}\,u$ действует как нелинейный источник в случае $\gamma > 0$ и как поглощение, если $\gamma < 0$.

Во всех этих случаях устанавливаются условия на диапазон изменения показателя q, размерность d и данные задачи для глобального и локального по времени существования и единственности слабого решения изучаемой задачи.
Ключевые слова: обратная задача, нелинейное псевдопараболическое уравнение, интегральный член, разрешимость.

1 Introduction

A coefficient inverse problems for differential equations have been called problems in which, together with the solution of the corresponding differential equation, it is also necessary to determine coefficient of the equation itself or the coefficient of the right-hand side (external influence). Such problems naturally arise in the mathematical modeling of physical, biological, etc. processes occurring in environments with previously unknown characteristics, since it is the characteristics of the environment that determine coefficient of the corresponding differential equation. This work devoted to study one of these kind of problem.

The statement of problem. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with smooth boundary $\partial\Omega$, and $Q_T = \{(x,t): \ x \in \Omega, \ 0 < t \leq T\}$ is a cylinder with lateral Γ_T. Let us consider the following inverse problem of finding the pair of functions $(u(x,t), f(t))$, which satisfy the pseudoparabolic equation with memory term and damping

$$u_t - \kappa\Delta u_t - \lambda\Delta u - \int\limits_0^t K(t-s)\Delta u(x,s)ds = \gamma\,|u|^{q-2}\,u + f(t)\cdot g(x,t), \ \text{ in } \ Q_T, \tag{1}$$

the initial condition

$$u(x,0) = u_0(x) \text{ in } \Omega, \tag{2}$$

the boundary condition

$$u(x,t) = 0 \ \text{ on } \ \Gamma_T, \tag{3}$$

and the integral overdetermination condition

$$\int\limits_\Omega u(x,t)\omega(x)dx = h(t), \ \ t \in [0,T]. \tag{4}$$

Here, the coefficient κ, λ are given positive numbers, γ is the coefficient of the damping term might be positive $\gamma > 0$ either negative $\gamma < 0$. The functions $g(x,t), \ u_0(x), \ \omega(x)$ and $h(t)$ are given. The exponent q is given positive number such that

$$1 < q < \infty. \tag{5}$$

Pseudo-parabolic equations can be used to describe various important physical processes, such as hydrodynamics, filtration theory, continuum mechanics, the heat conduction involving two temperature systems, dispersive, viscous flow in materials with memory and so on. One of the examples is furnished by the Kelvin-Voigt (Navier-Stokes-Voigt) equations. We refer the reader to the works [1–7] and the references therein, in which these issues were discussed in detail for the model equation (1).

In the absence of the memory term ($K(t) = 0$), the equation in (1) reduces to the pseudo-parabolic equation with damping. In corresponding equation if the coefficient of the external term $f(t)$ is given, then we will obtain an initial boundary value (IBV) problem. Various IBV problems for nonlinear pseudo-parabolic equation have been extensively studied in [4, 8–14] and results concerning existence and uniqueness of solution, and asymptotic behavior like blow up have been established. In the presence of the memory term ($K(t) \neq 0$), the various IBV problems have been considered and many results were obtained, such as the existence and uniqueness of classical and weak solutions, and finite-time blow up, asymptotic behavior of solutions in [15–19] and so on.

Next, we focus on the inverse problems posed for the pseudo-parabolic equation and their different modification. Since the pioneer works of [20–23] in the field of the inverse problem brought the authors international fame. In [24], a class of abstract pseudoparabolic equations of the form

$$\begin{cases} A_0 u_t(t) - A_1 u(t) = k * A_2 u(t) + f(t), \quad t \in [0, T] \\ u(0) = u_0 \end{cases}$$

for the operators $A_j, j = 0, 1, 2$, were investigated. The main focus was to pay attention to the recoverving the kernel and finding solution in the Volterra operator integral equation. Lyubanova and coauthors in [25, 26] proved existence and uniqueness, regularity results for strong solutions to the pseudoparabolic equation of the operator form

$$u_t + \eta M u_t + k(t) M u_t + g(x, t) u = f,$$

where M is a linear differential operator of the second order in the space variables. Yaman [27] discussed the coefficient inverse problem for Eq.(1) with $K(t) = 0$ and the special external source term

$$F(x, t) := f(t) \left(\omega - \Delta \omega \right), \tag{6}$$

where the test function ω replaced by $\omega - \Delta\omega$ in the overdetermination condition (4). It may restrict the statement of the problem from both of mathematical and physical view, he derived the upper bound for the blow-up time under some assumptions about the initial data. The equation consisting relation between the damping term and p-Laplacian was considered by Antontsev and et. [29] with the special right-hand side and overdetermination condition such as in [27]. The authors proved in [29] the local existence of weak solution (without the uniqueness). This work was later improved by Khompysh et al. [30] established global and local in time existence and uniqueness result. Recently, Aitzhanov and et. in [28] considered Eq. (1) with $\gamma = b(x, t)$ variable coefficients and instead of (4) assumed overdetermination condition (6). The authors showed that existence and uniqueness of weak and strong solutions under certain conditions and initial data of the corresponding inverse problem. In the present work overdetermination condition (4) cause some difficulties, thus author has to develop other techniques to overcome these difficulties.

The present paper is organized as follows. In Section 2, we introduce some auxiliary lemmas that we use in this work. In Section 3, we prove that the initial inverse problem (1)-(4) is equivalent to the direct problem (14)-(17) containing the nonlinear nonlocal operator of u. The global and local in time existence of a weak solution to the direct problem (14)-(17) is established in Section 4 in the case $\gamma > 0$, and in Section 5 in the case $\gamma \leq 0$. For that we construct Galerkin's approximations u^n and derive their priori estimates. Next using compactness arguments we realize a passage to the limit as $n \to \infty$. The Section 6 is devoted to the study the uniqueness of the weak solution to the problem (14)-(17) in both case of $\gamma > 0$ and $\gamma \leq 0$.

2 Preliminaries

In this section, we introduce some auxiliary lemmas that will be used throughout the paper. For the definitions, notations of the function spaces and for their properties, we address the reader to the monographs [31,32]. In particular, the norm in the Lebesgue spaces $L^p(\Omega)$ and $L^p(Q_T)$ are denoted as follows, respectively:

$$\|u\|_{p,\Omega} \equiv \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}}, \quad \|u\|_{p,Q_T} \equiv \left(\int_0^T \int_{\Omega} |u(x,t)|^p dx dt \right)^{\frac{1}{p}}.$$

We use the classical and the following nonlinear Gronwall's inequality ([31]) to establish the first and second local estimates.

Lemma 1 *If $y : \mathbb{R}^+ \longrightarrow [0, \infty)$ is a continuous function such that*

$$y(t) \leq C_1 \int_0^t y^{\mu}(s) ds + C_2, \quad t \in \mathbb{R}^+, \quad \mu > 1$$

for some positive constants C_1 and C_2, then

$$y(t) \leq C_2 \left(1 - (\mu - 1) C_1 C_2^{\mu-1} t\right)^{-\frac{1}{\mu-1}} \quad \text{for } \ 0 \leq t < t_{\max} := \frac{1}{(\mu-1) C_1 C_2^{\mu-1}}.$$

The following another very important auxiliary lemma (see [33] (Lemma 2.2., p. 1809.) will be used to prove the uniqueness and passage to the limit in the Galerkin approximation.

Lemma 2 *For all $p \in (1, \infty)$ and $\delta \geq 0$, there exist constants C_1 and C_2, depending on p and d, such that for all $\xi, \ \eta \in \mathbb{R}^d, \ d \geq 1$, it*

$$\left| |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \right| \leq C_1 |\xi - \eta|^{1-\delta} (|\xi| + |\eta|)^{p-2-\delta} \tag{7}$$

and

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta \right) \cdot (\xi - \eta) \geq C_2 |\xi - \eta|^{2+\delta} (|\xi| + |\eta|)^{p-2+\delta}. \tag{8}$$

3 Weak formulation.

Assume that the data of the problem satisfy the following conditions

$$u_0(x) \in W_0^{1,2}(\Omega) \cap L^q(\Omega), \tag{9}$$

$$|g_0(t)| := \left| \int_\Omega g(x,t)\omega(x)\,dx \right| \geq l_0 > 0 \quad \text{for all} \quad t \geq 0, \tag{10}$$

$$g(x,t) \in C(0,T; L^2(\Omega)), \tag{11}$$

$$h(t) \in W^{1,2}([0,T]) \text{ and } \int_\Omega u_0(x)\omega(x)dx = h(0), \tag{12}$$

$$\omega(x) \in W_0^{1,2}(\Omega) \cap L^q(\Omega). \tag{13}$$

Lemma 3 *Under the conditions (10) and (12)-(13), the inverse problem (1)-(4) is equivalent to the following problem for a nonlinear pseudoparabolic equation with nonlinear nonlocal operator of the function u*

$$u_t - \kappa\Delta u_t - \lambda\Delta u - \int_0^t K(t-s)\Delta u(x,s)ds = \gamma\,|u|^{q-2}\,u + f(t,u)\cdot g(x,t), \quad in \quad Q_T, \tag{14}$$

$$u(x,0) = u_0(x), \; x \in \Omega, \tag{15}$$

$$u(x,t) = 0 \quad on \quad \Gamma_T, \tag{16}$$

where

$$f(t,u) = \frac{1}{g_0(t)}\left(h'(t) + \kappa\int_\Omega \nabla u_t\nabla\omega dx + \lambda\int_\Omega \nabla u\nabla\omega dx + \int_0^t K(t-s)(\nabla u, \nabla\omega)_{2,\Omega}ds - \gamma\int_\Omega |u|^{q-2}\,u\omega dx\right) \tag{17}$$

Proof 1 *1. Let the pair $(u(x,t), f(t))$ be a solution of the inverse problem (1)-(4). Multiplying both sides of (1) by ω, and integrating over Ω and applying the formula of integrating by parts, we have*

$$\int_\Omega u_t\omega dx + \kappa\int_\Omega \nabla u_t\nabla\omega dx + \lambda\int_\Omega \nabla u\nabla\omega dx + \int_0^t K(t-s)\int_\Omega \nabla u(s)\nabla\omega dx ds = \gamma\int_\Omega |u|^{q-2}u\cdot\omega dx + f(t)\int_\Omega g(x,t)\omega dx. \tag{18}$$

Using

$$\int_\Omega u_t\omega dx = h'(t) \tag{19}$$

which follows from the overdetermination condition (1), and the assumption (10), we get from (18) the equality (17).

