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ON PROPER EXPANSIONS AND PROPER CONTRACTIONS OF NONLINEAR OPERATORS REPRESENTED IN THE FORM OF A PRODUCT

Today there are many works devoted to the questions of expansion and contraction of operators [1–12]. In all these works the questions of expansion of the additive “minimal” operator and the questions of contraction of the additive “maximal” operator are considered. In this paper it is shown that these restrictions on the additivity of the corresponding operators are not essential. In [10] the questions of proper contraction of a maximal operator represented as a product are considered, i.e., the relationship between the set of proper contractions of the operator $A = LM$ and the sets of proper contractions of the operators L and M is established. Here, an abstract theorem is proved which allows us to establish the relationship between the set of proper extensions of the operator $A_0 = L_0M_0$ and the sets of proper extensions of the operators L_0 and M_0. In this connection, we prove an abstract theorem that allows us to describe the correct contractions of one class of nonlinear operators represented as a product.
Key words: operator, correct expansion, correct contraction, regular expansion, Bitsadze-Samarskii type problem.

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Көбейтінді түрінде берілген сызықтық емес операторлардың дұрыс кеңейтілімдері мен дұрыс тарылымдары туралы.

Осы таңда операторлардың дұрыс тарылымы мен дұрыс кеңейтілімі бойынша көптеген жұмыстар жарық көрген [1–12]. Бұл жұмыстарда аддитивті "минимальды" операторлардың кеңейтілімдері мен аддитивті "максимальды" операторлардың тарылымдары қарастырылған. Бұл жұмыста қарастырылатын операторлардың аддитивтілігі маңызды болмайтыны көрсетілген. Автордың [10] еңбегінде көбейтінді түрінде берілген сызықтық максимальды операторлардың дұрыс тарылымдары қарастырылған, яғни аталмыш еңбекте $A = LM$ операторының барлық дұрыс тарылымдары жиыны мен L және M операторларының барлық дұрыс тарылымдары жиындары арасында өзара бірмәнді сәйкестік орнатылған. Бұл жұмыста $A_0 = L_0M_0$ операторының барлық дұрыс кеңейтілімдері мен L_0 және M_0 операторларының барлық дұрыс кеңейтілімдері арасында тығыз байланыс барын көрсететін абстракциялы теорема дәлелденген. Осы орайда дәлелденген теорема көбейтінді түрінде берілген қайсыбір сызықтық емес операторлар санатынаның дұрыс кеңейтілімдерін сипаттауға болатыны мысал арқылы көрсетілген.
Түйін сөздер: оператор, дұрыс кеңейтілім, дұрыс тарылым, регулярлы кеңейтілім, Бицадзе-Самарский типтес есептер.

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О корректных расширениях и коррекных сужениях нелинейных операторов, представленных в виде произведения

На сегодня имеются множество работ, посвященных вопросам расширения и сужения операторов [1–12]. Во всех этих работах рассматриваются вопросы расширения аддитивного "минимального" оператора и вопросы сужения аддитивного "максимального" оператора. В данной работе показано, что эти ограничения аддитивности соответствующих операторов не существенны. В работе [10] рассмотрены вопросы корректного сужения максимального оператора, представимого в виде произведения, т.е. установлено взаимосвязь между множеством правильных сужении оператора $A = LM$ и множествами правильных сужении операторов L и M. Здесь доказана абстрактная теорема, позволяющая установить взаимосвязь между множеством правильных расширении оператора $A_0 = L_0 M_0$ и множествами правильных расширении операторов L_0 и M_0. В этой связи, доказывается абстрактная теорема, позволяющая описать правильные сужения одного класса нелинейных операторов, представимых в виде произведения.
Ключевые слова: оператор, корректное расширения, корректное сужение, регулярное расширение, задача типа Бицадзе-Самарского.

1 Introduction

In this work we consider following PDE

$$u^{2n}\frac{\partial^2 u}{\partial x \partial y} + 2n \cdot u^{(2n-1)}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} = f(x,y), f(x,y) \in C(G)$$

A condition of univariate solvability of one problem of Bitsadze-Samarsky type is shown. Here $C([0,1]\times[0,1]) \equiv C(G)$.

Keywords: operator, proper extension, proper restriction, regular extension, Bitsadze-Samarsky type problem.

Let us briefly recall some provisions of [10].

Let the operator $A = LM$ act in the Banach space B. Here, L is a certain additive closed operator for which $D(L) \subset B$ and $R(L) = B$. In the domain of definition $D(L)$, we introduce the norm

$$\|u\|_{\mathfrak{M}} = \|u\|_{\mathbb{B}} + \|Lu\|_{\mathbb{B}}, \quad u \in D(L). \tag{1}$$

The closure of the manifold $D(L)$ in the norm (1) will be denoted by $\mathfrak{M}$. It is evident that $\mathfrak{M}$ is a Banach space. Now, let the operator $\mathfrak{M}$ map the manifold $D(M)$ onto the space $\mathfrak{M}$, i.e., $R(M) = \mathfrak{M}$. Then, we define the operator A by the equality $A = LM$. Clearly, $D(A) = D(M)$ and $R(A) = R(L) = B$. If $\tilde{L}$ and $\tilde{M}$ are certain proper restrictions of the operators L and M, respectively, then the following holds:

Theorem 1 *The operator $\tilde{A}^{-1} = \tilde{M}^{-1}\tilde{L}^{-1}$ is invertible, and its inverse $\tilde{A}$ is a proper restriction of the operator A.*

Additionally, the following lemma is proven:

Лемма 1 $\ker A = B$, *where*

$$\mathfrak{B} = \{g = \tilde{M}^{-1}g_1 + g_2, g_1 \in \ker L,\ g_2 \in \ker M\}.$$

Using this lemma, an abstract theorem is proved, which provides a complete description of the set of all proper restrictions of the operator A in terms of the sets of all proper restrictions of the operators L and M.

Now, we will show that this method can also be applied to a certain class of nonlinear operators that can be represented as a product.

Within the previously used notation, let us additionally consider a (generally nonlinear) bijective mapping $N : \mathbb{B} \to \mathbb{B}$ such that $N(0) = 0$. Then, we can define the product

$$A = LMN \tag{2}$$

Clearly, $D(A) = D(MN) = D(M)$ and $R(A) = R(L) = \mathbb{B}$. Let $\tilde{L}$ and $\tilde{M}$ still be certain proper restrictions of the operators L and M, respectively. Then, the following holds:

Theorem 2 *The operator $\tilde{A}^{-1} = N^{-1}\tilde{M}^{-1}\tilde{L}^{-1}$ is invertible, and its inverse $\tilde{A}$ is a proper restriction of the operator A.*

Proof 1 *The product $N^{-1}\tilde{M}^{-1}\tilde{L}^{-1}$ defines a certain operator $\tilde{A}^{-1}$. Indeed, by definition, we have $R(\tilde{L}) = D(\tilde{L}^{-1}) = \mathbb{B}$ and $R(\tilde{L}^{-1}) = D(\tilde{L}) \subset \mathfrak{M}, D(N^{-1}\tilde{M}^{-1}) = D(\tilde{M}^{-1}) = \mathfrak{M}$. Therefore, the operator $\tilde{A}^{-1} = N^{-1}\tilde{M}^{-1}\tilde{L}^{-1}$ is well-defined with domain $D(\tilde{A}^{-1}) = \mathbb{B}$ and range $R(\tilde{A}^{-1}) = N^{-1}\tilde{M}^{-1}D(\tilde{L}) \subset \mathbb{B}$.*

Now, if for some $f \in \mathbb{B}$ the equality $\tilde{A}^{-1}f = 0$ holds, then

$$f = LMN(N^{-1}\tilde{M}^{-1}\tilde{L}^{-1}f) = A(\tilde{A}^{-1}f) = 0,$$

which means that the operator $\tilde{A}^{-1}$ has an inverse operator $\tilde{A}$. Since the operator $\tilde{A}^{-1}$ is continuous, it follows that $\tilde{A}$ is a proper restriction of the operator A. The theorem is proved.

Previously, we considered proper restrictions of operators that can be represented as a product. Now, we will show that it is also possible to consider proper extensions of such operators.

Let L be a certain closed additive operator with domain $D(L) \subset \mathbb{B}$ and range $R(L) = \mathbb{B}$, where $\mathbb{B}$ is a Banach space. Let L_0 be a restriction of the operator L, which has a continuous inverse L_0^{-1} on $R(L_0)$ and satisfies $\overline{R(L_0)} \neq \mathbb{B}$, i.e., the operator L_0 has a continuous left inverse.

By taking the closure of the manifold $D(L)$ in the norm (4), we obtain the Banach space $\mathfrak{M}$. Let $\mathfrak{M}_0$ denote the closure of the manifold $D(L_0)$ in the norm (1).

Let the operator M_0 satisfy the following conditions:

a) $D(M_0) \subset B, R(M_0) \subset M_0$;

b) On the set $R(M_0)$, the operator M_0 has a continuous inverse M_0^{-1}.

Then, the product $A_0 = L_0M_0$ is well-defined, and we have

$D(A_0) = D(M_0) \subset B, \quad R(A_0) = R(L_0) \subset B.$

Clearly, the inverse operator $A_0^{-1} = M_0^{-1}L_0^{-1}$ is well-defined on the set $R(A_0)$.

Let $\tilde{L}$ be a regular extension of the operator L_0, i.e., $L_0 \subset \tilde{L} \subset L$. Let $\tilde{M}$ be a proper extension of the operator M_0. Then, the following holds:

Theorem 3 *The operator $\tilde{A}^{-1} = \tilde{M}^{-1}\tilde{L}^{-1}$ is invertible, and its inverse $\tilde{A}$ is the correct extension of the operator A_0.*

Proof 2 *It is evident that the operator* $\tilde{A}^{-1}$ *is defined on the entire space* B. *Now, let us show that the operator* $\tilde{A}^{-1}$ *is invertible. Indeed, if for some* $f \in B$ *we have* $\tilde{A}^{-1}f = 0$, *then* $\tilde{M}^{-1}\tilde{L}^{-1}f = 0$. *Since the operators* $\tilde{M}^{-1}$ *and* $\tilde{L}^{-1}$ *have inverses, we obtain* $\tilde{L}^{-1}f = 0$, *which implies* $f = 0$. *Therefore, the operator* $\tilde{A} = (\tilde{A}^{-1})^{-1}$ *exists. Now, it is sufficient to show that* $A_0 \subset \tilde{A}$.

Indeed, if $u_0 \in D(A_0)$, *then* $L_0 u_0 \in D(M_0)$, *i.e., there exists an element* $f_0 \in R(A_0)$ *such that* $f_0 = A_0 u_0$ *and* $u_0 = A_0^{-1} f_0$. *Then,*

$$\tilde{A}^{-1} f_0 = \tilde{M}^{-1}\tilde{L}^{-1} f_0 = \tilde{M}^{-1} L_0^{-1} f_0 = M_0^{-1} L_0^{-1} f_0 = u_0.$$

Therefore, $u_0 \in D(\tilde{A})$, *i.e.,* $\tilde{A}u_0 = f_0$. *The theorem is proven.*

Next, using this theorem, as an example, let's consider the proper extensions of a certain nonlinear differential operator.

In the space $C([0,1] \times [0,1]) \equiv C(G)$, we consider the following differential equation:

$$u^{2n}\frac{\partial^2 u}{\partial x \partial y} + 2n \cdot u^{2n-1}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in C(G), \tag{3}$$

Let L denote the operator acting as the differential expression u'_y with the domain of definition:

$$D(L) = \{u \in C(G) : \frac{\partial u}{\partial y} \in C(G)\}$$

Let M denote a Banach space obtained by closing the manifold $D(L)$ with respect to the norm:

$$\|u\|_M = \|u\|_{C(G)} + \|Lu\|_{C(G)}. \tag{4}$$

Let L_0 be the restriction of the operator L with the domain of definition:

$$D(L_0) = \{u \in D(L) : u(x,0) = 0, u(x,1) = 0\}.$$

Then,

$$R(L_0) = \{f(x,y) \in C(G) : \int_0^1 f(x,\tau)d\tau = 0\} \subset C(G).$$

In the set $R(L_0)$, there exists a continuous inverse L_0^{-1}:

$$L_0^{-1} f = \int_0^y f(x,\tau)d\tau.$$

Let M_0 denote the Banach space obtained by closing $D(L_0)$ with respect to the norm (4).

Let M_0 denote the operator $M_0 : D(M_0) \to R(M_0)$, where $D(M_0) \subset C(G), R(M_0) \subset B_0$, and:

$$D(M_0) = \{u \in C(G) : u^{2n}\frac{\partial u}{\partial x} \in B_0, u(x,0) = u(x,1) = 0\},$$

$$R(M_0) = \{f \in B_0 : \int_0^1 f(t,y)dt = 0\}.$$

In the set $R(M_0)$, there exists a continuous inverse M_0^{-1}

$$M_0^{-1} f = \sqrt{2n+1}\left[\int_0^x f(t,y)dt\right]^{1/(2n+1)}.$$

Let the operator $\tilde{L}$ be generated by the following boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in C(G), \\ u(x,0) = 0, \end{cases} \tag{5}$$

The operator $\tilde{L}$ is a regular extension of the operator L_0: $L_0 \subset \tilde{L} \subset L$, and:

$$\tilde{L}^{-1} f = \int_0^x f(x,\tau)d\tau.$$

Also, the operator $\tilde{M}$, generated by the following boundary value problem:

$$\begin{cases} u^{2n}\frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in B, \\ u(0,y) = 0, \end{cases} \tag{6}$$

is a proper extension of the operator M_0, and its inverse is:

$$\tilde{M}^{-1} f = \sqrt{2n+1}\left[\int_0^x f(t,y)dt\right]^{1/(2n+1)}.$$

Лемма 2 *The unique solution to the boundary value problem*

$$\begin{cases} u^{2n}\frac{\partial^2 u}{\partial x \partial y} + 2n \cdot u^{2n-1}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in C(G), \\ u(x,0) = 0, \quad u(0,y) = 0, \end{cases} \tag{7}$$

has the form:

$$u(x,y) = \sqrt{2n+1}\left[\int_0^x \int_0^y f(t,\tau)dtd\tau\right]^{1/(2n+1)}. \tag{8}$$

Proof 3 *According to Theorem 3, the operator*

$$\tilde{A}^{-1} = \tilde{M}^{-1}\tilde{L}^{-1}$$

is invertible, and its inverse operator $\tilde{A}$ is a proper extension of the operator $A_0 = L_0 M_0$. Therefore, from (5) and (7), we conclude that the boundary value problem

$$u^{2n}\frac{\partial^2 u}{\partial x \partial y} + 2n \cdot u^{2n-1}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = f(x,y), \quad f(x,y) \in C(G), \tag{9}$$

$$u(0,y) = 0, \tag{10}$$

$$\frac{\partial}{\partial x}u^{2n+1}(x,0)=0 \tag{11}$$

has a unique solution of the form (8). The boundary value problems (7) and (9)–(11) are equivalent. Indeed, integrating the condition (11) with respect to x, we get $u^{2n+1}(x,0)-u^{2n+1}(0,0)=0$, *and from (8), we have* $u(0,0)=0$, *so we obtain that* $u(x,0)=0$.

Now, let L_{10} *be the operator generated by the following boundary value problem:*

$$\frac{\partial u}{\partial x}=f(x,y), \quad f(x,y)\in C(G),$$

$$a_1(y)u(0,y)+a_2(y)u(\phi(y),y)+a_3(y)u(1,y)=0,$$

$$\frac{\partial u(0,y)}{\partial x}=0,$$

where $x=\phi(y)$ *is a smooth curve located in the region* G, $a_i(y)\in C[0,1]$, *and*

$$a^*(y)=a_1(y)+a_2(y)+a_3(y)\neq 0, y\in[0,1] \tag{12}$$

It is clear that:

$$L_{10}^{-1}f=\int_0^x f(t,y)dt-\frac{a_2(y)}{a^*(y)}\int_0^{\phi(y)} f(t,y)dt-\frac{a_3(y)}{a^*(y)}\int_0^1 f(t,y)dt, \tag{13}$$

and

$$R(L_{10})=\{f(x,y)\in C(G): f(0,y)=0\}.$$

As a regular extension of the operator L_{10}, *we take the operator* L_1, *generated by the following boundary value problem:*

$$\frac{\partial u}{\partial x}=f(x,y), \quad f(x,y)\in C(G),$$

$$a_1(y)u(0,y)+a_2(y)u(\phi(y),y)+a_3(y)u(1,y)=0.$$

For this problem to have a unique solution, it is necessary and sufficient to fulfill the condition (12) (i.e., $a^*(y)\neq 0$*), and the unique solution is given by (13).*

As the operator M_0, *we take the previously considered operator, i.e., the operator* $M_0: D(M_0)\to R(M_0)$, *where* $D(M_0)\subset C(G)$, $R(M_0)\subset B_0$, *and:*

$$D(M_0)=\{u\in C(G): u^{2n}\frac{\partial u}{\partial x}\in B_0, u(x,0)=u(x,1)=0\}.$$

We also consider the operator $\tilde{M}$, *generated by the boundary value problem (6). This operator is a proper extension of the operator* M_0, *and:*

$$\tilde{M}^{-1}f=\sqrt{2n+1}\left[\int_0^x f(t,y)dt\right]^{1/(2n+1)}.$$

Then, by Theorem 3, we have that the operator $A_1=L_1\tilde{M}$ *is a proper extension of the operator* $A_{10}=L_{10}M_0$. *Thus, we have proven the following theorem:*

Theorem 4 *In order for the problem*

$$u^{2n}\frac{\partial^2 u}{\partial x\partial y}+2n\cdot u^{2n-1}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=f(x,y),\quad f(x,y)\in C(G),$$

$$a_1(y)u(0,y)+a_2(y)u(\phi(y),y)+a_3(y)u(1,y)=0,$$

$$u(x,0)=0,$$

to be uniquely solvable, it is necessary and sufficient to fulfill the inequality $a^*(y)\neq 0$*,* $y\in[0,1]$*, and its unique solution is given by:*

$$u(x,y)=\sqrt{2n+1}\left[\int_0^y\int_0^x f(t,\tau)dtd\tau-\int_0^y\frac{a_2(\tau)}{a^*(\tau)}\int_0^{\phi(\tau)} f(t,\tau)dtd\tau\right]^{1/(2n+1)}-$$

$$-\sqrt{2n+1}\left[\int_0^y\frac{a_3(\tau)}{a^*(\tau)}\int_0^1 f(t,\tau)dtd\tau\right]^{1/(2n+1)},$$

where

$$a^*(y)=a_1(y)+a_2(y)+a_3(y)\neq 0,\quad y\in[0,1].$$

Remark 1 *: The results of Theorem 4 can also be obtained by applying Theorem 2. In this case, as the bijective map* $N:C(G)\to C(G)$*, we take the operator* N *acting as* $N(u)=u^{2n+1}$*,* $u\in C(G)$*.*

2 Conclusion

In this work an abstract theorem is proved which allows us to establish the relationship between the set of proper extensions of the operator $A_0=L_0M_0$ and the sets of proper extensions of the operators L_0 and M_0. In this connection, author proves an abstract theorem that allows us to describe the correct contractions of one class of nonlinear operators represented as a product.

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OPTIMAL APPROXIMATION OF SOLUTIONS OF POISSON EQUATION BY INITIAL DATA IN THE FORM OF ACCURATE AND INACCURATE INFORMATION OF TRIGONOMETRIC FOURIER COEFFICIENTS

Partial differential equations along with a function, derivative, and integral are basic mathematical models. Therefore, the problem of their approximation by accurate and inaccurate information with the construction of optimal computational aggregates (approximation methods) of approximation is relevant and many articles are devoted to this issue.
In the article is considered the problem of approximation of solutions of Poisson equation with the right-hand side f from the Nikol'skii classes $H_2^r(0,1)^s$ in the Lebesgue metrics $L^2(0,1)^s$ and $L^\infty(0,1)^s$. The orders of error of approximation of solutions of the Poisson equation by accurate and inaccurate information in the form of trigonometric Fourier coefficients of f are obtained. Namely, a lower bound for the approximation error based on accurate information is found for all possible computational agregates using an arbitrary finite set of trigonometric Fourier coefficients. A computational agregate (approximation method) by the trigonometric Fourier coefficients of the right-hand side f of the equation is constructed that confirms this lower bound. The boundaries of $\tilde{\varepsilon}_N$ of inaccurate information preserving the order of error of approximation by accurate information are established.
Key words: Poisson equation, approximation by accurate and inaccurate information, Nikol'skii classes, optimal computational aggregate, boundaries of inaccurate information.

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Тригонометриялық Фурье коэффициенттерінен алынған дәл және дәл емес бастапқы ақпарат бойынша Пуассон теңдеуінің шешімдерін оптималды жуықтау

Дербес туындылы дифференциалдық теңдеулер функция, туынды және интегралмен қатар негізгі математикалық модельдер қатарына жатады. Сондықтан, дәл және дәл емес ақпарат бойынша оларды жуықтаудың оптималды есептеу агрегаттарын (жуықтау әдістерін) құру мәселесі өзекті болып табылады және осы мәселеге көптеген мақалалар арналған.
Мақалада f оң жағы $H_2^r(0,1)^s$ Никольский класында жататын Пуассон теңдеуінің шешімдерін $L^2(0,1)^s$ және $L^\infty(0,1)^s$ Лебег метрикаларында жуықтау есебі қарастырылады. f функциясының тригонометриялық Фурье коэффициенттері түрінде берілген дәл және дәл емес ақпарат бойынша Пуассон теңдеуінің шешімдерін жуықтау қателігінің реті алынды. Атап айтқанда, тригонометриялық Фурье коэффициенттерінің кез келген ақырлы жиынын қолданып, барлық мүмкін есептеу агрегаттары үшін дәл ақпараттарға негізделген жуықтау қателігінің төменнен бағалауы алынды. Төменнен бағалауды растайтын есептеуіш агрегат (жуықтау әдісі) теңдеудің оң жақ тригонометриялық Фурье коэффициенттері бойынша құрылды. Дәл ақпарат бойынша жуықтау қателігінің ретін сақтайтын дәл емес ақпараттың $\tilde{\varepsilon}_N$ шекаралары анықталды.
Түйін сөздер: Пуассон теңдеуі, дәл және дәл емес ақпарат бойынша жуықтау, Никольский кластары, тиімді есептеу агрегат, дәл емес ақпарат шекаралары.

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Оптимальное приближение решений уравнений Пуассона по исходным данным в виде точных и приближенных значений тригонометрических коэффициентов Фурье

Дифференциальные уравнения в частных производных наряду с функцией, производной, интегралом относятся к основным математическим моделям.
Следовательно задача их приближения по точным и неточным данным с построением оптимальных вычислительных агрегатов (методов приближения) является актуальной и данному вопросу посвещено множество статей. В статье изучается задача приближения решений уравнения Пуассона с правой частью f из классов Никольского $H_2^r(0,1)^s$ в Лебеговой метриках $L^2(0,1)^s$ и $L^\infty(0,1)^s$. Получены порядки погрешности приближения решений уравнения Пуассона по точным и неточным данным в виде тригонометрических коэффициентов Фурье функции f. Именно, найдена оценка снизу погрешности приближения по точным данным по всем возможным вычислительным агрегатам, использующим конечный набор тригонометрических коэффициентов Фурье. Построен вычислительный агрегат (метод приближения) по тригонометрическим коэффициентам Фурье правой части f уравнения, подтверждающий данную оценку снизу. Установлены границы $\tilde{\varepsilon}_N$ неточной информации, сохраняющие порядок убывание по точной информации.
Ключевые слова: уравнение Пуассона, приближение по точным и неточным данным, классы Никольского, оптимальный вычислительный агрегат, границы неточной информации.

1 Introduction

Solutions of partial differential equations, even when expressed explicitly by means of Fourier series in the eigenfunctions of the corresponding differential operator or convolution with the corresponding kernels, being represented by series or integrals, in fact again infinite objects. Therefore, the problem of approximating them with finite objects again arises. In the article is considered the problem of approximation of solutions of Poisson equations in the sence of computational (numerical) diameter (denoted by C(N)D). Poisson equation has an various applications. One of them is that it describes the distribution of an electrostatistics, potential theory, scalar field, such as an electric potential or gravitational potential, in space. Thus, its physical meaning is that it relates the distribution of field sources to the field itself. Therefore, it is important to take this equation into account. Let at first consider the definition of computational (numerical) diameter problem.

In computational (numerical) diameter the initial definition is (see, for example, [1]- [2])

$$\delta_N(\varepsilon_N; D_N)_Y \equiv \delta_N(\varepsilon_N; T; F; D_N)_Y \equiv \inf_{(l^{(N)};\varphi_N)\in D_N} \delta_N(\varepsilon_N; (l^{(N)}; \varphi_N))_Y$$

where

$$\delta_N(\varepsilon_N; (l^{(N)}; \varphi_N))_Y \equiv \delta_N(\varepsilon_N; T; F; (l^{(N)}; \varphi_N))_Y \equiv$$
$$\equiv \sup_{\substack{f\in F,\\ \left\{\gamma_N^{(\tau)}\right\}_{\tau=1}^N, \left|\gamma_N^{(\tau)}\right|\le 1,\\ (\tau=1,\dots,N)}} \|Tf(\cdot) - \varphi_N(l_N^{(1)}(f) + \gamma_N^{(1)}\varepsilon_N^{(1)}; \dots, l_N^{(N)}(f) + \gamma_N^{(N)}\varepsilon_N^{(N)}; \cdot)\|_Y.$$

Here, a *mathematical model* is given by the operator $T : F \to Y$. X and Y are the normalized spaces of functions defined on Ω_X and Ω_Y respectively, $F \subset Y$ is a class of functions. Numerical information $l^{(N)} \equiv l^{(N)}(f) = (l_N^{(1)}(f), ..., l_N^{(N)}(f))$ of volume $N(N = 1, 2, ...)$ about f from class F is taken by linear functionals $l_N^{(1)}(f), ..., l_N^{(N)}(f)$ (in general, not necessarily linear). An *information processing algorithm* $\varphi_N(z_1, ..., z_N; \cdot) : C^N \times \Omega_X \to C$ is a correspondence, which for every fixed $(z_1, ..., z_N) \in C^N$ as a function of $(\cdot)$ is an element of Y and $\varphi_N(0, ..., 0; \cdot) = 0$. If the class of functions under consideration is centrally symmetric, then the last condition $\varphi_N(0, ..., 0; \cdot) = 0$ can be ignored. The record $\varphi_N \in Y$ means that φ_N satisfies all the conditions listed above, and $\{\varphi_N\}_Y$ is a set composed of all $\varphi_N \in Y$. Further, $(l^{(N)}; \varphi_N)$ is a *computational aggregate* of recovery from accurate information for the function $f \in F$ acting according to the rule $\varphi_N(l_N^{(1)}, ..., l_N^{(N)}; \cdot)$. The recovery of Tf by inaccurate information is proceeding as follows. At first the boundaries of the inaccuracy are set: a vector $\varepsilon_N = (\varepsilon_N^{(1)}, ..., \varepsilon_N^{(N)})$ with non-negative components. Then, the accurate values $l_N^{(\tau)}(f)$ are replaced with a given accuracy $\varepsilon_N^{(\tau)} \geq 0$ by approximate values $z_\tau \equiv z_\tau(f)$, $|z_\tau - l_N^{(\tau)}(f)| \leq \varepsilon_N^{(\tau)}$ $(\tau = 1, ..., N)$, numbers $z_\tau \equiv z_\tau(f)$ $(\tau = 1, ..., N)$ are processed using the algorithm φ_N up to the function $\varphi_N(z_1(f), ..., z_N(f); \cdot)$, which will constitute the computational aggregate $(l^{(N)}; \varphi_N) \equiv \varphi_N(z_1(f), ..., z_N(f); \cdot)$ constructed according to information of the precision $\varepsilon_N = (\varepsilon_N^{(1)}, ..., \varepsilon_N^{(N)})$.

Let $D_N \equiv D_N(F)_Y$ be a given set of complexes $(l_N^{(1)}, ..., l_N^{(N)}; \varphi_N) \equiv (l^{(N)}, \varphi_N)$, we emphasize, operators of recovery by accurate information.

For nonnegative sequences $\{A_N\}$ and $\{B_N\}$, we write $A_N \ll B_N$ (or, equivalently $A_N = O(B_N)$) if there exists a positive constant $c > 0$ such that, for all $N(N = 1, 2, \dots)$ hold $A_N \leq cB_N$. Furthermore, we write $A_N \asymp B_N$ if both $A_N \ll B_N$ and $B_N \ll A_N$ hold simultaneously.

Within the framework of given notations and definitions, the problem of optimal recovery by inaccurate information, framed under the name *computational (numerical) diameter*, according to the [1]- [2], consists in a collective sense in sequential solution of the following three problems: C(N)D-1, C(N)D-2 and C(N)D-3.

For given $T; F; Y; D_N$:

C(N)D-1: an order of $\asymp \delta_N(0; D_N)_Y \equiv \delta_N(0; T; F; D_N)_Y$ is found with the construction of a specific computional aggregate $(\bar{l}^{(N)}, \overline{\varphi}_N)$ from $D_N \equiv D_N(F)_Y$ supporting ordering

$$\asymp \delta_N(0; D_N)_Y;$$

C(N)D-2: for $(\bar{l}^{(N)}, \overline{\varphi}_N)$ is considered the problem of existence and finding a sequence $\widetilde{\varepsilon}_N \equiv \widetilde{\varepsilon}_N(D_N; (\bar{l}^{(N)}; \overline{\varphi}_N))_Y$ with non-negative components such that

$$\delta_N(0; D_N)_Y \asymp \delta_N(\widetilde{\varepsilon}_N; (\bar{l}^{(N)}; \overline{\varphi}_N))_Y \equiv$$

$$\equiv \sup\{\|Tf(\cdot) - \overline{\varphi}_N(z_1, ..., z_N; \cdot)\|_Y : f \in F, |\bar{l}_\tau(f) - z_\tau| \leq \widetilde{\varepsilon}_N^{(\tau)}(\tau \in \{1, ..., N\})\}$$

with simultaneous satisfying the following expression

$$\forall \eta_N \uparrow +\infty (0 < \eta_N < \eta_{N+1}, \eta_N \to +\infty) :$$

$$\overline{\lim_{N \to +\infty}} \delta_N(\eta_N \widetilde{\varepsilon}_N; (\bar{l}^{(N)}, \overline{\varphi}_N))_Y / \delta_N(0; D_N)_Y = +\infty;$$

C(N)D-3: *massiveness* of limiting error $\widetilde{\varepsilon}_N$ is set: as huge as possible set $M_N(\bar{l}^{(N)};\overline{\varphi}_N)$ from D_N (usually associated with the structure of the $(\bar{l}^{(N)};\overline{\varphi}_N)$) of computional aggregates $(l^{(N)},\varphi_N)$ is found, such that for each of them the following relation holds

$$\forall\eta_N\uparrow+\infty(0<\eta_N<\eta_{N+1},\eta_N\to+\infty):$$

$$\overline{\lim_{N\to+\infty}}\delta_N(\eta_N\widetilde{\varepsilon}_N;(l^{(N)},\varphi_N))_Y/\delta_N(0;D_N)_Y=+\infty.$$

In the article is considered the following concretization of computational (numerical) diameter problem. $Tf=u(x,f)$ – the solution of Dirichlet problem of Poisson equations

$$\Delta u\equiv\frac{\partial^2u}{\partial x_1^2}+\frac{\partial^2u}{\partial x_2^2}+...+\frac{\partial^2u}{\partial x_s^2}=f(x), \tag{1}$$

on a unit cube $[0,1]^s$, where $f(x)=f(x_1,...,x_s)\in F=H_2^r$ – Nikol'skii class, Y are Lebesgue metrics L^2 and L^∞ and recovery is performed over all computational aggregates, in which numerical information is specified by trigonometric Fourier coefficients with an arbitrary spectrum:

$$D_N=\Phi_N=\{l_N^{(1)}(f)=\widehat{f}(m^{(1)}),...,l_N^{(N)}(f)=\widehat{f}(m^{(N)}):m^{(j)}\in Z^s(j=1,...,N)\}\times\{\varphi_N\}_Y,$$

where Y is L^2 or L^∞,

$$\widehat{f}(m)=\int_{[0,1]^s}f(x)e^{-2\pi i(m,x)}dx$$

are trigonometric Fourier coeficients, $(m,x)=m_1x_1+...+m_sx_s$, $m=(m_1,...,m_s)$, $x=(x_1,...,x_s)$.

In this article, the computational (numerical) diameter problem in the specified concretization is solved in parts C(N)D-1 and the first part of C(N)D-2. Let's move on to a brief overview of the issue.

One of the first result, when f is odd, approximation of solution to Poisson equation is considered by N.M.Korobov in [3, p. 187-189]. There are approximation operator is constructed on the value of the function f (initial condition) at the points $\left(\left\{\frac{a_1k}{N}\right\},...,\left\{\frac{a_sk}{N}\right\}\right),k\in 1,...,N$, ($\{b\}$ – fractional part of b). If $a_1,...,a_s$ are the optimal coefficients (see definition of optimal coeffitients in [3, p. 96]) modulo N and β index, then the approximation of error is $O\left(\frac{(lnN)^{\frac{r\beta}{2}+s}}{N^{\frac{r}{2}-\frac{1}{2}+\frac{1}{s}}}\right)$.

The authors of [4] were achieved sharp estimates in the power scale for the approximation error, which is almost square times better in comparison with previous result of Korobov. More precisely, with accuracy $O\left(\frac{(lnN)^{(r+2/s)(s-1)}}{N^{r-(1-1/p-2/s)}}\right)$ and $O\left(\frac{(lnN)^{r(\beta+s)+s}}{N^r}\right)$ in cases $1-\frac{1}{p}-\frac{2}{s}>0$ and $1-\frac{1}{p}-\frac{2}{s}\leq 0$ respectively.

For practical purposes, however, in [5] got the result about sampling on sparse grids by the Smolyak's algorithm. In [6] considered the approximation of a function in the Besov class and used it to approximate solutions of Laplace equation. As well as, approximate the solution of 2D and 3D Poisson's equations by the Haar wavelet method is considered in [7]. Research on the problem of approximating solutions of the Poisson equation with accurate information in anisotropic Korobov classes $E^{r_1,\dots,r_s}(0,1)^s$ has been studied recently in the papers [8]-[9]. The problem of approximation of solutions to Poisson equation with right-hand side from Nikol'skii-Besov classes $B^r_{2,\theta}(0,1)^s$ and anisotropic Korobov classes $E^{r_1,\dots,r_s}(0,1)^s$ by the value of the function at the points $\left(\left\{\frac{a_1k}{N}\right\},\dots,\left\{\frac{a_sk}{N}\right\}\right), k\in 1,\dots,N$ is considered in [10]. Approximation by inaccurate information of solutions of Poisson equations with right-hand side $f\in E^{r_1,\dots,r_s}$ is considered in [11] and $f\in E^r_s$ and W^r_2 cases are considered in [12]-[13] respectively. There are obtained upper bound of error of approximation by innacurate information from values at the points of f in uniform metric. In [12], the author approximates the solutions of the Poisson equation in the L^2 metric using an approximation operator constructed from a finite set of Fourier coefficients of the function with right hand side $f\in E^r_s$. Here is given a complete solution for C(N)D problem.

2 Necessary definitions and statements

Definition 1 (see [14], p. 75-76). *The Nikol'skii class* $H^r_q(0,1)^s$ $(s=1,2,\dots;r>0;1<q<+\infty)$ is the set of all functions $f(x)\in L^q(0,1)^s$ that 1-periodic in each of their variable satisfying the inequality

$$\sup_{j=0,1,\dots} 2^{jr}\left\|\sum_{[2^{j-1}]\le\|m\|<2^j}\widehat{f}(m)\cdot e^{2\pi i(m,x)}\right\|_{L^q(0,1)^s}\le 1, \tag{2}$$

where the square bracket [...] means the integer part. For everywhere below for $m=(m_1,\dots,m_s)$ we set $\|m\|=\max_{j=1,\dots,s}|m_j|$.

Let F be some class of $f(x)=f(x_1,\dots,x_s)$ functions 1-periodic in each variable whose trigonometric Fourier series converges absolutely.

Assume that $\widehat{f}(0)\ne 0$. It is easy to verify that, for any boundary condition there exists a function $\omega(x)$ depending on this condition such that $\omega(x)$ is continuous on $[0,1]^s$ and $\Delta\omega\equiv 1$ on $[0,1]^s$. Moreover, solution of (1) has the form

$$u_\omega(x,f)=\omega(x)\cdot\widehat{f}(0)-\frac{1}{4\pi^2}\sum_{m\in Z^s}{}^{*}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}. \tag{3}$$

If $\widehat{f}(0)=0$, then for a solution of (1) to exist, it is necessary that the boundary condition $u|_G=h(x)$ on the boundary of G satisfies

$$h(x)=-\frac{1}{4\pi^2}\sum_{m\in Z^s}{}^{*}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}(x\in G).$$

If $f(x_1,\dots,x_s)$ is odd in each of the variables $x_1,\dots,x_s$ then the function (see, [3], p.187-189)

$$u(x,f)=-\frac{1}{4\pi^2}\sum_{m\in Z^s}{}^{*}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}$$

is a solution of (1) with zero boundary condition on $[0,1]^s$. Here and everywhere below the asterisk "*" over the sum means that $m=(0,\dots,0)$ is dropped in the summation.