2. Let now $u(x,t)$ be a solution to the direct problem (14)-(16) with (17). It means that the pair of functions (u,f) is satisfied the equations (1)-(3). Thus, the pair (u,f) to be a solution of the inverse problem (1)-(4) it is sufficient to prove that the function $u(x,t)$ satisfies the overdetermination condition (4). Let us assume that for contradiction, i.e. the overdetermination condition (4) doesn't hold. Suppose that

$$\int_\Omega u\omega dx = h_1(t), \quad t\geq 0 \tag{20}$$

where $h_1(t)\neq h(t)$ for all $t\geq 0$. Thus, by the conditions (4) and (12), we have $h_1(t)\in W_2^1([0,T])$ and

$$h_1(0) = \int_\Omega u_0\omega dx = h(0) \tag{21}$$

Multiply (14) by ω and integrating by parts and using (17),we get

$$h_1'(t) + \kappa\int_\Omega \nabla u_t\cdot\nabla\omega dx + \lambda\int_\Omega \nabla u\cdot\nabla\omega dx + \int_0^t K(t-s)(\nabla u,\nabla\omega)_{2,\Omega}ds - \gamma\int_\Omega |u|^{q-2}u\cdot\omega dx = f(t,u)g_0(t), \tag{22}$$

where $f(t,u)$ is defined in (17). Plugging (17) into (22), we obtain

$$h_1'(t) + \kappa\int_\Omega \nabla u_t\,\nabla\omega dx + \lambda\int_\Omega \nabla u\,\nabla\omega dx + \int_0^t K(t-s)(\nabla u,\nabla\omega)_{2,\Omega}ds - \gamma\int_\Omega |u|^{q-2}u\,\omega dx =$$

$$h'(t) + \kappa\int_\Omega \nabla u_t\,\nabla\omega dx + \lambda\int_\Omega \nabla u\,\nabla\omega dx + \int_0^t K(t-s)(\nabla u,\nabla\omega)_{2,\Omega}ds - \gamma\int_\Omega |u|^{q-2}u\,\omega dx.$$

$$(23)$$

(23) implies that the following Cauchy problem for $H(t) = h_1(t) - h(t)$:

$$H'(t) = 0,\ H(0) = h_1(0) - h(0) = 0 \tag{24}$$

which yields that $h_1(t) \equiv h(t)$ *for all* $t > 0$.

Definition 1 *A function* $u(x,t)$ *is a weak solution to the problem (14)-(17), if:*

1. $u \in L^\infty(0,T; W_0^{1,2}(\Omega) \cap L^q(\Omega)), u_t \in L^2(0,T; W_0^{1,2}(\Omega))$.
2. $u(0) = u_0$ *a.e. in* Ω
3. *The following identity*

$$\begin{aligned} \frac{d}{dt}\int_\Omega (u\varphi + \kappa\nabla u\nabla\varphi)dx + \lambda\int_\Omega \nabla u\nabla\varphi dx + \int_0^t K(t-s)(\nabla u, \nabla\varphi)_{2,\Omega}ds = \\ \gamma\int_\Omega |u|^{q-2}\, u\varphi dx + \int_\Omega f(t,u)g\varphi dx \end{aligned} \tag{25}$$

holds for every $\varphi \in W_0^{1,2}(\Omega) \cap L^q(\Omega)$ *and for a.a.* $t \in [0,T]$.

4 Global and local existence: a nonlinear source case.

In this section we consider the problem (14)-(17). Let

$$1 < q < \infty. \tag{26}$$

Now we present our mean result for

$$\gamma > 0 \tag{27}$$

First we state the global existence theorem.

Theorem 1 *(Global existence) Let the conditions (9)-(13), (27) are fulfilled and assume, that*

$$1 < q \leq 2 \tag{28}$$

Also exists a positive constant m *such that*

$$\frac{\kappa}{l_0^2}\sup_{t\in[0,T]} \|g(t)\|_{2,\Omega}^2 \|\nabla\omega\|_{2,\Omega}^2 \leq m < 2. \tag{29}$$

Then there exists a global weak solution to the problem (14)-(17) in the sense of Definition 1. Moreover, the weak solutions satisfy the following estimates

$$\sup_{t\in[0,T]} \left(\|\nabla u(t)\|_{2,\Omega}^2 + \|u(t)\|_{q,\Omega}^q\right) + \|u_t\|_{2,Q_T}^2 + \|\nabla u_t\|_{2,Q_T}^2 \leq C, \tag{30}$$

where C *is positive constant depending on data of the problem.*

In this theorem we establish local existence of the weak solution to the problem (14)-(17).

Theorem 2 *(Local existence) Let the conditions (9)-(13), (27), (29) are fulfilled and assume that the following condition holds*

$$2 < q \leq 2^*, 2^* = \frac{2d}{d-2} \ \textit{if} \ d > 2; 2^* = (1, \infty) \ \textit{if} \ d = 2 \tag{31}$$

Then there exists a time $T_1 \in (0, T)$ defined at (48), below such that the problem (14)-(17) has, at least, a weak solution $u(x,t)$ in the sense of Definition 1, with T_1 instead of T. Moreover, these weak solutions satisfy the estimate (30) for all $t \in (0, T_1]$ with another positive constant C depending on data of the problem.

Remark 1 *The condition (31) assures the passage to the limit as $n \to \infty$ below, see (4.4). We have assumed that the condition (31) is fulfilled, because we return to the statement of the 1 in case $q \leq 2$.*

Proof 2 *The proof of these theorems consists of the steps: construction of Galerkin's approximations; obtain energy estimates; passage to limit.*

4.1 Galerkin's approximations.

Let $\{\psi_k\}_{k\in N}$ be an orthonormal family in $L^2(\Omega)$ and their linear combinations are dense in $V := W_0^{1,2}(\Omega) \cap L^q(\Omega)$. Given $n \in N$, let us consider the n-dimentional space V^n spanned by $\psi_1, ..., \psi_n$. for each $n \in N$, we search for approximate solutions

$$u^n(x,t) = \sum_{j=1}^{n} c_j^n(t)\psi_j(x), \quad \psi_j \in V^n, \tag{32}$$

where the coefficients $c_1^n(t), ..., c_n^n(t)$ are defined as the solutions of the following n ordinary differential equations derived from

$$\begin{aligned} \int_\Omega (u_t^n \psi_k + \kappa \nabla u_t^n \nabla \psi_k)\, dx + \lambda \int_\Omega \nabla u^n \nabla \psi_k \, dx + \int_0^t K(t-s)(\nabla u^n, \nabla \psi_k)_{2,\Omega} ds = \\ \gamma \int_\Omega |u^n|^{q-2} u^n \cdot \psi_k \, dx + f(t, u^n) \int_\Omega g \psi_k dx \end{aligned} \tag{33}$$

for $k = 1, 2, \ldots, n$. The system (33) of ODEs is supplemented with the following Cauchy data

$$u^n(0) = u_0^n \quad \text{in} \quad \Omega \tag{34}$$

and assume that

$$u_0^n \to u_0(x) \text{ as } n \to \infty \ \text{ in } \ W_0^{1,2}(\Omega) \cap L^q(\Omega). \tag{35}$$

According to the general theory of nonlinear ODE, the problem (33)-(34) has a solution $c_j^n(t)$ in $[0, t_0]$, where $t_0 \in (0, T]$. The solution can be extended to $[0, T]$ by a priory estimate which we shall obtain below.

4.2 Global priori estimates

Let us consider the case (27). In this case we obtain the global a priori estimates.