3 Main result and its proof

Theorem 1. Let are given positive integer s and $r > s/2$. Then the following statements hold ($N = (2^{n+1}+1)^s$, $n = 1, 2, ...$)

C(N)D-1:

$$\delta_N(0; D_N)_{L^2} \equiv$$

$$\equiv \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s;\\ \varphi_N}} \sup_{f\in H_2^r} \left\| u_\omega(\cdot, f) - \varphi_N(\widehat{f}(m^{(1)}), ..., \widehat{f}(m^{(N)}); \cdot) \right\|_{L^2(0,1)^s} \asymp N^{-\frac{r+2}{s}}, \quad (4)$$

upper bound is sharps on computational aggregate

$$\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x) = \omega(x)\widehat{f}(0) - \frac{1}{4\pi^2} \sum_{m\in I_{2^n}}{}^{*} \frac{\widehat{f}(m)}{(m,m)} e^{2\pi i(m,x)}, \quad (5)$$

here in (5) the set $\{\overline{m}^{(1)} = 0, \overline{m}^{(2)}, ..., \overline{m}^{(N)}\}$ is some ordering of the set I_{2^n}, i.e.

$$I_{2^n} = \{m = (m_1, ..., m_s) \in Z^s : |m_j| \le 2^n (j = 1, 2, ..., s)\} = \{\overline{m}^{(1)} = 0, \overline{m}^{(2)} ..., \overline{m}^{(N)}\}. \quad (6)$$

C(N)D-2 (first part): For computational aggregates $\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}), x)$ from (5) and for the numerical sequence

$$\widetilde{\varepsilon}_N \asymp \begin{cases} N^{-\frac{r+2}{s}}, ifs < 4, \\ (lnN)^{-\frac{1}{2}} \cdot N^{-\frac{r+2}{4}}, ifs = 4, \\ N^{-\frac{r}{s}-\frac{1}{2}}, ifs > 4. \end{cases} \quad (7)$$

satisfy

$$\delta_N(0; D_N)_{L^2} \asymp \delta_N\left(\tilde{\varepsilon}_N; D_N\right)_{L^2} \asymp$$

$$\asymp \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s,\\ \varphi_N}} \sup_{\substack{f\in H_2^r,\\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N, |\gamma_N^{(\tau)}|\le 1,\\ (\tau=1,...,N)}} \Big\| u_\omega(x, f) - \varphi_N(\widehat{f}(m^{(1)}) + \widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)}, ..., \widehat{f}(m^{(N)}) +$$

$$+ \widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)}; x)\Big\|_{L^2} \asymp \sup_{\substack{f\in H_2^r,\\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N, |\gamma_N^{(\tau)}|\le 1,\\ (\tau=1,...,N)}} \Big\| u_\omega(x, f) - \varphi_N(\widehat{f}(\overline{m}^{(1)}) + \widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)}, ..., \widehat{f}(\overline{m}^{(N)}) +$$

$$+ \widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)}; x)\Big\|_{L^2} \asymp N^{-\frac{r+2}{s}}. \quad (8)$$

Proof. Let f belongs to Nikol'skii classes H_2^r. Then since $r > s/2$ from the definition of class H_2^r follows $u_\omega(x, f) \in L^2(0,1)^s$. Let n be a given positive integer, we set $N = |I_{2^n}| = (2^{n+1}+1)^s$. According to the definition of D_N, we set

$$B_N = \{\overline{m}^{(1)} = 0; \overline{m}^{(2)}; ...; \overline{m}^{(N)}\}, B_N = I_{2^n},$$

$$l_N^{(\overline{m}^{(1)})}(f) = \widehat{f}(\overline{m}^{(1)}) = \widehat{f}(0), l_N^{(\overline{m}^j)}(f) = \widehat{f}(\overline{m}^{(j)}), j = 2, 3, ..., N.$$

Let's start with an upper bound for the value of $\delta_N\left(\tilde{\varepsilon}_N; D_N\right)_{L^2}$ from C(N)D-2. Let are given $\{\gamma_N^{(\tau)}\}_{\tau=1}^N \equiv \{\gamma_N^{(m)}\}_{m\in I_{2^n}}$, $|\gamma_N^{(\tau)}| \leq 1(\tau = 1, ..., N)$. By (3) and (5), we have ($L^2 \equiv L^2(0,1)^s$)

$$\left\|u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}) + \widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)}, ..., \widehat{f}(\overline{m}^{(N)}) + \widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)}; x)\right\|_{L^2} \leq$$

$$\leq \left\|u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x)\right\|_{L^2} + \left\|\omega(x)\widetilde{\varepsilon}_N\gamma_N^{(0)} - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}}{}^{*}\frac{\widetilde{\varepsilon}_N\gamma_N^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2} \leq$$

$$\leq \left\|-\frac{1}{4\pi^2}\sum_{m\in Z^s/I_{2^n}}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2} + \left\|\omega(x)\widetilde{\varepsilon}_N\gamma_N^{(0)} - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}}{}^{*}\frac{\widetilde{\varepsilon}_N\gamma_N^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2} \equiv$$

$$\equiv \|I_1\|_{L^2} + \|I_2\|_{L^2}.$$

Estimating from above for $\|I_1\|_{L^2}$ gives upper bound for $\delta_N(0; D_N)_{L^2}$ in C(N)D-1. We will evaluate upper bound of the error of approximation in L^2 metric by using (2), (3), (5) and Parseval's equality:

$$\|I_1\|_{L^2}^2 \equiv \left\|u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x)\right\|_{L^2(0,1)^s}^2 =$$

$$= \left\|-\frac{1}{4\pi^2}\sum_{m\in Z^s/I_{2^n}}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2}^2 = \sum_{j=n+1}^{+\infty}\sum_{2^j\leq\|m\|<2^{j+1}}\frac{|\widehat{f}(m)|^2}{16\pi^4(m,m)^2} \ll$$

$$\ll \sum_{j=n+1}^{+\infty}\sum_{2^j\leq\|m\|<2^{j+1}}\frac{|\widehat{f}(m)|^2}{(m_1^2+...+m_s^2)^2} \ll \sum_{j=n+1}^{+\infty}\sum_{2^j\leq\|m\|<2^{j+1}}\frac{|\widehat{f}(m)|^2}{(\max_{j=1,...,s}|m_j|^2)^2} \ll$$

$$\ll \sum_{j=n+1}^{+\infty}\frac{1}{2^{4j}}\left(\sum_{2^j\leq\|m\|<2^{j+1}}|\widehat{f}(m)|^2\right)\cdot 2^{2(j+1)r}\cdot 2^{-2(j+1)r} \ll \frac{1}{2^{4n}}\sum_{j=n+1}^{+\infty}2^{-2(j+1)r} \ll$$

$$\ll 2^{-4n-2nr} \asymp N^{-\frac{2(r+2)}{s}}.$$

Further,

$$\sup_{f\in H_2^r}\left\|u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}), ..., \widehat{f}(\overline{m}^{(N)}); x)\right\|_{L^2} \ll N^{-\frac{r+2}{s}}$$

and

$$\delta_N(0; D_N)_{L^2} \equiv \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s,\\ \varphi_N}}\sup_{f\in H_2^r}\left\|u_\omega(x,f) - \varphi_N(\widehat{f}(m^{(1)}), ..., \widehat{f}(m^{(N)}); x)\right\|_{L^2} \ll N^{-\frac{r+2}{s}}.$$

which is the upper bound in (4).

Then let's evaluate $\|I_2\|_{L^2}$ (see(7))

$$\|I_2\|_{L^2} \equiv \left\|\omega(x)\widetilde{\varepsilon}_N\gamma_N^{(0)} - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}}{}^{*}\frac{\widetilde{\varepsilon}_N\gamma_N^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2} \ll \widetilde{\varepsilon}_N + \left(\sum_{m\in I_{2^n}}{}^{*}\widetilde{\varepsilon}_N^2\frac{1}{(m,m)^2}\right)^{\frac{1}{2}} \ll$$

$$\ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}\sum_{2^j\leq\|m\|<2^{j+1}}\frac{1}{(m_1^2+...+m_s^2)^2}\right)^{\frac{1}{2}}\right) \ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}\frac{1}{2^{4j}}\sum_{2^j\leq\|m\|<2^{j+1}}1\right)^{\frac{1}{2}}\right) \ll$$

$$\ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}2^{-4j}\cdot 2^{js}\right)^{\frac{1}{2}}\right) \asymp \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}2^{j(s-4)}\right)^{\frac{1}{2}}\right).$$

If $s<4$, then

$$\|I_2\|_{L^2} \ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}2^{j(s-4)}\right)^{\frac{1}{2}}\right) \asymp \widetilde{\varepsilon}_N \asymp N^{-\frac{r+2}{s}}.$$

If $s=4$, then

$$\|I_2\|_{L^2} \ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}1\right)^{\frac{1}{2}}\right) \asymp \widetilde{\varepsilon}_N\cdot n^{\frac{1}{2}} \asymp \widetilde{\varepsilon}_N\cdot(lnN)^{\frac{1}{2}} \asymp (lnN)^{-\frac{1}{2}}\cdot N^{-\frac{r+2}{4}}\cdot(lnN)^{\frac{1}{2}} \asymp N^{-\frac{r+2}{4}}.$$

If $s>4$, then

$$\|I_2\|_{L^2} \ll \widetilde{\varepsilon}_N \left(1+\left(\sum_{j=0}^{n-1}2^{j(s-4)}\right)^{\frac{1}{2}}\right) \asymp \widetilde{\varepsilon}_N\cdot 2^{\frac{n(s-4)}{2}} \asymp \widetilde{\varepsilon}_N\cdot N^{\frac{1}{2}-\frac{2}{s}} \asymp N^{-\frac{r}{s}-\frac{1}{2}}\cdot N^{\frac{1}{2}-\frac{2}{s}} \asymp N^{-\frac{r+2}{s}}.$$

Then, for $f\in H_2^r$ and $\{\gamma_N^{(\tau)}\}_{\tau=1}^N$, such that $|\gamma_N^{(\tau)}|\leq 1$ $(\tau=1,...,N)$ satisfies

$$\left\|u_\omega(x,f)-\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)})+\widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)},...,\widehat{f}(\overline{m}^{(N)})+\widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)};x)\right\|_{L^2} \ll \|I_1\|_{L^2}+\|I_2\|_{L^2} \ll N^{-\frac{r+2}{s}}.$$

Further, by the arbitrariness of the function $f\in H_2^r$ and$\{\gamma_N^{(\tau)}\}_{\tau=1}^N$, $|\gamma_N^{(\tau)}|\leq 1$ $(\tau=1,...,N)$

$$\sup_{\substack{f\in H_2^r,\\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N,|\gamma_N^{(\tau)}|\leq 1,\\ (\tau=1,...,N)}} \left\|u_\omega(x,f)-\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)})+\widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)},...,\widehat{f}(\overline{m}^{(N)})+\widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)};x)\right\|_{L^2} \ll N^{-\frac{r+2}{s}}.$$

In the end, we obtain the required upper bound in C(N)D-2

$$\delta_N(\widetilde{\varepsilon}_N;D_N)_{L^2} \equiv \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s;\\ \varphi_N}} \sup_{\substack{f\in H_2^r,\\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N,|\gamma_N^{(\tau)}|\leq 1,\\ (\tau=1,...,N)}} \left\|u_\omega(x,f)-\varphi_N(\widehat{f}(m^{(1)})+\widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)},...,\right.$$

$$\left.\widehat{f}(m^{(N)})+\widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)};x)\right\|_{L^2} \ll N^{-\frac{r+2}{s}}$$

and, by the definition of $\delta_N(0;D_N)_{L^2}$ and $\delta_N(\tilde{\varepsilon}_N;D_N)_{L^2}$,

$$\delta_N(0;D_N)_{L^2} \ll \delta_N(\tilde{\varepsilon}_N;D_N)_{L^2} \ll N^{-\frac{r+2}{s}}. \tag{9}$$

Let's evaluate lower bound for $\delta_N(0, D_N)_{L^2}$. Now, let us prove the lower bound in the case of approximation from accurate information. Let are given an integer $N \geq 1$ and set $A_N = \{m^{(1)}, ..., m^{(N)} : m^{(j)} \in Z^s (j = 1, ..., N)\}$. According to the choice of D_N, we define the functionals $l_N^{(1)}(f) = \widehat{f}(m^{(1)}), ..., l_N^{(N)}(f) = \widehat{f}(m^{(N)})$. Let $\varphi_N(\tau_1, ..., \tau_N; x)$ also be an arbitrary algorithm for processing information, such that $\varphi_N(0, ..., 0; x) = 0$. We define an integer $n = n(s, N) \geq 1$ from the conditions $|I_{2^n}| \geq 2N$ and $|I_{2^n}| \asymp N$.

Let consider the function

$$\overline{g}(x) = \sum_{m \in I_{2^n} \setminus A_N}{}^{*} \overline{a}_n^{(m)} e^{2\pi i (m,x)}, \tag{10}$$

where $\overline{a}_n^{(m)} = k_j(m) \equiv k(j, n, s)$ when $[2^{j(m)-1}] \leq \|m\| < 2^{j(m)}$, $m \in I_{2^n} \setminus A_N$, $j = 0, 1, ..., n$. For the number of points of $I_{2^n} \setminus A_N$:

$$N \asymp |I_{2^n}| \geq |I_{2^n} \setminus A_N| \geq |I_{2^n}| - |A_N| \geq 2N - N = N,$$

therefore

$$|I_{2^n} \setminus A_N| \asymp N.$$

By using Parseval's equality, let define the norm of $\overline{g}$

$$\|\overline{g}\|_{H_2^r} = \sup_{j=0,1,...,n} 2^{jr} \| \sum_{\substack{[2^{j-1}] \leq \|m\| < 2^j, \\ m \notin A_N}}{}^{*} k_j e^{2\pi i (m,x)} \|_{L^2} =$$

$$= \sup_{j=0,1,...,n} 2^{jr} \left(\sum_{\substack{[2^{j-1}] \leq \|m\| < 2^j, \\ m \notin A_N}}{}^{*} |k_j|^2 \right)^{\frac{1}{2}} = \sup_{j=0,1,...,n} 2^{jr} \cdot k_j \left(\sum_{\substack{[2^{j-1}] \leq \|m\| < 2^j, \\ m \notin A_N}}{}^{*} 1 \right)^{\frac{1}{2}} \ll$$

$$\ll \sup_{j=0,1,...,n} 2^{jr} \cdot k_j \cdot 2^{\frac{sj}{2}} = \sup_{j=0,1,...,n} 2^{j(r+\frac{s}{2})} \cdot k_j.$$

k_j is defined from the condition $\|\overline{g}\|_{H_2^r} \ll \sup_{j=0,1,...,n} 2^{j(r+\frac{s}{2})} \cdot k_j \asymp 1$ (in that case $\overline{g}$ belong to H_2^r class)

$$k_j = 2^{-j(r+\frac{s}{2})}, j = 0, 1, ..., n. \tag{11}$$

By putting (11) into (10), there are exist a positive constant $c(s)$ such that

$$g(x) = c(s)\overline{g}(x) = c(s) \sum_{m \in I_{2^n} \setminus A_N}{}^{*} \overline{a}_n^{(m)} e^{2\pi i (m,x)} = c(s) \sum_{j=0}^{n} \sum_{\substack{m \in I_{2^n} \setminus A_N, \\ [2^{j-1}] \leq \|m\| < 2^j}}{}^{*} 2^{-j(r+\frac{s}{2})} e^{2\pi i (m,x)}. \tag{12}$$

Then, according to definition of $g(x)$ satisfies $l_N^{(1)}(g) = \widehat{g}(m^{(1)}) = 0, ..., l_N^{(N)}(g) = \widehat{g}(m^{(N)}) = 0$, so it should be $\varphi_N(\widehat{g}(m^{(1)}), ..., \widehat{g}(m^{(N)}); \cdot) = 0$. Then for the lower bound of error of approximation by accurate information we have

$$\sup_{f \in H_2^r(0,1)^s} \left\| u_\omega(x, f) - \varphi_N \left(\widehat{f}(m^{(1)}), ..., \widehat{f}(m^{(N)}); x \right) \right\|_{L^2} \geq$$

$$\geq \left\| u_\omega(x,g) - \varphi_N\left(\widehat{g}(m^{(1)}),...,\widehat{g}(m^{(N)});x\right)\right\|_{L^2} = \|u_\omega(x,g)\|_{L^2}.$$

By definition of function g satisfies $\widehat{g}(0)=0$. Let calculate the error by using Parseval's equality.

$$\|u_\omega(x,g)\|_{L^2}^2 = \left\| \omega(x)\cdot\widehat{g}(0) - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}\setminus A_N}{}^{*}\frac{\overline{a}_n^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2(0,1)^s}^2 \asymp$$

$$\asymp \left\| -\frac{1}{4\pi^2}\sum_{m\in I_{2^n}\setminus A_N}{}^{*}\frac{\overline{a}_n^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^2(0,1)^s}^2 \asymp \sum_{m\in I_{2^n}\setminus A_N}{}^{*}\frac{|\overline{a}_n^{(m)}|^2}{(m,m)^2} \asymp$$

$$\asymp \sum_{j=0}^{n-1}\sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*}\left(2^{-2j(r+\frac{s}{2})}\frac{1}{m_1^2+...+m_s^2}\right)^2 \gg$$

$$\gg 2^{-2n(r+\frac{s}{2})}\sum_{j=0}^{n-1}\sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*}\left(\frac{1}{\max_{j=1,...,s}|m_j|^2}\right)^2 \gg$$

$$\gg 2^{-2n(r+\frac{s}{2})}\sum_{j=0}^{n-1}\frac{1}{2^{4(j+1)}}\sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*}1 \gg$$

$$\gg 2^{-2n(r+\frac{s}{2})-4n-4}\cdot\sum_{j=0}^{n-1}\sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*}1 \asymp 2^{-2n(r+\frac{s}{2})-4n-4}\cdot|I_{2^n}\setminus A_N| \asymp$$

$$\asymp 2^{-2nr-ns-4n}\cdot 2^{ns} \asymp 2^{-2nr-4n} \asymp N^{-\frac{2r}{s}-\frac{4}{s}}.$$

Finally, for (4) we have

$$\sup_{f\in H_2^r}\left\|u_\omega(x,f)-\varphi_N(\widehat{f}(m^{(1)}),...,\widehat{f}(m^{(N)});x)\right\|_{L^2} \gg N^{-\frac{r+2}{s}}. \tag{13}$$

Then, due to the arbitrariness of $m^{(1)},...,m^{(N)}$ from Z^s and the information processing algorithm φ_N, satisfies

$$\delta_N(0,D_N)_{L^2} \equiv \inf_{\substack{m^{(1)}\in Z^s,...,m^{(N)}\in Z^s;\\ \varphi_N}}\sup_{f\in H_2^r}\left\|u_\omega(x,f)-\varphi_N(\widehat{f}(m^{(1)}),...,\widehat{f}(m^{(N)});x)\right\|_{L^2} \gg N^{-\frac{r+2}{s}}. \tag{14}$$

As a result, by (9) and (14) we have (8)

$$\delta_N(0,D_N)_{L^2} \asymp \delta_N(\widetilde{\varepsilon}_N,D_N)_{L^2} \asymp N^{-\frac{r+2}{s}}.$$

Theorem 1 is proven.

Theorem 2. Let are given positive integer s and $r>s/2$. Then the following statements hold($N=(2^{n+1}+1)^s$, $n=1,2,...$)

C(N)D-1:

$$\delta_N(0; D_N)_{L^\infty} \equiv$$

$$\equiv \inf_{\substack{m^{(1)}\in Z^s,\dots,m^{(N)}\in Z^s;\\ \varphi_N}} \sup_{f\in H_2^r} \left\| u_\omega(\cdot, f) - \varphi_N(\widehat{f}(m^{(1)}),\dots,\widehat{f}(m^{(N)});\cdot)\right\|_{L^\infty(0,1)^s} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}. \quad (15)$$

C(N)D-2 (first part): For the computational aggregates $\overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}),\dots,\widehat{f}(\overline{m}^{(N)}),x)$ from (5) and for the numerical sequence

$$\widetilde{\varepsilon}_N \asymp \begin{cases} N^{-r-\frac{3}{2}}, ifs = 1, \\ (lnN)^{-1}\cdot N^{-\frac{r+1}{2}}, ifs = 2, \\ N^{-\frac{r}{s}-\frac{1}{2}}, ifs > 2. \end{cases} \quad (16)$$

satisfy

$$\delta_N(0; D_N)_{L^\infty(0,1)^s} \asymp \delta_N\left(\tilde{\varepsilon}_N; D_N\right)_{L^\infty(0,1)^s} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}. \quad (17)$$

Proof. The proof will be carried out similarly by Theorem 1. Let are given $f \in H_2^r$ positive integer n, $N = |I_{2^n}| = (2^{n+1}+1)^s$ and $\{\gamma_N^{(\tau)}\}_{\tau=1}^N \equiv \{\gamma_N^{(m)}\}_{m\in I_{2^n}}$, such that $|\gamma_N^{(\tau)}| \le 1$. Then for the error of approximation by computational aggregates (5)-(6) by inaccurate information ($L^\infty \equiv L^\infty(0,1)^s$)

$$\left\| u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}) + \widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)},\dots,\widehat{f}(\overline{m}^{(N)}) + \widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)};x)\right\|_{L^\infty} \le$$

$$\le \left\| u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}),\dots,\widehat{f}(\overline{m}^{(N)});x)\right\|_{L^\infty} +$$

$$+\left\|\omega(x)\widetilde{\varepsilon}_N\gamma_N^{(0)} - \frac{1}{4\pi^2}\sum_{m\in I_{2^n}}{}^{*}\frac{\widetilde{\varepsilon}_N\gamma_N^{(m)}}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^\infty} \equiv \|I_3\|_{L^\infty} + \|I_4\|_{L^\infty}.$$

Let's estimate from above $\|I_3\|_{L^\infty}$ and $\|I_4\|_{L^\infty}$

$$\|I_3\|_{L^\infty} \equiv \left\| u_\omega(x,f) - \overline{\varphi}_N(\widehat{f}(\overline{m}^{(1)}),\dots,\widehat{f}(\overline{m}^{(N)});x)\right\|_{L^\infty} =$$

$$= \left\| -\frac{1}{4\pi^2}\sum_{m\in Z^s\setminus I_{2^n}}\frac{\widehat{f}(m)}{(m,m)}e^{2\pi i(m,x)}\right\|_{L^\infty} \ll \sum_{j=n+1}^{+\infty}\sum_{2^j\le\|m\|<2^{j+1}}\frac{|\widehat{f}(m)|}{|(m,m)|} \ll$$

$$\ll \sum_{j=n+1}^{+\infty}\sum_{2^j\le\|m\|<2^{j+1}}\frac{|\widehat{f}(m)|}{m_1^2+\dots+m_s^2} \ll \sum_{j=n+1}^{+\infty}\sum_{2^j\le\|m\|<2^{j+1}}\frac{|\widehat{f}(m)|}{\max_{j=1,\dots,s}|m_j|^2} \ll$$

$$\ll \sum_{j=n+1}^{+\infty} 2^{-2j}\sum_{2^j\le\|m\|<2^{j+1}}|\widehat{f}(m)|.$$

Applying Holder's inequality and (2), we will get required upper bound

$$\|I_3\|_{L^\infty} \ll 2^{-2n}\sum_{j=n+1}^{+\infty}\left(\sum_{2^j\le\|m\|<2^{j+1}}|\widehat{f}(m)|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{2^j\le\|m\|<2^{j+1}}1\right)^{\frac{1}{2}} \ll$$

$$\ll 2^{-2n}\sum_{j=n+1}^{+\infty}2^{-j\left(r-\frac{s}{2}\right)}\cdot 2^{jr}\left\|\sum_{2^j\le\|m\|<2^{j+1}}\widehat{f}(m)e^{2\pi i(m,x)}\right\|_{L^2}\ll 2^{-2n-nr+\frac{ns}{2}}=N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}.$$

From the upper bound for $\|I_3\|_{L^\infty}$, we obtain the upper bound for (15) the approximation by the accurate information

$$\delta_N(0;D_N)_{L^\infty}\equiv$$

$$\equiv\inf_{\substack{m^{(1)}\in Z^s,\dots,m^{(N)}\in Z^s;\\ \varphi_N}}\sup_{f\in H_2^r}\left\|u_\omega(\cdot,f)-\varphi_N(\widehat{f}(m^{(1)}),\dots,\widehat{f}(m^{(N)});\cdot)\right\|_{L^\infty(0,1)^s}\asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}.(18)$$

Then evaluate of $\|I_4\|_{L^\infty}$ (see also (16))

$$\|I_4\|_{L^\infty}\ll\widetilde{\varepsilon}_N+\sum_{m\in I_{2n}}{}^{*}\widetilde{\varepsilon}_N\frac{1}{4\pi^2(m,m)}\ll\widetilde{\varepsilon}_N\left(1+\sum_{j=0}^{n-1}\frac{1}{2^{2j}}\sum_{2^j\le\|m\|<2^{j+1}}1\right)\ll\widetilde{\varepsilon}_N\left(1+\sum_{j=0}^{n-1}2^{j(s-2)}\right).$$

If $s=1$, then

$$\|I_4\|_{L^\infty}\ll\widetilde{\varepsilon}_N\asymp N^{-r-\frac{3}{2}}.$$

If $s=2$, then

$$\|I_4\|_{L^\infty}\ll\widetilde{\varepsilon}_N\left(1+\sum_{j=0}^{n-1}1\right)\asymp\widetilde{\varepsilon}_N\cdot n\asymp\widetilde{\varepsilon}_N\cdot lnN\asymp(lnN)^{-1}\cdot N^{-\frac{r}{2}-\frac{1}{2}}\cdot lnN\asymp N^{-\frac{r}{2}-\frac{1}{2}}.$$

If $s>2$, then

$$\|I_4\|_{L^\infty}\ll\widetilde{\varepsilon}_N\left(1+\sum_{j=0}^{n-1}2^{j(s-2)}\right)\asymp\widetilde{\varepsilon}_N\cdot 2^{n(s-2)}\asymp\widetilde{\varepsilon}_N\cdot N^{1-\frac{2}{s}}\asymp N^{-\frac{r}{s}-\frac{1}{2}}\cdot N^{1-\frac{2}{s}}\asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}.$$

Finally, by estimation from above $\|I_3\|_{L^\infty}$ and $\|I_4\|_{L^\infty}$ we obtain the required upper bounds in (17)

$$\delta_N(0,D_N)_{L^\infty}\ll\delta_N(\widetilde{\varepsilon}_N,D_N)_{L^\infty}\equiv$$

$$\equiv\inf_{\substack{m^{(1)}\in Z^s,\dots,m^{(N)}\in Z^s;\\ \varphi_N}}\sup_{\substack{f\in H_2^r,\\ \{\gamma_N^{(\tau)}\}_{\tau=1}^N,|\gamma_N^{(\tau)}|\le 1,\\ (\tau=1,\dots,N)}}\left\|u_\omega(x,f)-\varphi_N(\widehat{f}(m^{(1)})+\widetilde{\varepsilon}_N^{(1)}\gamma_N^{(1)},\dots,\widehat{f}(m^{(N)})+\right.$$

$$\left.+\widetilde{\varepsilon}_N^{(N)}\gamma_N^{(N)};x)\right\|_{L^\infty}\ll N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}. \qquad (19)$$

A lower bound in the case of approximation from accurate information gives the desired relation. Suppose we are given an integer $N\ge 1$, N linear functionals $l_N^{(1)}(f)=\widehat{f}(m^{(1)}),\dots,l_N^{(N)}(f)=\widehat{f}(m^{(N)}),\{m^{(1)},\dots,m^{(N)}\}\in Z^s$ and a function $\varphi_N(\tau_1,\dots,\tau_N;x)$, $\varphi_N(0,\dots,0;x)=0$. We define an integer $n=n(s,N)\ge 1$ from the conditions $|I_{2^n}|\ge 2N$ and $|I_{2^n}|\asymp N$.

Let consider the function

$$g(x) = c(s)N^{-\frac{r}{s}-\frac{1}{2}} \sum_{m\in I_{2^n}\setminus A_N}{}^{*} e^{2\pi i(m,x)} \in H_2^r.$$

where $c(s)$ is a positive constant, defined so that $g(x) \in H_2^r$.

Then, for the lower bound of error of approximation by accurate information

$$\sup_{f\in H_2^r} \left\| u_\omega(x,f) - \varphi_N\left(\widehat{f}(m^{(1)}),...,\widehat{f}(m^{(N)});x\right)\right\|_{L^\infty} \geq$$

$$\geq \sup_{f\in H_2^r} \left\| u_\omega(x,g) - \varphi_N\left(\widehat{g}(m^{(1)}),...,\widehat{g}(m^{(N)});x\right)\right\|_{L^\infty} \geq$$

$$\geq \left\| u_\omega(x,g) - \varphi_N(0,...,0;x)\right\|_{L^\infty} = \left\| u_\omega(x,g)\right\|_{L^\infty}.$$

Let estimate from below the norm of the solution.

$$\left\| u_\omega(x,g)\right\|_{L^\infty} = \left\| -\frac{1}{4\pi^2} \sum_{m\in I_{2^n}\setminus A_N}{}^{*} \frac{N^{-\frac{r}{s}-\frac{1}{2}}}{(m,m)} e^{2\pi i(m,x)} \right\|_{L^\infty} =$$

$$= \sup_{x\in[0,1]^s} \left| -\frac{1}{4\pi^2} \sum_{m\in I_{2^n}\setminus A_N}{}^{*} \frac{N^{-\frac{r}{s}-\frac{1}{2}}}{(m,m)} e^{2\pi i(m,x)} \right| \geq \left| -\frac{1}{4\pi^2} \sum_{m\in I_{2^n}\setminus A_N}{}^{*} \frac{N^{-\frac{r}{s}-\frac{1}{2}}}{(m,m)} \right| \gg$$

$$\gg 2^{-n(r+\frac{s}{2})} \sum_{j=0}^{n-1} \frac{1}{2^{2(j+1)}} \sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*} 1 \asymp$$

$$\asymp 2^{-n(r+\frac{s}{2})-2n-2} \cdot \sum_{j=0}^{n-1} \sum_{\substack{2^j\leq\|m\|<2^{j+1},\\ m\notin A_N}}{}^{*} 1 \asymp 2^{-n(r+\frac{s}{2})-2n-4} \cdot |I_{2^n}\setminus A_N| \asymp$$

$$\asymp 2^{-nr-2n+\frac{ns}{2}} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}.$$

As a result,

$$\delta_N(0,D_N)_{L^\infty} \gg N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}. \tag{20}$$

Then by (19) and (20) we have

$$\delta_N(0,D_N)_{L^\infty} \asymp \delta_N(\widetilde{\varepsilon}_N,D_N)_{L^\infty} \asymp N^{-\frac{r}{s}-\frac{2}{s}+\frac{1}{2}}.$$

Theorem 2 is proven.

4 Conclusion

In the present paper, the problem of the approximation of solutions of the Poisson equation with right-hand side from the Nikol'skii classes $H_2^r(0,1)^s$ by accurate and inaccurate information of the trigonometric Fourier coefficients in the sense of C(N)D-1 and the first part of C(N)D-2 is considered.

Firstly, two-sided estimates for the error $\delta_N(0; D_N)_Y$ ($Y = L^(0,1)^s$ and $Y = L^\infty(0,1)^s$) of approximation by accurate informmation were obtained (C(N)D-1 problem) with indicating a computational aggregate that confirms the lower bound. For this computational aggregate, bounds arises of inaccurate information that preserve the order of the error of approximation by accurate information were found—the first part of problem C(N)D-2.

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SOLUTION OF MULTILAYER PROBLEMS FOR THE HEAT EQUATION BY THE FOURIER METHOD

The multilayer problems for the heat equation arise in many areas of heat and mass transfer applications. There are two main approaches to finding exact solutions to multilayer diffusion problems: separation of variables and integral transformations. The difficulty of applying the Laplace transform method is redoubled by the difficulty of finding the inverse transform. The inverse Laplace transform is often performed numerically. The most popular analytical approach to multilayer problems for the heat equation is the method of separation of variables. It is very important to obtain analytical solutions to such problems as they provide a higher level of understanding of the solution behavior and can be used for comparative analysis of numerical solutions. In this paper, the solution of the multilayer problem for the heat equation by the Fourier method is substantiated. The solution of the initial-boundary value problem for the heat equation with discontinuous coefficients by the method of separation of variables is reduced to the corresponding non-self-adjoint spectral Sturm-Liouville eigenvalue problem. Such eigenvalue problems do not belong to the ordinary type of Sturm-Liouville problems due to the discontinuity of the heat conductivity coefficients. In addition, the non-self-adjointness of the corresponding spectral problem also complicates the solution of the problem. Using the replacement, the problem is reduced to a self-adjoint spectral problem and the eigenfunctions of this problem forming an orthonormal basis are constructed. The considered problem models the process of heat propagation of the temperature field in a thin rod of finite length, consisting of several sections with different thermal-physical characteristics. In this problem, in addition to the boundary conditions of the Sturm type, the conditions of conjugation at the point of contact of different media are specified. The existence and uniqueness of the classical solution of the considered multilayer problem for the heat conduction equation are proved.

Keywords: Heat equation, Fourier method, spectral problem, orthonormal basis, classical solution.

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Жылуөткізгіштік теңдеу үшін көпқабатты есептерді Фурье әдісімен шешу

Жылуөткізгіштік теңдеуіне арналған көпқабатты есептер жылу және масса алмасудың көптеген салаларында туындайды. Көпқабатты диффузиялық есептердің дәл шешімдерін табудың екі негізгі әдісі бар: айнымалыларды ажырату және интегралдық түрлендірулер. Лаплас түрлендіруі әдісін қолданудың қиындығы кері түрлендіруді табудың күрделілігімен шиеленіседі. Көбінесе кері Лаплас түрлендіруі сандық түрде орындалады. Жылуөткізгіштік теңдеу үшін көпқабатты есептерге ең танымал аналитикалық тәсіл айнымалыларды ажырату әдісі болып табылады. Мұндай есептердің аналитикалық шешімдері өте құнды, өйткені олар шешім тәртібін түсінудің жоғары деңгейін қамтамасыз етеді және сандық шешімдерді салыстырмалы түрде талдау үшін пайдаланылуы мүмкін. Бұл ғылыми мақалада Фурье әдісі арқылы жылуөткізгіштік теңдеуінің көпқабатты есебінің шешімі негізделеді. Коэффициенттері үзілісті жылуөткізгіштік теңдеу үшін бастапқы-шекаралық есеп айнымалылар ажырату әдісі бойынша өзіне-өзі түйіндес емес спектрлік Штурм-Лиувилль меншікті мән есебіне келтіріледі. Мұндай меншікті мәндер есептері жылуөткізгіштік коэффициенттерінің үзілуіне байланысты Штурм-Лиувилль есептерінің әдеттегі түріне жатпайды.

Сонымен қатар, спектрлік есептің өзіне-өзі түйіндес емес болуы да есепті шешуді қиындатады. Алмастыру арқылы берілген есеп өзіне-өзі түйіндес спектрлік есепке келтіріледі және осы есептің ортонормалдық базисі болатын меншікті функциялары құрылады. Қарастырылып отырған мәселе әртүрлі термофизикалық сипаттамалары бар бірнеше бөліктен тұратын, ұзындықтары ақырлы жіңішке таяқшадағы температуралық өрістің жылу таралу процесін моделдейді. Штурм типіндегі шекаралық шарттарға қосымша, әртүрлі орталардың жанасу нүктесіндегі түйіндес шарттары көрсетілген. Жылуөткізгіштік теңдеу үшін қарастырылып отырған көпқабатты есептің классикалық шешімінің бар және жалғыз екендігі дәлелденді.

Тү йін сөздер: Жылуөткізгіштік теңдеуі, Фурье әдісі, спектрлік есеп, ортонормалдық базис, классикалық шешім.

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Решение многослойных задач для уравнения теплопроводности методом Фурье

Проблемы многослойных задач для уравнения теплопроводности возникают во многих областях применения процессов тепло- и массообмена. Существует два основных подхода к поиску точных решений задач многослойной диффузии: разделение переменных и интегральные преобразования. Трудность применение метода преобразование Лапласа усугубляется из-за сложности нахождение обратного преобразование. Часто обратное преобразование Лапласа выполняется численно. Наиболее популярным аналитическим подходом к многослойным задачам для уравнения теплопроводности является метод разделение переменных. Аналитические решения таких задач очень ценны, поскольку они обеспечивают более высокий уровень понимания поведения решения и могут быть использованы для сравнительного анализа численных решений. В данной научной статье обосновано решение методом Фурье многослойной задачи для уравнения теплопроводности. Решения методом разделение переменных начально-краевые задачи для уравнения теплопроводности с разрывными коэффициентами сводится к соответствующей не самосопряженной спектральной задаче Штурма-Лиувилля на собственные значения. Такие задачи на собственные значения не относится к обычному типу задач Штурма-Лиувилля из-за разрыва коэффициентов теплопроводности. Кроме того не самосопряженность соответствующей спектральной задачи также усложняет решение поставленной задачи. С помощью замены поставленная задача сведена к самосопряженной спектральной задаче и построена собственные функции этой задачи, которая образует ортонормированный базис. Рассматриваемая задача моделирует процесс распространения тепла температурного поля в тонком стержне конечной длины, состоящем из нескольких участков с различными теплофизическими характеристиками. Дополнительно к граничным условиям типа Штурма задаются условия сопряжения в точке контакта различных сред. Доказано существование и единственность классического решения рассматриваемой многослойной задачи для уравнения теплопроводности.

Ключевые слова: Уравнение теплопроводности, метод Фурье, спектральная задача, ортонормированный базис, классическое решение.

1 Introduction

Parabolic equations with discontinuous coefficients with one point of discontinuity have been extensively studied [1]-[3]. In these works, the correctness of various initial-boundary value problems for parabolic equations with discontinuous coefficients has been proved by using the Green function and thermal potential methods. In [4]-[8], some boundary value problems for the heat equation with a discontinuous coefficient, with one and two points of discontinuity,

have been considered by the method of separation of variables.
The papers [9]-[13] are devoted to the solution of multilayer diffusion problems. Mathematical models of diffusion in layered materials arise in many industrial, environmental, biological and medical applications, such as thermal conductivity in composite materials, transport of polluting chemicals and gases in layered porous media, growth of brain tumors, thermal conductivity through skin, transdermal drug delivery and greenhouse gas emissions[14]-[17]. The considered problem may arise in describing the process of particle diffusion in turbulent plasma, as well as in modeling the process of heat propagation of a temperature field in a thin rod of finite length, consisting of several sections with different thermophysical characteristics. In addition to the boundary conditions, the conjugation conditions (ideal contact condition) at the contact boundary of these media with different thermophysical characteristics are specified. It is a theoretical paper, however, the obtained analytical solution can be used for numerical calculations.

2 Statement of problem

We consider the initial?boundary value problem for the heat equation with piecewise constant coefficients

$$\frac{\partial u_i}{\partial t} = k_i^2 \frac{\partial^2 u_i}{\partial x^2}, \qquad i = 1, 2, \dots, m, \tag{1}$$

in the domain

$$\Omega = \bigcup_{i=1}^{m} \Omega_i, \qquad \Omega_i = \{(x,t) : \, l_{i-1} < x < l_i, \; 0 < t < T\},$$

with the initial condition

$$u(x,0) = \varphi(x), \qquad l_0 \le x \le l_m. \tag{2}$$

The boundary conditions are of the form

$$\begin{cases} \alpha_1 \dfrac{\partial u_1}{\partial x}(l_0, t) + \beta_1 u_1(l_0, t) = 0, \\ \alpha_2 \dfrac{\partial u_m}{\partial x}(l_m, t) + \beta_2 u_m(l_m, t) = 0, \end{cases} \qquad 0 \le t \le T. \tag{3}$$

The conjugation conditions are

$$\begin{cases} u_i(l_i - 0, t) = u_{i+1}(l_i + 0, t), \\ k_i \dfrac{\partial u_i}{\partial x}(l_i - 0, t) = k_{i+1} \dfrac{\partial u_{i+1}}{\partial x}(l_i + 0, t), \end{cases} \qquad 0 \le t \le T, \quad i = 1, 2, \dots, m-1, \tag{4}$$

where the coefficients satisfy $k_i > 0$ and $\alpha_j, \beta_j \in \mathbb{R}$ for $i = 1, 2, \dots, m$ and $j = 1, 2$. In addition, $|\alpha_1| + |\beta_1| > 0$ and $|\alpha_2| + |\beta_2| > 0$.

3 Method of Solution

To solve problem (1)?(4) we employ the Fourier method and seek a separated solution

$$u_i(x,t) = X_i(x)\,T(t) \not\equiv 0.$$

Substituting into equation (1) and conditions (2)-(4), and separating the variables, we obtain the following spectral problem:

$$k_i^2\,X_i''(x) + \lambda\,X_i(x) = 0, \qquad l_{i-1} < x < l_i, \quad i = 1, 2, \ldots, m. \tag{5}$$

The boundary conditions

$$\begin{cases} \alpha_1\,X_1'(l_0) + \beta_1\,X_1(l_0) = 0, \\ \alpha_2\,X_m'(l_m) + \beta_2\,X_m(l_m) = 0, \end{cases} \tag{6}$$

and the conjugation conditions are

$$\begin{cases} X_i(l_i - 0, t) = X_{i+1}(l_i + 0, t), \\ k_i\,X_i'(l_i - 0, t) = k_{i+1}\,X_{i+1}'(l_i + 0, t), \end{cases} \qquad i = 1, 2, \ldots, m-1. \tag{7}$$

The function $T(t)$ satisfies the ordinary differential equation

$$T'(t) + \lambda\,T(t) = 0.$$

Lemma 1. The spectral problem (5)-(7) is non-self-adjoint in $L_2(l_0, l_m)$.
The proof is carried out by direct calculation.

After the following change of variables

$$X_i(x) = Y_i(y), \qquad i = 1, 2, \ldots, m, \tag{8}$$

where

$$y = \begin{cases} \dfrac{x - l_0}{k_1}, & l_0 < x < l_1, \\ \dfrac{x - l_1}{k_2}, & l_1 < x < l_2, \\ \ldots \\ \dfrac{x - l_{m-1}}{k_m}, & l_{m-1} < x < l_m, \end{cases} \tag{9}$$

Under the change of variables (8)-(9), the spectral problem (5)-(7) takes the form

$$Y_i''(y) + \lambda\,Y_i(y) = 0, \qquad 0 < y < h_i, \quad i = 1, 2, \ldots, m, \tag{10}$$

with boundary conditions

$$\begin{cases} \dfrac{\alpha_1}{k_1}\,Y_1'(0) + \beta_1\,Y_1(0) = 0, \\ \dfrac{\alpha_2}{k_m}\,Y_m'(h_m) + \beta_2\,Y_m(h_m) = 0, \end{cases} \tag{11}$$

and conjugation conditions

$$\begin{cases} Y_i(h_i-0)=Y_{i+1}(0+), \\ Y_i'(h_i-0)=Y_{i+1}'(0+), \end{cases} \qquad i=1,2,\dots,m-1, \tag{12}$$

where

$$h_i=\frac{l_i-l_{i-1}}{k_i}, \qquad i=1,2,\dots,m.$$

Lemma 2. The spectral problem (10)-(12) is self-adjoint in

$$H=L_2(0,h_1)\oplus L_2(0,h_2)\oplus\cdots\oplus L_2(0,h_m).$$

The proof is carried out by direct calculation.

Next, we determine the eigenvalues and construct the eigenfunctions of (10)-(12). The general solution of (10) has the form

$$\begin{cases} Y_1(y)=c_1\cos(\sqrt{\lambda}\,y)+c_2\sin(\sqrt{\lambda}\,y), & 0<y<h_1, \\ Y_2(y)=c_3\cos(\sqrt{\lambda}\,y)+c_4\sin(\sqrt{\lambda}\,y), & 0<y<h_2, \\ \quad\dots \\ Y_{m-1}(y)=c_{2m-3}\cos(\sqrt{\lambda}\,y)+c_{2m-2}\sin(\sqrt{\lambda}\,y), & 0<y<h_{m-1}, \\ Y_m(y)=c_{2m-1}\cos(\sqrt{\lambda}\,y)+c_{2m}\sin(\sqrt{\lambda}\,y), & 0<y<h_m, \end{cases}$$

where c_{2i-1}, c_{2i} are arbitrary constants, $i=1,2,\dots,m$.

From the boundary conditions (11) we obtain

$$\begin{cases} \dfrac{\alpha_1}{k_1}\sqrt{\lambda}\,c_2+\beta_1\,c_1=0, \\ \Big(\beta_2\cos(\sqrt{\lambda}\,h_m)-\dfrac{\alpha_2}{k_m}\sqrt{\lambda}\,\sin(\sqrt{\lambda}\,h_m)\Big)c_{2m-1}+ \\ \Big(\beta_2\sin(\sqrt{\lambda}\,h_m)+\dfrac{\alpha_2}{k_m}\sqrt{\lambda}\,\cos(\sqrt{\lambda}\,h_m)\Big)c_{2m}=0. \end{cases} \tag{13}$$

From the conjugation conditions (12) we obtain

$$\begin{cases} c_1\cos(h_1\sqrt{\lambda})+c_2\sin(h_1\sqrt{\lambda})=c_3, \\ -\,c_1\sin(h_1\sqrt{\lambda})+c_2\cos(h_1\sqrt{\lambda})=c_4, \\ c_3\cos(h_2\sqrt{\lambda})+c_4\sin(h_2\sqrt{\lambda})=c_5, \\ -\,c_3\sin(h_2\sqrt{\lambda})+c_4\cos(h_2\sqrt{\lambda})=c_6, \\ \dots \\ c_{2m-3}\cos(h_{m-1}\sqrt{\lambda})+c_{2m-2}\sin(h_{m-1}\sqrt{\lambda})=c_{2m-1}, \\ -\,c_{2m-3}\sin(h_{m-1}\sqrt{\lambda})+c_{2m-2}\cos(h_{m-1}\sqrt{\lambda})=c_{2m}. \end{cases} \tag{14}$$

Successively eliminating the constants c_i from (14) gives

$$c_1 \cos\big((h_1+h_2+\cdots+h_{m-1})\sqrt{\lambda}\big) + c_2 \sin\big((h_1+h_2+\cdots+h_{m-1})\sqrt{\lambda}\big) = c_{2m-1},$$
$$- c_1 \sin\big((h_1+h_2+\cdots+h_{m-1})\sqrt{\lambda}\big) + c_2 \cos\big((h_1+h_2+\cdots+h_{m-1})\sqrt{\lambda}\big) = c_{2m}.$$

Substituting the obtained c_{2m-1}, c_{2m} into system (13), we arrive at

$$\begin{cases} \beta_1 c_1 + \dfrac{\alpha_1}{k_1}\sqrt{\lambda}\, c_2 = 0, \\ \Big(\beta_2 \cos(s_m\sqrt{\lambda}) - \dfrac{\alpha_2}{k_m}\sqrt{\lambda}\, \sin(s_m\sqrt{\lambda})\Big)c_1 + \Big(\beta_2 \sin(s_m\sqrt{\lambda}) + \dfrac{\alpha_2}{k_m}\sqrt{\lambda}\, \cos(s_m\sqrt{\lambda})\Big)c_2 = 0, \end{cases}$$

where

$$s_m = \sum_{i=1}^{m} h_i = \sum_{i=1}^{m} \frac{l_i - l_{i-1}}{k_i}.$$

The characteristic determinant of the last system has the form

$$\Delta(\lambda) = \big(\alpha_1\alpha_2\lambda + \beta_1\beta_2 k_1 k_m\big)\, \sin\big(s_m\sqrt{\lambda}\big) + \big(\alpha_2\beta_1 k_1 - \alpha_1\beta_2 k_m\big)\,\sqrt{\lambda}\, \cos\big(s_m\sqrt{\lambda}\big) = 0. \quad (15)$$

We now consider all possible special cases.
1) Suppose $\alpha_1\alpha_2 \neq 0$, $\alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1 = 0$, $\beta_1\beta_2 = 0$ (that is, $Y_1'(0) = 0$, $Y_m'(h_m) = 0$). Then from (15) we obtain

$$\alpha_1\alpha_2\,\lambda\, \sin\big(s_m\sqrt{\lambda}\big) = 0.$$

From $\sin\big(s_m\sqrt{\lambda}\big) = 0$ we find the eigenvalues

$$\lambda_n = \left(\frac{\pi n}{s_m}\right)^2, \qquad n \in \mathbb{Z}.$$

The corresponding eigenfunctions are

$$Y_n(y) = C \cdot \begin{cases} y_{1n} = (-1)^n \cos\big(\frac{\pi n}{s_m}\, y\big), & 0 < y < h_1, \\ y_{2n} = \cos\big(\frac{\pi n}{s_m}\,(h_2 - y + h_3 + \cdots + h_m)\big), & 0 < y < h_2, \\ y_{3n} = \cos\big(\frac{\pi n}{s_m}\,(h_3 - y + h_4 + \cdots + h_m)\big), & 0 < y < h_3, \\ \dots \\ y_{m-1,n} = \cos\big(\frac{\pi n}{s_m}\,(h_{m-1} - y + h_m)\big), & 0 < y < h_{m-1}, \\ y_{mn} = \cos\big(\frac{\pi n}{s_m}\,(h_m - y)\big), & 0 < y < h_m, \end{cases}$$

where C is an arbitrary constant.
2) Suppose $\alpha_1\alpha_2 = 0$, $\alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1 = 0$, and $\beta_1\beta_2 \neq 0$ (i.e., $Y_1(0) = 0$ and $Y_m(h_m) = 0$). Then, similarly, the eigenvalues are

$$\lambda_n = \left(\frac{\pi n}{s_m}\right)^2, \qquad n \in \mathbb{Z},$$

with the corresponding eigenfunctions

$$Y_n(y) = C \cdot \begin{cases} y_{1n} = (-1)^{n+1} \sin\big(\frac{\pi n}{s_m}\, y\big), & 0 < y < h_1, \\ y_{2n} = \sin\big(\frac{\pi n}{s_m}\, (h_2 - y + h_3 + \dots + h_m)\big), & 0 < y < h_2, \\ y_{3n} = \sin\big(\frac{\pi n}{s_m}\, (h_3 - y + h_4 + \dots + h_m)\big), & 0 < y < h_3, \\ \dots \\ y_{m-1,n} = \sin\big(\frac{\pi n}{s_m}\, (h_{m-1} - y + h_m)\big), & 0 < y < h_{m-1}, \\ y_{mn} = \sin\big(\frac{\pi n}{s_m}\, (h_m - y)\big), & 0 < y < h_m, \end{cases}$$

where C is an arbitrary constant.
3) Now let $\alpha_1\alpha_2 = 0,\ \alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1 \neq 0,\ \beta_1\beta_2 = 0$. Then from (15) we obtain

$$\big(\alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1\big)\sqrt{\lambda}\,\cos\big(s_m\sqrt{\lambda}\big) = 0.$$

At $\lambda = 0$ equation (10) has only the trivial solution. From $\cos\big(s_m\sqrt{\lambda}\big) = 0$ we find the eigenvalues

$$\lambda_n = \left(\frac{\pi(2n+1)}{2s_m}\right)^2, \qquad n \in \mathbb{Z}.$$

To determine the eigenfunctions, consider two possible cases.

Case 3.1: $\alpha_1 = 0$, $\alpha_2 \neq 0$, $\beta_1 \neq 0$, $\beta_2 = 0$ (i.e., $Y_1(0) = 0$, $Y_m'(h_m) = 0$). The corresponding eigenfunctions are

$$Y_n(y) = C \cdot \begin{cases} y_{1n} = (-1)^{n} \sin\big(\frac{\pi(2n+1)}{2s_m}\, y\big), & 0 < y < h_1, \\ y_{2n} = \cos\big(\frac{\pi(2n+1)}{2s_m}\, (h_2 - y + h_3 + \dots + h_m)\big), & 0 < y < h_2, \\ y_{3n} = \cos\big(\frac{\pi(2n+1)}{2s_m}\, (h_3 - y + h_4 + \dots + h_m)\big), & 0 < y < h_3, \\ \dots \\ y_{m-1,n} = \cos\big(\frac{\pi(2n+1)}{2s_m}\, (h_{m-1} - y + h_m)\big), & 0 < y < h_{m-1}, \\ y_{mn} = \cos\big(\frac{\pi(2n+1)}{2s_m}\, (h_m - y)\big), & 0 < y < h_m, \end{cases}$$

where C is an arbitrary constant.
Case 3.2: $\alpha_1 \neq 0$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 \neq 0$ (i.e., $Y_1'(0) = 0$, $Y_m(h_m) = 0$). Then the

eigenfunctions are

$$Y_n(y) = C \cdot \begin{cases} y_{1n} = (-1)^n \cos\Big(\dfrac{\pi(2n+1)}{2s_m}\, y\Big), & 0 < y < h_1, \\ y_{2n} = \sin\Big(\dfrac{\pi(2n+1)}{2s_m}\, (h_2 - y + h_3 + \cdots + h_m)\Big), & 0 < y < h_2, \\ y_{3n} = \sin\Big(\dfrac{\pi(2n+1)}{2s_m}\, (h_3 - y + h_4 + \cdots + h_m)\Big), & 0 < y < h_3, \\ \dots \\ y_{m-1,n} = \sin\Big(\dfrac{\pi(2n+1)}{2s_m}\, (h_{m-1} - y + h_m)\Big), & 0 < y < h_{m-1}, \\ y_{mn} = \sin\Big(\dfrac{\pi(2n+1)}{2s_m}\, (h_m - y)\Big), & 0 < y < h_m. \end{cases}$$

4) Consider the case $\alpha_1\alpha_2 \neq 0$, $\alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1 \neq 0$, and $\beta_1\beta_2 = 0$. Then from (15) we have

$$\alpha_1\alpha_2\, \lambda \sin\big(s_m\sqrt{\lambda}\big) - \big(\alpha_1\beta_2 k_m - \alpha_2\beta_1 k_1\big)\sqrt{\lambda}\cos\big(s_m\sqrt{\lambda}\big) = 0.$$

It is easy to check that for $\lambda = 0$ equation (10) admits only the trivial solution. Hence the eigenvalues are given by the roots of

$$\tan\big(s_m\sqrt{\lambda}\big) = \begin{cases} \dfrac{k_m\,\beta_2}{\alpha_2\,\sqrt{\lambda}}, & \beta_1 = 0,\ \beta_2 \neq 0 \quad \Big(Y_1'(0) = 0,\ \dfrac{\alpha_2}{k_m} Y_m'(h_m) + \beta_2 Y_m(h_m) = 0\Big), \\ -\dfrac{k_1\,\beta_1}{\alpha_1\,\sqrt{\lambda}}, & \beta_1 \neq 0,\ \beta_2 = 0 \quad \Big(\dfrac{\alpha_1}{k_1} Y_1'(0) + \beta_1 Y_1(0) = 0,\ Y_m'(h_m) = 0\Big). \end{cases}$$

It is not possible to write the eigenvalues in explicit form. However, by Rouche's theorem one can obtain their asymptotics. Clearly, the zeros of the equation $\tan\big(s_m\sqrt{\lambda}\big) = 0$ are $\sqrt{\lambda} = \dfrac{\pi n}{s_m}$. Hence, by Rouche's theorem the zeros of

$$\tan\big(s_m\sqrt{\lambda}\big) = \frac{\alpha_1 k_m\beta_2 - \alpha_2 k_1\beta_1}{\alpha_1\alpha_2\sqrt{\lambda}}$$

have the form

$$\lambda_n = \left(\frac{\pi n}{s_m} + \delta_n\right)^2, \qquad n \in \mathbb{Z},$$

where $|\delta_n| \leq M$ and, moreover, $\delta_n = O\left(\dfrac{1}{n}\right)$.

If $\beta_1 \neq 0$ and $\beta_2 = 0$, the eigenfunctions are

$$Y_n(y) = C \cdot \begin{cases} \cos\big((h_1 - y + h_2 + \cdots + h_m)\sqrt{\lambda_n}\big), & 0 < y < h_1, \\ \cos\big((h_2 - y + h_3 + \cdots + h_m)\sqrt{\lambda_n}\big), & 0 < y < h_2, \\ \cos\big((h_3 - y + h_4 + \cdots + h_m)\sqrt{\lambda_n}\big), & 0 < y < h_3, \\ \dots \\ \cos\big((h_{m-1} - y + h_m)\sqrt{\lambda_n}\big), & 0 < y < h_{m-1}, \\ \cos\big((h_m - y)\sqrt{\lambda_n}\big), & 0 < y < h_m, \end{cases}$$

corresponding to the boundary conditions $\dfrac{\alpha_1}{k_1}Y_1'(0)+\beta_1 Y_1(0)=0$ and $Y_m'(h_m)=0$.

If $\beta_1=0$ and $\beta_2\neq 0$, the eigenfunctions are

$$Y_n(y)=C\cdot\begin{cases}\cos\bigl(y\sqrt{\lambda_n}\bigr), & 0<y<h_1,\\ \cos\bigl((h_1+y)\sqrt{\lambda_n}\bigr), & 0<y<h_2,\\ \cos\bigl((h_1+h_2+y)\sqrt{\lambda_n}\bigr), & 0<y<h_3,\\ \dots & \\ \cos\bigl((h_1+h_2+\dots+h_{m-2}+y)\sqrt{\lambda_n}\bigr), & 0<y<h_{m-1},\\ \cos\bigl((h_1+h_2+\dots+h_{m-1}+y)\sqrt{\lambda_n}\bigr), & 0<y<h_m,\end{cases}$$

corresponding to the boundary conditions $Y_1'(0)=0$ and $\dfrac{\alpha_2}{k_m}Y_m'(h_m)+\beta_2 Y_m(h_m)=0$.

5) In the case $\alpha_1\alpha_2=0,\ \ \alpha_1\beta_2 k_m-\alpha_2\beta_1 k_1\neq 0,\ \ \beta_1\beta_2\neq 0$, an argument analogous to the previous one shows that the eigenvalues are the solutions of

$$\cot\bigl(s_m\sqrt{\lambda}\bigr)=\begin{cases}-\dfrac{k_m\beta_2}{\alpha_2\sqrt{\lambda}}, & \alpha_1=0,\ \alpha_2\neq 0,\\ \dfrac{k_1\beta_1}{\alpha_1\sqrt{\lambda}}, & \alpha_1\neq 0,\ \alpha_2=0.\end{cases}$$

explicit forms for the eigenvalues are not available. By Rouche's theorem we can, however, obtain their asymptotics. Since the zeros of $\cot\bigl(s_m\sqrt{\lambda}\bigr)=0$ are $\sqrt{\lambda}=\dfrac{\pi(2n+1)}{2s_m}$, it follows from Rouche's theorem that the zeros of

$$\cot\bigl(s_m\sqrt{\lambda}\bigr)=\frac{\alpha_2 k_1\beta_1-\alpha_1 k_m\beta_2}{\alpha_1\alpha_2\sqrt{\lambda}}$$

have the form

$$\lambda_n=\left(\frac{\pi(2n+1)}{2s_m}+\delta_n^*\right)^2,\qquad |\delta_n^*|\le M,\qquad \delta_n^*=O\left(\frac{1}{n}\right).$$

If $\alpha_1\neq 0$ and $\alpha_2=0$, the eigenfunctions are

$$Y_n(y)=C\cdot\begin{cases}\sin\bigl((h_1-y+h_2+\dots+h_m)\sqrt{\lambda_n}\bigr), & 0<y<h_1,\\ \sin\bigl((h_2-y+h_3+\dots+h_m)\sqrt{\lambda_n}\bigr), & 0<y<h_2,\\ \sin\bigl((h_3-y+h_4+\dots+h_m)\sqrt{\lambda_n}\bigr), & 0<y<h_3,\\ \dots & \\ \sin\bigl((h_{m-1}-y+h_m)\sqrt{\lambda_n}\bigr), & 0<y<h_{m-1},\\ \sin\bigl((h_m-y)\sqrt{\lambda_n}\bigr), & 0<y<h_m,\end{cases}$$

corresponding to the boundary conditions

$$\frac{\alpha_1}{k_1}Y_1'(0)+\beta_1 Y_1(0)=0, \qquad Y_m(h_m)=0.$$

If $\alpha_1=0$ and $\alpha_2\neq 0$, the eigenfunctions are

$$Y_n(y)=C\cdot\begin{cases}\sin\big(y\sqrt{\lambda_n}\big), & 0<y<h_1,\\ \sin\big((h_1+y)\sqrt{\lambda_n}\big), & 0<y<h_2,\\ \sin\big((h_1+h_2+y)\sqrt{\lambda_n}\big), & 0<y<h_3,\\ \dots \\ \sin\big((h_1+h_2+\dots+h_{m-2}+y)\sqrt{\lambda_n}\big), & 0<y<h_{m-1},\\ \sin\big((h_1+h_2+\dots+h_{m-1}+y)\sqrt{\lambda_n}\big), & 0<y<h_m,\end{cases}$$

corresponding to the boundary conditions $Y_1(0)=0$
and $\dfrac{\alpha_2}{k_m}Y_m'(h_m)+\beta_2 Y_m(h_m)=0$.

6) In the case $\alpha_1\alpha_2\neq 0$, $\alpha_1\beta_2k_m-\alpha_2\beta_1k_1=0$, and $\beta_1\beta_2\neq 0$, equation (15) reduces to

$$\Big(\alpha_1\alpha_2\lambda+\beta_1\beta_2k_1k_m\Big)\ \sin\big(s_m\sqrt{\lambda}\big)=0 \quad \Big(\text{equivalently } \big(\tfrac{\alpha_1\alpha_2}{k_1k_m}\lambda+\beta_1\beta_2\big)\sin\big(s_m\sqrt{\lambda}\big)=0\Big).$$

Thus, if $\sin\big(s_m\sqrt{\lambda}\big)=0$, the eigenvalues are

$$\lambda_n=\left(\frac{\pi n}{s_m}\right)^2.$$

The corresponding eigenfunctions have the form: The corresponding eigenfunctions (for $\sin(s_m\sqrt{\lambda})=0$) are

$$Y_n(y)=C\cdot\begin{cases} y_{1n}=\cos\Big(\dfrac{\pi n}{s_m}y\Big)-\dfrac{\beta_1k_1s_m}{\alpha_1\pi n}\ \sin\Big(\dfrac{\pi n}{s_m}y\Big), & 0<y<h_1,\\ y_{2n}=\cos\Big(\dfrac{\pi n}{s_m}(h_1+y)\Big)-\dfrac{\beta_1k_1s_m}{\alpha_1\pi n}\ \sin\Big(\dfrac{\pi n}{s_m}(h_1+y)\Big), & 0<y<h_2,\\ y_{3n}=\cos\Big(\dfrac{\pi n}{s_m}(h_1+h_2+y)\Big)-\dfrac{\beta_1k_1s_m}{\alpha_1\pi n}\ \sin\Big(\dfrac{\pi n}{s_m}(h_1+h_2+y)\Big), & 0<y<h_3,\\ \dots \\ y_{m-1,n}=\cos\Big(\dfrac{\pi n}{s_m}(h_1+h_2+\dots+h_{m-2}+y)\Big)- \\ \frac{\beta_1k_1s_m}{\alpha_1\pi n}\ \sin\Big(\frac{\pi n}{s_m}(h_1+h_2+\dots+h_{m-2}+y)\Big), & 0<y<h_{m-1},\\ y_{mn}=\cos\Big(\dfrac{\pi n}{s_m}(h_1+h_2+\dots+h_{m-1}+y)\Big)- \\ \frac{\beta_1k_1s_m}{\alpha_1\pi n}\ \sin\Big(\frac{\pi n}{s_m}(h_1+h_2+\dots+h_{m-1}+y)\Big), & 0<y<h_m.\end{cases}$$

Here we have used the relation

$$\alpha_1\beta_2k_m-\alpha_2\beta_1k_1=0 \quad\Longrightarrow\quad \frac{\alpha_1}{\beta_1k_1}=\frac{\alpha_2}{\beta_2k_m}.$$

If

$$\frac{\alpha_1\alpha_2}{k_1k_m}\,\lambda + \beta_1\beta_2 = 0, \qquad \text{i.e.} \quad \lambda = -\frac{\beta_1\beta_2k_1k_m}{\alpha_1\alpha_2},$$

and taking into account that

$$\frac{\alpha_1\beta_2}{k_1} - \frac{\alpha_2\beta_1}{k_m} = 0 \quad \Longrightarrow \quad \frac{\beta_1k_1}{\alpha_1} = \frac{\beta_2k_m}{\alpha_2},$$

we obtain the special eigenvalue

$$\lambda = \left(\frac{\beta_1k_1}{\alpha_1}\right)^2 = \left(\frac{\beta_2k_m}{\alpha_2}\right)^2.$$

(The explicit form of the associated eigenfunction is given next.) For the special eigenvalue

$$\lambda = \left(\frac{\beta_1k_1}{\alpha_1}\right)^2 = \left(\frac{\beta_2k_m}{\alpha_2}\right)^2,$$

an associated eigenfunction can be chosen as

$$Y(y) = C\cdot\begin{cases} e^{-\frac{\beta_1k_1}{\alpha_1}\,y}, & 0<y<h_1,\\ e^{-\frac{\beta_1k_1}{\alpha_1}\,(h_1+y)}, & 0<y<h_2,\\ e^{-\frac{\beta_1k_1}{\alpha_1}\,(h_1+h_2+y)}, & 0<y<h_3,\\ \dots & \\ e^{-\frac{\beta_1k_1}{\alpha_1}\,(h_1+h_2+\dots+h_{m-2}+y)}, & 0<y<h_{m-1},\\ e^{-\frac{\beta_1k_1}{\alpha_1}\,(h_1+h_2+\dots+h_{m-1}+y)} & 0<y<h_m. \end{cases}$$

7) In the last case, $\alpha_1\alpha_2 \neq 0$, $\dfrac{\alpha_1\beta_2}{k_1} - \dfrac{\alpha_2\beta_1}{k_m} \neq 0$, and $\beta_1\beta_2 \neq 0$, equation (15) applies. Introduce the functions

$$g(\lambda) = \alpha_1\alpha_2\lambda\,\sin\!\big(s_m\sqrt{\lambda}\big),$$

$$\psi(\lambda) = \big(\alpha_1\beta_2k_m - \alpha_2\beta_1k_1\big)\sqrt{\lambda}\,\cos\!\big(s_m\sqrt{\lambda}\big) - \beta_1\beta_2k_1k_m\,\sin\!\big(s_m\sqrt{\lambda}\big).$$

By Rouche's theorem, if $|g(\lambda)| \geq |\psi(\lambda)|$ for large λ, then $g(\lambda)$ and $g(\lambda)+\psi(\lambda)$ have the same number of zeros.

The eigenfunctions can be written as

$$Y_n(y) = C_n\cdot\begin{cases} \Phi\big(y\sqrt{\lambda_n}\big), & 0<y<h_1,\\ \Phi\big((s_1+y)\sqrt{\lambda_n}\big), & 0<y<h_2,\\ \Phi\big((s_2+y)\sqrt{\lambda_n}\big), & 0<y<h_3,\\ \dots & \\ \Phi\big((s_{m-2}+y)\sqrt{\lambda_n}\big), & 0<y<h_{m-1},\\ \Phi\big((s_{m-1}+y)\sqrt{\lambda_n}\big), & 0<y<h_m, \end{cases} \tag{16}$$

where $s_j = \sum_{i=1}^{j} h_i$ (with $s_0 = 0$) and

$$\Phi(z) = \alpha_1 \cos z - \beta_1 \frac{k_1}{\sqrt{\lambda_n}} \sin z. \tag{17}$$

A explicit-form expression for the eigenvalues is not available, but Rouche's theorem yields their asymptotics. Since the zeros of $\tan\big(s_m\sqrt{\lambda}\big) = 0$ are $\sqrt{\lambda} = \dfrac{\pi n}{s_m}$, it follows that the zeros of

$$\tan\big(s_m\sqrt{\lambda}\big) = \frac{\alpha_1 k_m \beta_2 - \alpha_2 k_1 \beta_1}{\alpha_1 \alpha_2 \sqrt{\lambda}} - \frac{\beta_1 \beta_2 k_1 k_m}{\lambda} \tan\big(s_m\sqrt{\lambda}\big)$$

have the form

$$\lambda_n = \left(\frac{\pi n}{s_m} + \delta_n\right)^2, \qquad |\delta_n| \le M, \qquad \delta_n = O\left(\frac{1}{n}\right).$$

Since $\{Y_n(y)\}$ are the eigenfunctions of the self-adjoint problem (10)-(12) (see Lemma 2), they form an orthonormal basis [18]. We choose C_n from the normalization condition; equivalently,

$$C_n = \left(\sum_{i=1}^{m} \frac{1}{k_i^2} \int_{l_{i-1}}^{l_i} \Phi^2\left(\left(s_{i-1} + \frac{x - l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right) dx\right)^{-\frac{1}{2}}.$$

Then the solution to problem (1)?(4) has the form

$$u_i(x,t) = \sum_{n=1}^{\infty} \varphi_n X_i(x)\, e^{-\lambda_n t} = \sum_{n=1}^{\infty} \varphi_n Y_n(y)\, e^{-\lambda_n t},$$

where

$$\varphi_n = \sum_{i=1}^{m} \int_0^{h_i} \varphi_i(k_i\eta + l_{i-1})\, Y_n(\eta)\, d\eta, \qquad y \text{ is defined by (5).}$$

Making the change of variables

$$\xi = k_i\eta + l_{i-1}, \qquad d\eta = \frac{d\xi}{k_i},$$

in the last integral we obtain

$$\varphi_n = \sum_{i=1}^{m} \frac{1}{k_i} \int_{l_{i-1}}^{l_i} \varphi_i(\xi)\, Y_n\left(\frac{\xi - l_{i-1}}{k_i}\right) d\xi. \tag{18}$$

Therefore, rewriting formula (16) we get

$$Y_n\left(\frac{x - l_{i-1}}{k_i}\right) = C_n \cdot \begin{cases} \Phi\big(\frac{x-l_{i-1}}{k_i}\sqrt{\lambda_n}\big), & l_0 < x < l_1, \\ \Phi\big((s_1 + \frac{x-l_{i-1}}{k_i})\sqrt{\lambda_n}\big), & l_1 < x < l_2, \\ \Phi\big((s_2 + \frac{x-l_{i-1}}{k_i})\sqrt{\lambda_n}\big), & l_2 < x < l_3, \\ \dots \\ \Phi\big((s_{m-2} + \frac{x-l_{i-1}}{k_i})\sqrt{\lambda_n}\big), & l_{m-2} < x < l_{m-1}, \\ \Phi\big((s_{m-1} + \frac{x-l_{i-1}}{k_i})\sqrt{\lambda_n}\big), & l_{m-1} < x < l_m, \end{cases} \tag{19}$$

where Φ is given by (17) and $s_j = \sum_{p=1}^{j} h_p$ (with $s_0 = 0$).

We now proceed to prove the main theorem.

Theorem 1. Let $\varphi(x)$ be a twice continuously differentiable function satisfying the boundary conditions (3) and the conjugation conditions (4), namely,

$$\alpha_1\varphi_1'(l_0) + \beta_1\varphi_1(l_0) = 0, \qquad \alpha_2\varphi_m'(l_m) + \beta_2\varphi_m(l_m) = 0, \tag{20}$$

$$\varphi_i(l_i - 0) = \varphi_{i+1}(l_i + 0), \qquad k_i\varphi_i'(l_i - 0) = k_{i+1}\varphi_{i+1}'(l_i + 0), \quad i = 1, 2, \ldots, m-1. \tag{21}$$

Then the function

$$u_i(x,t) = \sum_{n=1}^{\infty} \varphi_n\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) e^{-\lambda_n t}, \tag{22}$$

where the coefficients φ_n are defined by (18), is the unique classical solution of problem (1)-(4).

Proof. First we prove existence of the solution (22). Since $\left\{ Y_n\left(\frac{x-l_{i-1}}{k_i}\right) \right\}$ are the eigenfunctions and $\{\lambda_n\}$ are the eigenvalues of problem (1)-(4), it is straightforward to verify that the function $u(x,t)$ defined by (22) satisfies the equation, the initial condition, the boundary conditions, and the conjugation conditions of (1)-(4). The series (22) is a sum of the functions

$$u_n(x,t) = \varphi_n\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) e^{-\lambda_n t}. \tag{23}$$

We show that for any fixed $\varepsilon > 0$ the series

$$\sum_{n=1}^{\infty} u_n(x,t), \qquad \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t}(x,t), \qquad \sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2}(x,t)$$

converge uniformly on $\{(x,t) : l_0 < x < l_m,\ t \geq \varepsilon\}$. Clearly, $|\varphi| \leq K_1$, hence from (18) it follows that $|\varphi_n| \leq K_2$. Using (23) and the equalities

$$\frac{\partial u_n}{\partial t} = -\lambda_n\, \varphi_n\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) e^{-\lambda_n t}, \qquad \frac{\partial^2 u_n}{\partial x^2} = -\frac{\lambda_n}{k_i^2}\, \varphi_n\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) e^{-\lambda_n t},$$

we obtain, for $t \geq \varepsilon$,

$$|u_n(x,t)| \leq K_3\, e^{-\lambda_n \varepsilon}, \qquad \left\{ \left|\frac{\partial u_n}{\partial t}\right|, \left|\frac{\partial^2 u_n}{\partial x^2}\right| \right\} \leq K_4\, \lambda_n\, e^{-\lambda_n \varepsilon},$$

where the constants $K_i > 0$ $(i = 1, 2, 3, 4)$ do not depend on n.

Therefore, using the asymptotics $\lambda_n \sim (\pi n / s_m)^2$, we have

$$\left\{ \sum_{n=1}^{\infty} |u_n(x,t)|,\ \sum_{n=1}^{\infty} \left|\frac{\partial u_n}{\partial t}(x,t)\right|,\ \sum_{n=1}^{\infty} \left|\frac{\partial^2 u_n}{\partial x^2}(x,t)\right| \right\} \leq \sum_{n=1}^{\infty} K\, n^2\, e^{-\left(\frac{\pi n}{s_m}\right)^2 \varepsilon},$$

for some constant $K > 0$ independent of n. Since the series on the right-hand side converges absolutely, the Weierstrass M test implies that the series for u, u_t, and u_{xx} converge uniformly for $t \geq \varepsilon$; hence $u(x,t)$, $\dfrac{\partial u(x,t)}{\partial t}$, and $\dfrac{\partial^2 u(x,t)}{\partial x^2}$ are continuous for $t \geq \varepsilon$. Now we must show that the series (22) converges uniformly on the whole domain Ω. Note that the n-th term of (22) is majorized by $|\varphi_n|$. Integrating by parts the integral in (18), we obtain

$$\begin{aligned}
\varphi_n = C_n\Bigg[&-\frac{\beta_1 k_1\,\varphi_1(l_0)}{\lambda_n} + \frac{\varphi_1(l_0)}{\sqrt{\lambda_n}}\,\widetilde{\Phi}\big(s_1\sqrt{\lambda_n}\big) - \int_{l_0}^{l_1}\frac{\varphi_1'(\xi)}{\sqrt{\lambda_n}}\,\widetilde{\Phi}\left(\frac{\xi-l_0}{k_1}\sqrt{\lambda_n}\right)d\xi\Bigg] + \\
&C_n\frac{\varphi_2(l_2-0)}{\sqrt{\lambda_n}}\,\Phi\big(s_2\sqrt{\lambda_n}\big) - \frac{\varphi_2(l_1+0)}{\sqrt{\lambda_n}}\,\Phi\big(s_1\sqrt{\lambda_n}\big) - \\
&\int_{l_1}^{l_2}\frac{\varphi_2'(\xi)}{\sqrt{\lambda_n}}\,\Phi\left(\left(s_1+\frac{\xi-l_1}{k_2}\right)\sqrt{\lambda_n}\right)d\xi \\
&\quad+\cdots \\
&C_n\left[\frac{\varphi_{m-1}(l_{m-1}-0)}{\sqrt{\lambda_n}}\,\Phi\big(s_{m-1}\sqrt{\lambda_n}\big) - \frac{\varphi_{m-1}(l_{m-2}+0)}{\sqrt{\lambda_n}}\,\Phi\big(s_{m-2}\sqrt{\lambda_n}\big)\right] - \\
&\int_{l_{m-2}}^{l_{m-1}}\frac{\varphi_{m-1}'(\xi)}{\sqrt{\lambda_n}}\,\Phi\left(\left(s_{m-2}+\frac{\xi-l_{m-2}}{k_{m-1}}\right)\sqrt{\lambda_n}\right)d\xi + \\
&C_n\left[\frac{\varphi_m(l_m)}{\sqrt{\lambda_n}}\,\Phi\big(s_m\sqrt{\lambda_n}\big) - \frac{\varphi_m(l_{m-1}+0)}{\sqrt{\lambda_n}}\,\Phi\big(s_{m-1}\sqrt{\lambda_n}\big)\right] - \\
&\int_{l_{m-1}}^{l_m}\frac{\varphi_m'(\xi)}{\sqrt{\lambda_n}}\,\Phi\left(\left(s_{m-1}+\frac{\xi-l_{m-1}}{k_m}\right)\sqrt{\lambda_n}\right)d\xi,
\end{aligned}$$

where Φ is given by (17) and

$$\widetilde{\Phi}(z) = \alpha_1 \sin z + \beta_1 \frac{k_1}{\sqrt{\lambda_n}}\cos z. \tag{24}$$

Taking into account the first relation in (21), $\varphi_i(l_i-0) = \varphi_{i+1}(l_i+0)$ for $i = 1, 2, \ldots, m-1$, and integrating once more, we obtain

$$\begin{aligned}
\varphi_n = C_n\Bigg[&-\frac{\beta_1 k_1}{\lambda_n}\,\varphi_1(l_0) - \frac{k_1\alpha_1}{\lambda_n}\,\varphi_1'(l_0) + \frac{k_1}{\lambda_n}\,\varphi_1'(l_1-0)\,\Phi(s_1\sqrt{\lambda_n}) - \\
&\int_{l_0}^{l_1}\frac{k_1\varphi_1''(\xi)}{\lambda_n}\,\Phi\left(\frac{\xi-l_0}{k_1}\sqrt{\lambda_n}\right)d\xi\Bigg] + \\
&C_n\Bigg[\frac{k_2}{\lambda_n}\,\varphi_2'(l_2-0)\,\Phi(s_2\sqrt{\lambda_n}) - \frac{k_2}{\lambda_n}\,\varphi_2'(l_1+0)\,\Phi(s_1\sqrt{\lambda_n}) - \\
&\int_{l_1}^{l_2}\frac{k_2\varphi_2''(\xi)}{\lambda_n}\,\Phi\left(\left(s_1+\frac{\xi-l_1}{k_2}\right)\sqrt{\lambda_n}\right)d\xi\Bigg] \\
&\quad+\cdots
\end{aligned} \tag{25}$$

$$C_n\left[\frac{k_{m-1}}{\lambda_n}\,\varphi'_{m-1}(l_{m-1}-0)\,\Phi(s_{m-1}\sqrt{\lambda_n})-\frac{k_{m-1}}{\lambda_n}\,\varphi'_{m-1}(l_{m-2}+0)\,\Phi(s_{m-2}\sqrt{\lambda_n})-\right.$$
$$\left.\int_{l_{m-2}}^{l_{m-1}}\frac{k_{m-1}\varphi''_{m-1}(\xi)}{\lambda_n}\,\Phi\left(\left(s_{m-2}+\frac{\xi-l_{m-2}}{k_{m-1}}\right)\sqrt{\lambda_n}\right)d\xi\right]+$$
$$C_n\left[\frac{\varphi_m(l_m)}{\sqrt{\lambda_n}}\,\widetilde{\Phi}(s_m\sqrt{\lambda_n})+\frac{k_m}{\lambda_n}\,\varphi'_m(l_m)\,\Phi(s_m\sqrt{\lambda_n})-\right.$$
$$\left.\frac{k_m}{\lambda_n}\,\varphi'_m(l_{m-1}+0)\,\Phi(s_{m-1}\sqrt{\lambda_n})-\int_{l_{m-1}}^{l_m}\frac{k_m\varphi''_m(\xi)}{\lambda_n}\,\Phi\left(\left(s_{m-1}+\frac{\xi-l_{m-1}}{k_m}\right)\sqrt{\lambda_n}\right)d\xi\right],$$

where Φ is given by (17) and $\widetilde{\Phi}(z)=\alpha_1\sin z+\beta_1\dfrac{k_1}{\sqrt{\lambda_n}}\cos z$ (cf. (24)). Using the second relation in (21) together with (17) and (24), one checks that

$$\lambda_n\,\varphi_m(l_m)\,\widetilde{\Phi}\big(s_m\sqrt{\lambda_n}\big)+k_m\sqrt{\lambda_n}\,\varphi'_m(l_m)\,\Phi\big(s_m\sqrt{\lambda_n}\big)=\Delta(\lambda_n), \tag{26}$$

i.e., the left-hand side coincides with the characteristic equation evaluated at λ_n. Hence, using (20)-(21) and (26) in (25), we obtain

$$\varphi_n=-C_n\sum_{i=1}^{m}\frac{k_i}{\lambda_n}\int_{l_{i-1}}^{l_i}\varphi''_i(\xi)\,\Phi\left(\left(s_{i-1}+\frac{\xi-l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right)d\xi.$$

From this representation we derive the estimate

$$|\varphi_n|\le K\,\frac{|\alpha_n|}{n^2},\qquad K=\max_{1\le i\le m}k_i^2, \tag{27}$$

where α_n are the Fourier coefficients of the function $\varphi''(x)$ on the interval $[l_0,l_m]$ with respect to the orthonormal system of eigenfunctions $Y_n\big(\frac{x-l_{i-1}}{k_i}\big)$ defined by (19). From (27) it follows that

$$\sum_{n=1}^{\infty}|\varphi_n|\le K.$$

Thus the majorizing series converges absolutely; hence the series (22) converges uniformly on Ω and defines a continuous function $u(x,t)$ on Ω. This proves existence of a solution.