Proof 3 *Multiplying both sides of (33) by* $\frac{dc_k^n(t)}{dt}$*, and summing on* k*, and adding* $\frac{1}{\gamma}\frac{d}{dt}\|u^n\|_{q,\Omega}^q$ *on both side, we have*

$$\frac{d}{dt}\left(\frac{\lambda}{2}\|\nabla u^n\|_{2,\Omega}^2+\frac{\gamma}{q}\|u^n\|_{q,\Omega}^q\right)+\|u_t^n\|_{2,\Omega}^2+\kappa\|\nabla u_t^n\|_{2,\Omega}^2=\frac{2\gamma}{q}\frac{d}{dt}\|u^n\|_{q,\Omega}^q+I_1+I_2 \tag{36}$$

where

$$I_1=-\int_0^t K(t-s)(\nabla u^n(s),\nabla u_t^n(t))_{2,\Omega}ds, \tag{37}$$

$$\begin{aligned}I_2=\frac{1}{g_0(t)}&\left(h'(t)+\kappa\int_\Omega\nabla u_t^n\cdot\nabla\omega dx+\lambda\int_\Omega\nabla u^n\cdot\nabla\omega\ dx+\right.\\&\left.\int_0^t K(t-s)(\nabla u^n,\nabla\omega)_{2,\Omega}ds-\gamma\int_\Omega|u^n|^{q-2}u^n\cdot\omega dx\right)(g(t),u_t^n(t))_{2,\Omega}\,.\end{aligned} \tag{38}$$

Now we estimate each term on the right hand side of (36) by using the Hölder and Cauchy inequalities with ε_0 *together with the assumptions in (28). Thus, we have*

$$\begin{aligned}&\frac{2\gamma}{q}\frac{d}{dt}\|u^n\|_{q,\Omega}^q=2\gamma\int_\Omega|u|^{q-1}u_tdx\le2\gamma\|u_t^n\|_{2,\Omega}\|u^n\|_{2(q-1),\Omega}^{q-1}\le\\&\frac{\varepsilon_0}{2}\|u_t^n\|_{2,\Omega}^2+C(\varepsilon_0,\gamma,q)\|u^n\|_{2(q-1),\Omega}^{2(q-1)}\le\frac{\varepsilon_0}{2}\|u_t^n\|_{2,\Omega}^2+C(\varepsilon_0,\gamma,q,\Omega)\left(\|\nabla u^n\|_{2,\Omega}^2\right)^{q-1}.\end{aligned} \tag{39}$$

Also the Cauchy inequality with some ε*, gives us*

$$|I_1|\le\int_0^t|K(t-s)|\,\|\nabla u^n(s)\|_{2,\Omega}\|\nabla u_t^n\|_{2,\Omega}ds\le\frac{\varepsilon_1}{2}\|\nabla u_t^n\|_{2,\Omega}^2+\frac{K_0^2}{2\varepsilon_1}\int_0^t\|\nabla u^n\|_{2,\Omega}^2ds. \tag{40}$$

Exploiting the Hölder integral inequality and Cauchy inequality with $\varepsilon_2>0$*, we estimate the third term* i_2 *in (36)*

$$\begin{aligned}|I_2|\le\frac{1}{l_0}&\left(|h'(t)|+\kappa\|\nabla u_t^n\|_{2,\Omega}\|\nabla\omega\|_{2,\Omega}+\lambda\|\nabla u^n\|_{2,\Omega}\|\nabla\omega\|_{2,\Omega}+\right.\\&\left.\int_0^t|K(t-s)|\,\|\nabla u^n(s)\|_{2,\Omega}\|\nabla\omega\|_{2,\Omega}ds+\gamma\|\omega\|_{q,\Omega}\|u^n\|_{q,\Omega}^{q-1}\right)\|g\|_{2,\Omega}\|u_t^n\|_{2,\Omega}dt\le\\&\frac{\varepsilon_2}{4}\int_0^t\|u_t^n\|_{2,\Omega}^2d\tau+\frac{\kappa^2}{l_0^2\varepsilon_2}\|\nabla\omega\|_{2,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\|\nabla u_t^n\|_{2,\Omega}+\frac{\varepsilon_2}{4}\|u_t^n\|_{2,\Omega}^2+\end{aligned}$$

$$\begin{aligned}
&\frac{1}{l_0^2\varepsilon_2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\Big(|h'(t)|^2+\lambda^2\|\nabla u^n\|_{2,\Omega}^2\|\nabla\omega\|_{2,\Omega}^2+\\
&K_0^2\|\nabla\omega\|_{2,\Omega}^2\int_0^t\|\nabla u^n(s)\|_{2,\Omega}^2\,ds+\gamma^2\|\omega\|_{q,\Omega}^2\|u^n\|_{q,\Omega}^{2(q-1)}\Big)\le\\
&\frac{\varepsilon_2}{2}\|u_t^n\|_{2,\Omega}^2+\frac{\kappa^2}{l_0^2\varepsilon_2}\|\nabla\omega\|_{2,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\|\nabla u_t^n\|_{2,\Omega}^2+\frac{1}{l_0^2\varepsilon_2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\big(|h'(t)|^2+\\
&\lambda^2\|\nabla u^n\|_{2,\Omega}^2\|\nabla\omega\|_{2,\Omega}^2+K_0^2\|\nabla\omega\|_{2,\Omega}^2\int_0^t\|\nabla u^n(s)\|_{2,\Omega}^2\,ds+C\gamma^2\|\omega\|_{q,\Omega}^2\|\nabla u^n\|_{2,\Omega}^{2(q-1)}\Big).
\end{aligned}\tag{41}$$

Plugging (39)-(41) into (36), we get

$$\begin{aligned}
&\frac{d}{dt}\left(1+\frac{\lambda}{2}\|\nabla u^n\|_{2,\Omega}^2+\frac{\gamma}{q}\|u^n\|_{q,\Omega}^q\right)+\alpha\|u_t^n\|_{2,\Omega}^2+\beta\|\nabla u_t^n\|_{2,\Omega}^2\le\\
&C_1\|\nabla u^n\|_{2,\Omega}^2+C_2\left(\|\nabla u^n\|_{2,\Omega}^2\right)^{q-1}+C_3\int_0^t\|\nabla u^n(s)\|_{2,\Omega}^2\,ds+C_4\le\\
&C_1\left(1+\frac{\lambda}{2}\|\nabla u^n\|_{2,\Omega}^2+\frac{\gamma}{q}\|u^n\|_{q,\Omega}^q\right)+C_2\left(1+\frac{\lambda}{2}\|\nabla u^n\|_{2,\Omega}^2+\frac{\gamma}{q}\|u^n\|_{q,\Omega}^q\right)^{q-1}+\\
&C_3\int_0^t\left(1+\frac{\lambda}{2}\|\nabla u^n\|_{2,\Omega}^2+\frac{\gamma}{q}\|u^n\|_{q,\Omega}^q\right)ds+C_4,
\end{aligned}\tag{42}$$

where $\alpha:=1-\frac{\varepsilon_0+\varepsilon_2}{2}$; $\quad\beta:=\kappa-\frac{\varepsilon_1}{2}-\frac{\kappa^2}{l_0^2\varepsilon_2}\|\nabla\omega\|_{2,\Omega}^2\sup\limits_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2$;

$$C_1:=\frac{\lambda^2}{l_0^2\varepsilon_2}\|\nabla\omega\|_{2,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2,\ C_2:=\frac{K_0^2}{2\varepsilon_1}+\frac{K_0^2}{l_0^2\varepsilon_2}\|\nabla\omega\|_{2,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2,$$

$$C_3:=C(\varepsilon_0,\gamma,q,\Omega)+\frac{C\gamma^2}{l_0^2\varepsilon_2}\|\omega\|_{q,\Omega}^2\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2,\ C_4:=\frac{1}{l_0^2\varepsilon_2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2|h'(t)|^2.$$

Now we choose $\varepsilon_i,\ i=0,1,2$ *such that* $\alpha,\beta>0$, *and* $C_j,\ j=1,2,3,4$ *to be finite. It is possible by (29).*

However $\varepsilon_1,\varepsilon_2$ *cannot be chosen such that* $m>2$, *because* $\varepsilon_2<2$ *due to* $\alpha>0$. *Thus, choosing* $\varepsilon_i,\ i=0,1,2$ *with suitable values and in case* $q-1\le 1$ *integrating (42) with respect to* τ *from* 0 *to* t *and using (35), we obtain*

$$y(t)+\int_0^t\left(\alpha\|u_\tau^n\|_{2,\Omega}^2+\beta\|\nabla u_\tau^n\|_{2,\Omega}^2\right)d\tau\le C_5\int_0^t y(\tau)d\tau+C_6,\tag{43}$$

where $y(t) := 1 + \frac{\lambda}{2} \|\nabla u^n\|_{2,\Omega}^2 + \frac{\gamma}{q} \|u^n\|_{q,\Omega}^q$,

$$C_5 := C_1 + C_2 + C_3 T; \quad C_6 := C_4 T + \frac{\lambda}{2} \|\nabla u_0\|_{2,\Omega}^2 + \frac{\gamma}{q} \|u_0\|_{q,\Omega}^q .$$

Omitting the integral terms on left hand side and applying classical Grönwall's lemma, inequality (43) implies that

$$y(t) \le C_6 e^{C_5 T}. \tag{44}$$

Thus, substituting (44) into (43) and taking suppremum, we obtain the estimate (30)

$$\sup_{t\in[0,T]} \left(\frac{\lambda}{2} \|\nabla u^n\|_{2,\Omega}^2 + \frac{\gamma}{q} \|u^n\|_{q,\Omega}^q \right) + \int_0^t \left(\alpha \|u_\tau^n\|_{2,\Omega}^2 + \beta \|\nabla u_\tau^n\|_{2,\Omega}^2 \right) ds \le M_1 < +\infty, \tag{45}$$

where $M_1 := M_1(T, C_5, C_6)$.

4.3 Local a priori estimates

Let now be $q \le 2^*$. In this case we obtain the local a priori estimates.