Uniqueness. Assume there are two solutions $\widetilde{u}(x,t)$ and $\widehat{u}(x,t)$. Let $v(x,t)=\widetilde{u}(x,t)-\widehat{u}(x,t)$. Then v solves

$$\frac{\partial v_i}{\partial t}=k_i^2\frac{\partial^2 v_i}{\partial x^2},\qquad (x,t)\in\Omega_i,\ i=1,2,\dots,m, \tag{28}$$

with the initial condition

$$v(x,0)=0,\qquad l_0\le x\le l_m, \tag{29}$$

the boundary conditions

$$\begin{cases} \alpha_1 \dfrac{\partial v_1}{\partial x}(l_0,t) + \beta_1 v_1(l_0,t) = 0, \\ \alpha_2 \dfrac{\partial v_m}{\partial x}(l_m,t) + \beta_2 v_m(l_m,t) = 0, \end{cases} \qquad 0 \le t \le T, \tag{30}$$

and the conjugation conditions

$$\begin{cases} v_i(l_i - 0,t) = v_{i+1}(l_i + 0,t), \\ k_i \dfrac{\partial v_i}{\partial x}(l_i - 0,t) = k_{i+1} \dfrac{\partial v_{i+1}}{\partial x}(l_i + 0,t), \end{cases} \qquad 0 \le t \le T, \quad i = 1,2,\ldots,m-1. \tag{31}$$

The solution of (28)-(31) can be expanded in the basis $Y_n\big(\frac{x-l_{i-1}}{k_i}\big)$; namely,

$$v_i(x,t) = \sum_{n=1}^{\infty} v_n(t)\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right), \tag{32}$$

where

$$v_n(t) = \sum_{i=1}^{m} \frac{1}{k_i} \int_{l_{i-1}}^{l_i} v_i(x,t)\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) dx. \tag{33}$$

Transforming (33) and differentiating with respect to t, we obtain

$$v_n'(t) = C_n \sum_{i=1}^{m} k_i \int_{l_{i-1}}^{l_i} \frac{\partial^2 v_i(x,t)}{\partial x^2}\, \Phi\left(\left(s_{i-1} + \frac{x - l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right) dx,$$

where Φ is given by (17) and C_n are the normalization constants.

Proceeding similarly, integrate twice by parts, using the boundary conditions (30), the conjugation conditions (31), and the identity

$$\Phi''\left(\left(s_{i-1} + \frac{x - l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right) = -\frac{\lambda_n}{k_i^2}\, \Phi\left(\left(s_{i-1} + \frac{x - l_{i-1}}{k_i}\right)\sqrt{\lambda_n}\right), \qquad i = 1,2,\ldots,m.$$

We obtain

$$v_n'(t) = -\lambda_n v_n(t), \qquad \text{hence} \qquad v_n(t) = c_n e^{-\lambda_n t}, \quad n = 1,2,\ldots.$$

Substituting this $v_n(t)$ into (33) gives

$$v_n(t) = \sum_{i=1}^{m} \frac{1}{k_i} \int_{l_{i-1}}^{l_i} v_i(x,t)\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) dx = c_n e^{-\lambda_n t}. \tag{34}$$

Passing to the limit in (34) as $t \to 0$ (which is permitted by the continuity of $v(x,t)$ on $\overline{\Omega}$), we obtain

$$\lim_{t\to 0} \sum_{i=1}^{m} \frac{1}{k_i} \int_{l_{i-1}}^{l_i} v_i(x,t)\, Y_n\left(\frac{x - l_{i-1}}{k_i}\right) dx = v_n(0) = c_n,$$

and therefore $c_n = 0$ for all $n = 1,2,\ldots$. It follows from (32) that $v(x,t) \equiv 0$, whence $\widetilde{u}(x,t) = \widehat{u}(x,t)$. This completes the proof of the theorem.

4 Conclusion

In this paper, the solution of a multilayer problem for the heat equation with a discontinuous coefficient by the method of separation of variables is substantiated. The existence theorem of a unique classical solution of this problem is proved. The technique used here can also be applied to more general boundary problems and more general conjugation conditions.
Analytical solutions to such problems are very useful and necessary because they provide a higher level of understanding of the solution behavior and can be used for numerical solutions.

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ASYMPTOTIC SOLUTIONS TO INITIAL VALUE PROBLEMS FOR SINGULARLY PERTURBED QUASI-LINEAR IMPULSIVE SYSTEMS

This paper investigates a singularly perturbed quasi-linear impulsive differential system with singularities present both in the differential equations and in the impulse functions. The boundary function method is employed to derive the main results. A uniform asymptotic approximation with higher accuracy is constructed and a complete asymptotic expansion is obtained. Theoretical findings are supported by illustrative examples and numerical simulations. The analysis reveals the presence of boundary and interior layers caused by the singular perturbation and impulsive effects. Sufficient conditions for the existence and uniqueness of the solution are established. The results contribute to the theoretical understanding of impulsive systems with complex singular structures and may be applicable to various problems in applied mathematics.
Key words:singularly perturbed systems, impulsive differential equations with singularities, small parameter, the boundary function method.

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Сингулярлы ауытқыған квазисызықты импульсті жүйелер үшін бастапқы есептің асимптотикалық шешімі

Бұл мақала дифференциалдық теңдеуімен қатар импульстік функциясында кіші параметрі бар сингулярлы ауытқыған квази-сызықты импульстік дифференциалдық жүйені қарастырады. Негізгі нәтижелерді алу үшін шекаралық функциялар әдісі қолданылады. Шешімнің кез-келген дәлдіктегі асимптотикалық жуықтауы алынды және толық асимптотикалық жіктелуі құрылады. Теориялық тұжырымдар иллюстрациялық мысалдармен және сандық модельдеу нәтижелерімен расталады. Зерттеу нәтижесі сингулярлық ауытқу мен импульстік әсерлерден туындайтын шекаралық және ішкі қабаттардың болуын анықтайды. Шешімнің бар және жалғыз болуының жеткілікті шарттары анықталады. Алынған нәтижелер күрделі сингулярлық құрылымы бар импульстік жүйелер туралы теориялық түсініктің дамуына ықпал етеді және қолданбалы математика мәселелерінде қолдануға болады.
Түйін сөздер: сингулярлы ауытқыған жүйелер, сингулярлы ауытқыған импульсті дифференциалдық теңдеулер, кіші параметр, шекаралық функциялар әдісі.

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Асимптотические решения начальных задач для сингулярно возмущённых квазилинейных импульсных систем

В данной работе исследуется сингулярно возмущённая квазилинейная импульсная дифференциальная система, в которой сингулярности присутствуют как в дифференциальных уравнениях, так и в импульсных функциях. Для получения основных результатов применяется метод граничных функций. Построено равномерное асимптотическое приближение повышенной точности и получено полное асимптотическое разложение. Теоретические выводы подтверждаются иллюстративными примерами и результатами численного моделирования. Анализ выявляет наличие как граничных, так и внутренних слоёв, возникающих в результате сингулярных возмущений и импульсных эффектов.

Установлены достаточные условия существования и единственности решения. Полученные результаты способствуют развитию теоретического понимания импульсных систем со сложной сингулярной структурой и могут быть применимы в задачах прикладной математики.

Ключевые слова: сингулярно возмущённые системы, импульсные дифференциальные уравнения с сингулярностями, малый параметр, метод граничных функций.

1 Introduction

Perturbation methods deal with problems that contain a small parameter, usually denoted by ε, which perturbs or slightly modifies a simpler, well-understood problem. These problems arise frequently in applied mathematics [1, 2], physics and engineering [3]. There are two main types of perturbation problems: Regular perturbation problems – the solution varies smoothly with ε.

Singular perturbation problems – the small parameter multiplies the highest derivative [4], causing drastic changes in the nature of the solution as $\varepsilon \to 0$.

Singularly perturbed differential equations represent a challenging and fascinating class of problems where small parameters significantly impact the solution behavior. These equations require specialized methods like matched asymptotic expansions to accurately capture the full dynamics of the solution across different scales.

This work is associated with one of the effective asymptotic methods in the theory of singular perturbations about the method of boundary functions, the mathematical foundations of which were laid in the works [5, 6]. The boundary layer method is a powerful analytical technique used to study differential equations with rapid changes in a small region of the domain — typically near a boundary. This method is especially useful in fluid dynamics, applied mathematics, and singular perturbation theory. In many physical systems (especially fluid flow), variables like velocity or temperature change very sharply near boundaries (e.g., surfaces), but slowly elsewhere. The thin region of rapid change is called the boundary layer. Outside this layer, the solution varies smoothly this is the outer region.

Impulse effects describe the response or reaction of a system to a sudden, short-duration force or signal. These effects are critical in understanding how systems behave under rapid or transient conditions. Impulse differential equations (or impulsive differential equations) are used to model systems that experience sudden changes (impulses) at specific moments in time [7]. These equations combine continuous dynamics (ordinary differential equations) with discrete jumps or instantaneous changes.

Singularly perturbed impulsive systems present significant difficulties. An exact solution of impulsive differential equations with singular perturbations is elusive, which explains the relatively small number of studies in this area. Major works in this field were performed before 2000 (see [8–13]), including the research of Kulev (1992) and Bainov et al. (1996) on uniform asymptotic stability, as well as the work of Zhu et al. (2007) on the exponential stability of singularly perturbed equations with impulsive delay.

In [14–17], singularly perturbed Tikhonov-type systems with impulsive effects are studied. These systems are distinguished by the presence of both slow and fast dynamics, as well as by discrete state discontinuities occurring at fixed time instants. The combination of multi-scale behavior and impulsive phenomena provides a rigorous mathematical framework for the analysis and modeling of complex processes exhibiting rapid transitions and time-scale separation induced by a small perturbation parameter.

Akhmet and Çağ [18–20] extended the Tikhonov theorem to a class of singularly perturbed impulsive systems of the form

$$\begin{aligned} \mu\dot{z} &= f(z,y,t), \qquad \dot{y} = g(z,y,t), \\ \mu\Delta z|_{t=\theta_i} &= I(z,y,\mu), \qquad \Delta y|_{t=\eta_j} = J(z,y), \end{aligned} \tag{1}$$

with initial condition

$$z(0,\mu) = z^0, \quad y(0,\mu) = y^0, \tag{2}$$

where z, f and I are m-dimensional vector valued functions, y, g and J are n-dimensional vector valued functions, $\theta_{i}{}_{i=1}^{p}, 0 < \theta_1 < \theta_2 < ... < \theta_p < T$, and $\eta_{j}{}_{j=1}^{k}$, are distinct discontinuity moments in $(0,T)$.

Unlike the study referenced in [10], the authors considered systems in which not only the differential part but also the impulsive parts are singularly perturbed. In this framework, the impulsive function depends explicitly on the small parameter μ, and the moments of discontinuity for the functions z and y are not coincident. The extension of Tikhonov's theorem to such systems necessitates the treatment of additional complexities arising from the perturbation of impulses.

In [18], two types of singular behavior are analyzed: single-layer and multi-layers structures, both arising due to the nature of the impulse functions. The singularities in the impulsive part are addressed using techniques from singular perturbation theory. Stability of the reduced system in the fast (rescaled) time is established through Lyapunov's second method.

Papers [21–23] are devoted to the study of impulsive systems with singularities. Using the boundary layer method, the authors constructed a uniform asymptotic approximation of the solution for $0 < t < T$, and obtained higher-order approximations as well as complete asymptotic expansions for systems with singularly perturbed impulses.

2 Formalities of approximation

Let us consider on the segment $[0,T]$ the following system

$$\begin{aligned} \mu z' &= F(y,t)z + G(y,t), \quad \mu\Delta z|_{t=\theta_i} = I_1(y,\mu)z + I_2(y,\mu), \\ y' &= f(y,t)z + g(y,t), \qquad \Delta y|_{t=\theta_i} = J_1(y,\mu)z + J_2(y,\mu) \end{aligned} \tag{3}$$

with initial condition

$$z(0,\mu) = z^0, \quad y(0,\mu) = y^0, \tag{4}$$

where μ is a small positive real number, z^0 and y^0 are assumed to be independent of μ, $\theta_{i}{}_{i=1}^{p}, 0 < \theta_1 < \theta_2 < ... < \theta_p < T$, are distinct discontinuity moments in $(0,T)$. We define $\Delta x|_{t=\theta_i} = x(\theta_i+) - x(\theta_i)$, assuming that the right-hand limit $x(\theta_i+) = \lim\limits_{t\to\theta_i+} x(t)$ exists and that the left-hand limit satisfies $x(\theta_i-) = x(\theta_i)$.

Assume that $\mu = 0$ in equation (3). In this case, system (3) reduces to the following system

$$\begin{aligned} 0 &= F(\overline{y},t)\overline{z} + G(\overline{y},t), \qquad & 0 &= I_1(\overline{y},0)\overline{z} + I_2(\overline{y},0), \\ \overline{y}' &= f(\overline{y},t)\overline{z} + g(\overline{y},t), \qquad & \Delta\overline{y}|_{t=\theta_i} &= J_1(\overline{y},0)\overline{z} + J_2(\overline{y},0), \end{aligned} \tag{5}$$

which is called to as a degenerate system, since its order is lower than that of system (3). Therefore, for system (5) the number of initial conditions to be less than the number of initial conditions for (3). For system (5) we should retain only the initial condition for $\overline{y}$ since no initial condition for $\overline{z}$ is needed:

$$\overline{y}(0) = y^0. \tag{6}$$

In order to solve system (5), one needs to find $\overline{z}$ from the equations $0 = F(\overline{y},t)\overline{z} + G(\overline{y},t)$ and $0 = I_1(\overline{y},0)\overline{z} + I_2(\overline{y},0)$. Then, one selects a root of the system in the form $\overline{z} = \varphi(\overline{y}(t),t) = -\frac{G(\overline{y},t)}{F(\overline{y},t)}$, which satisfies the equations $0 = F(\overline{y},t)\varphi(\overline{y}(t),t) + G(\overline{y},t)$ and $0 = I_1(\overline{y},0)\varphi(\overline{y}(t),t) + I_2(\overline{y},0)$. Substituting this expression into equation (5) together with the initial condition (6) yield system

$$\begin{aligned} &\overline{y}' = f(\overline{y},t)\varphi(\overline{y}(t),t) + g(\overline{y},t), \qquad \Delta\overline{y}|_{t=\theta_i} = J_1(\overline{y},0)\varphi(\overline{y}(t),t) + J_2(\overline{y},0), \\ &\overline{y}(0) = y^0. \end{aligned} \tag{7}$$

The following conditions are assumed to hold.

(C1) The functions $F(y,t), G(y,t), f(y,t), g(y,t)$ and $I_i(y,\varepsilon), J_i(y,\varepsilon), i = 1,2$ are infinitely differentiable on the interval $0 \le t \le T$.

(C2) $F(y,t) < 0, 0 \le t \le T$.

(C3) The system (7) has a unique solution $\overline{y}(t)$ on $0 \le t \le T$.

(C4) $1 + \frac{\partial}{\partial \overline{y}}(J_1(\overline{y},0)\varphi(\overline{y}(t),t) + J_2(\overline{y},0)) \neq 0$.

(C5) $\lim\limits_{(z,y,)\to(\varphi,\overline{y},0)} \dfrac{I_1(y,\mu)z + I_2(y,\mu)}{\mu} = 0$, where $\overline{y} = \overline{y}(\theta_i)$ are the values for each impulse moment at the points $t = \theta_i, i = 1,2,...,p$.

An asymptotic approximation to the solution $z(t,\mu), y(t,\mu)$ of problem (3)–(4) will be sought in the form

$$\begin{aligned} &z(t,\mu) = \overline{z}(t,\mu) + \omega^{(i)}(\tau_i,\mu), \quad \tau_i = \frac{t-\theta_i}{\mu}, \quad i = 0,1,2,...,p, \\ &y(t,\mu) = \overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu), \quad \theta_i < t \le \theta_{i+1}, \quad \theta_0 \equiv 0, \theta_{p+1} \equiv T. \end{aligned} \tag{8}$$

where

$$\begin{aligned} \overline{z}(t,\mu) &= \sum_{k=0}^{\infty} \mu^k \overline{z}_k(t), \qquad & \overline{y}(t,\mu) &= \sum_{k=0}^{\infty} \mu^k \overline{y}_k(t), \\ \omega^{(i)}(\tau_i,\mu) &= \sum_{k=0}^{\infty} \mu^k \omega_k^{(i)}(\tau_i), \qquad & \nu^{(i)}(\tau_i,\mu) &= \sum_{k=0}^{\infty} \mu^k \nu_k^{(i)}(\tau_i). \end{aligned} \tag{9}$$

The coefficients $\omega_k^{(i)}(\tau_i)$ and $\nu_k^{(i)}(\tau_i)$ in (9) are called boundary functions, for which the following additional condition is imposed,

$$\omega_k^{(i)}(\infty) = 0,\ \nu_k^{(i)}(\infty) = 0, i = 0, 1, 2, ..., p. \tag{10}$$

By substituting the expansions (8) into system (3), we get at the following equalities

$$\begin{aligned}
&\mu\overline{z}'(t,\mu) + \dot{\omega}^{(i)}(\tau_i,\mu) = F(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t)(\overline{z}(t,\mu) + \omega^{(i)}(\tau_i,\mu)) - F(\overline{y}(t,\mu),t)\overline{z}(t,\mu) + \\
&+ F(\overline{y}(t,\mu),t)\overline{z}(t,\mu) + [G(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t) - G(\overline{y}(t,\mu),t)] + G(\tilde{y}(t,\mu),t),\\
&\overline{y}'(t,\mu) + \dot{\nu}^{(i)}(\tau_i,\mu) = f(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t)(\overline{z}(t,\mu) + \omega^{(i)}(\tau_i,\mu)) - f(\overline{y}(t,\mu),t)\overline{z}(t,\mu) + \\
&+ f(\overline{y}(t,\mu),t)\overline{z}(t,\mu) + [g(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t) - g(\overline{y}(t,\mu),t)] + g(\overline{y}(t,\mu),t).
\end{aligned}$$

Separating the expressions with respect to the variables t and τ_i, we derive two systems

$$\begin{aligned}
\mu\overline{z}'(t,\mu) &= F(\overline{y}(t,\mu),t)\overline{z}(t,\mu) + G(\overline{y}(t,\mu),t),\\
\overline{y}'(t,\mu) &= f(\overline{y}(t,\mu),t)\overline{z}(t,\mu) + g(\overline{y}(t,\mu),t),
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\dot{\omega}^{(i)}(\tau_i,\mu) &= F(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t)\omega^{(i)}(\tau_i,\mu) + [F(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t) - F(\overline{y}(t,\mu),t)]\overline{z}(t,\mu) + \\
&+ G(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t) - G(\overline{y}(t,\mu),t),\\
\dot{\nu}^{(i)}(\tau_i,\mu) &= f(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t)\omega^{(i)}(\tau_i,\mu) + [f(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t) - f(\overline{y}(t,\mu),t)]\overline{z}(t,\mu) + \\
&+ g(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t) - g(\overline{y}(t,\mu),t).
\end{aligned} \tag{12}$$

Let us express F, f, I_1 and I_2 in the form of power series in μ as follows:

$$\begin{aligned}
&F(\overline{y}(t,\mu),t)\overline{z}(t,\mu) + G(\overline{y}(t,\mu),t) = \\
&= F(\overline{y}_0(t) + \mu\overline{y}_1(t) + ...,t)\overline{z}(t,\mu) + G(\overline{y}_0(t) + \mu\overline{y}_1(t) + ...,t) = \\
&= \left(F(\overline{y}_0(t),t) + \mu F_y(\overline{y}_0(t),t)\overline{y}_1(t) + ... + \mu^k F_y(\overline{y}_0(t),t)\overline{y}_k(t) + ...\right)(\overline{z}_0(t) + \mu\overline{z}_1(t) + ...) + \\
&+ \left(G(\overline{y}_0(t),t) + \mu G_y(\overline{y}_0(t),t)\overline{y}_1(t) + ... + \mu^k G_y(\overline{y}_0(t),t)\overline{y}_k(t) + ...\right) = \\
&= F(\overline{y}_0(t),t)\overline{z}_0(t) + G(\overline{y}_0(t),t) + \mu[F(\overline{y}_0(t),t)\overline{z}_1(t) + (F_y(t)\overline{z}_0(t) + G_y(t))\overline{y}_1(t)] + \\
&+ \mu^k[F(\overline{y}_0(t),t)\overline{z}_k(t) + (F_y(t)\overline{z}_0(t) + G_y(t))\overline{y}_k(t) + H_k(t)] + ... = \\
&= F(\overline{y}_0(t),t)\overline{z}_0(t) + G(\overline{y}_0(t),t) + \mu\overline{H}_1(t) + \ldots \mu^k\overline{H}_k(t) + ...,
\end{aligned}$$

where functions $F_y(t)$ and $G_y(t)$ are calculated at the point $(\overline{y}_0(t),t)$ and $H_k(t)$ are defined recursively in terms of $\overline{z}_j(t)$ and $\overline{y}_j(t)$ for $j < k$,

$$\begin{aligned}
&F(\overline{y}(t,\mu) + \mu\nu^{(i)}(\tau_i,\mu),t) - F(\overline{y}(t,\mu),t) = \\
&= F(\overline{y}_0(\theta_i + \mu\tau_i) + \mu\overline{y}_1(\theta_i + \mu\tau_i) + ... + \mu\nu_0^{(i)}(\tau_i) + \mu^2\nu_1^{(i)}(\tau_i) + ...,\theta_i + \mu\tau_i) - \\
&- F(\overline{y}_0(\theta_i + \mu\tau_i) + \mu\overline{y}_1(\theta_i + \mu\tau_i) + ...,\theta_i + \mu\tau_i) = \\
&= \mu F_y(\overline{y}_0(\theta_i),\theta_i)\nu_0^{(i)}(\tau_i) + \mu^2[F_y(\overline{y}_0(\theta_i),\theta_i)\nu_1^{(i)}(\tau_i) + F_2(\theta_i)] + ... + \\
&+ \mu^k[F_y(\overline{y}_0(\theta_i),\theta_i)\nu_{k-1}^{(i)}(\tau_i) + F_k(\theta_i)] + ... = \mu\Pi_1F(\tau_i) + \ldots + \mu^k\Pi_kF(\tau_i) + ...,
\end{aligned}$$

$$F(\overline{y}(t,\mu)+\mu\nu^{(i)}(\tau_i,\mu),t)\omega^{(i)}(\tau_i,\mu)+[F(\overline{y}(t,\mu)+\mu\nu^{(i)}(\tau_i,\mu),t)-F(\overline{y}(t,\mu),t)]\overline{z}(t,\varepsilon)+$$
$$+G(\overline{y}(t,\mu)+\mu\nu^{(i)}(\tau_i,\mu),t)-G(\overline{y}(t,\mu),t)=$$
$$=F(\overline{y}(\theta_i+\mu\tau_i,\mu)+\mu\nu^{(i)}(\tau_i,\mu),\theta_i+\mu\tau_i)\omega^{(i)}(\tau_i,\mu)+[\mu\Pi_1F(\tau_i)+\ldots+\mu^k\Pi_kF(\tau_i)+...]\overline{z}(\theta_i+\mu\tau_i,\mu)+$$
$$+\mu\Pi_1G(\tau_i)+\ldots+\mu^k\Pi_kG(\tau_i)+...=F(\overline{y}_0(\theta_i),\theta_i)\omega_0^{(i)}(\tau_i)+\mu F(\overline{y}_0(\theta_i),\theta_i)\omega_1^{(i)}(\tau_i)+...+$$
$$+[\mu\Pi_1F(\tau_i)+\ldots+\mu^k\Pi_kF(\tau_i)+...](\overline{z}_0(\theta_i)+\mu\overline{z}_1(\theta_i)+...)+\mu\Pi_1G(\tau_i)+\ldots+\mu^k\Pi_kG(\tau_i)+...=$$
$$=F(\overline{y}_0(\theta_i),\theta_i)\omega_0^{(i)}(\tau_i)+\mu[F(\overline{y}_0(\theta_i),\theta_i)\omega_1^{(i)}(\tau_i)+\Pi_1F(\tau_i)\overline{z}_0(\theta_i)+\Pi_1G(\tau_i)]+...+$$
$$+\mu^k[F(\overline{y}_0(\theta_i),\theta_i)\omega_k^{(i)}(\tau_i)+\Pi_kF(\tau_i)\overline{z}_0(\theta_i)+\Pi_kG(\tau_i)]+...=$$
$$=\Pi_0H(\tau_i)+\mu\Pi_1H(\tau_i)+\ldots+\mu^k\Pi_kH(\tau_i)+...,$$

$$\overline{z}(\theta_i+\mu\tau_i,\mu)=\overline{z}_0(\theta_i+\mu\tau_i)+\mu\overline{z}_1(\theta_i+\mu\tau_i)+...=\overline{z}_0(\theta_i)+\mu\tau_i\overline{z}_0'(\theta_i)+...+$$
$$+\mu(\overline{z}_1(\theta_i)+\mu\tau_i\overline{z}_1'(\theta_i)+...)+...=\overline{z}_0(\theta_i)+\mu[\overline{z}_1(\theta_i)+\overline{z}_0'(\theta_i)\tau_i]+$$
$$+\mu^2[\overline{z}_2(\theta_i)+\overline{z}_1'(\theta_i)\tau_i+\overline{z}_0''(\theta_i)\frac{\tau_i}{2}]+...=\overline{\omega}_0(\tau_i)+\mu\overline{\omega}_1(\tau_i)+\mu^2\overline{\omega}_2(\tau_i)+...$$

where the functions $F_k(\theta_i)$ are calculated at the point $(\overline{y}_0(\theta_i),\theta_i), i=1,2,...,p,$ and $\Pi_kF(\tau_i),\Pi_kG(\tau_i), i=1,2,...,p,$ are defined recursively in terms of $\omega_j^{(i)}(\tau_i)$ and $\nu_j^{(i)}(\tau_i)$ for $j<k$. Analogously, one can get that

$$I_1(y(\theta_i,\mu),\mu)z(\theta_i,\mu)+I_2(y(\theta_i,\mu),\mu)=I_1(y(\theta_i-,\mu),\mu)z(\theta-,\mu)+I_2(y(\theta_i-,\mu),\mu)=$$
$$=I_1\left(\overline{y}(\theta_i,\mu)+\mu\nu^{(i-1)}\left(\frac{\theta_i-\theta_{i-1}}{\mu},\mu\right),\mu\right)\left(\overline{z}(\theta_i,\mu)+\omega^{(i-1)}\left(\frac{\theta_i-\theta_{i-1}}{\mu},\mu\right)\right)+$$
$$+I_2\left(\overline{y}(\theta_i,\mu)+\mu\nu^{(i-1)}\left(\frac{\theta_i-\theta_{i-1}}{\mu},\mu\right),\mu\right)=$$
$$=I_1(\overline{y}(\theta_i,\mu),\mu)\overline{z}(\theta_i,\mu)+I_2(\overline{y}(\theta_i,\mu),\mu)=I_1(\overline{y}_0(\theta_i),0)\overline{z}_0(\theta_i)+I_2(\overline{y}_0(\theta_i),0)+ \tag{13}$$
$$+\mu[I_1(\overline{y}_0(\theta_i),0)\overline{z}_1(\theta_i)+I_{1y}(\theta_i)\overline{z}_0(\theta_i)\overline{y}_1(\theta_i)+I_{1\varepsilon}(\theta_i)]+\varepsilon[I_{2y}(\theta_i)\overline{y}_1(\theta_i)+I_{2\varepsilon}(\theta_i)]+...+$$
$$+\mu^k[I_1(\overline{y}_0(\theta_i),0)\overline{z}_k(\theta_i)+I_{1y}(\theta_i)\overline{z}_0(\theta_i)\overline{y}_k(\theta_i)+I_{1k}(\theta_i)]+\mu^k[I_{2y}(\theta_i)\overline{y}_k(\theta_i)+I_{2k}(\theta_i)]+...=$$
$$=I_1(\overline{y}_0(\theta_i),0)\overline{z}_0(\theta_i)+I_2(\overline{y}_0(\theta_i),0)+$$
$$+\mu[I_1(\overline{y}_0(\theta_i),0)\overline{z}_1(\theta_i)+(I_{1y}(\theta_i)\overline{z}_0(\theta_i)+I_{2y}(\theta_i))\overline{y}_1(\theta_i))+\overline{I}_\mu(\theta_i)]+...+$$
$$+\mu^k[I_1(\overline{y}_0(\theta_i),0)\overline{z}_k(\theta_i)+(I_{1y}(\theta_i)\tilde{z}_0(\theta_i)+I_{2y}(\theta_i))\tilde{y}_k(\theta_i))+\overline{I}_k(\theta_i)]+...=$$
$$=\overline{T}_0(\theta_i)+\mu\overline{T}_1(\theta_i)+\ldots+\mu^k\overline{T}_k(\theta_i)+\ldots,$$

where the terms $I_{1y}(\theta_i)$, $I_{2y}(\theta_i), I_{1k}(\theta_i)$ and $I_{2k}(\theta_i)$ are calculated at the point $(\overline{y}_0(\theta_i),0), i=1,2,...,p,$ and $I_{1k}(\theta_i),I_{2k}(\theta_i)$ are defined recursively in terms of $\overline{z}_j(\theta_i)$ and $\overline{y}_j(\theta_i)$ for $j<k$. Analogous expansions hold for the expression $J_1(y,\mu)z+J_2(y,\mu)$.

The problems (3), (4) with (11) and (12) can be rewritten in the following form

$$\mu(\overline{z}_0'(t)+\mu\overline{z}_1'(t)+\ldots+\mu^k\tilde{z}_k'(t)+\ldots)=\overline{H}_0(t)+\mu\overline{H}_1(t)+\ldots\mu^k\overline{H}_k(t)+\ldots,$$
$$\overline{y}_0'(t)+\mu\overline{y}_1'(t)+\ldots+\mu^k\overline{y}_k'(t)+\ldots=\overline{h}_0(t)+\mu\overline{h}_1(t)+\ldots\mu^k\overline{h}_k(t)+\ldots,$$
$$\dot{\omega}_0^{(i)}(\tau_i)+\mu\dot{\omega}_1^{(i)}(\tau_i)+\ldots+\mu^k\dot{\omega}_k^{(i)}(\tau_i)+\ldots=\Pi_0H(\tau_i)+\mu\Pi_1H(\tau_i)+\ldots+\mu^k\Pi_kH(\tau_i)+\ldots,$$
$$\dot{\nu}_0^{(i)}(\tau_i)+\varepsilon\dot{\nu}_1^{(i)}(\tau_i)+\ldots+\varepsilon^k\dot{\nu}_k^{(i)}(\tau_i)+\ldots=\Pi_0h(\tau_i)+\varepsilon\Pi_1h(\tau_i)+\ldots+\mu^k\Pi_kh(\tau_i)+\ldots,$$
$$\mu(\sum_{k=0}^{\infty}\mu^k\Delta\overline{z}_k|_{t=\theta_i}+\sum_{k=0}^{\infty}\mu^k\omega_k^{(i)}(0))=\overline{T}_0(\theta_i)+\mu\overline{T}_1(\theta_i)+\ldots+\mu^k\tilde{T}_k(\theta_i)+\ldots,$$
$$\sum_{k=0}^{\infty}\mu^k\Delta\overline{y}_k|_{t=\theta_i}+\mu\sum_{k=0}^{\infty}\mu^k\nu_k^{(i)}(0)=\overline{S}_0(\theta_i)+\mu\overline{S}_1(\theta_i)+\ldots+\mu^k\overline{S}_k(\theta_i)+\ldots.$$

By inserting the expansion (9) into conditions (4), we get

$$z(0,\mu)=\sum_{k=0}^{\infty}\mu^k\overline{z}_k(0)+\sum_{k=0}^{\infty}\mu^k\omega_k^{(0)}(0)=z^0,$$

and

$$y(0,\mu)=\sum_{k=0}^{\infty}\mu^k\overline{y}_k(0)+\mu\sum_{k=0}^{\infty}\mu^k\nu_k^{(0)}(0)=y^0.$$

The expansions are performed up to order n and the coefficients are equated by powers of μ. For the zero-order approximation $\overline{z}_0(t),\overline{y}_0(t),\omega_0^{(i)}(\tau_i)$ and $\nu_0^{(i)}(\tau_i), i=1,2,...,p$, the following systems are obtained:

$$\begin{aligned}0&=F(\tilde{y}_0(t),t)\overline{z}_0(t)+G(\overline{y}_0(t),t),\\ \overline{y}_0'(t)&=f(\overline{y}_0(t),t)\overline{z}_0(t)+g(\overline{y}_0(t),t),\end{aligned}\tag{14}$$

$$\begin{aligned}\dot{\omega}_0^{(i)}(\tau_i)&=F(\overline{y}_0(\theta_i),\theta_i)\omega_0^{(i)}(\tau_i)=\Pi_0H(\tau_i),\\ \dot{\nu}_0^{(i)}(\tau_i)&=f(\overline{y}_0(\theta_i),\theta_i)\omega_0^{(i)}(\tau_i)=\Pi_0h(\tau_i),\end{aligned}\tag{15}$$

$$0=\frac{I_1(\overline{y}_0(\theta_i),0)\overline{z}_0(\theta_i)+I_2(\overline{y}_0(\theta_i),0)}{\mu},\tag{16}$$

$$\begin{aligned}&\Delta\overline{z}_0|_{t=\theta_i}+\omega_0^{(i)}(0)=I_1(\overline{y}_0(\theta_i),0)\overline{z}_1(\theta_i)+(I_{1y}(\theta_i)\overline{z}_0(\theta_i)+I_{2y}(\theta_i))\overline{y}_1(\theta_i)+\overline{I}_\mu(\theta_i)=\overline{T}_1(\theta_i),\\ &\Delta\overline{y}_0|_{t=\theta_i}=J_1(\overline{y}_0(\theta_i),0)\overline{z}_0(\theta_i)+J_2(\overline{y}_0(\theta_i),0)=\overline{S}_0(\theta_i),\\ &\overline{z}_0(0)+\omega_0^{(0)}(0)=z^0,\ \ \overline{y}_0(0)=y^0.\end{aligned}\tag{17}$$

To find the coefficients of $\mu^k(k\geq 1)$, the following equations are used

$$\begin{aligned}\overline{z}_{k-1}'(t)&=F(\overline{y}_0(t),t)\overline{z}_k(t)+(F_y(t)\overline{z}_0(t)+G_y(t))\overline{y}_k(t)+H_k(t),\\ \overline{y}_k'(t)&=f(\overline{y}_0(t),t)\overline{z}_k(t)+(f_y(t)\overline{z}_0(t)+g_y(t))\overline{y}_k(t)+h_k(t),\end{aligned}\tag{18}$$

$$\dot{\omega}_k^{(i)}(\tau_i) = F(\overline{y}_0(\theta_i), \theta_i)\omega_k^{(i)}(\tau_i) + \Pi_k F(\tau_i)\overline{z}_0(\theta_i) + \Pi_k G(\tau_i) = \Pi_k H(\tau_i),$$
$$\dot{\nu}_k^{(i)}(\tau_i) = f(\overline{y}_0(\theta_i), \theta_i)\omega_k^{(i)}(\tau_i) + \Pi_k f(\tau_i)\overline{z}_0(\theta_i) + \Pi_k g(\tau_i) = \Pi_k h(\tau_i),$$

$$\begin{aligned} &\Delta\overline{z}_k|_{t=\theta_i} + \omega_k^{(i)}(0) = I_1(\overline{y}_0(\theta_i), 0)\overline{z}_{k+1}(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_{k+1}(\theta_i)) + \overline{I}_{k+1}(\theta_i), \\ &\Delta\overline{y}_k|_{t=\theta_i} + \nu_{k-1}^{(i)}(0) = J_1(\overline{y}_0(\theta_i), 0)\overline{z}_k(\theta_i) + (J_{1y}(\theta_i)\overline{z}_0(\theta_i) + J_{2y}(\theta_i))\overline{y}_k(\theta_i)) + \overline{J}_k(\theta_i), \\ &\overline{z}_k(0) + \omega_k^{(0)}(0) = 0, \;\; \overline{y}_k(0) + \nu_{k-1}^{(0)}(0) = 0. \end{aligned} \tag{19}$$

Now we consider the interval $t \in [0, \theta_1]$. To obtain the leading-order approximations $\overline{z}_0(t) = \overline{z}(t)$ and $\overline{y}_0(t) = \overline{y}(t)$, we solve system

$$0 = F(\overline{y}_0(t), t)\overline{z}_0(t) + G(\overline{y}_0(t), t),$$
$$\overline{y}_0'(t) = f(\overline{y}_0(t), t)\overline{z}_0(t) + g(\overline{y}_0(t), \; \overline{y}_0(0) = y^0.$$

By virtue of the first equation in (14), equation (15) can be rewritten in the form

$$\dot{\omega}_0^{(0)}(\tau_0) = F(\overline{y}_0(0), 0)\omega_0^{(0)}(\tau_0).$$

From the last equation, together with the initial condition

$$\omega_0^{(0)}(0) = z^0 - \overline{z}_0(0)$$

the function $\omega_0^{(0)}(\tau_0)$ can be determined. According to condition $(C5)$, $\omega_0^{(0)}(\tau_0)$ admits the exponential estimate

$$|\omega_0^{(0)}(\tau_0)| \leq c\exp(-\kappa\tau_0), \tag{20}$$

where $c > 0$ and $\kappa > 0$.

The final step is to solve equation

$$\dot{\nu}_0^{(0)}(\tau_0) = F(\overline{y}_0(0), 0)\omega_0^{(0)}(\tau_0) \equiv \Pi_0 h(\tau_0).$$

In view of condition (10), the initial condition is given by

$$\nu_0^{(0)}(0) = -\int_0^\infty \Pi_0 h(s)ds,$$

from which it follows that

$$\nu_0^{(0)}(\tau_0) = -\int_{\tau_0}^\infty \Pi_0 h(s)ds.$$

Since $\Pi_0 f(\tau_0)$ decays exponentially, i.e., $|\Pi_0 f(\tau_0)| \leq c\exp(-\kappa\tau_0)$ the same holds for $\nu_0^{(0)}(\tau_0):$

$$|\nu_0^{(0)}(\tau_0)| \leq c\exp(-\kappa\tau_0).$$

The coefficients of μ^k in the approximations $\overline{z}_k(t)$ and $\overline{y}_k(t)$ are obtained by applying system

$$\overline{z}_{k-1}'(t) = F(\overline{y}_0(t), t)\overline{z}_k(t) + (F_y(t)\overline{z}_0(t) + G_y(t))\overline{y}_k(t) + H_k(t),$$
$$\overline{y}_k'(t) = f(\overline{y}_0(t), t)\overline{z}_k(t) + (f_y(t)\overline{z}_0(t) + g_y(t))\overline{y}_k(t) + h_k(t),$$
$$\overline{y}_k(0) + \nu_{k-1}^{(0)}(0) = 0.$$

To get $\omega_k^{(0)}(\tau_0)$, the following system must be solved

$$\dot{\omega}_k^{(0)}(\tau_0) = F(\overline{y}_0(0), 0)\omega_k^{(0)}(\tau_0) + \Pi_k F(\tau_0)\overline{z}_0(0) + \Pi_k G(\tau_0) = \Pi_k H(\tau_0),$$
$$\omega_k^{(0)}(0) = -\overline{z}_k(0).$$

The remaining task is to solve the equation

$$\dot{\nu}_k^{(0)}(\tau_0) = f(\overline{y}_0(0), 0)\omega_k^{(0)}(\tau_0) + \Pi_k f(\tau_0)\overline{z}_0(0) + \Pi_k g(\tau_0) = \Pi_k h(\tau_0)$$

Taking into account condition (10), the initial condition is given by

$$\nu_k^{(0)}(0) = -\int_0^{\infty} \Pi_k h(s)ds,$$

from which it follows that

$$\nu_k^{(0)}(\tau_0) = -\int_0^{\tau_\infty} \Pi_k h(s)ds.$$

Both $\Pi_k H(\tau_0)$ and $\Pi_k h(\tau_0)$ satisfy exponential estimates of the type given in (20). As a consequence, the following inequalities are satisfied,

$$|\omega_k^{(0)}(\tau_0)| \leq c\exp(-\kappa\tau_0),$$
$$|\nu_k^{(0)}(\tau_0)| \leq c\exp(-\kappa\tau_0).$$

Let us now consider the interval $t \in (\theta_i, \theta_{i+1}], i = 1, 2, ..., p$. To obtain the leading-order terms $\overline{z}_0(t) = \overline{z}(t)$ and $\overline{y}_0(t) = \overline{y}(t)$, corresponding to the power ε^0, we make use of system

$$\begin{aligned} 0 &= F(\overline{y}_0(t), t)\overline{z}_0(t) + G(\overline{y}_0(t), t), & 0 &= I_1(\overline{y}_0(\theta_i), 0)\overline{z}_0(\theta_i) + I_2(\overline{y}_0(\theta_i), 0), \\ \overline{y}_0'(t) &= f(\overline{y}_0(t), t)\overline{z}_0(t) + g(\overline{y}_0(t), t), & \Delta\overline{y}_0|_{t=\theta_i} &= J_1(\overline{y}_0(\theta_i), 0)\overline{z}_0(\theta_i) + J_2(\overline{y}_0(\theta_i), 0). \end{aligned}$$

In view of the first equation in (14), equation (15) takes the form

$$\dot{\omega}_0^{(i)}(\tau_i) = F(\overline{y}_0(\theta_i), \theta_i)\omega_0^{(i)}(\tau_i), i = 1, 2, ..., p.$$

Based on the last equation and the initial condition

$$\omega_0^{(i)}(0) = I_1(\overline{y}_0(\theta_i), 0)\overline{z}_1(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_1(\theta_i) + \overline{I}_\mu(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}, i = 1, 2, ..., p,$$

the function $\omega_0^{(i)}(\tau_i)$ is to be determined, where $\omega_0^{(i)}(0)$ represented in the modified form below. Differentiating both sides of the first equations in (14) and (16) yields the following

$$\begin{aligned} F_y(\overline{y}_0(\theta_i), \theta_i)\overline{z}_0(\theta_i) + G_y(\overline{y}_0(\theta_i), \theta_i) &= -F(\overline{y}_0(\theta_i), \theta_i)\frac{dz}{dy}, \\ I_{1y}(\overline{y}_0(\theta_i), 0)\overline{z}_0(\theta_i) + I_{2y}(\overline{y}_0(\theta_i), 0) &= -I_1(\overline{y}_0(\theta_i), 0)\frac{dz}{dy}. \end{aligned} \tag{21}$$

Inserting the first equation of (21) into (18) results in

$$\overline{z}_0'(\theta_i) = F(\overline{y}_0(\theta_i), \theta_i)\overline{z}_1(\theta_i) + (F_y(\theta_i)\overline{z}_0(\theta_i) + G_y(\theta_i))\overline{y}_1(\theta_i) + H_1(\theta_i).$$

Hence, it follows that

$$\overline{z}_1(\theta_i) - \overline{y}_1(\theta_i)\frac{dz}{dy} = \frac{\overline{z}_0'(\theta_i) - H_1(\theta_i)}{F(\overline{y}_0(\theta_i), \theta_i)}. \tag{22}$$

Inserting the second equation of (21) into (19) gives

$$\omega_0^{(i)}(0) = I_1(\overline{y}_0(\theta_i), 0)[\overline{z}_1(\theta_i) - \overline{y}_1(\theta_i)\frac{dz}{dy}] + \overline{I}_\mu(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

Substituting equation (22) in place of the square bracket yields

$$\omega_0^{(i)}(0) = \frac{I_1(\overline{y}_0(\theta_i), 0)}{F(\overline{y}_0(\theta_i), \theta_i)}(\overline{z}_0'(\theta_i) - H_1(\theta_i)) + \overline{I}_\mu(\theta_i) - \Delta\overline{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

According to condition $(C5)$, the function $\omega_0^{(i)}(\tau_i)$ satisfies an exponential estimate of the form

$$|\omega_0^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), i = 1, 2, ..., p, \tag{23}$$

where c and κ denote positive constants, which values may differ across various inequalities.

The remaining task is to solve the following equation

$$\dot{\nu}_0^{(i)}(\tau_i) = f(\overline{y}_0(\theta_i), \theta_i)\omega_0^{(i)}(\tau_i) = \Pi_0 h(\tau_i), i = 1, 2, ..., p.$$

Using condition (10), we determine the initial condition as follows

$$\nu_0^{(i)}(0) = -\int_0^\infty \Pi_0 h(s)ds.$$

Consequently, the following result is derived

$$\nu_0^{(i)}(\tau_i) = -\int_{\tau_i}^\infty \Pi_0 h(s)ds.$$

Since $|\Pi_0 h(\tau_i)| \le c\exp(-\kappa\tau_i)$, it holds that

$$|\nu_0^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), i = 1, 2, ..., p.$$

The coefficients of ε^k in the approximations $\overline{z}_k(t)$ and $\overline{y}_k(t)$ are determined from the following system

$$\begin{aligned} \overline{z}_{k-1}'(t) &= F(\overline{y}_0(t), t)\overline{z}_k(t) + (F_y(t)\overline{z}_0(t) + G_y(t))\overline{y}_k(t) + H_k(t), \\ \overline{y}_k'(t) &= f(\overline{y}_0(t), t)\overline{z}_k(t) + (f_y(t)\overline{z}_0(t) + g_y(t))\overline{y}_k(t) + h_k(t), \\ \Delta\overline{y}_k|_{t=\theta_i} + \nu_{k-1}^{(i)}(0) &= J_1(\overline{y}_0(\theta_i), 0)\overline{z}_k(\theta_i) + (J_{1y}(\theta_i)\overline{z}_0(\theta_i) + J_{2y}(\theta_i))\overline{y}_k(\theta_i)) + \overline{J}_k(\theta_i). \end{aligned}$$

The functions $\omega_k^{(i)}(\tau_i)$ are determined as the solutions of the following system

$$\begin{aligned} &\dot{\omega}_k^{(i)}(\tau_i) = F(\overline{y}_0(\theta_i), \theta_i)\omega_k^{(i)}(\tau_i) + \Pi_k F(\tau_i)\overline{z}_0(\theta_i) + \Pi_k G(\tau_i) = \Pi_k H(\tau_i), \\ &\omega_k^{(i)}(0) = I_1(\overline{y}_0(\theta_i), 0)\overline{z}_{k+1}(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_{k+1}(\theta_i) + \overline{I}_{k+1}(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i}, \end{aligned}$$

where the initial value $\omega_k^{(i)}(0)$ can be represented in an equivalent form below

$$\omega_k^{(i)}(0) = \frac{I_1(\overline{y}_0(\theta_i), 0)}{F(\overline{y}_0(\theta_i), \theta_i)}(\overline{z}_k'(\theta_i) - H_{k+1}(\theta_i)) + \overline{I}_{k+1}(\theta_i) - \Delta\overline{z}_k|_{t=\theta_i},$$

Finally, it is necessary to solve the equation

$$\dot{\nu}_k^{(i)}(\tau_i) = f(\overline{y}_0(\theta_i), \theta_i)\omega_k^{(i)}(\tau_i) + \Pi_k f(\tau_i)\overline{z}_0(\theta_i) + \Pi_k g(\tau_i) = \Pi_k h(\tau_i), i = 1, 2, ..., p.$$

By applying condition (10), we obtain

$$\nu_k^{(i)}(0) = -\int_0^\infty \Pi_k h(s)ds,$$

and

$$\nu_k^{(i)}(\tau_i) = -\int_{\tau_i}^\infty \Pi_k h(s)ds.$$

The functions $\Pi_k H(\tau_i)$ and $\Pi_k h(\tau_i)$ admit exponential estimates of the form (23). Accordingly, one can prove that the following inequalities are satisfied,

$$\begin{aligned} &|\omega_k^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), i = 1, 2, ..., p, \\ &|\nu_k^{(i)}(\tau_i)| \le c\exp(-\kappa\tau_i), i = 1, 2, ..., p. \end{aligned} \tag{24}$$

Hence, the expansions in (9) are constructed at least up to the terms of order $k = n$.

3 Main Results

In this section, we prove Theorems 1 and Theorem 2, which address two different behaviors: a single layer singularity and a multi-layers singularity. The first behavior corresponds to a layer concentrated near $t = 0$, while the second deals with the presence of multiple layers near $t = 0$ and at the points $t = \theta_i$, $i = 1, 2, \ldots, p$. It is demonstrated that the partial sums of the series (8) form a sequence of uniform approximations to the solution of the problem (3)–(4).

3.1 Asymptotic expansion of singularity with a single layer

We consider the case in which the convergence of the solution is non-uniform in a neighborhood of $t = 0$, as a result of the initial condition $z(0, \mu) = z^0$ satisfying $z^0 \neq \varphi$ for all $\mu > 0$. The interval where this non-uniformity occurs is referred to as the *initial layer*.

In accordance with condition $(C5)$ of (13), the following identity holds,

$$I_1(\overline{y}_0(\theta_i), 0)\overline{z}_1(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_1(\theta_i) + \overline{I}_\mu(\theta_i) = 0, i = 1, 2, ..., p.$$

As a result, the first equation of (17) becomes

$$\omega_0^{(i)}(0) = -\Delta\overline{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

Substituting the above expression into (8), we obtain

$$z(\theta_i+,\mu) = \overline{z}_0(\theta_i+) + \omega_0^{(i)}(0) + O(\mu) = \overline{z}_0(\theta_i) + O(\mu), i = 1,2,...,p.$$

It can be concluded that the region of non-uniform convergence has a thickness of order $O(\mu)$, since for $t > 0$ the estimate $|z(t,\mu) - \varphi| = O(\mu)$ holds and can be made arbitrarily small by choosing sufficiently small μ. This indicates that, for sufficiently small values of μ, the solution $z(t,\mu)$ to the problem (3), (4) does not exhibit boundary layer behavior in the vicinity of the points $t = \theta_i$, $i = 1,2,\dots,p$.

Theorem 1 *Let conditions $(C1) - (C4)$ and $(C5)$ be satisfied. Then there exist positive constants μ_0 and c such that, for all $\mu \in (0,\mu_0]$, the problem (3), (4) admits a unique solution $z(t,\mu), y(t,\mu)$ that satisfies the inequality*

$$\begin{aligned} &|z(t,\mu) - Z_n(t,\mu)| \le c\mu^{n+1}, \quad 0 \le t \le T, \\ &|y(t,\mu) - Y_n(t,\mu)| \le c\mu^{n+1}, \quad 0 \le t \le T, \end{aligned} \tag{25}$$

where

$$\begin{aligned} &Z_n(t,\mu) = Z_n^{(i)}(t,\mu), Y_n(t,\mu) = Y_n^{(i)}(t,\mu), \theta_i < t \le \theta_{i+1}, \\ &Z_n^{(i)}(t,\mu) = \sum_{k=0}^{n} \mu^k \overline{z}_k(t) + \sum_{k=0}^{n} \mu^k \omega_k^{(i)}(\tau_i), \tau_i = \frac{t-\theta_i}{\mu}, \\ &Y_n^{(i)}(t,\mu) = \sum_{k=0}^{n} \mu^k \overline{y}_k(t) + \mu \sum_{k=0}^{n} \mu^k \nu_k^{(i)}(\tau_i), i = 1,2,...,p. \end{aligned}$$

Proof 1 *Substituting the expressions $z(t,\mu) = u(t,\mu) + Z_n(t,\mu)$ and $y(t,\mu) = v(t,\mu) + Y_n(t,\mu)$ into equations (3) and (4), we derive the following system*

$$\begin{aligned} \mu\frac{du}{dt} &= F(Y_0,t)u + [F_y(Y_0,t)Z_0 + G_y(Y_0,t)]v + T_1(u,v,t,\mu), \\ \frac{dv}{dt} &= f(Y_0,t)u + [f_y(Y_0,t)Z_0 + g_y(Y_0,t)]v + T_2(u,v,t,\mu), \\ \mu\Delta u|_{t=\theta_i} &= I_1(Y_0,0)u + [I_{1y}(Y_0,0)Z_0 + I_{2y}(Y_0,0)]v + S_1(u,v,\theta_i,\mu), \\ \Delta v|_{t=\theta_i} &= J_1(Y_0,0)u + [J_{1y}(Y_0,0)Z_0 + J_{2y}(Y_0,0)]v + S_2(u,v,\theta_i,\mu), \end{aligned} \tag{26}$$

with initial condition

$$u(0,\mu) = 0, \quad v(0,\mu) = 0, \tag{27}$$

where the components of the functions F_z, F_y, f_z and f_y are calculated at the points $(\overline{z}_0(t) +$

$\omega_0^{(i)}(\tau_i), \overline{y}_0(t), 0), i = 1, 2, ..., p,$

$$\begin{aligned}
T_1(u, v, t, \mu) =& F(v + Y_n, t)(u + Z_n) + G(v + Y_n, t) - F(Y_0, t)u- \\
&- [F_y(Y_0, t)Z_0 + G_y(Y_0, t)]v - \mu \frac{dZ_n}{dt}, \\
T_2(u, v, t, \mu) =& f(v + Y_n, t)(u + Z_n) + g(v + Y_n, t) - f(Y_0, t)u- \\
&- [f_y(Y_0, t)Z_0 + g_y(Y_0, t)]v - \frac{dY_n}{dt}, \\
S_1(u, v, \theta_i, \mu) =& I_1(v + Y_n^{(i-1)}, \mu)(u + Z_n^{(i-1)}) + I_2(v + Y_n^{(i-1)}, \mu) - I_1(Y_0, 0)u- \\
&- [I_{1y}(Y_0, 0)Z_0 + I_{2y}(Y_0, 0)]v + \mu Z_n^{(i-1)} - \mu Z_n^{(i)}, \\
S_2(u, v, \theta_i, \mu) =& J_1(v + Y_n^{(i-1)}, \mu)(u + Z_n^{(i-1)}) + J_2(v + Y_n^{(i-1)}, \mu) - J_1(Y_0, 0)u- \\
&- [J_{1y}(Y_0, 0)Z_0 + J_{2y}(Y_0, 0)]v + Y_n^{(i-1)} - Y_n^{(i)}.
\end{aligned}$$

The functions $T(u, v, t, \mu)$ possess the following two properties,

1) $T_1(0, 0, t, \mu) = O(\mu^{n+1}), T_2(0, 0, t, \mu) = O(\mu^{n+1})$.

2) *For any $\mu > 0$, there exist constants $c_2 > 0$ and $\mu_0 > 0$ such that, for all $\mu \in (0, \mu_0)$ and for $u_i, v_i, i = 1, 2$, the following inequalities are satisfied,*

$|T_i(u_1, v_1, t, \mu) - T_i(u_2, v_2, t, \mu)| \leq c_2 \mu(|u_2 - u_1| + |v_2 - v_1|),\ i = 1, 2.$

We now proceed to prove property 1). For $t \in (\theta_i, \theta_{i+1}]$, it follows that

$$\begin{aligned}
&T_1(0, 0, t, \mu) = F(v + Y_n, t)Z_n + G(v + Y_n, t) - \mu \frac{dZ_n}{dt} = G(\sum_{k=0}^{n} \mu^k(\overline{y}_k(t) + \mu\nu_k^{(i)}(\tau_i)), t)+ \\
&+ F(\sum_{k=0}^{n} \mu^k(\overline{y}_k(t) + \mu\nu_k^{(i)}(\tau_i)), t)(\sum_{k=0}^{n} \mu^k(\overline{z}_k(t) + \omega_k^{(i)}(\tau_i))) - \sum_{k=0}^{n} \mu^k(\overline{z}'_k(t) + \dot{\omega}_k^{(i)}(\tau_i)) = \\
&= F(\sum_{k=0}^{n} \mu^k(\overline{y}_k(t), t) \sum_{k=0}^{n} \mu^k \overline{z}_k(t) + G(\sum_{k=0}^{n} \mu^k(\overline{y}_k(t), t) - \sum_{k=0}^{n} \mu^k \overline{z}'_k(t)+ \\
&F(\overline{y}(\theta_i + \mu\tau_i, \mu) + \mu\nu^{(i)}(\tau_i, \mu), \theta_i + \mu\tau_i) \sum_{k=0}^{n} \mu^k \omega_k^{(i)}(\tau_i) + \sum_{k=0}^{n} \mu^k(\Pi_k F(\tau_i)\overline{z}_0(\theta_i) + \Pi_k G(\tau_i))- \\
&- \sum_{k=0}^{n} \mu^k \dot{\omega}_k^{(i)}(\tau_i) = [\sum_{k=0}^{n} \mu^k \overline{H}_k(t) + O(\mu^{n+1}) - \sum_{k=0}^{n} \mu^k \overline{z}'_k(t)]+ \\
&+ [\sum_{k=0}^{n} \mu^k \Pi_k H(\tau_i) + O(\mu^{n+1}) - \sum_{k=0}^{n} \mu^k \dot{\omega}_k^{(i)}(\tau_i)] = O(\mu^{n+1}),
\end{aligned}$$

similarly to that for the functions $\overline{y}_k(t), \nu_k^{(i)}(\tau_i), i = 1, 2, ..., p$. The validity of the second property of the functions $T_j, j = 1, 2$, can be derived by applying the mean value theorem. In fact,

$$T_i(u_1, v_1, t, \mu) - T_i(u_2, v_2, t, \mu) = \sup_{[0;T]} |\partial_u^* T| \cdot (u_1 - u_2) + \sup_{[0;T]} |\partial_v^* T| \cdot (v_1 - v_2),$$

where $\partial_u^* T = \partial_u^* T(u^*(s), v^*(s), t, \mu)$, $\partial_v^* T = \partial_v^* T(u^*(s), v^*(s), t, \mu)$, $u^*(s) = u_2 + s(u_1 - u_2)$, $v^*(s) = u^*(s) = v_2 + s(v_1 - v_2)$, $0 < s < 1$. *But*

$$\partial_u T_i(u^*(s), v^*(s), t, \mu) = F(v + Y_n, t) - F(Y_0, t),$$
$$\partial_v T_i(u^*(s), v^*(s), t, \mu) = F_y(v + Y_n, t)(u + Z_n) - F_y(Y_0, t)Z_0 + G_y(v + Y_n, t) - G_y(Y_0, t),$$

and

$$|u^*(s) + Z_n(t, \mu) - Z_0(t)| \le |u^*(s)| + C\mu,$$
$$|v^*(s) + Y_n(t, \mu) - Y_0(t)| \le |v^*(s)| + C\mu.$$

The continuity of the first-order partial derivatives of the functions $F(y,t), G(y,t), f(y,t)$ *and* $g(y,t)$ *ensures the validity of property* 2). *The functions* $S_i(u, v, \theta_i, \mu)$, $i = 1, 2$, *possess the following two properties,*

1^*) *For* $0 < \mu < \mu_0$

$$S_1(0, 0, \theta_i, \mu) = O(\mu^{n+1}), S_2(0, 0, \theta_i, \mu) = O(\mu^{n+1}).$$

2^*) *For any* $\mu > 0$, *there exist constants* $c_2 > 0$ *and* $\mu_0 > 0$ *such that, for all* $\mu \in (0, \mu_0)$ *and for* $u_i, v_i, i = 1, 2$, *the following inequalities are satisfied,*

$$|S_i(u_1, v_1, t, \mu) - S_i(u_2, v_2, t, \mu)| \le c_2\mu(|u_2 - u_1| + |v_2 - v_1|), \ i = 1, 2.$$

The proofs of properties 1^*) *and* 2^*) *follow analogously to those of properties* 1) *and* 2), *respectively.*

We now reformulate the impulsive system (26)–(27) *as an equivalent system of integral equations*

$$u(t, \mu) = \frac{1}{\mu}\int_0^t \Phi(t, s, \mu)[(F_y(Y_0, s)Z_0 + G_y(Y_0, s))v(s, \mu) + T_1(u, v, s, \mu)]ds + \tag{28}$$
$$+ \sum_{0<\theta_i<t} \Phi(t, \theta_i, \mu)(1 + \frac{I_1(Y_0, 0)}{\mu})^{-1}([I_{1y}(Y_0, 0)Z_0 + I_{2y}(Y_0, 0)]v(\theta_i, \mu) + S_1(u, v, \theta_i, \mu)),$$
$$v(t, \mu) = \int_0^t \Psi(t, s, \mu)[f(Y_0, s)u(s, \mu) + T_2(u, v, t, \mu)]ds + \tag{29}$$
$$+ \sum_{0<\theta_i<t} \Psi(t, \theta_i, \mu)(1 + J_{1y}(Y_0, 0)Z_0 + J_{2y}(Y_0, 0))^{-1}(J_1(Y_0, 0)u(\theta_i, \mu) + S_2(u, v, \theta_i, \mu)),$$

where $\Phi(t, s, \mu)$ *and* $\Psi(t, s, \mu)$ *denote the fundamental matrices of the corresponding system*

$$\mu\frac{d\Phi}{dt} = F(Y_0, t)\Phi, \ t \ne \theta_i, \ \mu\Delta\Phi|_{t=\theta_i} = I_1(Y_0, 0)\Phi, \ \Phi(s, s, \mu) = 1,$$
$$\frac{d\Psi}{dt} = (f_y(Y_0, t)Z_0 + g_y(Y_0, t))\Psi, \ t \ne \theta_i, \ \Delta\Psi|_{t=\theta_i} = (J_{1y}(Y_0, 0)Z_0 + J_{2y}(Y_0, 0))\Psi, \ \Psi(s, s, \mu) = 1.$$

The following holds for the fundamental matrix $\Phi(t, s, \mu)$

$$|\Phi(t, s, \mu)| \le c\exp(-\frac{\kappa}{\mu}(t - s)), \ \ 0 \le s \le t \le T.$$

By inserting the representation of $v(t,\mu)$ from equation (29) into the first equation, we derive

$$u(t,\mu) = \int_0^t H(t,s,\mu)u(s,\mu)ds + N_1(u,v,t,\mu),$$

where H denotes a bounded kernel, and the function N_1 satisfies the same two properties as the function $T(u,v,t,\mu)$. The last equation may be replaced by an equivalent one of the form

$$u(t,\mu) = \int_0^t R(t,s,\mu)N_1(u,v,s,\mu)ds + N_1(u,v,t,\mu) = M_1(u,v,t,\mu), \tag{30}$$

where R is the resolvent corresponding to the kernel H. Substituting the representation (30) for $u(t,\mu)$ into equation (29) yields

$$\begin{aligned} v(t,\mu) = & \int_0^t \Psi(t,s,\mu)[f(Y_0,s)M_1(u,v,s,\mu) + T_2(u,v,s,\mu)]ds+ \\ & + \sum_{0<\theta_i<t} \Psi(t,\theta_i,\mu)(1 + J_{1y}(Y_0,0)Z_0 + J_{2y}(Y_0,0))^{-1}(J_1(Y_0,0)M_1(u,v,\theta_i,\mu)+ \\ & + S_2(u,v,\theta_i,\mu)) = M_2(u,v,t,\mu). \end{aligned} \tag{31}$$

The functions M_1 and M_2 possess the same two properties as the function $T(u,v,t,\mu)$. The method of successive approximations applied to systems (30) and (31) yields a unique solution that fulfills the corresponding estimates

$$\begin{aligned} |u(t,\mu)| &= |z(t,\mu) - Z_n(t,\mu)| \le c\mu^{n+1}, \ \ 0 \le t \le T, \\ |v(t,\mu)| &= |y(t,\mu) - Y_n(t,\mu)| \le c\mu^{n+1}, \ \ 0 \le t \le T. \end{aligned}$$

The theorem is proven.

3.2 Asymptotic expansion of singularity with multi-layers

In the previous subsection, it was shown that there exists a single initial layer. Using an impulse function, the convergence can be nonuniform near several points, that is to say, that *multi-layers* emerge. These layers occur on the neighborhoods of $t = 0$ and $t = \theta_{ii=1}^{p}$. In the preceding subsection, the existence of a single initial layer was demonstrated. The introduction of an impulse function leads to nonuniform convergence in the vicinity of multiple points, resulting in the formation of multi-layer structures. These layers are localized near $t = 0$ and $t = \theta_i, i = 1, 2, \ldots, p$.

In order to generate a singularity exhibiting a multi-layer structure, we examine system (3) subject to conditions(C1)–(C4) along with the additional requirement condition

(C6) $$\lim_{(z,y,\mu)\to(\varphi,\overline{y},0)} \frac{I_1(y,\mu)z + I_2(y,\mu)}{\mu} = l_i \neq 0,$$

where l_i is a constant, $\varphi(\overline{y}(\theta_i),\theta_i)+l_i, i = 1, 2, ..., p$, are the values for each impulse moment at the points $t = \theta_i, i = 1, 2, ..., p$. By virtue of condition (C6) from equation (13), the following equality holds

$$I_1(\overline{y}_0(\theta_i),0)\overline{z}_1(\theta_i) + (I_{1y}(\theta_i)\overline{z}_0(\theta_i) + I_{2y}(\theta_i))\overline{y}_1(\theta_i) + \overline{I}_\mu(\theta_i) = l_i \neq 0, i = 1, 2, ..., p.$$

Accordingly, the first equation of system (17) can be rewritten in the following form

$$\omega_0^{(i)}(0) = l_i - \Delta\overline{z}_0|_{t=\theta_i}, i = 1, 2, ..., p.$$

By substituting the previously derived expression into (8), we arrive at

$$z(\theta_i+, \mu) = \overline{z}_0(\theta_i+) + \omega_0^{(i)}(0) + O(\mu) = \overline{z}_0(\theta_i) + l_i + O(\mu), i = 1, 2, ..., p.$$

According to condition $(C6)$, after each impulse moment θ_i, the difference $|z(\theta_i+, \mu) - \varphi| = l_i + O(\mu)$ does not vanish as $\mu \to 0$. Consequently, the convergence is nonuniform. Therefore, it can be concluded that the solution $z(t, \mu)$ of system (3) with the initial condition (4) exhibits a multi-layer structure, with layers forming in the neighborhoods of $t = 0$ and $t = \theta_i$ for $i = 1, 2, \ldots, p$.

The proof of the next theorem follows by analogy with the proof of Theorem 1.

Theorem 2 *Let conditions $(C1) - (C4)$ and $(C6)$ be satisfied. Then there exist positive constants μ_0 and c such that, for all $\mu \in (0, \mu_0]$, the problem (3), (4) admits a unique solution $z(t, \mu), y(t, \mu)$ that satisfies the inequality*

$$|z(t, \mu) - Z_n(t, \mu)| \le c\mu^{n+1}, \quad 0 \le t \le T,$$
$$|y(t, \mu) - Y_n(t, \mu)| \le c\mu^{n+1}, \quad 0 \le t \le T,$$

where

$$Z_n(t, \mu) = Z_n^{(i)}(t, \mu), Y_n(t, \mu) = Y_n^{(i)}(t, \mu), \theta_i < t \le \theta_{i+1},$$
$$Z_n^{(i)}(t, \mu) = \sum_{k=0}^{n} \mu^k \overline{z}_k(t) + \sum_{k=0}^{n} \mu^k \omega_k^{(i)}(\tau_i), \tau_i = \frac{t - \theta_i}{\mu},$$
$$Y_n^{(i)}(t, \mu) = \sum_{k=0}^{n} \mu^k \overline{y}_k(t) + \mu \sum_{k=0}^{n} \mu^k \nu_k^{(i)}(\tau_i), i = 1, 2, ..., p.$$

4 Numerical examples

4.1 Example 1

Consider the impulsive system with singularities

$$\begin{aligned} \mu z' &= -y^2 z + y^2 - 5\mu^2 y, \qquad & \mu\Delta z|_{t=\theta_i} &= zy - y - 2\mu^2 y^3, \\ y' &= 2zy - 8y, & \Delta y|_{t=\theta_i} &= 2yz - 8y, \end{aligned} \tag{32}$$

initial conditions

$$z(0, \mu) = 2, \quad y(0, \mu) = 3, \tag{33}$$

where $\theta_i = i/5, i = 1, 2, ...7$. Assume that $\mu = 0$ in the considered problem. In this case, the first equation of system (32) reduces to the form $-\overline{y}^2\overline{z} + \overline{y}^2 = 0, \overline{zy} - \overline{y} = 0$. which yields the solution $\overline{z} = \varphi = 1$. Nevertheless, according to condition $(C2)$, the root $\overline{z} = 1$ is uniformly asymptotically stable. Inserting the value $\overline{z} = 1$ into the second equation of (32) yields the following result

$$\begin{aligned} &\overline{y}' = -6\overline{y}, \quad \Delta\overline{y}|_{t=\theta_i} = \overline{y} + 1, \\ &\overline{y}(0) = 3. \end{aligned} \tag{34}$$

This system possesses a unique solution $\overline{y}(t)$. Next, we examine the validity of condition $(C5)$

$$\lim_{(z,y,\mu)\to(\varphi,\overline{y},0)} \frac{zy - y - 2\mu^2 y^3}{\mu} = 0.$$

The solution $z(t,\mu)$ of system (32) with the initial condition (33) exhibits a single initial layer at $t = 0$. The simulation results presented in Figure 1 confirm the presence of this single-layer behavior. As $\mu \to 0$, Figure 2 shows that the solution to problem (32), (33) converges to the solution of the corresponding degenerate system (34).

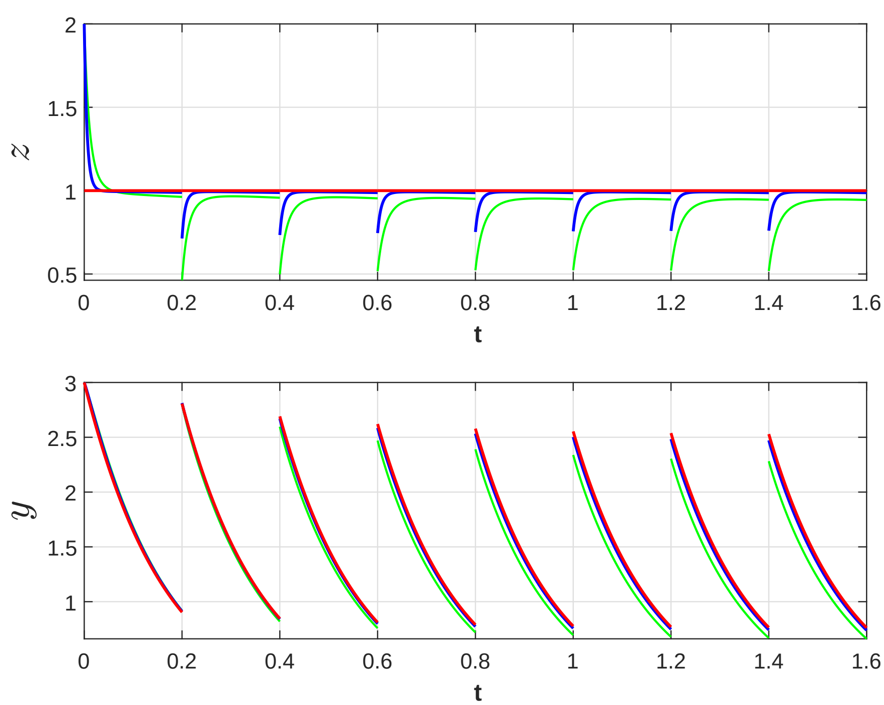

Figure 1: The blue and green curves illustrate the solutions of system (32) with initial conditions (33), corresponding to the values $\mu = 0.1$ and $\mu = 0.05$, respectively. The red line represents the solution to problem (34).

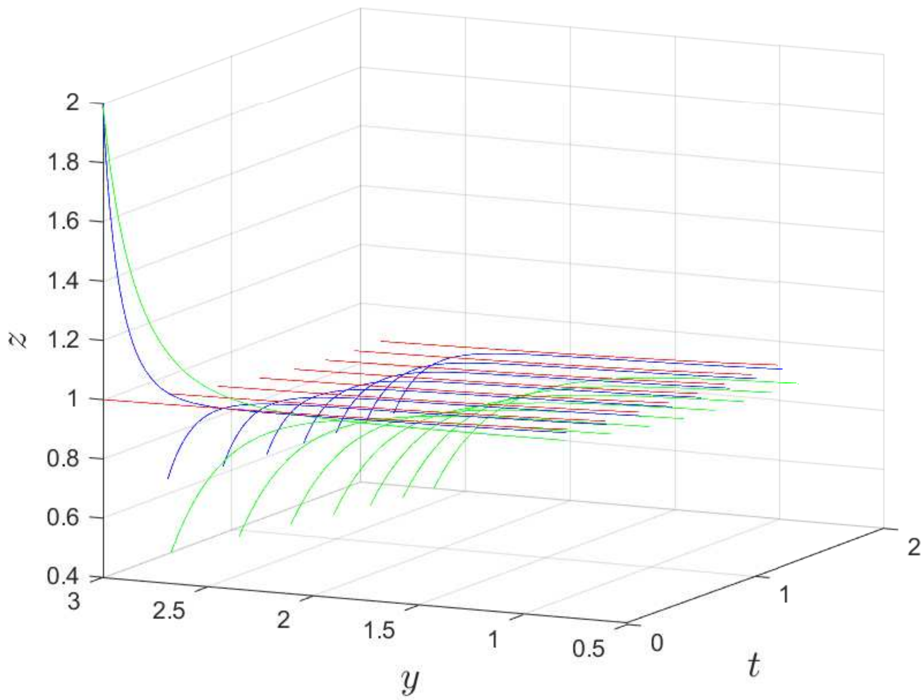

Figure 2: The blue and green curves illustrate the solutions of system (32) with initial conditions (33), corresponding to the values $\mu = 0.1$ and $\mu = 0.05$, respectively. The red line represents the solution to problem (34).

4.2 Example 2

Now, we now consider the following system

$$\begin{aligned} &\mu z' = -y^2 z - 3y^2 - 4\mu^2 yz, &&\mu\Delta z|_{t=\theta_i} = zy + 3y - 6\mu^2 yz - 4sin(2\mu),\\ &y' = 2zy - 8y, &&\Delta y|_{t=\theta_i} = 2y - z, \end{aligned} \tag{35}$$

initial conditions

$$z(0,\mu) = -1, \quad y(0,\mu) = 3, \tag{36}$$

where $\theta_i = i/5, i = 1, 2, ...7$. Setting $\mu = 0$ in (35) transforms the first equation into $-\overline{y}^2\overline{z} - 3\overline{y}^2 = 0$, which simplifies to $\overline{zy} + 3\overline{y} = 0$. This yields the root $\overline{z} = -3$. The corresponding root $\varphi = -3$ is uniformly asymptotically stable, as it satisfies condition $(C2)$. Inserting $\overline{z} = -3$ into the second equation of system (35) yields

$$\begin{aligned} &\overline{y}' = -14\overline{y}, \ \ \Delta\overline{y}|_{t=\theta_i} = 2\overline{y} + 3,\\ &\overline{y}(0) = 3. \end{aligned} \tag{37}$$

One can confirm that condition $(C6)$ is satisfied

$$\lim_{(z,y,\mu)\to(\varphi,\overline{y},0)} \frac{zy + 3y - 6\mu^2 yz - 4sin(2\mu)}{\mu} = -8 \neq 0.$$

The solution $z(t,\mu)$ of system (35) with initial condition (36) exhibits multi-layers near $t=0$ and at each point $t=\theta_i^+, i=1,2,...,7$. Figure 3 reveals the presence of multi-layer behavior in the solution, while Figure reffig4 shows that, as $\mu \to 0$, the solution of the original problem (35), (36) approaches the solution of the degenerate system (37).