Proof 4 *Choosing* $\varepsilon_i, i = 0, 1, 2$ *with suitable values as we did in obtaining a priory estimates above and in case* $q - 1 > 1$ *integrating (42) with respect to* $\tau \in (0, t)$ *and using (35), we have*

$$z(t) + \int_0^t \left(\alpha \|u_\tau^n\|_{2,\Omega}^2 + \beta \|\nabla u_\tau^n\|_{2,\Omega}^2 \right) d\tau \le C_5 \int_0^t z^{q-1}(\tau) d\tau + C_6. \tag{46}$$

Omitting the second and third terms on left hand side (46) and applying Grönwall's Lemma 1, we obtain

$$z(t) \le C_6 \left[1 - (q-2) C_5 C_6^{q-2} t \right]^{-\frac{1}{q-2}} \tag{47}$$

for

$$0 \le t < T_1 := \frac{1}{(q-2) C_5 C_6^{q-2}}. \tag{48}$$

Using (47) and maximizing (46) by $t \in (0, T_1]$*, we have*

$$\sup_{t\in[0,T_1]} \left(\frac{\lambda}{2} \|\nabla u^n\|_{2,\Omega}^2 + \frac{\gamma}{q} \|u^n\|_{q,\Omega}^q \right) + \alpha \|u_t^n\|_{2,Q_{T_1}}^2 + \beta \|\nabla u_t^n\|_{2,Q_{T_1}}^2 \le M_2 < \infty, \tag{49}$$

where $M_2 := M_2(T_1, C_5, C_6)$.

4.4 Passage to the limit

By means of reflexivity and up to some subsequences, the estimate (30) implies that

$$u^n \rightharpoonup u \quad \text{weakly-}* \text{ in } L^\infty(0,T;W_0^{1,2}(\Omega)\cap L^q(\Omega)) \quad \text{as } n\to\infty, \tag{50}$$

$$u_t^n \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;W_0^{1,2}(\Omega)) \quad \text{as } n\to\infty. \tag{51}$$

On the other hand, (30) implies the existence of function R such that

$$|u^n|^{q-2}u^n \rightharpoonup R \quad \text{weakly in } L^{q'}(Q_T), \quad \text{as } n\to\infty, \tag{52}$$

where $q'=\dfrac{q}{q-1}$ is the Hölder conjugate of q. By the compact and continuous embedding

$$W_0^{1,2}(\Omega)\hookrightarrow\hookrightarrow L^r(\Omega)\hookrightarrow L^2(\Omega), \qquad \forall\, r: 2\le r<2^*$$

and by Aubin-Lions compactness lemma, (50) and (51) imply that

$$u^n \longrightarrow u \quad \text{strongly in } L^2(0,T;L^r(\Omega)), \quad 2\le r<2^* \quad \text{as } n\to\infty, \tag{53}$$

and in particular,

$$u^n \longrightarrow u \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \quad \text{as } n\to\infty, \tag{54}$$

where 2^* is the Sobolev conjugate of 2, i.e. $2^*=\frac{2d}{d-2}$ with $d>2$.

As a consequence of (54) and Riesz-Fischer's theorem, we have up to some subsequence,

$$u^n \longrightarrow u \quad \text{a.e. in } Q_T \quad \text{as } n\to\infty, \tag{55}$$

which together with (52) yields (see Lemma 1.3 in [32, p. 12])

$$|u^n|^{q-2}u^n \rightharpoonup |u|^{q-2}u \quad \text{weakly in } L^{q'}(Q_T), \quad \text{as } n\to\infty. \tag{56}$$

Under the assumption (31) and (51), (53), we have also that

$$u^n \longrightarrow u \quad \text{strongly in} \quad L^q(Q_T) \quad \text{as} \quad n\to\infty, \quad \text{for} \quad q<2^*$$

and consequently

$$\|u^n\|_{q,Q_t} \longrightarrow \|u\|_{q,Q_t} \quad \text{as} \quad n\to\infty. \tag{57}$$

Let $\eta(t)$ be a continuously differentiable function on $[0,T]$, where T is the maximal time such that above first and second estimates are hold. Multiplying (33) by η and integrating by $t\in[0,T]$, we obtain

$$\begin{aligned}
&\int\limits_{Q_T}(u_t^n z_k+\kappa\nabla u_t^n\nabla z_k)dxdt+\lambda\int\limits_{Q_T}\nabla u^n\nabla z_k\,dxdt+\int\limits_0^T\int\limits_0^t K(t-s)(\nabla u^n,\nabla z_k)_{2,\Omega}dsdt=\\
&\gamma\int\limits_{Q_T}|u^n|^{q-2}u^n z_k\,dxdt+\int\limits_0^T\frac{1}{g_0(t)}\left(h'(t)+\kappa\int\limits_\Omega\nabla u_t^n\nabla\omega\;dx+\lambda\int\limits_\Omega\nabla u^n\nabla\omega\;dx+\right.\\
&\left.\int\limits_0^t K(t-s)(\nabla u^n,\nabla\omega)_{2,\Omega}ds-\gamma\int\limits_\Omega|u^n|^{q-2}u^n\omega dx\right)\int\limits_\Omega g z_k\,dxdt
\end{aligned}\tag{58}$$

and using above convergence results (50), (51) and (56), we obtain

$$\int_{Q_T}(u_t z_k+\kappa\nabla u_t\nabla z_k)dxdt+\lambda\int_{Q_T}\nabla u\nabla z_k\,dxdt+\int_0^T\int_0^t K(t-s)(\nabla u,\nabla z_k)_{2,\Omega}dsdt=$$
$$\gamma\int_{Q_T}|u|^{q-2}uz_k\,dxdt+\int_0^T\frac{1}{g_0(t)}\left(h'(t)+\kappa\int_\Omega\nabla u_t\nabla\omega\;dx+\lambda\int_\Omega\nabla u\nabla\omega\;dx+\right.\tag{59}$$
$$\left.\int_0^t K(t-s)(\nabla u,\nabla\omega)_{2,\Omega}ds-\gamma\int_\Omega|u|^{q-2}u\omega dx\right)\int_\Omega gz_k\,dxdt$$

for all $z_k=\psi_k(x)\eta(t),\quad k\in\{1,\dots,n\}$. By linearity and by a continuity argument, the equation (59) is still true for any

$$z\in Z:=\{z=\psi\zeta:\psi\in\mathcal{V},\ \zeta\in C_0^\infty(0,T)\}.$$

5 Global and local existence: an absorption case.

In this section we present existence of weak solution to the problem (14)-(17) for

$$\gamma\le 0.\tag{60}$$

For existence of the weak solution the following theorems hold.

Theorem 3 *(Global existence) Let the conditions (9)-(13), (28), (29), (60) are fulfilled. Then there exists a global weak solution to the problem (14)-(17) in the sense of Definition 1. Moreover, the weak solutions satisfy the estimate (30) for all $t\in[0,T]$ with another positive constant C depending on data of the problem.*

Theorem 4 *(Local existence) Let the conditions (9)-(13), (29), (31), (60) are fulfilled. Then there exists $T_2\in(0,T]$ and at least one weak solution to the problem (14)-(17) in the sense of Definition 1 and satisfies the estimate (30) in Q_{T_2}, where T_2 depending on data of the problem.*

Proof 5 *The proof of Theorems 3 and 4 are similar to the Theorems 1 and 2.*

6 Uniqeness

Theorem 5 *Assume that the following conditions*

$$\nabla\omega\in L^2(\Omega),\tag{61}$$

$$2\le q\le\frac{2d}{d-2},\ d>2\tag{62}$$

hold. Moreover, there exists a positive constant m such that

$$\frac{\kappa}{l_0^2}\sup_{t\in[0,T]}\|g(t)\|_{2,\Omega}^2\|\nabla\omega\|_{2,\Omega}^2\le m<2. \tag{63}$$

If $\gamma\le 0$, assume addition to (61)-(62) that all conditions of Theorems 1 and 2 are fulfilled. If $\gamma>0$, assume addition to (61)-(62) that all conditions of Theorem 3 and 4 are fulfilled.

Then the weak solution of (14)-(17) is unique.

Proof 6 *Let u_1 and u_2 be two weak solutions to the problem (14)-(17) in the sense of Definition 1. Using $\partial_t u:=\partial_t u_1-\partial_t u_2$ as a test function in (25), it follows, by subtracting the equation for u_2 to the equation for u_1, that*

$$\frac{\lambda}{2}\frac{d}{dt}\|\nabla u\|_{2,\Omega}^2+\|u_t\|_{2,\Omega}^2+\kappa\|\nabla u_t\|_{2,\Omega}^2=D+G+F, \tag{64}$$

where

$$D=-\int_0^t K(t-s)(\nabla u(s),\nabla u_t(t))_{2,\Omega}ds, \tag{65}$$

$$G=\gamma\int_\Omega\left(|u_1|^{q-2}u_1-|u_2|^{q-2}u_2\right)\cdot u_t\,dx, \tag{66}$$

$$\begin{aligned}F=&\frac{1}{g_0(t)}\left(\kappa\int_\Omega\nabla u_t\cdot\nabla\omega\,dx+\lambda\int_\Omega\nabla u\cdot\nabla\omega\,dx+\int_0^t K(t-s)(\nabla u,\nabla\omega)_{2,\Omega}ds\right.\\&\left.-\gamma\int_\Omega\left(|u_1|^{q-2}u_1-|u_2|^{q-2}u_2\right)\cdot\omega\,dx\right)\int_\Omega g(x,t)u\,dx.\end{aligned} \tag{67}$$