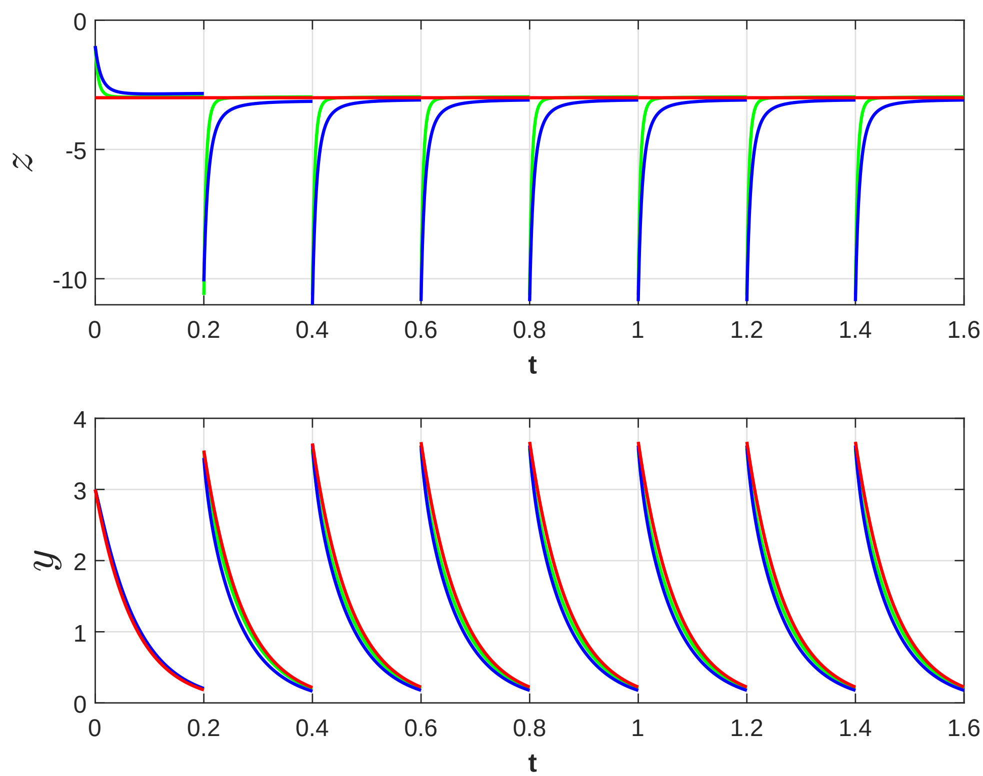

Figure 3: The green and blue curves illustrate the solutions of system (35) with initial conditions (36), corresponding to the values $\mu = 0.1$ and $\mu = 0.05$, respectively. The red line represents the solution to problem (37).

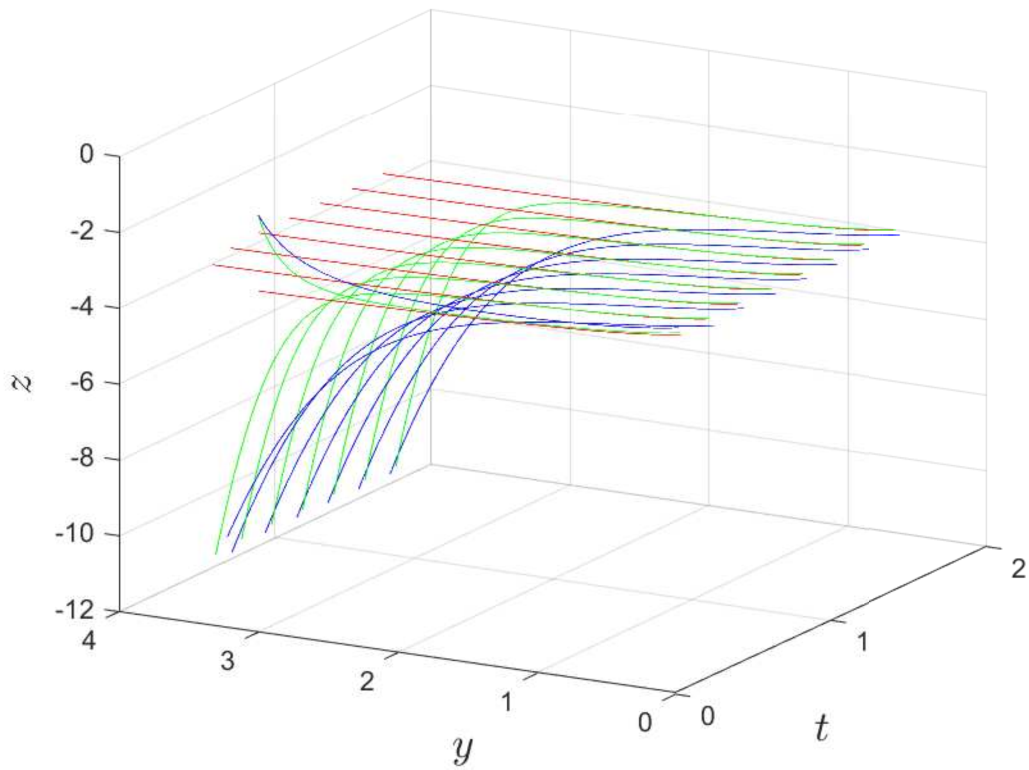

Figure 4: The green and blue curves illustrate the solutions of system (35) with initial conditions (36), corresponding to the values $\mu = 0.1$ and $\mu = 0.05$, respectively. The red line represents the solution to problem (37).

5 Conclusion

In this paper, the singularly perturbed quasi-linear impulsive differential equation is considered. The boundary function method is employed to construct asymptotic solutions with arbitrary accuracy. Both single-layer and multi-layers phenomena are analyzed within the framework of asymptotic expansions. This approach allows a detailed description of the solution behavior in regions characterized by rapid transitions and boundary layers. The theoretical results are supported by illustrative examples and numerical simulations.

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ON EXPERIMENTAL PROOF OF "P VERSUS NP"THEOREM

We propose a simple and intuitive algorithm for solving md-DFA problem using algorithm concepts within extended operators, our approach shows quadratic polynomial time and hence proves the equivalence between polynomial and non-polynomial classes, we have also shown that minimal non-emptiness of automata problem can be solved in polynomial time with help of modified subset construction, rather that building a product automaton, which lead to factorial size of the memory and time, in this work we also have used many non-tractable existing examples and computed them in polynomial time, which guarantees that our algorithm solves NP-complete problem in almost linear polynomial time, we have also avoided the problem of product automata by an algorithmic approach, we are also giving the starting ground for the proof of back-reference problem which was discussed before, notion to the globally local increment is also given as the main argument towards the resolution of "P versus NP"theorem, which coincides with the finitarity term in general mathematics.
Keywords: P versus NP, complexity, theorem, experimental, proof.

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Об экспериментальном доказательстве теоремы «P против NP»

Мы предлагаем простой и интуитивно понятный алгоритм для решения задачи md-DFA с использованием концепций алгоритма в расширенных операторах, наш подход показывает квадратичное полиномиальное время и, следовательно, доказывает эквивалентность между полиномиальными и неполиномиальными классами, мы также показали, что минимальная непустота автомата может быть решена за полиномиальное время с помощью модифицированной конструкции подмножества, а не построения автомата-произведения, что приводит к факториальному размеру памяти и времени, в этой работе мы также использовали множество неразрешимых существующих примеров и вычислили их за полиномиальное время, что гарантирует, что наш алгоритм решает NP-полную задачу за почти линейное полиномиальное время, мы также избежали проблемы автоматов-произведений с помощью алгоритмического подхода, мы также даем отправную точку для доказательства проблемы обратных ссылок, которая обсуждалась ранее, понятие глобально локального приращения также приводится в качестве основного аргумента к разрешению теоремы «P против NP», которая совпадает с термином финитарности в общей математике.
Ключевые слова: P против NP, теорема, доказательство, сложность, прикладная математика.

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«P қарсы NP» теоремасын тәжірибе жүзінде дәлелдеу туралы

Біз кеңейтілген операторлар ішінде алгоритм ұғымдарын пайдалана отырып, md-DFA есебін шешудің қарапайым және интуитивті алгоритмін ұсынамыз, біздің әдіс квадраттық көпмүшелік уақытты көрсетеді және осылайша көпмүшелік және көпмүшелік емес сыныптар арасындағы эквиваленттілігін дәлелдейді, сонымен қатар біз автоматтандырылған есептің минималды бос еместігін модификацияланған ішкі уақыт өнімін құру арқылы шешуге болатындығын көрсеттік. жады мен уақыттың факторлық өлшеміне әкелетін автомат, бұл жұмыста біз сонымен қатар көптеген транзакцияланбайтын бар мысалдарды қолдандық және оларды көпмүшелік уақытта есептедік, бұл біздің алгоритміміз NP-толық есепті дерлік сызықтық көпмүшелік уақытта шешетініне кепілдік береді, сонымен қатар алгоритмдік тәсіл арқылы өнімнің автоматтары мәселесін болдыртпадық, біз бұған дейін дәлелдеме берген мәселені де талқыладық. жаһандық жергілікті өсімге жалпы математикадағы соңғылық терминімен сәйкес келетін «P қарсы NP» теоремасын шешудің негізгі дәлелі ретінде де берілген.
Түйін сөздер: P қарсы NP, теорема, дәлелдеу, күрделілік, қолданбалы математика.

1 Introduction

The NP-hardness was first defined in [1], also there's a defined lower linear bound for deciding arbitrary non-deterministic finite automata on regular languages or even other arbitrary [2]. The problem was first seen on the partial case of non-deterministic automata [3, 4]. The problem of md-DFA is to find a minimal finite automaton which is a subset of any given automata and isn't included in others [5].

The relationship between P vs. NP is one of the greatest open problems in computer science. The central challenge is whether problems whose solutions can be efficiently verified (NP) can also be efficiently computed (P). Here, we propose a new perspective: Is there a deeper mathematical structure that enables a more efficient computation of NP problems? We investigate whether parallels exist between quantum mechanics, the fractal structure of the Riemann zeta function, and the superposition of NP problems.

NP-Completeness in DFA Problems An important class of NP-complete problems arises from automata-based computation. Two key problems are: - Minimal Distinguishing DFA Problem: Determining the smallest deterministic finite automaton (DFA) that distinguishes between two regular languages. - DFA Non-Emptiness Problem: Deciding whether the intersection of multiple DFAs is non-empty. Both problems are NP-complete [5, 7]. - The product construction for DFAs has a complexity of $O(|A_1| * ... * |A_n|)$, which grows exponentially with the number of automata. - A modified subset construction can help reduce the complexity, but a fundamental lower bound remains. Question: Is there a hidden structure that allows for more efficient computation?

Superposition of NP Problems In classical computation, NP problems are solved sequentially: all possible solutions must be explicitly checked one by one. In quantum mechanics, states exist in superposition: - A quantum system can exist in multiple states simultaneously until a measurement collapses it to a single definite state. - Quantum computers could solve NP problems more efficiently by evaluating all solutions simultaneously and amplifying the optimal one (e.g., using Grover's algorithm). Hypothesis: NP problems are not randomly distributed but follow a hidden mathematical structure that enables a more efficient computation.

1.1 Connection to the Zeta Function Fractal Structure & Superposition

According to Kardeis [14], the analytic continuation of the Riemann zeta function exhibits remarkable symmetry: the function has poles at s = 1 and possibly at s = 0, as supported by the functional equation; the critical line $R(s) = 0.5$, $R(s) = 0.5$ contains infinitely many nontrivial zeros, reflecting the structure of prime numbers; the self-similarity of the zeta function suggests a fractal order in its structure; a key point in Kardeis' work is the hypothesis that the structure of the zeta function resembles a superposition of states: the statement "0 = 1 simultaneously like a superposition suggests that the zeros of the zeta function represent a simultaneous existence of multiple solutions, this directly corresponds to the idea that NP problems do not need to be solved sequentially but can be structured within a higher-order fractal framework.

Hypothesis: The zeta function may reflect a deeper order in NP problems, enabling a more efficient computation.

Connecting DFA, Quantum Mechanics, and the Zeta Function DFA & NP problems are exponentially complex: classical algorithms require sequential computation, superposition in quantum mechanics allows for parallel states.

The zeta function exhibits a fractal order: the self-similarity of its zeros and their reflection symmetry could serve as a mathematical analogue to quantum superposition, this suggests that NP problems are not randomly distributed but follow a fractal structure.

Implication for P vs. NP: if a deeper structure in NP problems can be identified, this could break the exponential complexity barrier, the fractal organization could provide an alternative ordering principle for search algorithms, similar to how quantum algorithms already offer advantages today.

The fractal structure of the zeta function could provide a new perspective on NP problems. Superposition in quantum mechanics could serve as a natural mathematical analogy for the distribution of zeta function zeros. The existence of a fractal order in NP problems could open new pathways for efficient computation.

No strict mathematical proof yet: a formal demonstration is needed to show that NP problems can indeed be described by a fractal structure, specific quantum algorithms must be developed to leverage this structure for efficient computations.

Future question: "Could the mathematical structure of the zeta function contribute to a new theory of P vs. NP?"

2 Re-writing algorithm

We give the subtraction re-writing method to solve this problem in linear-logarithmic time as per our previous research. The technique known as re-writing is summarized, for the &-operator we use the overridden state with logical consumers as well as for the subtraction operator which is defined within the same terms, however, differing only in logical statement, the complement operator in this manner is also re-written using the alphabet star and subtraction from the operand. Thus, the mix of re-writing and logical state composition gives the way to the modified subset construction, which rather than visiting every possible composition, produces exact answer on each iteration, thus giving the polynomial running time, rather than exponential or even factorial. The DFA constructed from the subtraction of DFA1 and DFA2 was constructed experimentally on the example in [5]. The first DFA corresponds to the regular expression $((aaaa)*a|(aaaa)*aa|(aaaa)*aaa)$, as the second one to $(a|aa|aaa|aa(aaa)*|aaa(aaa)*)$, the figure above shows the result produced by Regex+ software package. Thus, the decision problems based upon the extended operators can be solved in full and more efficiently if we will choose the strategy of computing locally optimal solution which gives the optimal step to the global one - this technique we will call as the globally local increment (GLI). This decision problems which we encountered are only two: minimal distinguishing DFA (md-DFA) and non-emptiness DFA of the given automata: both of which has the state space of factorial size, which, in turn, means that we have to choose the better strategy and solution in order to get to the certificate of acceptance in non-polynomial problem within the visible time limit, in our experiments, it didn't exceed more than minute. In the next section we will give the compound benchmarks for the derived examples.

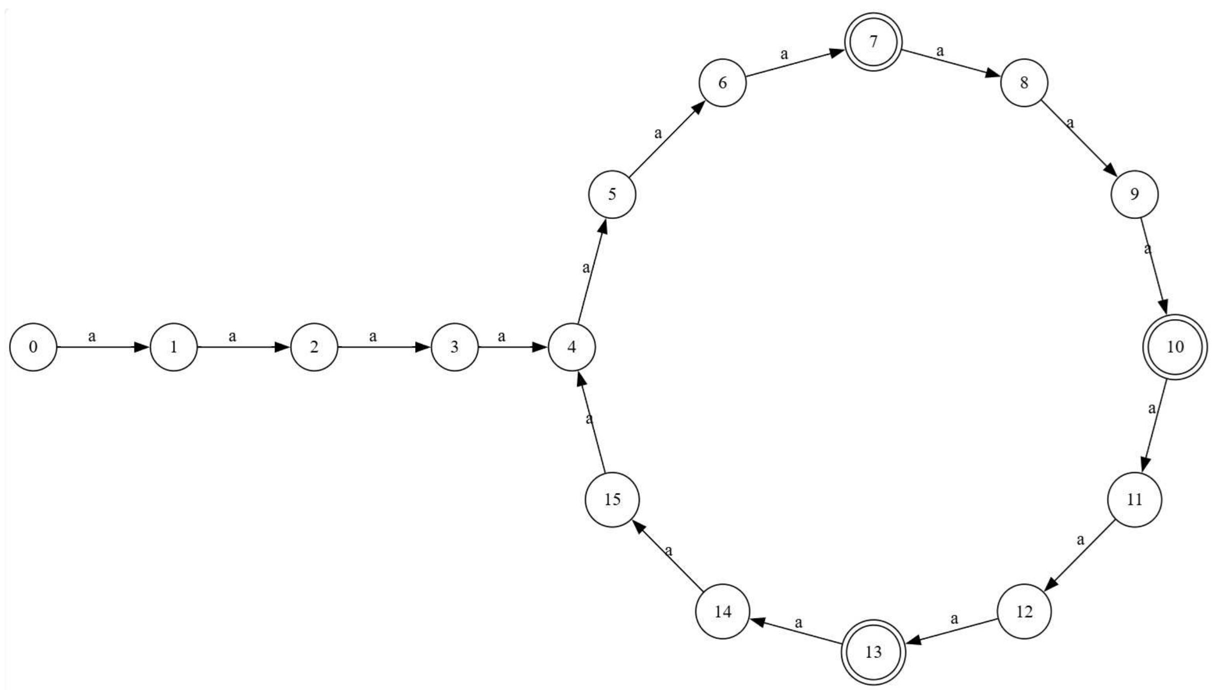

Figure 1: The distinguishing DFA for example in [5]

Re-writing works as it was outlined, thus, we can state that the DFA corresponding to the expression $R_1 - R_2$ is a subset of DFA(R_1) and isn't contained in DFA(R_2).

3 Proof by Product Automata

As it was presented in [6], the maximal complexity of product construction is the product of its operands, which can be factorial due to the number of automata, however, we can use same methodology as we have presented before using re-writing and event call, thus, giving polynomial solution to various number of operands with variable cardinality. As it was present in [7–9] the minimal intersection of arbitrary automata cannot be approximated using product construction as it gives the factorial number of solution to be searched, otherwise,

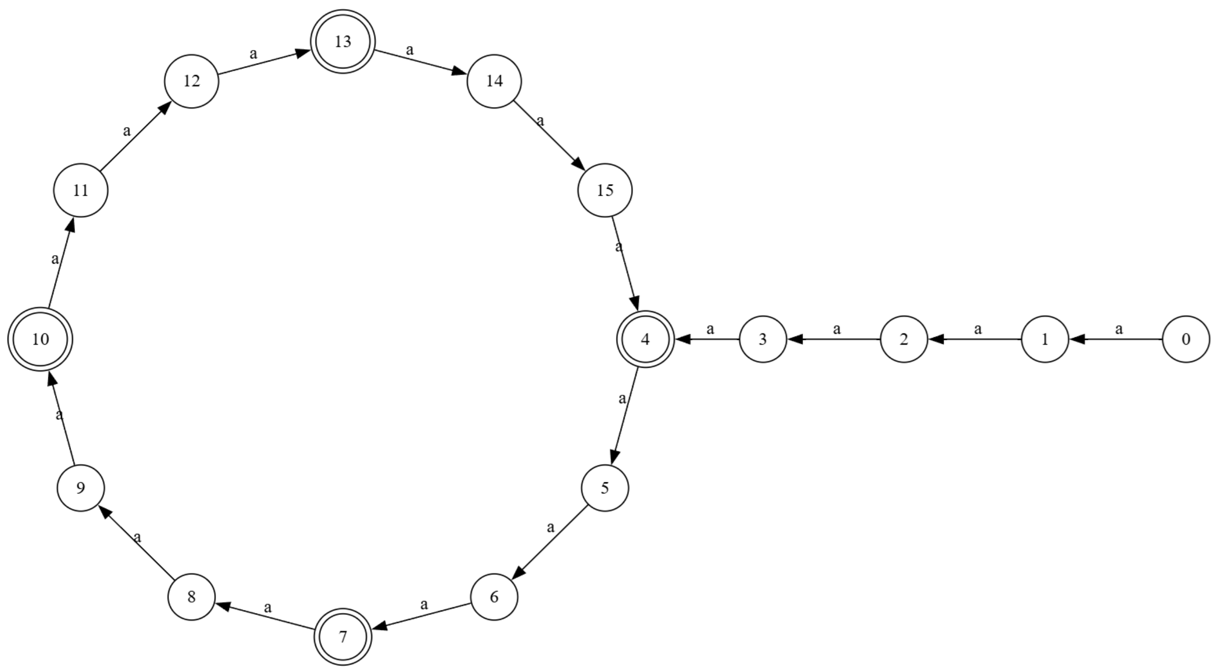

Figure 2: Example md-DFA for the expression $((aaaa) * a\{1,4\})\{4000\} - (a\{1,3\}(aaa)*)\{4000\}$

the better strategy is to use modified subset construction approach. The term "minimal distinguishing"can be also viewed from the approach we invented before as we can compute the minimal possible automaton by simply computing the shortest path between starting and ending points using Dijkstra algorithm. The non-emptiness problem which is known to be NP-hard will be also proved to be solvable by a polynomial algorithm in the next section. Another counter-example is from [10], where the expression in the form $(ab)*\&...$ was studied, we have shown in the next section that it can be computed in time of several seconds for the string of length 8000 containing 1000 intersection expressions - this gives the contrary towards the minimal DFA recognizing this language which has an exponential complexity. During the review of the present results we haven't met other counter examples which could get the running experimental program to work with errors or in not observable time frame. Another case we get from [11] for intersection operator, which gives the exponential estimation of the time and space complexity, our results give the reasonable amount of time not greater than 12 seconds for one thousand intersections. The proof of correctness of re-writing algorithm and modified subset construction can be done by viewing the cut, as it was presented much more earlier. We also point the regression of complexity to almost linear and quadratic with respect to the back-referencing problem in extended regular expressions, one can see that number of decompositions decreases as we proceed further with the given search and, thus, certificate of acceptance is achieved almost "on the fly". The co-NP complete problem [12] can be by analogy solved in reasonable time as of our experimentation procedure, which states that co-NP classes lie within polynomial P classes.

4 Benchmarks and Experimentation

In table 1 there is a summary of the tests on the expression in the form of $((a^k) * a^{1..k})^k - (a^{1..k}(a^k)*)^k$.

k	String length	DFA Number of States	Time (sec)
0	63	16	0.155
1	93	25	0.018
2	129	36	0.032
3	171	49	0.042
4	219	64	0.053
5	273	81	0.071
6	333	100	0.085
7	399	121	0.093
8	471	144	0.185
9	549	169	0.261
10	633	196	0.817
11	723	225	0.403
12	819	256	0.389
13	921	289	0.509
20	1803	576	2.808
30	3573	1156	13.06
40	5943	1936	41.488
50	8913	2916	188.674
60	12483	4096	276.88
70	16653	5476	585.398
80	21423	7056	1222.708

Figure 3: The results of expression tests using re-writing algorithm

On the next figure there is a visualization of the data in Table 1, as it can be seen results converge to quadratic polynomial function.

With respect to the term fixed-parameter tractability (FPT) as our alphabet before in tests consisted of only letter, we have run tests for arbitrary alphabet "abcd"for cases in form $((a|b|c|d)*)^k(a|b|c|d)^k - ((a|b|c|d)^k((a|b|c|d)*)^k$:

In the other test we will use the regular expression in the form $a^k * \&.., k = 1..n$, the results are as follows.

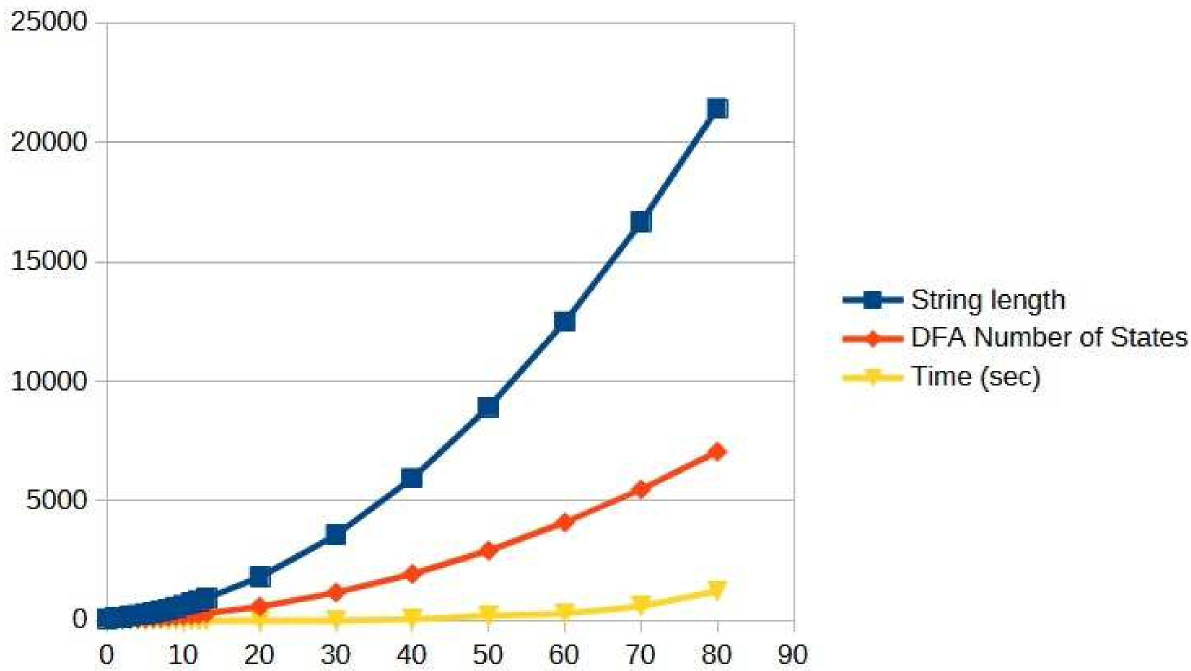

Figure 4: Visualization of the benchmarks in first table

5 Discussions

5.1 On Effective Algorithms and Cook's Conjecture

This section is a review of the advancement methods in modern combinatorial optimization within some major results in usage of dynamic programming on trees as well as main conjectures in graph theory and theory of computational complexity, which in recent time is studied more as we get in time within modern trends like social networks and publicly available hubs, most of which rely on artificial intelligence, however, this work won't deal with AI, better we will propose several fundamental approaches, conjectures and questions as per which we can give a clear and positive answer that this problem isn't an ending case and, thus, can be probed on the particular basis which include the deep review of the newest papers on graph theory and other conforming topics, which are, in turn, become popular during the past decade of the research within tractability, application and generalization progress, we also give the important relation to chromatic numbers in graphs.

In this preamble paper we give the definition of some effective algorithms like subset construction and variable maximum flow problem using potentials which was better studied before as the analogy by Malhotra-Kumar and Maheshwari; we will also go further and show that the Stephen Cook's conjecture of the NP-complete problem implies the uncertain complexity classes which were classified by us before as to be impractical while the certificate of correctness remains of polynomial complexity.

As it was proposed and defined before in a seminal paper the NP-complete problem is a problem whose verifying certificate is linear, however, it was incomplete to define the number

k	String length	DFA Number of States	Time (sec)
0	104	14	0.284
1	158	12	0.08
2	224	13	0.059
3	302	14	0.1
4	392	15	0.132
5	494	16	0.131
6	608	17	0.209
7	734	18	0.307
8	872	19	0.335
9	1022	20	0.338
10	1184	21	0.437
11	1358	22	0.394
12	1544	23	0.423
13	1742	24	0.529
20	3464	31	1.78
30	6944	41	7.442
40	11624	51	17.878

Figure 5: Tests for the quadratic alphabet

of possible solutions to form the multiplicative space over the operator (*).

Dana and Scott also remained many unspecified in their decision of the subset construction algorithm which was actually superseded by Berry-Sethi algorithm which produces the linear number of states in deterministic automaton with respect to the preliminary construction algorithm.

Since the definition of the networks and optimal flows on them, number of many algorithms was proposed – one of them is due to Malhotra, Kumar and Maheshwari which has a polynomial cubic complexity. We also give the notion to the super string problem as it's EXPTIME- and EXPSPACE-complete, thus gives the way of defining it as NP-hard.

As the efficient algorithm which includes both the optimization process as well as the conversion of non-deterministic automaton to deterministic one can be viewed as the splice between the initial and accepting states, thus giving the notion to the cases where the exponential blow of number of states occur which we have well studied before and gave the unary O(1)-time complexity check. Thus, this tendency gives the proof of the linear nature of the subset construction algorithm whose minimal upper bound is $O(n * log(n))$,

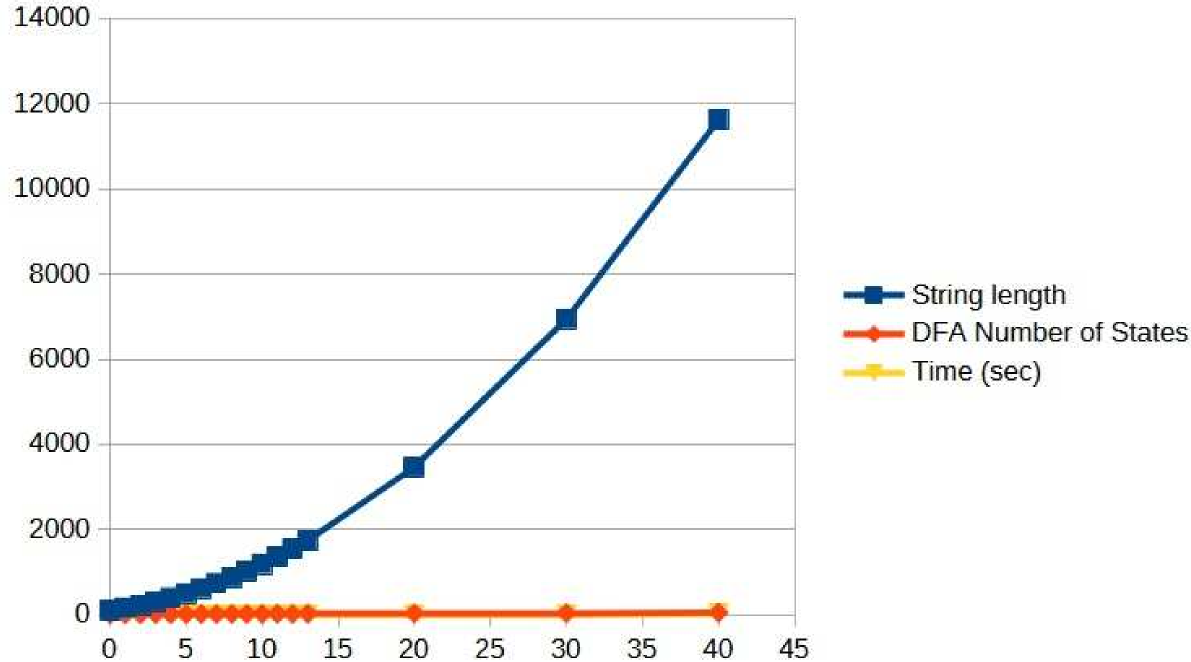

Figure 6: Visualization for example arbitrary alphabet

however the minimal one is $O(n)$. In this method we make the choice by the divide-and-conquer strategy from beginning to the end of the state graph describing the automaton – this gives the possibility of avoiding variety of optimization techniques which doesn't pass the dead point of the exponential growth of spaces, however, rather our algorithm makes it possible "on-the-fly" which predominant viewing on the combined techniques of construction and optimization applied together. On the figure below the basic idea is depicted which shows how the algorithm works on non-deterministic finite automaton while making it deterministic. The super string problem can be viewed from the minimal bound of exponential complexity as there is the minimal string of variable length $n = 2^{f(s)}$ – for each s in the set of all string S as the any minimal string containing all the strings in S as sub-string can be viewed on the other hand as a minimal string along the "trie" for which the dynamic optimization is applied and, thus, the correct composition is sought on the every node of the string forrest. Also Malhotra-Kumar-Maheshwari algorithm is a good sense of creating the variable algorithm with potentials defined for each of the element of the flow network, where the optimization is applied according to the hierarchical order. At least, this is true, for variety of networks and lead to the exponential blow-up for the networks of finding maximal pair whose algorithm uses maximal flow algorithm along the augmentation paths.

We have defined the optimal cases for the subset construction algorithm which was proved to be linear in complexity, also we have shown that the definition of NP-complete problem originally has to be expanded to the class of the uncountable spaces which cannot be realized in time of the arbitrary polynomial function. From the above it follows that DFA has same power as NFA and can be used practically in the testing or membership problem. Also, we

String length	DFA Number of States	Time (sec)
6	2	0.135
13	2	0.008
14	3	0.005
21	3	0.009
22	3	0.004
23	7	0.007
30	7	0.008
31	7	0.006
32	7	0.009
33	13	0.006
40	13	0.011
41	13	0.014
42	13	0.013
55	61	0.016
77	421	0.197
93	841	1.003
109	2521	6.094
125	2521	5.678

Figure 7: Results for non-emptiness test

have revisited the maximum-flow problem with the definition of arbitrary potentials of each of the vertex which is defined as its minimum of the incoming and outgoing flow. Also the superstring problem is actually is NP-hard as we have shown shortly in this paper due to the variable complexity of the string to encompass any other defined set of strings to be checked against the correct answer.

5.2 Argument towards Cook-Rabin-Scott Conjecture in Complexity Theory

We give the full proof of the equivalence of complexity classes like polynomial (P) and non-polynomial (NP) according to Cook-Rabin-Scott conjecture and our prior results of the subset construction which were first proposed by Berry-Sethi. The "P versus NP" has a long-lasting history of its interpretation and first appearance and definition . As it follows from the original

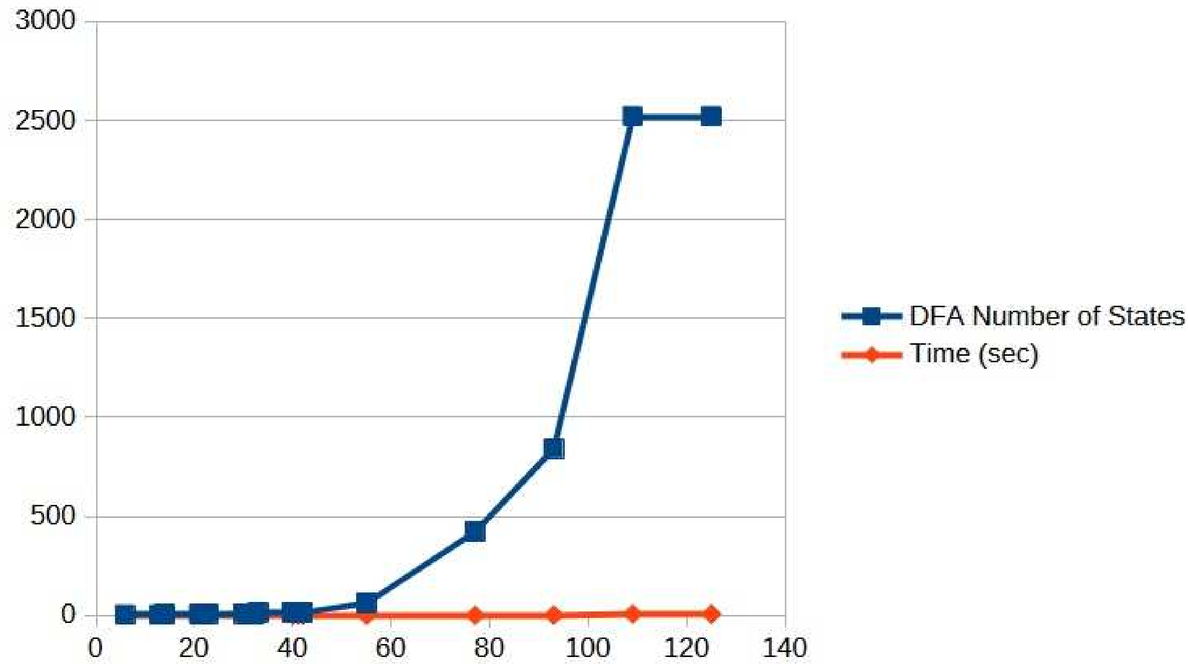

Figure 8: The visualization of the performance of the non-emptiness test algorithm

paper the problem can be classified as NP-complete if there's a defined subset of certified words in language DL(M), where M is a Turing tape automaton or non-deterministic finite automaton (NFA) as they are isomorphic due to our prior objective finding. As it was well presented and discussed by Scoot and Rabin, the NFA can be well converted to deterministic finite automaton (DFA) encoding arbitrary set of accepting words over the language L(M), also: DL(M) is a subset of L(M). Berry-Sethi gave the definition of the linear size automaton and the undefined complexity of the pre-computation stage on the abstract syntax trees of the input regular expression . Also it was shown before that complexity classes have the barriers of their weight along the computational space . It was shown before that linear programming (LP) can be used to solve NP-hard problems with the given customization of constraints .

Cook-Rabin-Scott conjecture can be obtained as a theorem proving the equivalence of P and NP-classes along the full proof of the linear pre-processing and main algorithm complexities when converting the sub-automaton DL(M) to deterministic.

As we have shown before the complexity of converting NFA to DFA is linear in time and space, also any DL(M) can be represented by the regular language and, thus, we have that P equals NP for the subset of the certificate language.

5.3 Homomorphism of Regular Languages is NP-complete

We consider the problem of homomorphism on regular languages by defining the mapping over the set of alphabets on two difference language, we will also show that this problem is actually NP-complete and can be solved by polynomial algorithm.

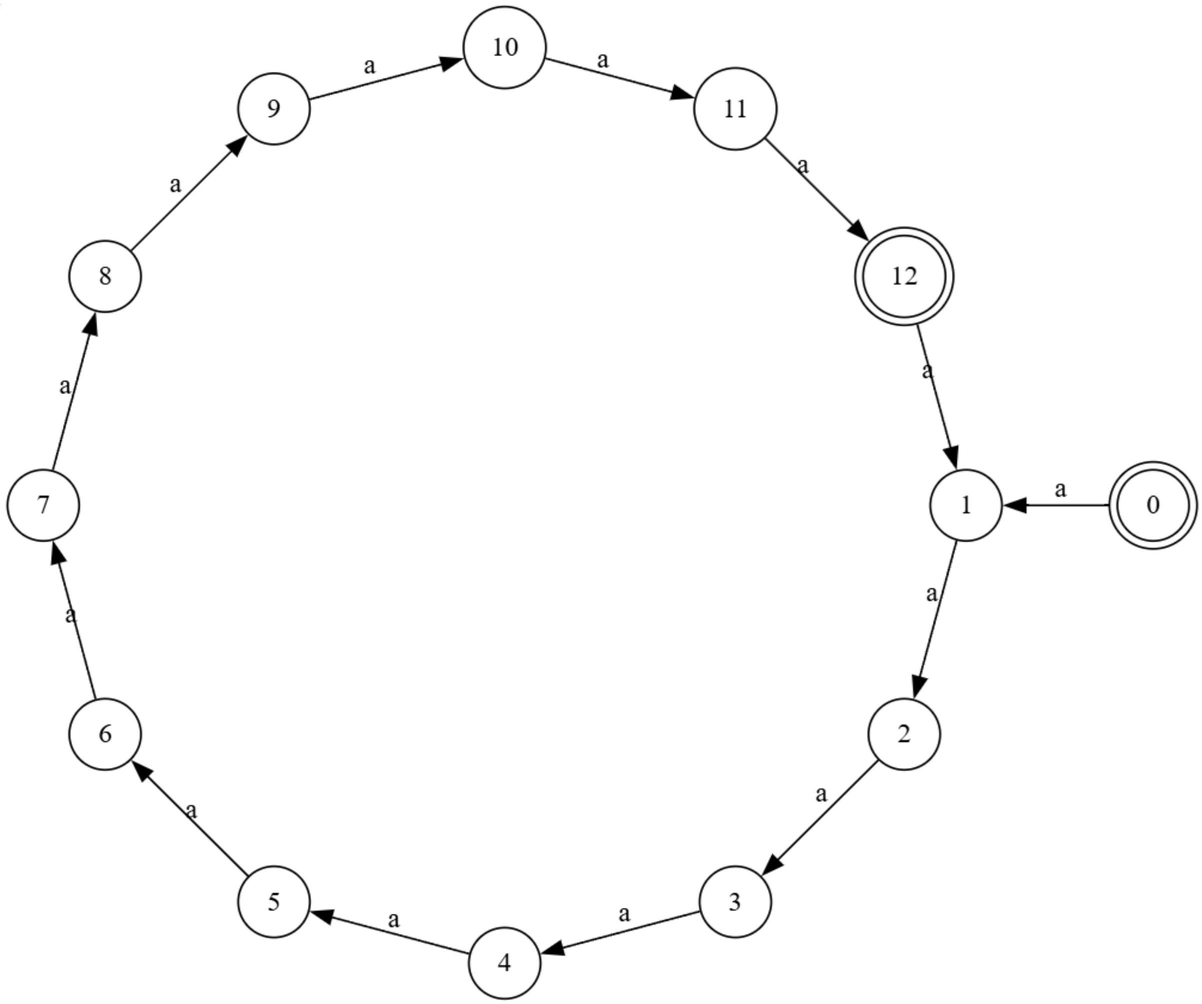

Figure 9: The non-emptiness DFA for the expression $((a)*)\&((aa)*)\&((aaa)*)\&((aaaa)*)\&((aaa)*)$

The Cook's statement of "P versus NP" still remains actual to the present day as many researchers tend to get the argumentative response on the practical meaning of the open problems which can give the open to the new applications of regular languages theory . The partial proof of the existence of the semiring homomorphism on the account of alphabet substitution problem was given in – this problem is to find the mapping between two regular languages for their alphabets, so that they are homomorphic. We solve this NP-complete problem by using maximal bipartite matching algorithm, which can be even parallel .

Obviously the certificate of acceptance of the Homomorphism Problem on Regular Languages (HPRL) stands against the undefined, infinite or arbitrary set of input, moreover the possible measure of matchings is of factorial complexity, thus we proof that HRPL, or HOM, is NP-complete.

We give the set of measures on the bipartite matching graph, where the left side is of one alphabet and right side is of another alphabet: at each time of iteration the matching weight is increased according to the relation difference function between two symbols in both set of

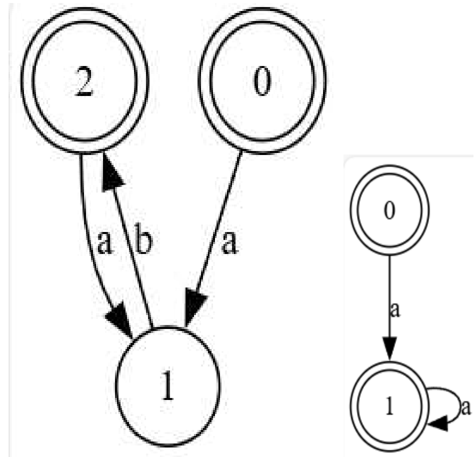

Figure 10: The resulting DFAs for expressions $((ab)*)\&...and(a*a|)\&...$

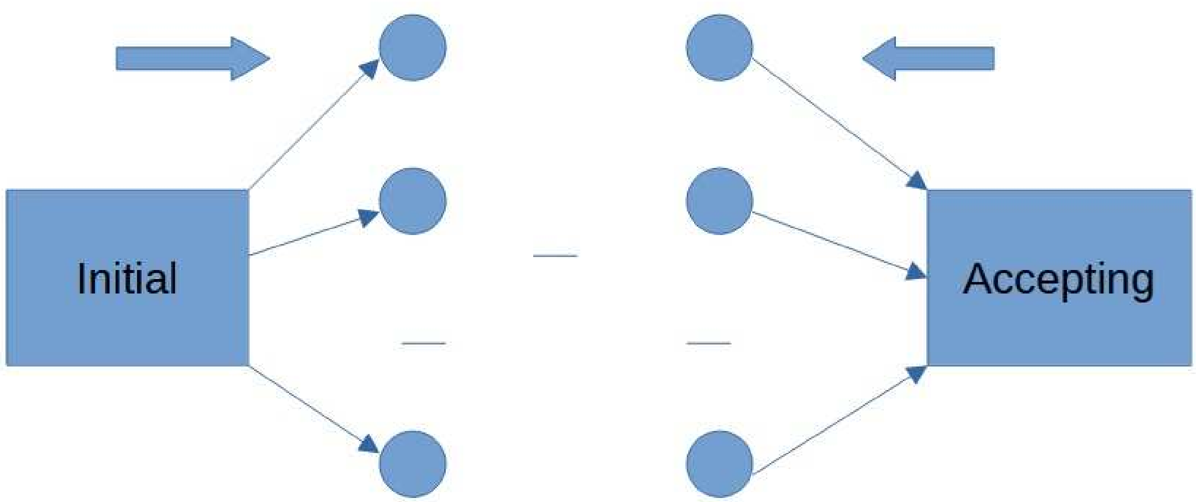

Figure 11: The viewing and strategy of optimal subset construction algorithm

symbols. As we know the algorithm complexity in this case is at most cubic for the relatively small set of letters.

5.4 Differentiating between Complexity Notation within Upper and Lower Classes

We present the final outcome on the account of upper and lower bounds for complexity classes like polynomial and non-polynomial, including exponential and factorial growth as per subset sum problem or classical Travelling Salesman Problem, the further distinguished relation can be used further in particular domains of application of complexity theory like Applied Mathematics and pure Mathematical way of expressing relations between Cook's main 3SAT-theorem and its partial cases like functional divergence and other related theorems and foundations.

When the “P versus NP” was first introduced, it was still unclear if there is at most a connection between complexity classes, their big-O notation and asymptotical complexity in mathematics, which states that there exist a limit between the complexity and its first definition by Cormen et al. Recent research also showed that even linear programming within additional constraints gives the profound solution to TSP and 3-SAT problems. We will show further the full relation between O-notation and asymptotics in terms of pure mathematics for its further application.

Thus, we get the following statement for P and NP complexity classes and their classification function like O(f(n)), where f is a function: $N^+ \rightarrow N^+$:

$O(f(x)) = \{f(x), f(x) \in NP; x^k, f(x) \in P \wedge f(x) < x^k\}$

While we have already classified the class of NP-complete functions to be starting from exponentiation and factorial, including Ackermann’s function, which is a super potentials over others. Polynomial functions are actually solvable functions and can be computed in the observable amount of time.

As it was presented in the partial function $O(ln(ln(x))$ is polynomial of the order 1, while $o(ln(ln(x)) = 0$, the middle and actual theta-function will be as follows – $theta(ln(ln(x)) = \frac{(O(f(x))+o(x))}{2} = \frac{1}{2}$, thus it’s obvious that series S converge to Riemann’s complex number z.

5.5 Algorithm Deciding Automata Ambiguity Problem

We give the proof of NP-completeness of arbitrary automata ambiguity problem which shows that according to our functional hypothesis, there’s a function spawning the polynomial algorithm to solve it, we will also show that it’s of affordable complexity.

We start from well-known “P versus NP” theorem, which proves the full invocation impractical content in order to decide any NP-hard problem, we will use further the final version of subset construction algorithm . Also it’s known that first definition of derivatives and extended operators was well-studied by Berry-Sethi . Orna Kupferman et al., later gave a cubic algorithm for the decision of extended operators like intersection, subtraction and complement . The problem itself is stated in .

The problem is to decide if the given set of initially non-deterministic finite automata (NFA) are equivalent, as well as their subsets like deterministic ones (DFA). Obviously, the problem lies in recurrent relation which leads to the undefined behavior of acceptance and search of the accepting states defined as certificates – this gives the full proof of the NP-completeness of this problem. In order to solve it we use the Modified Subset Construction (MSC) by using the subtraction operators in extended finite automata (EFA), the algorithm complexity, as it was shown, before is linearly logarithmic, $O(n * log(n))$ – to be exact.

5.6 On Account of Regular Automata Separability Problem

The recent research showed the new problems coinciding with our algorithm for extended operators including intersection, we will use it in order to solve the separability problem as it was stated before.

As we already know there’s a set decidable and non-solvable problems . The subset construction algorithm gives the determinisation of non-deterministic finite automata (NFA to DFA), also intersection operator was well studied in along the regular sets or, in other words, sets produced by any regular language. The separability problem was first presented, the problem is coNP-complete which is by definition the both sides of non-decidable or non-polynomial problems, we will give the polynomial method of computation which gives the answer of the verification against separability of arbitrary number of sets on the automata for extended regular expressions.

As we can conclude the only relation of intersection of the sets K and L in spawns arbitrary language which can be non-regular as regular languages are actually subsets of any language as a set of words by definition, thus we can simply test the intersection operator using Modified Subset Construction (MSC) with state activators in the way it was presented in our seminal paper. The complexity of algorithm lying in P-class is linear and constitutes the number $O(|K|+|L|)$ - this is by the way the lowest bound for any coNP-hard separability problem on regular sets or other non Parikh automata.

5.7 Disproof of Unsatisfiability of Boolean Circuits

We give the full disproof and shade the light towards generalized MAX-SAT problem, also classically known as 3-SAT, which cannot be solved on any Turing automata in observable amount of time even if there is a tie between polynomial and non-polynomial complexities.

In this preamble we follow the certain source of the foundations of Computational Complexity Theory by Stephen Cook, who also showed that 3-SAT problem and its general case MAX-SAT on boolean circuits cannot be handled by Turing tape automata or their isomorphisms like non-deterministic finite automata and deterministic one. For the past time the SAT problem was well applied and studied in-depp, however the main ridiculous challenge is about to build the universal SAT automata – the author of this work shows that there could be boolean function which can make the automata producing positive answer on some set of inputs and their co-variants.

Yes, the problem is still open and innovative as any boolean function on the mirror circuit can produce either positive or negative answer, however, this problem is a case of the generalized MAX-SAT problem which is known to be NP-complete and, thus, unable to be solved in reasonable amount of time using computational materials like present day hardware.

5.8 Full Proof of Universality of Regular Languages

We give the fool proof of the universality of regular languages, which states that any language is regular and every regular language can be arbitrary.

The proof problem of regular languages is actually NP-complete as there is no state automata which could be deterministic and descriptive at the same time – this fact is well-stated for any regular language and for any arbitrary language . We will show further that these languages are actually equivalent using Aho-Corasick algorithm .

Obviously, Aho-Corasick tries are linearly deterministic and can form a logarithmic regular language – this fact shows that both regular and arbitrary languages are equivalent.

5.9 Star Packing Problem Algorithm in Linear Time

We give the full exact algorithm to star packing problem.

The problem is considered to be NP-hard and obviously NP-complete due to the reduction to classical Vertex Cover Problem which has a parameterized invariant . Star packing graphs were studied before in and the actual statement of the problem is given in .

We build the tree from the graph and optimize it linearly using the leaf traversal strategy which gives full and exact answer.

5.10 On Consensus of Cardinality of Complexity Classes

We give the notion towards Louiz's partial conjecture about inequivalence in classical "P versus NP" theorem and other related research.

We are all well-known about undecidability of 3-SAT problem . The term 'cardinality' in computational complexity and its bordering applied sciences was recently raised upon the necessary level . Dr. Akram Louiz gave all the necessary partial solution towards axiomatization of non-conforming complexity classes like polynomials and non-polynomials . The critics behind the scene is completely wrong and further we will give the full shed of light on the mystery in science and its application.

Yes, indeed the exponential, factorial or even Ackermann's complexities cannot be considered as countable – and this is what gives the strong border of the non-existent classical solutions, and as we know until the present time none of the NP-complete problem was solved.

The author of the 'critical work' is completely wrong as he sees Louiz's conjecture as a first argument towards resolution of "P versus NP" concept and we show that these classes cannot be even comparable – this shows that Jamell Ivor Samuels is completely wrong in his critical work.

5.11 Polynomial Solution for Detour Problem

We give the full polynomial solution for finding the detour in graphs in its general case.

The detour problem is NP-complete, detours were used in many aspects of science on graphs . The detour problem was first initiated in . We give the full solution using dynamic programming . Our solution depends on the length of the detour among any pair vertices as we are using dynamic programming approach with memoization which gives the recurrence relation along the two connected vertices and the visited length. We have given the full general solution to graph detour problem using dynamic programming which runs in time $O(n * k)$, where n is a number of vertices in graph and k is the maximal length of detour.

5.12 Optimization Techniques on Automata and Graphs

We give the profound notion along algorithmic optimization techniques to manipulate with graph and automata structures.

The automata theory was well defined for the past time, the homomorphism of graphs and these automata and the application of this finding was also presented in the prior works. At the present time publication shows the connection between languages described by automata and their mapping to graph automata.

We have a novel technique based upon the strongly-connected components on arbitrary graphs like oriented and non-oriented in general – this technique shows a strong method of describing any graph by its underlying regular language and its finite non-deterministic or deterministic automaton.

5.13 Order on Trees and Hierarchical Logical Problem

We introduce the optimized algorithm solving optimization problems in linear polynomial time, also we give the notion towards the solution of hierarchical logical problem.

The graph theory remains still actual due to its wide range of applications, the problem of finding strongly connected components gives the solution towards the existence of the logical order relation between these components and hierarchy which can be seen on the depth or breadth first search. We are using Fenwick trees for range min query in order to give the full relation between the hierarchical logical system consisting of operands and comparison operators like less, greater or equal.

The optimization towards trees is computed using the directed edge and its subtree by the preserving dynamical programming and the ordered computation of not more than two values for the general case with star vertexes.

The hierarchy according to which our algebra can give the answer to the query in the form of relation between two any operands in a directed graph can be done using the range min query as of Bender-Farach-Colton method of finding least common ancestor in a graph before vertex labeling operation – we are using the same approach which gives the linear-logarithmic complexity of the solution.

5.14 Isomorphism Problem on Graphs within Regular Language Notation

We give the notion towards the algorithm of indentifying isomorphism on graphs using the finite automata and their regular languages.

The graph isomorphism problem is NP-complete . It has a long-lasting history and application. Regular languages were introduced previously in . We will give the notion towards the automata describing graphs upon their internal structure with respect to some of the values like degree of vertexes and their adjacency property.

5.15 Permutation Pattern Waves and Polynomial Solution

We give fully polynomial solution to the permutation waves problem.

Since Cook's first statement the problem is considered to be NP-complete due to the existence of certificate of criteria as per integer sets with permutation pattern waves. Permutations were well studied before, the first appearance of the permutation pattern wave problem is given in .

As we can see we can use any notation building the corresponding relation on an oriented graph and performing the valid labeling.

5.16 Configuration Swap Problem on Describing Trees

We give the definition of the exact and sub-optimal algorithm to the configuration swap problem on graphs.

The graph theory is a theory which has a long-lasting history and its application . The swapping problem was introduced in and is of very practical meaning.

We simply build the tree from the graph after which we apply the oriented edge optimization as per the given circumstance where all other sets are settled – this gives exact and optimal solution to the minimal swapping problem.

5.17 On Boolean Circuits and Optimal Prefix Codes

We give the full notion on the boolean circuits and their relation to finite automata as well as the definition of the optimal prefix codes for the binary encoding of the words in text.

The boolean circuits were defined in, with its prior statement on the solution of the defined function – as it can be seen they can be converted to the deterministic finite automata defining the language on which the boolean function will be satisfied or, in other words, equal to "true". Efficient encoding and prefix codes is a far more historical problem .

The boolean circuits can form a typical non-deterministic model which can be determined, thus giving the observation towards the solution on a random function and random configuration.

Optimal encoding prefix codes are to be formed from the assumption of the division of the sums of occurrences of the symbol in source text, thus together forming the combinatorial optimization problem, where the division strategy is due to pivot selection and obeys the certain subset sum problem.

5.18 Non-polynomial Complexity of Permutation Automata

We give the full coverage of the notion of the permutation automata deciding complexity to be NP-complete.

The definition of non-solvable problem was first introduced, permutation automata in general were also presented, the problem of permutation automata acceptance without weighted function was proposed in .

As we have already shown that there is a local bound for permutation automata which can be re-presented in regular languages with extensions like back-references, it's obvious that full optimization network complies with the classical NP-complete problem as Traveling Salesman Problem (TSP). The above fact shows that there could be non-countable number of pre-permutations before visiting the layer on the arbitrary state of automaton.

5.19 Token Sliding Problem on Graphs in Polynomial Time

The quadro-linear polynomial algorithm is given for the token sliding problem on graphs.

Graph theory has a long history and meaning as a model, the token sliding problem is well-known also, we will show further that this problem can be optimized on a produced tree from an arbitrary graph.

At each step of optimization we change the order of independent set series according to the orientation of the leaf and its sub-tree due to this orientation – this gives a quadratic worst case method to compute the number of swaps in order to get the right configuration of independent sets.

5.20 Solving Optimization Problems on Graphs using Automata Composition

We give the full definition of the optimization problem and its isomorphic transformation to the non-deterministic finite automata as per the order of traversal and its corresponding settlement as in undirected as well as in directed graphs.

The optimization problems on graphs are known to be NP-complete, since the optimization function can be easily verified and hard to get to the optimal point . Graph theory is a model on which even finite automata can be operated in a pre-defined method. The strongly connected components of directed graphs give the notion towards the ordered relation between each of the node which can be optimized as per the sample problem like .

We can construct the correct automaton recognizing the language of the paths in the graph after which we apply the optimization according to the order in the strongly connected components of the directed graph, for general case of undirected graphs we can consider the same option with respect to the search strategy.

5.21 Polynomial Algorithm for Clique Problem using Matrix Space

We give a polynomial algorithm in quadratic complexity for finding cliques in graphs according to the matrix space with single operation like boolean multiplication of the adjacency matrix of the given set of vertexes, we will show that this is a fully polynomial solution with the current lower bound on the number of operations in order to find the clique in graph of the defined rank.

The hardness of the problem is a key of its classification, graph theory is described, the clique problem is known to be NP-complete and, thus, is to be solved efficiently using conceptual algorithmic approach.

We give the matrix of adjacency an algebra with single closed operation like boolean multiplication, after forming the maximal independent set and applying the multiplication of this matrix and its transposition we can devise the sets for which this can be implied according to the matching within the kernel of the clique – this operation reduces the size of sought input up to the given order.

5.22 Subgraph Enumeration Problem on Graphs

The polynomial algorithm is presented along which the number of subgraphs of the given graph can be counted.

The NP-hardness of this problem is defined as there are many subgraphs and only isomorphic certify for the given subgraph in order to count all its isomorphisms in the given graph. Graph theory is well-defined during the past time, subgraphs and their isomorphisms are defined in . The counting problem is presented in .

We give the solution towards finding the number of subgraphs or simply enumerating them during the descent on a produced tree for the given graph and subgraph, thus, we can solve the problem by applying recurrent relation on the edge which divides the tree in several parts with respect to the structure of the subgraph.

5.23 Solution to Triangle Finding Problem in Graph

The fully quadratic algorithm in maximal number of edges is presented in order to find the number or enumerate all subgraph triangles in graph.

Graph theory is presented, the triangles are discussed, the problem of finding triangles in graphs is in .

We use at maximum quadratic space and time and number of edges at most which is the most optimal exact solution to the stated problem. At first we build the sub-tree of a graph and the adjacency of any two pairs for the pre-computed set of vertexes where the third vertex is a middle and, thus, has the adjacent two vertices in the edge of the tree.

5.24 Solution to Disjoint Paths Problem on Graphs

The linear algorithm in the number of edges and vertices in graph is given for finding k-disjoint paths.

Graph theory has a long-lasting history and application . The path or vertex disjoint set problem in a graph is given in .

We start from the set of each pairs from the left to right and from right to left by building the fully directed tree ascending in both directions so that there would be a cut of size more than k, thus, satisfying the condition of disjoint path on vertexes or edges.

5.25 Solution to Maximum Satisfiability Problem and Minimal Vertex Covers on Graphs

We give the solution to partial maximum satisfiability problem on the example of enumerating minimal vertex covers which coincide and are non-polynomial, our solution is fully polynomial and exact.

The problem of not more three variables in logical satisfiability problem was proved to be NP-complete as well as its case on graphs for finding the minimal vertex cover .

We build the tree in which we descend from the produced tree and use memory in order to store the bitmap of all satisfied conditions as well as per model of minimal vertex cover on graphs.

5.26 Solution to Even-Path Problem in Arbitrary Graphs

We give a polynomial solution to even-path problem on graphs between two given vertices in arbitrary graph as it can be either directed or undirected, our approach also states the minimal bound of number of edges and vertexes in graph.

The problem can be seen as NP-complete, graph theory was well described, the even-path problem is defined in .

5.27 Solution to Minimal Decomposition Problem on Graph

We give an algorithm and minimal bound for linear decomposition of the graph with given maximal degree of its vertex.

Graph theory is described, the linear arboricity conjecture is stated, according to which there is not more than half of the maximum degree of the paths.

6 Recurrent Diversification of Counting Alternation Permutations

We give the recurrence relation towards the counting of alternation permutations thus providing the exact formula in order to compute the number of alternating iterations within the insertion operation and the union of the sets.

The permutations are well presented, alternating permutations are permutations with pre-defined order .

Since implication we give the upper bound using the recurrent relation which is defined as the oracle function $PA(n)$: $PA(n) = f(n) * PA(n-1)$.

Where $f(n)$ is a function defined as the error factor for which the alternation decision holds true, obviously $f(0) = 1$ and $PA(0) = 1$.

To define the function $f(n)$, we are using each triplet consideration with respect to each triplet in the form: $a_i < a_{i+1} > a_{i+2}, a_i > a_{i+1} < a_{i+2}$,

The above definition is a result of the term alternating permutation in its canonical sense. As we see from above the second condition cannot hold true as we cannot insert the biggest element n in any of the position when this fact is satisfied. Let's consider this occurrence: $a_i < n > a_{i+1} > a_{i+2} < a_{i+3} : a_i < n > a_{i+2} < max(a_{i+1}, a_{i+3}) > min(a_{i+1}, a_{i+3})$.

For the second condition we have: $n > a_{i+1} < max(a_i, a_{i+2}) > min(a_i, a_{i+2}), n > min(a_i, a_{i+2}) < max(a_i, a_{i+2}) < a_{i+1}, min(a_i, a_{i+2}) < max(a_i, a_{i+2}) > a_{i+1} < n$.

Thus, we have four subsets to devise the function $f(n)$, thus giving us the following exact relation like: $f(n) = 4 * PA(n-1) : A_1 \cup A_2 \cup A_3 \cup A_4$.

Obviously: $4_n * P_{A_1 \cup A_2 \cup A_3 \cup A_4} \leq f(n) \leq 4_n$.

Where in above relation probability is the union of all the cases when the four insertion conditions hold true, which is recursive and can be counted.

6.1 Reductions of Graph Edge Coloring Problem and Chromatic Number

In this short note we are to give the note towards graph edge coloring problem (ECP) and its reduction to graph vertex coloring problem (VCP), which gives the significant result in deciding the minimal number of colors for edge coloring problem.

The graphs are widely studied, chromatic numbers in VCP denote the minimal number of colors required to color it so that no two adjacent vertices bear the same color. The latest research aims also towards Euler's lattices problem.

As we can construct the graph for the given graph $G(V, E) : G(E, (a, b), (a, i) \& (b, i) \in E \forall i)$, it follows that the chromatic number can encode the number of edge coloring in ECP, so that this number is at least greater than the same number of the initial graph by induction.

6.2 Relation between Chromatic Number and Length of Hamiltonian Paths in Graph

We give the strict computational relation between chromatic number of graph and sum of lengths of Hamiltonian paths using set exclusion theorem as well as the addition towards inverse graph.

Graph theory was steadily studied, the graph coloring problem and its chromatic number are known to be NP-complete, the partial relation between these numbers and the length of the maximal path were studied.

As per the set theory, graph can be considered as a set if we would at each step of iteration remove some 2-vertex graph with a single edge or not, this will look like as follows:

$|G(V, E)| = |G(V_1, E_1)| + |G(V_2, E_2)| - |G(V_1 \cap V_2, E_1 \cap E_2)|$.

The above relation can be approximated within any path if we would get at each iteration the pair of nodes (u, v), then our relation will look like:

$|G(V, E)| = |G(V - \{v\}, E - \{v\})| + \{1, 2\} - 1$.

As in both division operator we divide the parts along the maximal length and an optional edge in graph, obviously this function is to be minimal, thus we have to find a path of maximal length in the inverse graph $\neg G(V, E)$.

Thus, we get to the following relation:

$\chi(G(V, E)) = |V| - max\{\sum_{p \in H(\neg G)} |p|\}$

Where $H(\neg G(V, E))$ is a set of longest paths through the whole set of vertexes in inverse graph, the paths are to be disjoint.

The proof can be done by induction to the general graph G(V, E) as we approximate towards minimal possible number. This proof gives an evidence of the connection between Dirac's formula for graph containing Hamiltonian and the chromatic number of the inverse graph.

We have given the strict relation between longest paths which can be either Hamiltonian of size $|V| - 1$ or any other maximal possible of all the paths in inverse graph, thus, giving observation of the Hamiltonian cycle presence for Dirac's formula on general graphs. This fact gives us the observation of using the divide algorithm on inverse graph in order to find the maximal longest path of the maximal size within Dirac's equivalence relation. We will use our equivalence to establish connection between chromatic numbers of the graph and its inverse, thus we have:

$\chi(\neg G(V, E)) = |V| - max\{\sum_{p \in H(G)} |p|\}$

From this point, we get:

$\chi(G(V, E)) - \chi(\neg G(V, E)) = max\{\sum_{p \in H(G)} |p|\} - max\{\sum_{p \in H(\neg G)} |p|\}$

In addition we give the definition of the complete graphs or cliques Kn: the paths are actually Hamiltonians in these decomposition.

6.3 Optimal Labeling Algorithm for Vertex Coloring Problem in Graph

We present the labeling algorithm for Vertex Coloring Problem (VCP) which runs in product linear time on number of vertexes and edges in graph at minimum with the chromatic number as parameter.

VCP refers to graph theory, it's known to be NP-complete and, thus, optimal or approximate algorithm is to be applied .

We start from forming the system of inequalities between each of adjacent vertexes in graph $G(V, E)$. We start by labeling with choosing the minimal label index from adjacent vertexes with stabilization principle on the obtained index which can be reverted in time $O(m)$ using each iteration on maximal chromatic number with total complexity of $O(nm)$.

We claim that this algorithm is most optimal as to the consensus of simplex system formed by inequalities and the target function to be minimal possible. The stabilization, thus, runs, each time the node changes its correct labeling according to the selection rule.

6.4 Proof and Solution of Meels-Colnet Conjecture and Problem

We give the full proof towards Meels-Colnet conjecture and CFG problem, which is in finding the number of words of fixed size on grammars. The problem can be classified as P-complete and also states that it's strongly P-complete by Meels and Colnet [15].

We use our basic notation of extended regular expressions which can construct the deterministic automata for the expression in the form $(< CFG >)\&a^n$ - this gives the polynomial solution as we know that subset construction is P-complete which proofs that this problem belongs to the same class of computational complexity.

The solution is to construct non-deterministic finite automata (NFA) for the &-expression and subsequently convert it to deterministic finite automata (DFA) – this is a full solution for any case of the problem.

6.5 Proof of Equivalence of Complexity Classes and Other Relations

The notion of complexity classes was before presented by Stephen Cook, as we know functions can be polynomial and non-polynomial, as well as arbitrary.

Let $f(x)$ be the sought non-polynomial function, then we have:

$P =?NP$.

We also know due to our functional hypothesis or Rabin-Scott conjecture that:

$f(P) = NP$.

Let's assume that:

$P \neq NP$, then - $P = f^{-1}(NP) \neq NP\&NP = f(P) \neq P \Rightarrow f(f^{-1}(NP)) = f(P) = NP \neq NP$, which is a contradiction. For the second inequality we have: $f^{-1}(f(P)) = f^{-1}(NP) = P \neq P$, which is also a contradiction, then we get $NP \neq NP\&P \neq P \Rightarrow P = NP$.

The statement is due to the mathematician Akram Louiz [16] and is as follows: $P = NP \Leftrightarrow |P| = |NP|$, where P and NP are sought complexity classes.

As we know by Cook's definition: $P \subseteq NP \Rightarrow |P| \leq |NP|$. Let's assume that P equals NP, i.e. $P = NP$, then we have the following: $P = NP \Rightarrow |P| = |NP| \geq 0 \Rightarrow |NP| = 0\&|P| = 0 \Rightarrow NP = \{\}$, in other words the NP-class has no members and, thus, is actually an empty set. However, as we know, according to Cook's theorem there's at least one problem, also known as 3-SAT which belongs to the set of NP-complete problems – which is a contradiction and, thus: $P \neq NP$.

From what follows Louiz's equation: $NP = N + P$.

The solution of the system of equations: $NP = N \cdot P$ and $NP = N + P$, gives the function $f(x) = \frac{x}{x-1}$ both for P and N, thus stating that our conjecture is correct, since the function $f(x)$ exists.

Thus we have given definition to the following algebra: $< +, \cdot, \{N\}, \{P, NP\} >$ which can be used both as a regular language or any arithmetical expression.

As we have prior result of regular grammars over set of computational problems, we are to present the universal 'complexity automata' which can be used in solving any problem.

The "P versus NP"practice, theory and 3-SAT proof was well understood, however, there was an attention towards, as we suppose, the resolution of this statement. Kardeis was near the term as 'quantuam computing' and Zeta-Function, however, we use an automatic approach to give all the required framework foundation towards the solution of both polynomial and NP-hard, or non-polynomial, problems, provided both equality or inequality of P- and NP-classes of computational complexity.

We give the following definition of our automata based on obtained result, thus, 'complex automata' is defined as follows: $< +, \cdot, \{P, N\} >$, where + and · is a union and concatenation operation over the set of terminal symbols "P"and "N where "P"stands for the certificate and "N"is a problem itself. We have given all the necessary framework to operate on Complexity Theory for the definition of the problem as a regular expression and further converted to finite or 'complex' automata, thus, proving that any complex problem can be solved using our approach.

We present the solution to two classical problems like MAX-SAT, or its 3-SAT partial case, and TSP from computational complexity theory using subset construction. The problem was stated before, MAX-SAT is also well-studied, as well as TSP. We give the exact algorithms to these problems using subset construction.

For MAX-SAT problem we simply assign an alphabet symbol to each variable and its value from the set of $\{0, 1\}$, then we apply the &-operator for each of the clause in MAX-SAT, thus, solving it in polynomial time. For TSP we are using counter tags in finite automata and also apply &-operator as in our previous local search algorithm.

6.6 Artificial Neural Networks without Layering Concept

We present the basic abstract of the newly obtained results on class of non-layered artificial neural networks. Artificial neural networks is a well-known concept and solution, however due to the lack of performance they are less productive for practical approach and mainly are focused on artificial intelligence.

We define the prediction function as: $f(x) = \frac{1}{x+1}$ - which is decreasing.

Meanwhile the training sigmoid function is defined as well: $g(x) = \frac{1}{1+e^{-x}}$ - which is strongly increasing.

Both functions are defined on the set of the range [0, 1] or [1, 0] with respect to probability.

We present the algorithm of training and prediction during input interaction:
1. Find the set of feasible set in model using machine learning algorithm and function f(x) which can be presented as a binary tree.
2. Compute the prediction.
3. Get the input for the given prediction.
4. Train the model with the newly predicted fact using sigmoid function g(x).
5. Request the new input.
6. If input is empty, then halt.
7. Return to step 1 for the input from step 6.

6.7 Application and Theory of Several Aspects in Optimization

The 3SUM problem was viewed from a singleton point of view for the past time. In this work the experimental results along with proof are presented: the state explosion doesn't occur in specific cases after decomposition of regular expression into non-deterministic finite automata (NFA), thus, the P-complete procedure to take turn for converting NFA into deterministic finite automaton (DFA) with construction according to the De Morgan Law. We give the notion of the equivalence of the complexity classes due to the recent research according to Rabin-Scott subset construction. We also give the linear algorithm for lookahead and lookbehind assertions in regular expressions by implementing the intersection operator which was well studied before in our prior research, the work also includes: the experimental part of research in our investigation of the "P versus NP"theorem and the optimization principle within the physical layout, the full proof of inequivalence of P and NP complexity classes which can be addressed to the famous "P versus NP"theorem by Stephen Cook, our approach summarizes all the results obtained before in our prior research of this topic and its failure during the decades of its first appearance in the scientific press, definition of the single operation for giving the output to the new state in Berry-Sethi approach of building deterministic finite automata (DFA), address the output, produced by the Turing tape automaton, or Turing Machine, which is further divided as deterministic and non-deterministic, to the set of regular languages recognized by finite automata. It's known that the subset sum problem lies in the NP-class of complexity, however, due to the integer factorization of any number it states another argument towards P = NP. The unified system based upon Ford-Fulkerson maximum flow method for solving the civil engineering problems like flooding and human evacuation during earthquakes or other disasters is also presented. According to the present time the normal forms are consequent to the efficient data manipulation, still there's no universal method for solving this problem according to the criteria of data to be small and the modeled solution provides the sufficient normalization of the source data. The $X + Y$ sum problem as observed wasn't solved before, so we provide a fast and simple solution to this problem using algebraic properties of the vector as well as the general case. The recent research and study in the theory of Computational Complexity gives new perspectives in studying the "P versus NP"question.

We build the binary tree for each of the elements in binary notation in the given input array, after which we concord the search according to the valid combinations. The conversion of NFA to DFA, or subset construction, and its possibility proof first appeared in has an exponential complexity of $O(2^n)$ and thus is EXP or NP-complete.

Many techniques were done before in order to avoid the effect of state explosion, however, we present the De Morgan law for rewriting both union and intersection operators as well as in extended regular expressions, which leads to P-complete result.

For the past decade the "P versus NP"problem was well studied with conformance that P is a subset of NP. If this happens, then there's a set of problems which are strictly in NP and not in P assuming P not equal NP .

We simply close the circuit in our algorithm when converting non-deterministic finite automata (NFA) to deterministic finite automata (DFA). The same is true even for NFA constructions which give rise to the question of relation between regular language algebra and features like lookahead and lookbehind assertions.

As it was stated before the "P versus NP"question has a long-standing history in the theory of Computational Complexity and Mathematics as well, where the symbol of infinity isn't defined as operand due to the inconvenience of its relation to the operands in the mathematical expression in the algebra of numbers .

Thus, by showing that P equals NP we still cannot devise the relation in this algebra, however, if P not equals NP, we can proceed further with the modern aspects.

Before we have shown that there's a functional relation between complexity classes, i.e. there could exist the function $f(x)$, so that $f(P) = NP$.

We will proceed to the above publication further and give the strict proof of inequivalence of complexity classes and as it follows from this proof the hierarchy of classes which give consolidated proof of the relations between the variety of complexity classes in Computer Science and Theory of Computation .

Before our research the Theory of Complexity was well underlined and it follows that first we have to postulate and only then give the question of the relation between complexity classes, basically polynomial "P"and non-polynomial "NP"classes.

The main problem in the Berry-Sethi approach is the conformance of the new state to the states added before during each iteration of the algorithm. We use the single test application of our method for the equivalence of the states to the regular language they represent.

Since each of the Turing machines has a limited set of states during which it can transit to the next step, or iteration, of processing the input and, thus, going to the halting or accepting, or rejecting, state, we are to define the set of words which are written are well-defined as programs produced by this machine. The regular language is formed from the Deterministic Turing Machine (DTM) or Non-deterministic Turing Machine (NDTM) can arbitrarily produce the regular set of languages, known as programs of this machines according to the finite state of states in the transition diagram as it can be seen in various sources . On the account of "P versus NP"theorem we are to define the proved equality of P and NP-classes of complexity as Finite Automata (FA) are isomorphic with to the regular language they accept .

The "P versus NP"problem is the main problem stated before.

We simply factorize numbers within the prime number factorization algorithm and build on-level tree structure for finding the structure of the method. As we know the prime factorization is reminiscent of the tractable logic of computer numbers which tend to limit the Ackermann numbers . The prime factorization itself isn't studied nowadays and is well-known to be P-complete within the subquadratic algorithm which is still inefficient against big numbers which are met in cryptography . Still it's omitted that the subset sum problem can be devised from the whole set of problems within the multi-cubic trees along prime factorization and prime number notation system. Still it's posed that linear complexity of introduced parameters harms the overall magnitude of complexity, which is well-known and can be factored according to the optimal notation of consecutive prime numbers to which the parameter tends to grow linearly with predefined maximal speed of Ackermann numbers. We build the growing structure of the tree on each level having the prime number in prime notation of the parameter whose limit is to be deduced from even factorial decomposition which leads to blow and speed-up and, thus, makes the free parameter less playing the role.

The normalization problem is to be presented as the mathematical programming problem within the constraints and the main function as the size of data to be minimized. Earlier we have shown that the factorial number of possible data in each table depends on the number of columns and number of rows. This fact makes it possible to seed the data and store them in a fast and efficient way.

The "$X+Y$" problem is in general still unsolved and plays a role in effective optimization .

We simply sort the items by the normal vector distance to the line $X+Y$ going through the given point.

This simply gives the minimal possible running time on average as $O(n*log(n))$.

As it was pointed out by the modern research the "P versus NP"question is to be studied from a different point of view and the broad horizons of its decidability are to be omitted due to the insufficient approach in the formalization of this theorem.

6.8 Proof

We prove the above fact by the assumption already given in the statement before: thus, as we know any non-deterministic finite automata (NFA) can be converted to analogous deterministic finite automata (DFA).

According to our latest research the subset construction is P-complete and, thus, there exist no automata with the strict property given above, thus, it follows that $P = NP$.

We have devised the parallel computing law as follows: $lim_{N\to\infty}\frac{NP}{N} = P$.

Let's assume that the above law is correct according to the number of threads N, which operate on a Non-deterministic Turing Machine (NDTM). Also let's assume that $P = NP$, then it follows:

$lim_{N\to\infty}\frac{1}{N} = 1 \Rightarrow 0 = 1 \Rightarrow P \neq NP$.

The output above demonstrates the simplest way of proving the inequivalence of complexity classes according to the parallel computing law: $NP = N \cdot P$

6.9 Experiment

We build the integral circuit on a board and give the experimental projection of the abstract processes and processes which are put in a different environment, or physically. The VLSI devices are used today in many aspects. Hamiltonian is met in the Traveling Salesman Problem (TSP) where it defines the shortest possible path visiting each of the city once. TSP by itself was an argument of "P versus NP"theory and practice . On the experimental electrical board we build the layered circuit by using elements for satisfying the "visit-once"condition. Thus, the shortest path of a limited number of mediate elements can be found.

We develop the maximum flow network on map using any applicable source and present each cell with the incoming or outgoing edges as of the each of the bordering cells on the map.

The maximum flow applied to the above-described model gives the result of simple prediction scheme according to which the residue flow can be pushed forward as well as backward, and the possible dangerous zones with blocking flow can be detected and successfully mapped to the physical map where the ecological disaster happens.