Using Hölder's and Minkovskii inequalities and (7) in Lemma 2 with $\delta=0$, we estimate D, G and F

$$|D|\le\frac{\varepsilon_0}{2}\|\nabla u_t\|_{2,\Omega}^2+\frac{K_0^2}{2\varepsilon_0}\int_0^t\|\nabla u\|_{2,\Omega}^2ds, \tag{68}$$

$$\begin{aligned}|G|=&\left|\gamma\int_\Omega\left(|u_1|^{q-2}u_1-|u_2|^{q-2}u_2\right)u_t dx\right|\le|\gamma|\int_\Omega|u|\left(|u_1|+|u_2|\right)^{q-2}|u_t|\,dx\le\\&|\gamma|\,\|u\|_{2^*,\Omega}\,\||u_1|+|u_2|\|^{q-2}_{\frac{(q-2)d}{2},\Omega}\|u_t\|_{2^*,\Omega}\le|\gamma|C^2\left(\|\nabla u_1\|_{2,\Omega}+\|\nabla u_2\|_{2,\Omega}\right)^{q-2}\times\\&\|\nabla u\|_{2,\Omega}\|\nabla u_t\|_{2,\Omega}\le\frac{\varepsilon_2}{2}\|\nabla u_t\|_{2,\Omega}^2+\frac{\lambda}{2}b_0\|\nabla u\|_{2,\Omega}^2,\ q\le\frac{2d}{d-2}\end{aligned} \tag{69}$$

where $b_0 := \frac{2|\gamma|^2 C^2 C^{q-2}}{\lambda}$.

$$|F| \leq \frac{1}{l_0} \|g\|_{2,\Omega} \|u_t\|_{2,\Omega} \left(\kappa \|\nabla u_t\|_{2,\Omega} \|\nabla\omega\|_{2,\Omega} + \lambda \|\nabla u\|_{2,\Omega} \|\nabla\omega\|_{2,\Omega} + \int_0^t K(t-s) \|\nabla u\|_{2,\Omega} \|\nabla\omega\|_{2,\Omega} ds + |\gamma| \int_\Omega \left(|u| \left(|u_1| + |u_2|\right)^{q-2} |\omega|\, dx\right)\right) \leq$$

$$\frac{1}{l_0} \|g\|_{2,\Omega} \|u_t\|_{2,\Omega} \Big[\kappa \|\nabla u_t\|_{2,\Omega} \|\nabla\omega\|_{2,\Omega} + \lambda \|\nabla u\|_{2,\Omega} \|\nabla\omega\|_{2,\Omega} + \int_0^t K(t-s) \|\nabla u\|_{2,\Omega} \|\nabla\omega\|_{2,\Omega} ds + |\gamma| \|u\|_{2^*,\Omega} \left(\|u_1 + u_2\|_{\frac{(q-2)d}{2},\Omega}\right)^{q-2} \|\omega\|_{2^*,\Omega}\Big] \leq$$

$$\frac{\varepsilon_1}{2} \|u_t\|^2_{2,\Omega} + \frac{\kappa^2}{4l_0^2\varepsilon_1} \|\nabla\omega\|^2_{2,\Omega} \sup_{t\in[0,T]} \|g\|^2_{2,\Omega} \|\nabla u_t\|^2_{2,\Omega} + \frac{1}{4l_0^2\varepsilon_1} \|\nabla\omega\|^2_{2,\Omega} \sup_{t\in[0,T]} \|g\|^2_{2,\Omega} \times \left(\lambda^2 \|\nabla u\|^2_{2,\Omega} + K_0^2 \int_0^t \|\nabla u\|^2_{2,\Omega} ds + \gamma^2 C^2 \left(\|\nabla u_1\|_{2,\Omega} + \|\nabla u_2\|_{2,\Omega}\right)^{q-2} \|\nabla u\|^2_{2,\Omega}\right) \leq \frac{\varepsilon_1}{2} \|u_t\|^2_{2,\Omega} + \frac{\kappa^2}{4l_0^2\varepsilon_1} \|\nabla\omega\|^2_{2,\Omega} \sup_{t\in[0,T]} \|g\|^2_{2,\Omega} \|\nabla u_t\|^2_{2,\Omega} + \frac{\lambda}{2} b_1 \|\nabla u\|^2_{2,\Omega} + \frac{\lambda}{2} b_2 \int_0^t \|\nabla u\|^2_{2,\Omega} ds, \tag{70}$$

where

$$q \leq \frac{2d}{d-2}, \frac{(q-2)d}{2} \leq \frac{2d}{d-2} := 2^* \Leftrightarrow q \leq \frac{2d}{d-2},$$

$$b_1 = \frac{1}{2\lambda l_0^2 \varepsilon_1} \|\nabla\omega\|^2_{2,\Omega} \sup_{t\in[0,T]} \|g\|^2_{2,\Omega} \left(\lambda^2 + 2\gamma^2 C^4 C^{q-2}\right),$$

$$b_2 = \frac{K_0^2}{2\lambda l_0^2 \varepsilon_1} \|\nabla\omega\|^2_{2,\Omega} \sup_{t\in[0,T]} \|g\|^2_{2,\Omega}.$$

Furthermore, taking into account Sobolev's inequality we derive

$$\|\nabla u\|^2_{2,\Omega} \geq \frac{1}{C(\Omega)+1} \left(\|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega}\right) = \eta y(t), \tag{71}$$

where $y(t) := \|u\|^2_{2,\Omega} + \|\nabla u\|^2_{2,\Omega}$ *and* $\eta := \frac{1}{C(\Omega)+1}$.

Using estimates for $u_i,\ i = 1, 2,$ *and choosing* $\varepsilon_i, i = 0, 1, 2$ *with suitable values as we did as obtaining a priory estimates above, we can make* α, β *to be positive and finite constants, and it is possible due to the assumption* $\frac{\kappa}{l_0^2} \sup_{t\in[0,T]} \|g(t)\|^2_{2,\Omega} \|\nabla\omega\|^2_{2,\Omega} \leq m < 2$.

Plugging (68)–(70) into (64) in the case $\gamma \leq 0$, *and (68), (70) and (69) into (**??**) in the case* $\gamma > 0$, *using (71) and integrating by* $\tau \in (0, t)$ *we arrive to the following Cauchy problem*

$$\begin{cases} y(t) \le a \int\limits_0^t y(\tau)d\tau, \\ \\ y(0) = 0, \end{cases} \tag{72}$$

where

$$a := \frac{b_1 + b_2 T + b_0}{\eta}.$$

Due to the conditions to the Theorem 5 and then by the Gronwall's lemma, it follows from (72) that $y(t) \equiv 0$ *for all* $t \in [0, T_{max}]$*, and consequently that* $u_1 \equiv u_2$*, where* T_{max} *is a maximal time such that the weak solution to the problem (14)-(17) exists.*

7 Conclusion

In the paper, the space of a weak generalized solution of inverse problem for the pseudoparabolic equation with memory term and damping is defined. Under suitable conditions on the data of the problem, the global and local in time existence and uniqueness theorems are obtained and proved.

Acknowledgment

This research work has been funded by Grant number AP23486218 the Ministry of Science and Higher Education of the Republic of Kazakhstan.

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Received: March 5, 2025
Accepted: June 7, 2025

2-бөлім **Раздел 2** **Section 2**

Қолданбалы математика **Прикладная математика** **Applied mathematics**

IRSTI 30.19.21 DOI: https://doi.org/10.26577/JMMCS20251262012

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PREDICTION OF DRILL STRING VIBRATIONS USING MACHINE LEARNING TOOLS

The objective of this work is to apply machine learning algorithms for the analysis of dynamic vibrations of drill strings. As part of the research, a mathematical model describing the vibrations of the drill string was developed; a finite difference scheme was implemented, and numerical modeling was carried out using the three-point sweep method. Based on the data obtained from the numerical solution, a machine learning model was created. The numerical modeling was implemented using C++, while data collection and the construction of the machine learning model were performed using the Python programming language. As a result, a predictive model was obtained, capable of accurately forecasting the dynamic vibrations of the drill string and determining optimal parameters, thereby improving the efficiency and safety of drilling operations.

Key words: drill string, vibrations, machine learning, linear regression, random forest.

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Бұрғылау бағандарының тербелістерін машиналық оқыту құралдары арқылы болжау

Бұл жұмыстың мақсаты бұрғылау бағандарының динамикалық тербелістерін талдау үшін машиналық оқыту алгоритмдерін қолдану. Зерттеу барысында бұрғылау бағанының тербелістерін сипаттайтын математикалық модель әзірленді; ақырлы-айырымдық сұлбасы құрылып, үш нүктелі қуалау әдісін қолдану арқылы сандық модельдеу жүргізілді. Сандық шешімнен алынған мәліметтерге сүйене отырып, машиналық оқыту моделі құрылды. Сандық модельдеу үшін C++ тілінде бағдарламалық код жасалды, ал деректерді жинау және машиналық оқыту моделін құру процестері Python бағдарламалау тілі арқылы жүзеге асырылды. Нәтижесінде бұрғылау бағанасыныұ динамикалық тербелістерін жоғары дәлдікпен болжай алатын және оұтайлы параметрлерді анықтайтын болжамды модель алынды, бұл бұрғылау жұмыстарының тиімділігі мен қауіпсіздігін арттыруға ықпал етеді.

Түйін сөздер: бұрғылау бағаны, тербеліс, машиналық оқыту, сызықты регрессия, кездейсоқ орман.

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Прогнозирование колебаний бурильных колонн с применением инструментов машинного обучения

Целью данной работы является применение алгоритмов машинного обучения для анализа динамических колебаний бурильных колонн. В рамках проведённого исследования была разработана математическая модель, описывающая колебания бурильной колонны; реализована конечно-разностная схема и проведено численное моделирование с использованием метода трёхточечной прогонки. Основываясь на данных, полученных в результате численного решения, была создана модель машинного обучения. Для численного моделирования был разработан программный код на языке C++, а процессы сбора данных и построения модели машинного обучения были выполнены с использованием языка программирования Python.

В результате была получена прогнозная модель, способная с высокой точностью предсказывать динамические колебания бурильной колонны и определять оптимальные параметры, что способствует повышению эффективности и безопасности проводимых буровых операций.
Ключевые слова: бурильная колонна, колебания, машинное обучение, линейная регрессия, случайный лес.

1 Introduction

Oil and gas constitute a significant portion of the world's energy resources, and the drilling industry plays a key role in their extraction. Drilling is a complex technological process that involves penetrating rock formations to access hydrocarbon reservoirs. During the drilling process, various types of vibrations frequently occur in the drill string, which are an undesirable side effect. Excessive vibrations can damage drilling tools, cause premature equipment wear, reduce drilling efficiency, and increase overall costs. Therefore, monitoring, analyzing, and minimizing drill string vibrations have become a critical area of research in the drilling industry.

Machine learning, as part of modern technological advancements, offers extensive opportunities for addressing this challenge. These methods enable the analysis of large datasets, the identification of hidden patterns, and the prediction of complex system behavior in real time. In recent years, numerous studies have been conducted on the application of machine learning for vibration analysis and control in drill string operations.

For instance, Saadeldin et al. [1] developed machine learning models for detecting drill string vibrations during horizontal drilling using surface sensor data. The study utilized radial basis functions (RBF), support vector machines (SVM), adaptive neuro-fuzzy inference systems (ANFIS), and functional networks (FN). These models successfully identified axial, torsional, and lateral vibrations with high accuracy, achieving correlation coefficients above 0.9 and a mean absolute percentage error (MAPE) of less than 7.5%. The results demonstrated that using surface data for vibration monitoring can significantly reduce costs by eliminating the need for expensive downhole sensors.

Another study by Saadeldin et al. [2] focused on predicting vibrations during the drilling of curve sections using real field data from multiple wells. The models were built using the same algorithms (RBF, SVM, ANFIS, and FN), but the emphasis was placed on curved well sections. The models showed impressive results, with ANFIS and SVM models achieving correlation coefficients of up to 0.99 and an error rate of less than 2.8%. The study confirmed that machine learning applications can significantly improve drilling performance in challenging conditions.

In the work [3], the authors investigated the application of different neural network architectures for predicting drill string vibrations. The tested models included fully connected networks, physics-informed neural networks, and long short-term memory (LSTM) networks. The results indicated that incorporating physical constraints into neural network structures enhanced prediction reliability, particularly in the context of nonlinear interactions between drilling equipment and rock formations.

Etaje [4] employed principal component analysis (PCA) and decision tree algorithms to identify optimal drilling zones with minimal vibrations. The proposed approach allowed for real-time adjustments of drilling parameters, thereby minimizing vibration-related risks. Notably, the study demonstrated high efficiency when using only surface data without requiring additional downhole measurements.

Hegde et al. [5] developed a model for classifying stick-slip vibrations using a random forest algorithm. The model was trained on historical data, including drill string rotation speed, bit load, and torque measurements. During testing, the model achieved 90% accuracy, highlighting its potential for integration into rate of penetration (ROP) optimization systems to improve the safety and efficiency of drilling operations. The authors in [6] investigated the application of machine learning for predicting drilling complications in oil and gas wells. Using historical data from 67 wells, they analyzed key drilling parameters such as standpipe pressure, hook load, rotary table torque, and rate of penetration to classify potential risks. Eight machine learning algorithms with gradient boosting (GB) demonstrating the highest accuracy in anomaly detection were tested. The study concluded that ML-based models can significantly enhance drilling efficiency by providing early warnings of complications, reducing non-productive time, and optimizing decision-making for drilling engineers. Future work suggests integrating geomechanical parameters to improve prediction accuracy. The authors in [7] developed a new machine learning-based model for predicting the rate of penetration (ROP) in vertical wells. The study compared physic-based models and data-driven methods, concluding that data-driven techniques provide better accuracy.

Despite significant progress in applying machine learning to analyze drill string vibrations, research in this area remains relevant. The complex dynamic processes that arise during interactions between drilling equipment and rock formations require the development of more accurate models capable of considering a wide range of influencing factors. An essential task is to create efficient algorithms capable of reliably predicting vibrations in real production environments and assisting operators in making informed drilling decisions.

The outcome of this work is a predictive model capable of accurately forecasting drill string vibrations and determining optimal drilling parameters. The implementation of this model contributes to improving the efficiency and safety of drilling operations by reducing downtime, minimizing accident risks, and optimizing operational parameters in real-time production conditions.

2 Materials and methods

2.1 Linear mathematical model

The linear model of the dynamics of a rotating drill string, compressed from both ends with a longitudinal load $N(x,t)$, is given by [8]:

$$\rho S\frac{\partial^2 u}{\partial t^2} + EI_{x_1}\frac{\partial^4 u}{\partial x^4} - \rho I_{x_1}\frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\partial}{\partial x}\left(N(x,t)\frac{\partial u}{\partial x}\right) - \rho S\Omega^2 u = 0. \tag{1}$$

The parameters of the linear model are as follows: ρ is the density of the drill string material, which is typically made of steel or duralumin; S is the cross-sectional area of the transverse section; E is Young's modulus, a physical quantity that characterizes the elastic properties of the material, defining the relationship between stress and strain during tension or compression; I is the principal moment of inertia, describing the distribution of mass around the axis of rotation; $N(x,t)$ is the axial load that occurs during drilling, directed along the drill string and influenced by the string's weight, tensile forces, soil resistance, and dynamic loading; Ω is the angular speed of the drill string, representing its rotation around its own axis.

Taking into account the method of fixing the upper end of the drill string and its interaction with the rock at the bottom, as well as the fact that only rotational motion is possible while longitudinal and transverse movements are constrained, the boundary conditions are presented as follows:

$$u(x,t) = 0 \quad (x = 0, x = l), \tag{2}$$

$$EI_{x_1} \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \quad (x = 0, x = l). \tag{3}$$

These conditions correspond to a hinged support.

The initial conditions are defined as follows:

$$u(x,t) = 0 \quad (t = 0), \tag{4}$$

$$\frac{\partial u(x,0)}{\partial t} = C_1 \quad (t = 0), \tag{5}$$

where C_1 represents the displacement rate of the drill string in the Ox_1x_3 plane at the initial moment in time.

The numerical solution of equation (1) was obtained by discretizing it in a finite difference form. The finite difference scheme of equation (1) is given as follows:

$$\begin{aligned} &\rho S \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} + EI_{x_1} \frac{u_{i-2}^n - 4u_{i-1}^n + 6u_i^n - 4u_{i+1}^n + u_{i+2}^n}{\Delta x^4} \\ &- \rho I_{x_1} \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} - 2(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + u_{i+1}^{n-1} + 2u_i^{n-1} - u_{i-1}^{n-1}}{\Delta t^2 \Delta x^2} \\ &+ \frac{1}{4\Delta x^2} \left(N_{i+1}^n (u_{i+1}^n - u_i^n) - N_i^n (u_i^n - u_{i-2}^n) \right) - \rho S \Omega^2 u_i^{n+1} = 0. \end{aligned} \tag{6}$$

Since the scheme is semi-implicit, the three-point Thomas algorithm was applied.

2.2 Application of machine learning

2.2.1 Data collection and processing of vibration information

Machine learning data is a fundamental component of the training process for machine learning models and algorithms. The data was collected through numerical solutions of the mathematical model of drill string vibrations. Three parameters of the drilling process — initial velocity, longitudinal load, and angular velocity — were used as features, while the maximum vibration value at the exact center of the drill string was taken as the data point. The dataset consists of 300 data points and 3 features. The min-max normalization method was used to avoid difficulties when building the model. Data collection and processing were implemented using the Python programming language. The collected data was split into 75% training data and 25% test data using the train_test_split function from the Scikit-learn library. Figures 1 and 2 illustrate the training and test datasets.

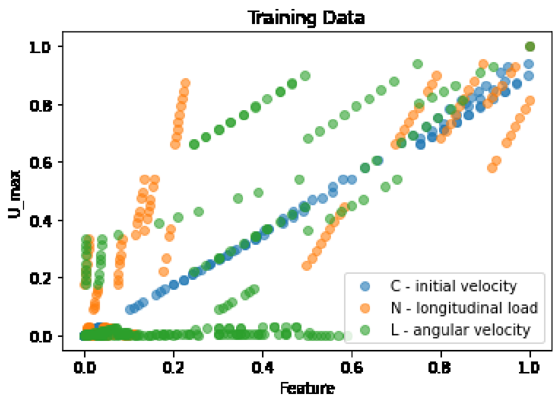


Figure 1: Training dataset

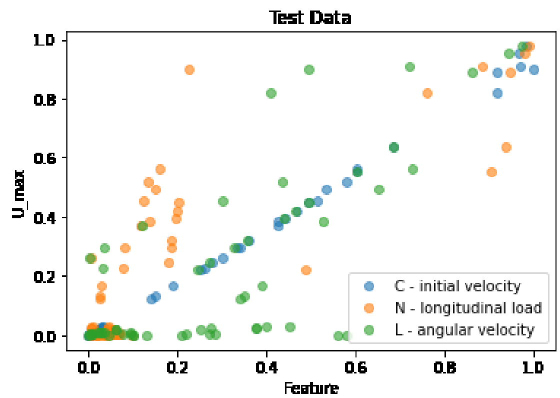


Figure 2: Test dataset

2.2.2 Random forest

The Random Forest algorithm is a machine learning method that uses an ensemble of decision trees for classification and regression tasks. The Random Forest algorithm operates as follows:

1. A random subset of objects (bootstrap sample) and a random subset of features (feature subset) are selected from the training dataset.

2. For each selected feature subset, a decision tree is built using an information criterion.

3. Steps 1 and 2 are repeated k times, where k is the number of trees in the forest.

4. For classifying a new object, each tree in the forest makes a prediction, and the final decision is determined by majority voting.

5. For regression tasks, each tree predicts a value, and the final prediction is calculated by averaging the results across all trees.

To build the Random Forest model, the number of trees (the `n_estimators` parameter) was set to 100. These trees are built independently of each other, and the algorithm selects a random subset of features for constructing each tree. To build a tree, a bootstrap sample is first created from the training dataset: from `n_samples` examples, `n_samples` examples are randomly selected with replacement. As a result, the sample has the same size as the original dataset, but some examples may be missing, while others may appear multiple times.

Next, a decision tree is built based on the generated bootstrap sample. The node-splitting algorithm selects a subset of features at each split and determines the best split using one of the selected features. The number of features to consider is controlled by the `max_features` parameter [9].

Let us visualize the decision boundaries of the first 10 trees and the aggregated prediction provided by the Random Forest model. Each plot in the graph corresponds to a decision tree trained on the training dataset. Including all three features results in a 3D graph (Figure 3).The model built using all decision trees is shown in Figure 4.

The "feature importance"function in the Random Forest algorithm was used to determine the importance of each feature. This metric indicates how significant each feature is in the decision-making process. The importance of the features in the dataset is shown in Figure 5.

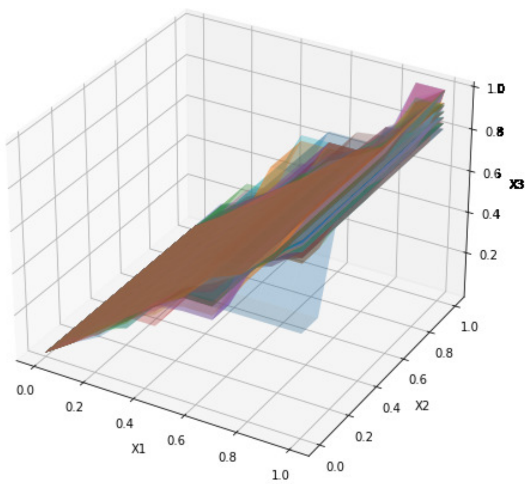


Figure 3: Decision trees model

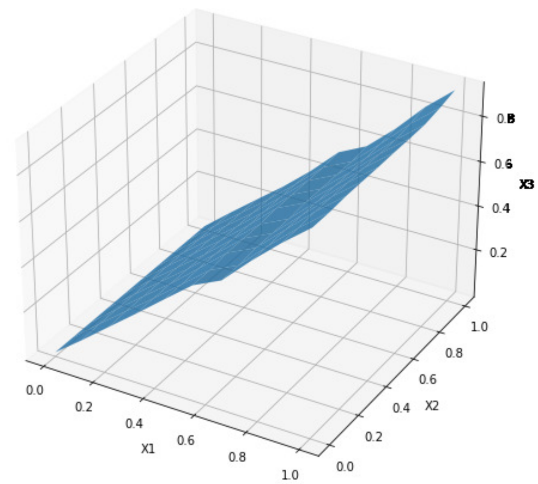


Figure 4: Random Forest model

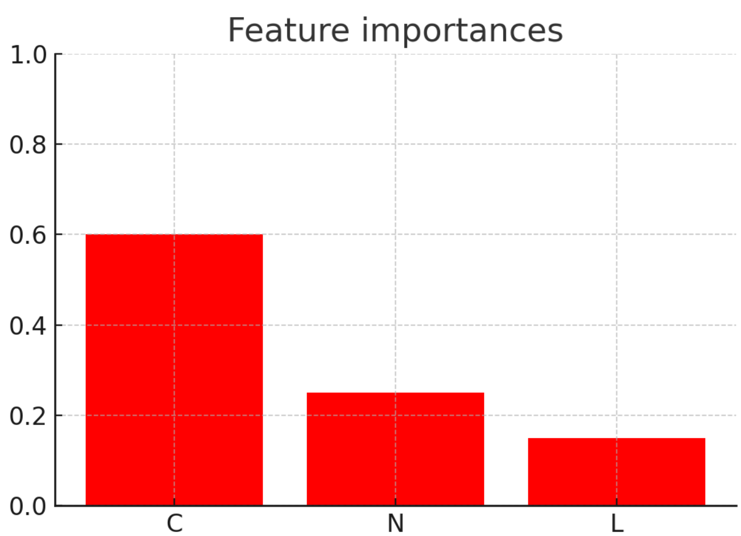


Figure 5: Visualization of feature importance

2.2.3 Linear regression

Linear regression is a machine learning method used to predict numerical values based on a linear relationship between features and the target variable. The general prediction formula for linear regression is as follows:

$$\hat{y} = w[0] \cdot x[0] + w[1] \cdot x[1] + w[p] \cdot x[p] + b \tag{7}$$

where $x[0]$ to $x[p]$ are the features. In our case, $p = 3$. w and b are the model parameters, while $\hat{y}$ is the prediction generated by the model [10].

The model is trained by finding the optimal coefficients w and b that minimize the mean squared error (MSE) between the predicted and actual target values. The parameters are adjusted to minimize the MSE, ensuring the best possible predictive performance. The 3D linear regression model is illustrated in Figure 6.

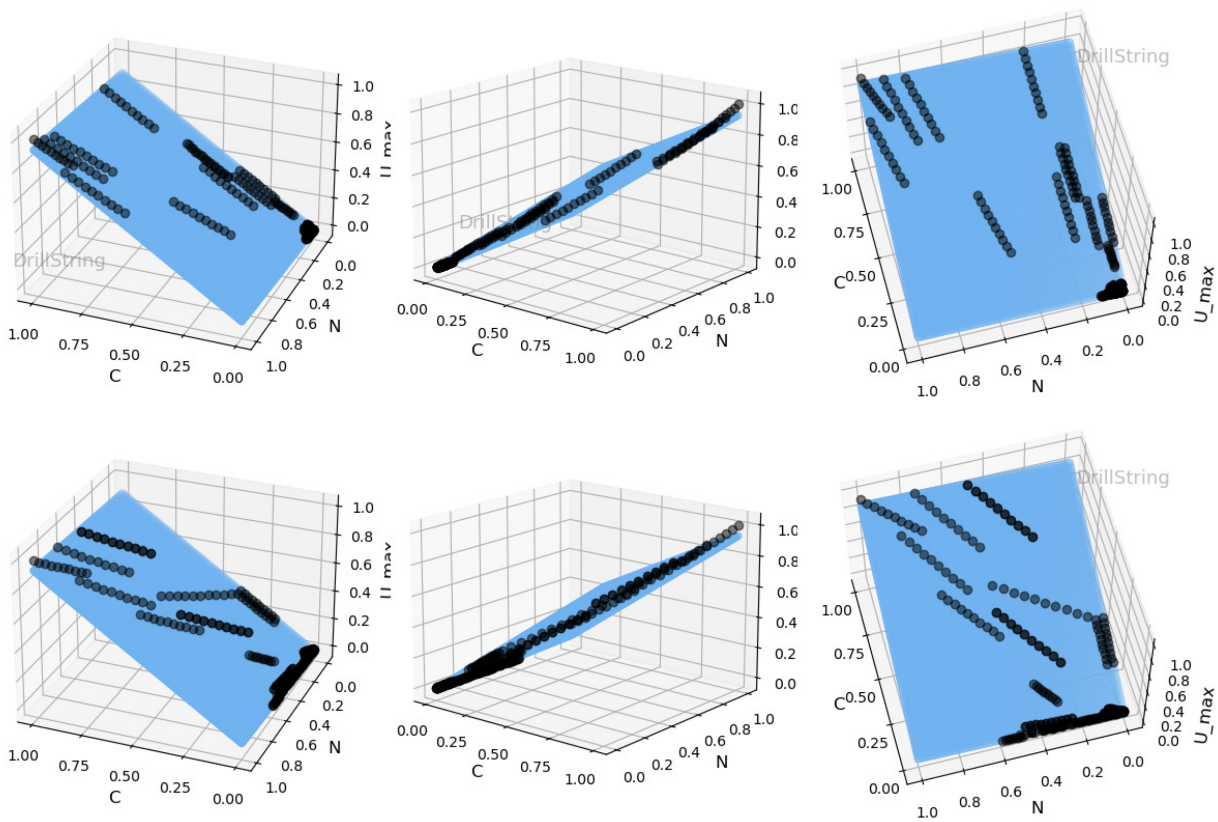


Figure 6: The 3D linear regression model

Figure 7 shows the visualization of the linear model predicting the initial velocity based on vibration values.

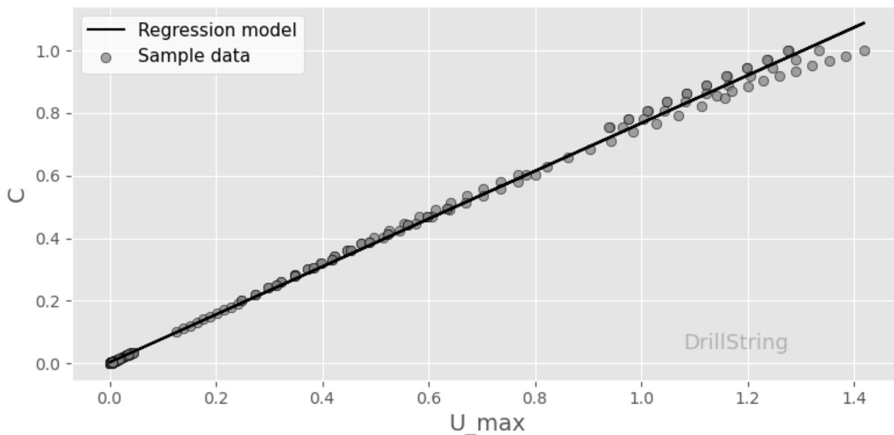


Figure 7: Visualization of the linear model

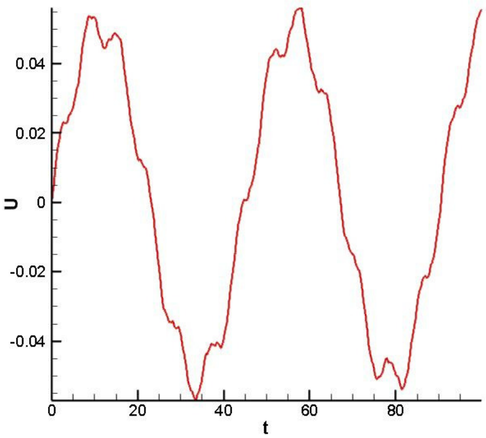
3 Results and discussions

The parameters obtained from [8] in Table 1 were used to obtain the results of the numerical solution of the linear mathematical model.

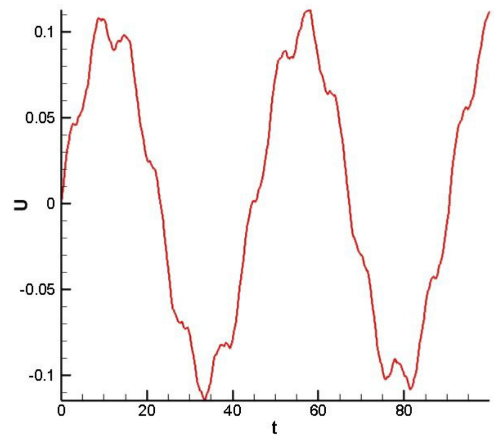
Table 1: Parameter values of the drilling system

Young's modulus	$2.1 \times 10^{11}\ Pa$
Drill string density	$7800\ kg/m^3$
Drill string length	$200\ m$
Inner diameter of the drill string	$0.12\ m$
Outer diameter of the drill string	$0.20\ m$
Cross-sectional area of the string	$2.1 \times 10^{-2}\ m^2$
Angular velocity of the string	$0.083\ rad/s$
Longitudinal load	$1200\ N$
Initial velocity	$0.01\ m/s$
Spatial step	$0.1\ m$
Time step	2×10^{-5}

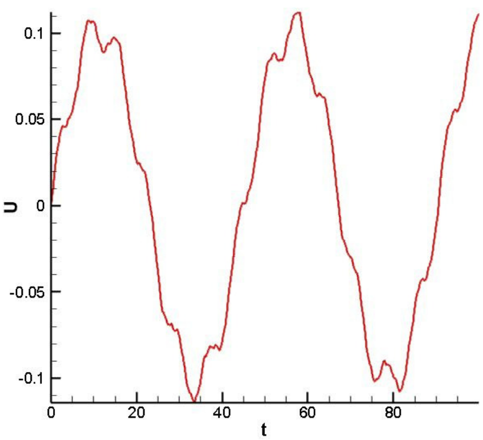
Figure 8 shows the vibration graphs at different parts of the drill string. As a result, we observe that the largest vibrations occur in the middle of the drill string.This is due to the resonance vibrations caused by the interaction of the elastic properties of the drill string material, gyroscopic forces, and external loads, including friction against the wellbore walls. The maximum vibration amplitudes occur in the middle of the drill string due to the distribution of mass, length, and boundary fixation conditions [11].



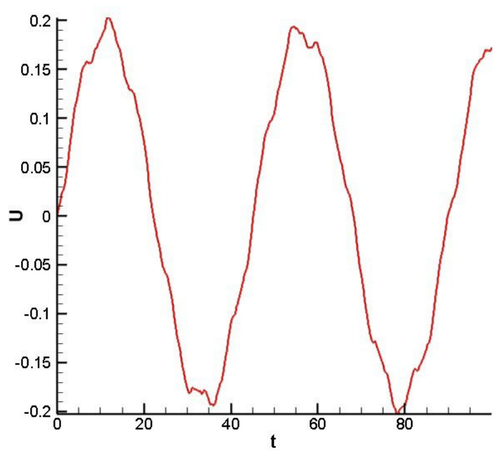
a.The time variation of $U\left[\frac{l}{2}\right]$



b.The time variation of $U\left[\frac{l}{4}\right]$



c.The time variation of $U\left[\frac{3l}{4}\right]$



d.The time variation of $U\left[\frac{l}{3}\right]$

Figure 8: The time variation of displacement U at different cross-sections

3.1 Evaluation of machine learning model performance

To evaluate the performance of the developed models, 25% of the dataset was set aside as test data before model training. This test dataset will be used to assess the model's performance by generating predictions for each test sample based on the provided features. The predicted values will then be compared with the actual observed values. For this purpose, the `score` method of the trained model object will be applied. Figures 9, 10 illustrate the comparison of the Random Forest model and the Linear Regression model with the test dataset.

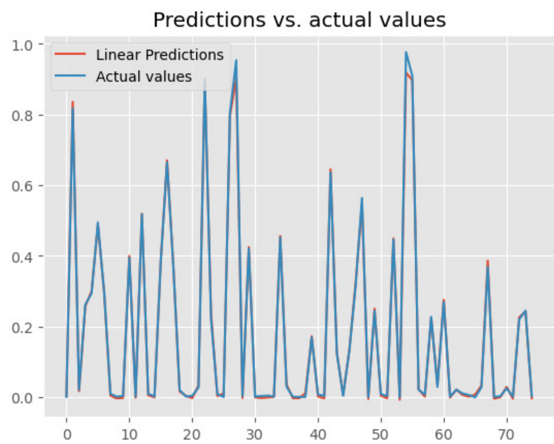


Figure 9: Comparison of the linear regression model with actual data

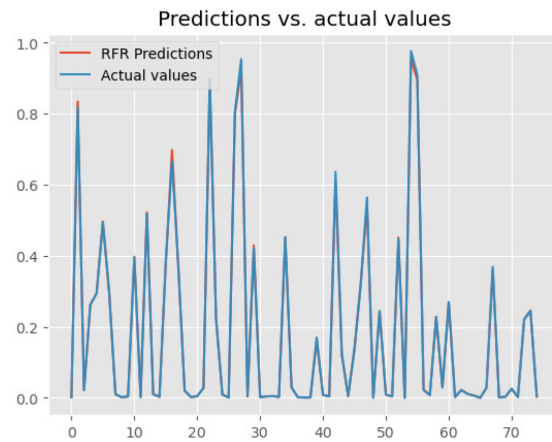


Figure 10: Comparison of the random forest model with actual data

As observed, the predicted values closely align with the actual values, with only minor discrepancies in certain areas. The x-axis represents the 75 test samples, while the y-axis corresponds to their respective values. The results indicate that the model demonstrates high accuracy in predicting vibration characteristics.

Table 2 presents a comparison of the results obtained from the Random Forest model, the Linear Regression model, and the numerical solution of the mathematical model under various conditions.

Table 2: Comparison of model predictions with actual values

C	N	L	Random Forest U_{max}	Linear Regression U_{max}	Actual Value U_{max}
0.46	3000	0.15	0.28344	0.286592	0.28610
0.30	2800	0.05	0.18170	0.18549	0.18639
0.15	3800	0.33	0.09693	0.09369	0.09355
0.70	2500	0.25	0.44384	0.438285	0.435873

Table 3 presents the results of the second linear regression analysis, where the initial velocity serves as the independent variable and the vibration amplitude as the dependent variable. Additionally, the table includes the corresponding results obtained from the mathematical model.

Table 3: Comparison of the model's predicted initial velocity with the actual value

U_max	Linear Regression, C	Actual Value, U_max
0.33	0.5	0.31
0.0025	0.0093	0.005
0.085	0.13	0.084
0.22	0.34	0.21
0.75	1.14	0.70

The linear regression model demonstrated better performance than the random forest model, making it the preferred choice for this task. Random forest offers high accuracy, robustness to large datasets, and feature importance evaluation but suffers from complex interpretability and high computational cost. Linear regression, on the other hand, provides simplicity, efficiency, and clear result interpretation while being sensitive to outliers and limited in capturing nonlinear relationships. Given its reliable performance on the given dataset, linear regression is recommended for predicting drill string vibrations.

4 Conclusion

The results of this study demonstrate that the application of machine learning techniques significantly improves the accuracy of drill string vibration predictions. Three models were developed and tested. Two models were designed to predict the maximum vibration amplitude based on input parameters: one using the random forest algorithm and the other using linear regression. Comparative analysis showed that linear regression performed better on new data due to the linear dependency between initial velocity and maximum vibration amplitude. However, with larger datasets, the performance of linear regression may decline, making the random forest algorithm more suitable for such cases. The third model predicts the initial velocity required to keep the vibration amplitude below a specified threshold, aiding in optimizing operational parameters and ensuring wellbore stability. Linear regression was chosen for this model due to its simplicity and accuracy with the given dataset. The use of machine learning techniques in this context offers several advantages, including faster and more accurate predictions, process optimization through reduced preparation time, and automated data analysis for real-time decision-making. However, the models require high-quality data and sufficient sample size for reliable performance, which can be challenging when dealing with complex drilling conditions. Future work could involve expanding the dataset with more diverse parameters and exploring advanced models to capture nonlinear relationships more effectively. Overall, the study confirms the potential of machine learning techniques for predicting drill string vibrations and optimizing drilling operations in the oil and gas industry.

5 Acknowledgement

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23490543).

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Received: February 26, 2025
Accepted: April 23, 2025

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