At each model we give the maximum capacity as the maximum capacity of the fluid or human factor.

7 Conclusion

We have given a fully polynomial algorithm for the md-DFA problem which is NP-complete - this fact gives the experimental proof of the equivalence of complexity classes like polynomial "P"and non-polynomial "NP"as the benchmarks above state as the argument as they are almost linear to the size of the expressions and running time depends also linearly, also, the problem described was proved to be NP-complete before. We have also shown the experimental proof of tractability of the problems like md-DFA and Non-emptiness-DFA which are known to lie in NP-class of complexity. We have also concluded that within the new proof, back-referencing problem can be computed fast within arbitrary number of capturing groups. Thus, we claim experimental proof of "P versus NP"theorem: $P = NP$, which could be used in solving other problems like Riemann hypothesis by Akram Louiz [13]. The reader is invited to use Regex+ software package and provide examples. I also have some notes about theories like fixed-parameter tractability and classification - all these theories and similar to them are all about the direct solution or final resolution of P-NP theorem by Cook, while our conjecture of functional hypothesis gives the final outcome with to the full statement of the problem: are there solution to NP-complete problem or not.

There could be parallels between the P vs. NP problem and quantum mechanics, particularly in relation to the concept of superposition. In quantum mechanics, a system can exist in multiple states simultaneously until a measurement is made. Similarly, NP problems could be seen as a kind of "superposition"of many possible solutions that exist at the same time until verification or computation collapses them into a final solution.

Thus, we have also proved that subset construction, or powerset construction, is polynomial, or P-complete, with respect to the prior obtained results.

The common misinterpretation of the "P versus NP"theorem lies between the fact that it can be solved effectively, still, it doesn't follow that from this consequence we can devise the relationship between two classes.

The potential of the method above gives the main result of the past research for efficient implementation of lookaround assertions. We have presented the experimentation theory for which there could be conjectured that each of the shortest paths found on the simulated integral circuit can form the Hamiltonian by itself in its decomposed state.

We have shown the inequivalence of complexity classes around our final proof which is given as a final contribution to the field of Computational Complexity.

The above method is expensive, however, in static mode it can be more productive.

We have come to the end of the "P versus NP"continued story and the positive output of the proof of equivalence of these complexity classes gives the horizons of the universality of automata and their isomorphic properties as well.

The overall can be considered as the other argument towards "P versus NP"theorem and the proof P = NP, as the subset sum is both in P and NP and NP-complete classes of complexity.

We presented the safe method for detecting possible bottlenecks during flooding and evacuation which can lead to a more humanistic approach in science and engineering.

As the much earlier works were towards the static structure of the database, for now we have defined the universal approach towards data normalization.

The above case can be extended to the problem where the normal vector of a line is given with arbitrary weights which open new horizons to the studying of the application and theoretical acquisition of this problem.

The sorting problem for dual coords can be solved in minimal possible time $O(n)$ by converting it to co-NP problem as integer sorting.

Thus, we came through the inequality of the question "P versus NP"which clearly gives the argument towards the preliminary axiomatization of the complexity classes which we name as "decidable"and "undecidable".

We have provided all the necessary conditions to operate on problem sets and more using the new complexity algebra.

We have given linear algorithm for MAX-SAT and poly-logarithmic for TSP.

We have also presented the evolutionary and mainly performing model for artificial intelligence and machine learning.

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ON A SPECTRAL PROBLEM FOR A FOURTH-ORDER DIFFERENTIAL OPERATOR

This paper considers a generalized spectral problem for a fourth-order differential operator. The primary goal of the research is to analyze the spectral properties of the operator arising in boundary value problems for the Stokes and Navier-Stokes equations, as well as to utilize the obtained eigenfunctions to construct a fundamental system in the space of solenoidal functions. The work combines theoretical analysis with practical applications, making it relevant for numerical modeling of hydrodynamic processes. The main methodology is based on the method of separation of variables and the use of curl operators for different domain dimensions. In particular, the paper proposes approaches to introducing curl operators for the three- and four-dimensional cases, which generalize the problem formulation. The key results include proving the existence and distribution of eigenvalues, as well as constructing an orthonormal basis in functional spaces. This study contributes to the development of spectral analysis of high-order operators and can be useful for developing efficient algorithms for solving hydrodynamic problems. The practical significance of the results lies in their application to numerical modeling of fluid flows in various fields of science and engineering.

Key words: spectral problem, curl operator, eigenvalues, eigenfunction.

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Төрінші ретті дифференциалдық оператор үшін бір спектралды есеп туралы

Бұл жұмыста төртінші ретті дифференциалдық оператор үшін жалпыланған спектрлік есеп қарастырылады. Зерттеудің негізгі мақсаты – Стокс және Навье-Стокс теңдеулері үшін шекаралық есептерді шешу барысында туындайтын оператордың спектрлік қасиеттерін талдау, сондай-ақ алынған меншікті функцияларды соленоидалды функциялар кеңістігінде іргелі жүйені құру үшін пайдалану. Жұмыс теориялық талдауды практикалық қолданумен үйлестіретіндіктен, бұл оны гидродинамикалық үдерістердің сандық модельдеуі үшін өзекті етеді. Негізгі әдіс айнымалыларды бөлу әдісіне және әртүрлі өлшемдегі облыстар үшін ротор операторларын қолдануға негізделген. Атап айтқанда, үш және төрт өлшемді жағдайлар үшін ротор операторларын енгізу тәсілдері ұсынылып, бұл өз кезегінде есептің қойылымын жалпылауға мүмкіндік береді. Негізгі нәтижелерге меншікті мәндердің бар екендігі мен орналасуын дәлелдеу, сондай-ақ функционалдық кеңістіктерде ортонормаланған базисті құру жатады. Бұл зерттеу жоғары ретті операторлардың спектрлік талдауының дамуына үлес қосып, гидродинамикалық есептерді шешудің тиімді алгоритмдерін әзірлеуге пайдалы болуы мүмкін. Жұмыстың практикалық маңызы – алынған нәтижелердің әртүрлі ғылыми және инженерлік салалардағы сұйықтық ағындарын сандық модельдеуде қолданылуында.

Түйін сөздер: спектрлік есеп, ротор операторы, меншікті мән, меншікті функция.

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Об одной спектральной задаче для дифференциального оператора четвертого порядка

В данной работе рассматривается обобщенная спектральная задача для одного дифференциального оператора четвертого порядка. Основной целью исследования является анализ спектральных свойств оператора, возникающего при решении краевых задач для уравнений Стокса и Навье-Стокса, а также использование полученных собственных функций для построения фундаментальной системы в пространстве соленоидальных функций. Работа сочетает теоретический анализ с практическим применением, что делает её актуальной для численного моделирования гидродинамических процессов. Основная методология основана на методе разделения переменных и использовании роторных операторов для различных размерностей области. В частности, предлагаются способы введения операторов ротор для трех- и четырехмерного случаев, что позволяет обобщить постановку задачи. Основными результатами являются доказательство существования и расположения собственных значений, а также построение ортонормированного базиса в функциональных пространствах. Данное исследование вносит вклад в развитие спектрального анализа операторов высокого порядка и может быть полезно для разработки эффективных алгоритмов решения гидродинамических задач. Практическая значимость результатов заключается в их применении в численном моделировании потоков жидкости в различных областях науки и техники.

Ключевые слова: спектральная задача, оператор ротор, собственные значения, собственные функции.

Introduction

In this paper, we consider a generalized spectral problem for a fourth-order differential operator.

By introducing a scalar or vector stream function, the spectral problem for the two-, three-, and four-dimensional Stokes operators can be reduced to a generalized spectral problem for the biharmonic operator.

Let us provide the mathematical formulations of the latter statement.

First, let us formulate the spectral problem for the d-dimensional Stokes operator. Let $x = (x_1, ..., x_d) \in \Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded simply connected domain with boundary $\partial\Omega$. The goal is to find nontrivial solutions $\{\vec{w}_k(x),\ p_k(x),\ x \in \Omega,\ \ k \in \mathbb{N}\}$ and the corresponding values of the parameter $\{\lambda_k^2,\ k \in \mathbb{N}\}$ for the following boundary value problem ([1], Chapter II, § 4; [2], Chapter I, § 6, Corollary 6.1; [3], Chapter I, § 2, Subsection 2.6):

$$\begin{cases} -\Delta\vec{w}(x) + \nabla p(x) &=& \lambda^2 w(x), & x \in \Omega, \\ \operatorname{div}\{\vec{w}(x)\} &=& 0, & x \in \Omega, \\ \vec{w}(x) &=& 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

Let $\dim\{\Omega\} = 2$, and consider the two-dimensional curl operator **curl** defined as follows:

$$\{w_1, w_2\} = \mathbf{curl}\{0, 0, U(x)\} = \{\partial_{x_2} U, -\partial_{x_1} U\}, \tag{2}$$

where $U(x)$ is a scalar function known as the stream function. From equation (1) using the formulas in (2), we can proceed as follows: first, by substituting the vector function $\vec{w}$ in (1) with $\operatorname{curl} U$; second, by applying the operator curl, to the resulting expressions; and third, by summing the results obtained after the second step. As a result, we obtain:

$$\begin{cases} (-\Delta)^2 U(x) &=& \lambda^2(-\Delta)U(x), & x \in \Omega, \\ U(x) &=& 0, & x \in \partial\Omega, \\ \partial_{\vec{n}} U(x) &=& 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

where $\vec{n}$ is the outward normal to the boundary $\partial\Omega$.

Since the differential equation in (3) contains the operator $-\Delta$ on the right-hand side, we will refer to problem (3) as a generalized spectral problem for the biharmonic operator $(-\Delta)^2$. It is evident that the key role in transforming problem (1) into the spectral problem (3) is played by the **curl** operator given in (2).

Let $\dim\{\Omega\} = 3$, and consider the three-dimensional **curl** operator defined as follows:

$$\operatorname{curl}\vec{U}(x_1,x_2,x_3) = \vec{w}(x_1,x_2,x_3), \;\; \operatorname{div}\vec{w}(x_1,x_2,x_3) = 0, \;\; (x_1,x_2,x_3) \in \Omega, \tag{4}$$

where $\vec{U} = \{U_1, U_2, U_3\}$, $\vec{w} = \{w_1, w_2, w_3\}$ are three-dimensional vector functions,

$$\vec{w} = \operatorname{curl}\vec{U} = \{\partial_{x_2}U_3 - \partial_{x_3}U_2,\, \partial_{x_3}U_1 - \partial_{x_1}U_3,\, \partial_{x_1}U_2 - \partial_{x_2}U_1\}. \tag{5}$$

If we assume that all three components of the vector $\vec{U}$ are equal, i.e., $U_1 = U_2 = U_3 = U(x_1,x_2,x_3)$ in Ω, then, similarly to the two-dimensional case, using equations (4)–(5), we can derive from (1) the following:

$$\begin{cases} -\Delta(-\Delta + S)U(x) &= \lambda^2(-\Delta + S)U(x), \quad x \in \Omega, \\ U(x) &= 0, \quad x \in \partial\Omega, \\ \partial_{\vec{n}}U(x) &= 0, \quad x \in \partial\Omega, \end{cases} \tag{6}$$

where $S = \partial^2_{x_1x_2} + \partial^2_{x_2x_3} + \partial^2_{x_3x_1}$. If we temporarily remove the operator S from the differential equation in (6), we once again obtain a spectral problem of the form (3), but now in the three-dimensional case.

Let $\dim\{\Omega\} = 4$, and consider the four-dimensional **curl** operator defined as follows:

$$\operatorname{curl}\vec{U}(x_1,x_2,x_3,x_4) = \vec{w}(x_1,x_2,x_3,x_4), \;\; \operatorname{div}\vec{w}(x_1,x_2,x_3,x_4) = 0, \;\; (x_1,x_2,x_3,x_4) \in \Omega, \tag{7}$$

where $\vec{U} = \{U_1, U_2, U_3, U_4, U_5, U_6\}$, $\vec{w} = \{w_1, w_2, w_3, w_4\}$,

$$\vec{w} = \operatorname{curl}\vec{U} = \begin{pmatrix} \partial_{x_4}U_1 + \partial_{x_3}U_5 - \partial_{x_2}U_6 \\ \partial_{x_4}U_2 + \partial_{x_1}U_6 - \partial_{x_3}U_4 \\ \partial_{x_4}U_3 + \partial_{x_3}U_4 - \partial_{x_1}U_5 \\ -\partial_{x_1}U_1 - \partial_{x_2}U_2 - \partial_{x_3}U_3 \end{pmatrix}, \;\; \operatorname{div}\operatorname{curl}\vec{U} = 0. \tag{8}$$

Remark 1 *The* curl *operator in equations* (7)–(8) *acts on a six-dimensional vector function $\vec{U}$, which, in particular, corresponds to the following vector composed of the electric $\vec{E}$ and magnetic $\vec{H}$ field intensity vectors: $\vec{E} = \{E^1, E^2, E^3\}$, $\vec{H} = \{H^1, H^2, H^3\}$ ([4], Chapter V, § 1, Chapter VII, § 1; [5], Chapter III, § 8 and § 9; [6], Chapter I, § 5), namely,*

$$\vec{U} = \{E^1, E^2, E^3, H^1, H^2, H^3\}.$$

From equation (1), using formulas (7)–(8), we can derive the following:

$$\begin{cases} (-\Delta)^2U(x) &= 3\lambda^2(-\Delta)U(x), \quad x \in \Omega, \\ U(x) &= 0, \quad x \in \partial\Omega, \\ \partial_{\vec{n}}U(x) &= 0, \quad x \in \partial\Omega. \end{cases} \tag{9}$$

If we disregard the factor of 3 in front of the spectral parameter λ^2, the spectral problem (9) fully coincides with problem (3), but now in the four-dimensional case, i.e., $\dim\Omega = 4$.

Once again, it is evident that the key role in transforming problem (1) into the spectral problem (9) is played by the **curl** operator, which is defined by formulas (7)–(8).

The aim of this work is to construct a fundamental system in the space of solenoidal functions. If we were able to solve spectral problems for the biharmonic operator (3) in domains of various dimensions $\dim\{\Omega\} = d,\ d \geq 2$, we would succeed in constructing such a fundamental system, which is important not only from a theoretical point of view but also for the development of computationally efficient algorithms for the approximate solution of boundary value problems for the Stokes and Navier–Stokes systems [7]. In this work, we will limit ourselves to solving a certain generalized spectral problem for a fourth-order differential operator.

It is worth noting that spectral problems for the Stokes operator (but with periodic boundary conditions) in a cubic domain have also been considered in the works [8], [9], and [10].

In [8], the spectra of the curl and Stokes operators in a cube are studied for functions satisfying periodic boundary conditions. The Cauchy problem for the 3D Navier-Stokes equations with periodic conditions in the spatial variable was investigated in [10].

Since our approach actively utilizes the properties of the **curl** operator, which is closely related to vortex theory, we refer to the foundational works on vortex theory [11], [12], [13], [14], [15], [16], and others. Some ideas from these works have been used in establishing our statements.

Let us introduce the main function spaces that will be used in this work. Let $x = (x_1, ..., x_d) \in \Omega \subset \mathbb{R}^d$ where $d \geq 2$, be an open bounded simply connected domain with a sufficiently smooth boundary $\partial\Omega$, and let $m \geq 0$ be an integer,

$$W_2^m(\Omega) = \left\{v |\ \partial_x^{|\alpha|} v \in L^2(\Omega),\ |\alpha| \leq m\right\}, \quad \text{where} \ \ \partial_x^{|\alpha|} = \partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d},\ |\alpha| = \sum_{j=1}^{d} \alpha_j,\ \partial_{x_j} = \frac{\partial}{\partial_{x_j}},$$

$$\overset{\circ}{W}{}_2^m(\Omega) = \left\{v |\ v \in W_2^m(\Omega),\ \partial_{\vec{n}}^j v = 0,\ j = 0, 1, 2..., m-1,\ \vec{n} \text{ is the outward normal to } \ \partial\Omega\right\}.$$

For the notation of function spaces, we will follow the monographs [17], [18], [19], and [20].

1 Formulation of the Spectral Problem

Let us consider the following spectral problem for a fourth-order differential operator.

Problem 1

$$\sum_{k=1}^{d} \partial_{x_k}^4 u(x) = \lambda^2 (-\Delta) u(x), \ \ x \in \Omega, \tag{10}$$

$$u(x) = \partial_{\vec{n}} u(x) = 0, \ \ x \in \partial\Omega, \tag{11}$$

where $\vec{n}$ is the outward normal to $\partial\Omega$.

Let us introduce the following spaces:

Определение 1 *Let us denote by* $V_1(\Omega)$ *and* $V_2(\Omega)$ *the Hilbert spaces with the corresponding inner products*

$$(\nabla u, \nabla v)_{L^2(\Omega)}, \qquad \forall\, u, v \in \overset{\circ}{W}{}_2^1(\Omega), \tag{12}$$

$$((u,v)) \overset{def}{=} \sum_{k=1}^{d} \left(\partial_{x_k}^2 u, \partial_{x_k}^2 v\right)_{L^2(\Omega)}, \qquad \forall\, u, v \in \overset{\circ}{W}{}_2^2(\Omega), \tag{13}$$

and norms

$$\|u\|_{V_1(\Omega)} = \sqrt{\|\nabla u\|_{L^2(\Omega)}^2}, \;\; \|u\|_{V_2(\Omega)} = \sqrt{\sum_{k=1}^{d} \|\partial_{x_k}^2 u\|_{L^2(\Omega)}^2}. \tag{14}$$

It is obvious that the norms (14), induced by the inner products (12)–(13), define equivalent norms in the spaces $\overset{\circ}{W}{}_2^1(\Omega)$ and $\overset{\circ}{W}{}_2^2(\Omega)$, respectively..

Предположение 1 *In the spectral problem* (10)–(11)*, the fourth-order operator is elliptic and possesses the properties of symmetry and positive definiteness in the space* $V_2(\Omega)$*. Therefore, the eigenvalues* $\{\lambda_n^2,\ n \in \mathbb{N}\}$ *of this problem are real and located on the positive semi-axis. Moreover, the smallest eigenvalue is bounded away from zero, i.e.,* $\lambda_1 \geq \delta > 0$*.*

The following statement holds true.

Предположение 2 *The spectral problem* (10)–(11) *possesses a set of "generalized eigenfunctions"* $\{u_n(x),\ n \in \mathbb{N}\}$*, which belong to the space* $V_2(\Omega)$ *and form an orthonormal basis in the space* $V_1(\Omega)$*.*

Let us formulate the main result of this work.

Theorem 1 (Main result) *The spectral problem* (10)–(11) *has the following solution*

$$u_n(x) = X_{1n}(x_1) X_{2n}(x_2) ... X_{dn}(x_d), \quad \lambda_n^2, \quad n \in \mathbb{N}, \tag{15}$$

where $X_{1n}(x_1) = \Phi_n(y)_{|y=x_1},\ X_{2n}(x_2) = \Phi_n(y)_{|y=x_2}, ..., X_{dn}(x_d) = \Phi_n(y)_{|y=x_d}$*:*

$$\begin{cases} \Phi_{2n-1}(y) = \sin^2 \frac{\lambda_{2n-1} y}{2},\ \lambda_{2n-1}^2 = \left(\frac{2\pi n}{l}\right)^2,\ n \in \mathbb{N}, \\ \Phi_{2n}(y) = [\lambda_{2n} l - \sin \lambda_{2n} l] \sin^2 \frac{\lambda_{2n} y}{2} - \sin^2 \frac{\lambda_{2n} l}{2} \left[\lambda_{2n} y - \sin \lambda_{2n} y\right], \\ \qquad\qquad \lambda_{2n}^2 = \left(\frac{2\nu_n}{l}\right)^2,\ n \in \mathbb{N}, \end{cases} \tag{16}$$

and $\{\nu_n,\ n \in \mathbb{N}\}$ *are the positive roots of the equation* $\tan \nu = \nu,\ n \in \mathbb{N}$*.*

The arrangement of the eigenvalues on the positive semi-axis is shown in Figure 1.1 (here $l = 2$).

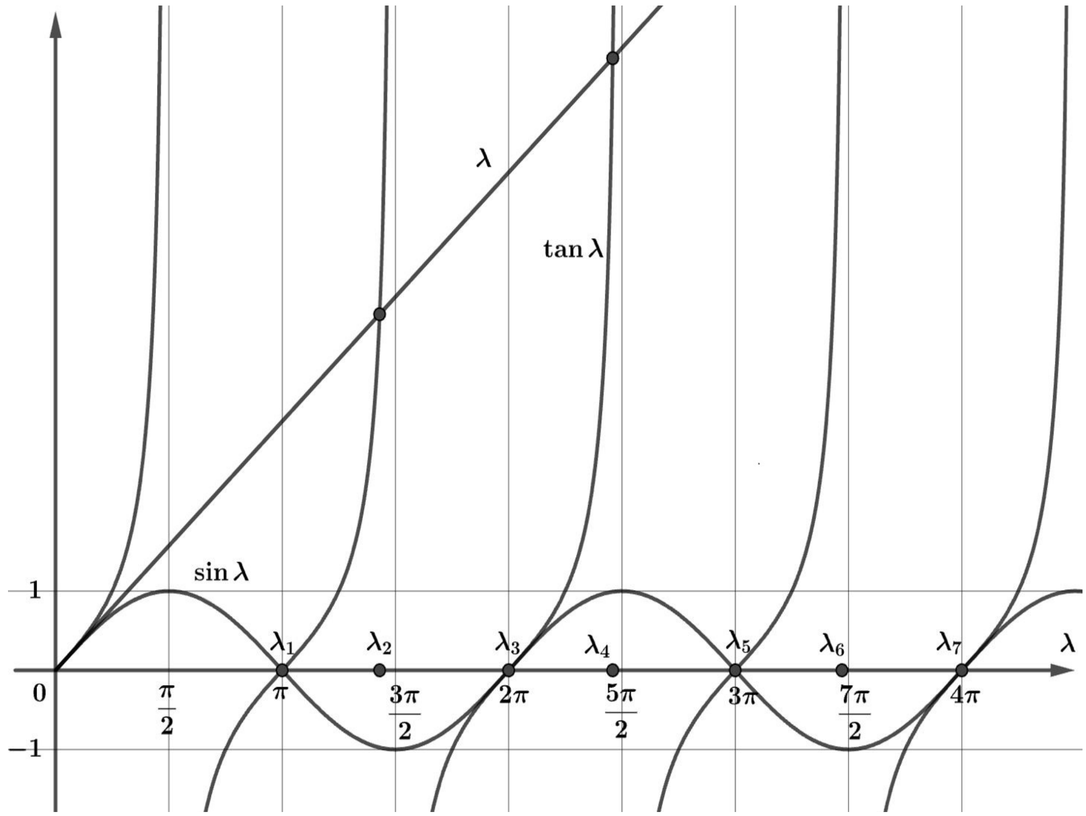

Figure 1.1. The positive roots of the equations (for $l = 2$):
$\tan\nu_n = \nu_n,\ \ \nu_n = \frac{\lambda_n l}{2} = \lambda_n;\ \ \sin\lambda_n = 0,\ \ n \in \mathbb{N}.$

From Figure 1.1 we have:

$$0 < \lambda_1 = \pi < \lambda_2 = \frac{3\pi}{2} - \varepsilon_1 < \lambda_3 = 2\pi < \lambda_4 = \frac{5\pi}{2} - \varepsilon_2$$

$$< \lambda_5 = 3\pi < \lambda_6 = \frac{7\pi}{2} - \varepsilon_3 < \lambda_7 = 4\pi < ...$$

Next, from Theorem 1, we obtain:

Следствие 1 *The eigenvalues* $\{\lambda_{2n},\ n \in \mathbb{N}\}$ *are ordered as follows:*

$$0 < \lambda_{2n} = \frac{2\nu_n}{l} < \frac{(2n+1)\pi}{2},\ \ \forall\, n \in \mathbb{N},$$

$$\lambda_{2n} = \frac{2\nu_n}{l} \to \frac{(2n+1)\pi}{2},\ \ n \to \infty,$$

where $\{\nu_n,\ n \in \mathbb{N}\}$ *are the positive roots of the equation* $\tan\nu = \nu$.

2 Proof of Theorem 1

We will use the method of separation of variables. Substituting the expression $u_n(x) = X_{1n}(x_1)X_{2n}(x_2)...X_{dn}(x_d)$ into the relations (10)–(11) for each $n \in \mathbb{N}$, we obtain:

$$\begin{cases} X_{kn}^{IV}(x_k) + \lambda_n^2 X_{kn}^{II}(x_k) - \alpha_{kn}\mu_n X_{kn}(x_k) = 0,\ \ x_k \in (0,l), \\ X_{1n}(0) = X_{1n}(l) = X_{1n}^{I}(0) = X_{1n}^{I}(l) = 0, \end{cases} \qquad (17)$$

where $k=1,..,d$, and $\{\alpha_{kn},\ k=1,...d\}$ are arbitrarily chosen numbers for each $n\in\mathbb{N}$ from the set $\{\alpha_{kn}\in\mathbb{R}^1\setminus\{0\},\ \sum\limits_{k=1}^{d}\alpha_{kn}=0\}$; moreover, $\mu_n\in\mathbb{C},\ n\in\mathbb{N}$, are (in the general case) unknown complex numbers.

Firstly, note that due to the positivity of the numbers λ_n^2 (as shown earlier in Proposition 1), the parameter μ_n can only take real values. Let us separately consider the following cases: (a) $\mu_n\neq 0$, (b) $\mu_n=0$.

(a) $\mu_n\neq 0$. The general solutions of the equations from (17) have the form

$$X_{kn}(x_k)=A_{kn}\sinh\theta_{(2k-1)n}x_k+B_{kn}\cosh\theta_{(2k-1)n}x_k+C_{kn}\sin\theta_{2kn}x_k+D_{kn}\cos\theta_{2kn}x_k, \quad (18)$$

where $\{A_{kn},B_{kn},C_{kn},D_{kn},\ k=1,...,d\}$ are constant values, and the constants $\{\theta_{kn},\ k=1,...,d\}$ must satisfy the equations:

$$2\theta_{(2k-1)n}\theta_{2kn}\left[1-\cosh\theta_{(2k-1)n}l\cdot\cos\theta_{2kn}l\right]=\left(\theta_{2kn}^2-\theta_{(2k-1)n}^2\right)\sinh\theta_{(2k-1)n}l\cdot\sin\theta_{2kn}l, \quad (19)$$

where $k=1,...,d$, and they ensure the fulfillment of the boundary conditions from (17).

In terms of the original constants λ_n^2 and $\sigma_{kn}=\alpha_{kn}\mu_n,\ k=1,...,d$, the equations (19) take the following form:

$$\pm4i\sqrt{\sigma_{kn}}\left[1-\cosh\left(l\sqrt{\frac{-\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}\right)\cdot\cos\left(l\sqrt{\frac{\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}\right)\right]$$
$$=\lambda_n^2\sinh\left(l\sqrt{\frac{-\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}\right)\cdot\sin\left(l\sqrt{\frac{\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}\right),\quad k=1,...,d, \quad (20)$$

where

$$\theta_{(2k-1)n}^2\theta_{2kn}^2=\sigma_{kn},\ \ \theta_{2kn}^2-\theta_{(2k-1)n}^2=\lambda_n^2,\ \ k=1,...,d.$$

(a1). Let $\sigma_{kn}>0$ for some fixed index k. If $\mu_n\neq 0$, then such an index k always exists! In this case, the relation (20) is equivalent to the equation:

$$\pm i4\sqrt{\sigma_{kn}}\left[1-\cosh\xi_{kn}\cos\eta_{kn}\right]=\lambda_n^2\sinh\xi_{kn}\sin\eta_{kn},\ \ \xi_{kn}\neq\eta_{kn},\ \ \xi_{kn},\ \eta_{kn}\in\mathbb{R}_+^1,$$

which cannot be satisfied, where the following notations are introduced:

$$\xi_{kn}=l\sqrt{\frac{-\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}},\ \ \eta_{kn}=l\sqrt{\frac{\lambda_n^2+\sqrt{\lambda_n^4+4\sigma_{kn}}}{2}}.$$

Thus, the remaining case is when $\mu_n=0$, i.e. $\sigma_{kn}=0,\ k=1,...,d$.

(b). Let $\mu_n=0$. In this case, the boundary value problems (17) take the following form:

$$\begin{cases} X_{kn}^{IV}(x_k)+\lambda_n^2X_{kn}^{II}(x_k)=0,\ \ x_k\in(0,l),\\ X_{kn}(0)=X_{kn}(l)=X_{kn}^{I}(0)=X_{kn}^{I}(l)=0,\end{cases}\quad k=1,...,d. \quad (21)$$

The general solutions of the equations from (21) are the following functions:

$$X_{kn}(x_k)=A_{kn}+B_{kn}x_k+C_{kn}\sin\lambda_nx_k+D_{kn}\cos\lambda_nx_k, \quad (22)$$

where the roots of the characteristic equations for (22) are respectively given by:

$$\theta_{kn1} = 0,\ \theta_{kn2} = 0,\ \theta_{kn3} = i\lambda_n,\ \theta_{kn4} = -i\lambda_n,\ k = 1, ..., d.$$

Moreover, the constant λ_n is a solution of the equation:

$$\lambda_n \left\{ 4\sin^4 \frac{\lambda_n l}{2} - [\lambda_n l - \sin\lambda_n l]\sin\lambda_n l \right\} = 0. \tag{23}$$

The equation (23) is equivalent to the following relations:

$$\lambda_n \neq 0, \quad \begin{cases} \sin\dfrac{\lambda_{2n-1} l}{2} = 0, & \lambda_{2n-1}^2 = \left(\dfrac{2\pi n}{l}\right)^2, \\ \tan\dfrac{\lambda_{2n} l}{2} = \dfrac{\lambda_{2n} l}{2}, & \lambda_{2n}^2 = \left(\dfrac{2\nu_n}{l}\right)^2, \end{cases} \quad n \in \mathbb{N}, \tag{24}$$

where $\{\nu_n,\ n \in \mathbb{N}\}$ are the positive roots of the equation $\tan\nu = \nu$.

By ensuring the fulfillment of the boundary conditions from (21) for the solutions (22) with the constants $A_{kn}, B_{kn}, C_{kn}, D_{kn},\ k = 1, ..., d$, we establish the statement of Theorem 1.

Conclusion

The paper solves the generalized spectral problem for a fourth-order differential operator in a domain Ω, which has dimension $\dim\{\Omega\} = d \geq 2$. In the future, it is assumed that the eigenfunctions of the generalized spectral problem will be used to construct a fundamental system in the space of solenoidal functions. It is worth noting that in the works [23] and [24], a solution to the spectral problem (3) for the biharmonic operator in the domain Ω, represented by a 3-D sphere, was found.

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LIMITING ERROR OF THE OPTIMAL COMPUTING UNIT FOR FUNCTIONS FROM THE CLASS $W_2^{r;\alpha}$

In the problem of optimal recovery of an infinite object (functions on a continuum, integrals of continuous functions, solutions of partial differential equations, derivative of functions,...) from finite numerical information about it, the problem of finding the limiting error of the optimal computing unit naturally arises, since the numerical information about the infinite object to be restored , as a rule, will not be accurate. In this article, the limiting error of the optimal computing unit is found in the problem of optimal recovery of periodic functions of many variables from the anisotropic Sobolev class $W_2^{r;\alpha}$ in a power-logarithmic scale in the space metric L^2. The actuality of this work is determined by the following factors: firstly, the found limiting error $\overline{\varepsilon}_N$ of the optimal computing unit preserves the exact order of the smallest recovery error , when replacing exact numerical information about a function $f \in W_2^{r;\alpha}$ with inaccurate information and is unimprovable in order; secondly, the problem of finding the limiting error of an optimal computing unit has not previously been studied in the class under consideration; thirdly, the anisotropic Sobolev class in the power-logarithmic scale is a finer scale of classification of periodic functions according to the rate of decrease of their trigonometric Fourier coefficients than the anisotropic Sobolev class in the power scale.

Key words: optimal recovery, optimal computing unit, linear functionals, exact order, anisotropic Sobolev class, trigonometric Fourier coefficients, limiting error

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$W_2^{r;\alpha}$ класы функциялары үшін оптималды есептеу агрегатының шектік қателігі

Ақырсыз объектіні (континуумда анықталған функцияны, үзіліссіз функциялар интегралдарын, дербес туындылы дифференциалдық теңдеулер шешімдерін, функция туындыларын,...) одан алынған саны ақырлы мәліметтер арқылы оптималды қалыптастыру есебінде табиғи түрде, қалыптастырылуға тиіс ақырсыз объекттен алынатын сандық мәліметтер әдетте дәл болмайтындықтан, оптималды есептеу агрегатының шектік қателігін табу есебі пайда болады. Бұл мақалада L^2 кеңістігі метрикасында дәреже – логарифмдік шкаладағы анизотропты Соболев $W_2^{r;\alpha}$ класына тиесілі көп айнымалылы периодты функцияларды оптималды қалыптастыру есебіндегі оптималды есептеу агрегатының шектік қателігі табылған. Осы жұмыстың өзектілігі оптималды есептеу агрегатының келесі факторлар арқылы қамтамасыз етіледі: біріншіден, оптималды есептеу агрегатының табылған $\overline{\varepsilon}_N$ шектік қателігі $f \in W_2^{r;\alpha}$ функциясынан алынған дәл сандық мәліметті дәл емес мәліметке ауыстырғанда да қалыптастырудың ең аз қателігінің дәл ретін сақтайды және реті бойынша жақсармайды; екіншіден, оптималды есептеу агрегатының шектік қателігін табу есебі осы күнге дейін қарастырып отырған класта зерттелмеген; үшіншіден, периодты функцияларды олардың тригонометриялық Фурье коэффициенттерінің кему жылдамдығы бойынша классификациялап сипаттауда логарифм-дәрежелік шкаладағы анизотропты Соболев класы дәрежелік шкаладағы анизотропты Соболев класымен салыстырғанда кең, әрі дәл сипаттама болып келеді.

Түйін сөздер: оптималды қалыптастыру, оптималды есептеу агрегаты, сызықтық функционалдар, дәл рет, анизотропты Соболев класы, тригонометриялық Фурье коэффициенттері, шектік қателік.

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Предельная погрешность оптимального вычислительного агрегата для функций из класса $W_2^{r;\alpha}$

В задаче оптимального восстановления бесконечного объекта (функции на континууме, интегралы от непрерывных функций, решения дифференциальных уравнений в частных производных, производной функций, ...) по конечной числовой информации о нем естественным образом возникает задача нахождения предельной погрешности оптимального вычислительного агрегата, поскольку числовая информация о подлежащем к восстановлению бесконечном объекте, как правило, не будет точной. В данной статье найдена предельная погрешность оптимального вычислительного агрегата в задаче оптимального восстановления периодических функций многих переменных из анизотропного класса Соболева $W_2^{r;\alpha}$ в степенно – логарифмической шкале в метрике пространства L^2. Актуальность настоящей работы обусловлена следующими факторами: во – первых, найденная предельная погрешность $\overline{\varepsilon}_N$ оптимального вычислительного агрегата сохраняет точный порядок наименьшей погрешности восстановления при замене точной числовой информации о функции $f \in W_2^{r;\alpha}$ на неточную и является неулучшаемой по порядку; во – вторых, ранее задача нахождения предельной погрешности оптимального вычислительного агрегата не изучалась на рассматриваемом классе; в – третьих, анизотропный класс Соболева в степенно – логарифмической шкале является более тонкой шкалой классификаций периодических функций по скорости убывания их тригонометрических коэффициентов Фурье, чем анизотропный класс Соболева в степенной шкале.

Ключевые слова: Оптимальное восстановление, оптимальный вычислительный агрегат, линейные функционалы, точный порядок, анизотропный класс Соболева, тригонометрические коэффициенты Фурье, предельная погрешность

1 Introduction

Using the notations of the articles [1] and [2], we present definitions of the computing unit, the exact order of error of the optimal recovery, the optimal computing unit and its limiting error. Let a natural number N, normalized spaces X and Y of numerical functions defined on sets Ω and Ω_1 respectively, a functional class $F \subset X$, an operator $T : F \mapsto Y$, a function

$$\varphi_N \equiv \varphi_N(z_1, \dots, z_N; y) : \mathbb{C}^N \times \Omega_1 \to \mathbb{C},$$

which for each fixed $(z_1, \dots, z_N)$ as a function of a variable y belongs to the space Y, are given. Further, the symbol $l^{(N)}$ will be used to denote a $N-$ dimensional vector $\left(l_N^{(1)}, \dots, l_N^{(N)}\right)$ with functionals $l_N^{(1)} : F \to \mathbb{C}, \dots, l_N^{(N)} : F \to \mathbb{C}$.

Definition 1. *For a given pair* $\left(l^{(N)}, \varphi_N\right)$, *a numerical function*

$$\varphi_N\left(l_N^{(1)}(f), \dots, l_N^{(N)}(f); y\right)$$

of a variable y is called a computing unit.

Every below, we will use $C(\alpha, \beta, \dots)$ to denote positive quantities that depend only on the parameters indicated in brackets. For positive sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ the notation $a_n \underset{\alpha,\beta,\dots}{\ll} b_n$ will mean the existence of some quantity $C(\alpha, \beta, \dots) > 0$ such that

$a_n \le C(\alpha,\beta,\ldots)b_n$ for all $n \in \mathbb{N}$. It should be taken into account that the values of $C(\alpha,\beta,\ldots) > 0$ in different expressions may be different. And the simultaneous fulfillment of the relations $a_n \underset{\alpha,\beta,\ldots}{\ll} b_n$ and $b_n \underset{\alpha,\beta,\ldots}{\ll} a_n$ is written as $a_n \underset{\alpha,\beta,\ldots}{\succ\prec} b_n$.

Further, for given F, Y, D_N and $T : F \mapsto Y$ we determine the quantity

$$\delta_N(D_N,T,F)_Y = \inf_{(l^{(N)},\varphi_N)\in D_N} \delta_N((l^{(N)},\varphi_N),T,F)_Y, \tag{1}$$

where D_N is a subset of the set of all pairs $(l^{(N)},\varphi_N)$,

$$\delta_N((l^{(N)},\varphi_N),T,F)_Y = \sup_{f\in F} \left\| (Tf)(\cdot) - \varphi_N(l_N^{(1)}(f),\ldots,l_N^{(N)}(f);\cdot) \right\|_Y.$$

Definition 2. *A positive sequence* $\{\psi_N\}_{N\ge 1}$ *such that*

$$\delta_N(D_N,T,F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \psi_N \tag{2}$$

is called the exact order of error of the optimal recovery of the operator $T : F \to Y$ *by computing units from* D_N *in the metric of the space* Y.

Definition 3. *A computing unit* $\left(\tilde{l}^{(N)},\widetilde{\varphi}_N\right) \equiv \widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f),\ldots,\tilde{l}_N^{(N)}(f);\cdot\right)$ *such that*

$$\delta_N(D_N,T,F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \delta_N((\tilde{l}^{(N)},\widetilde{\varphi}_N),T,F)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \psi_N \tag{3}$$

is called optimal.

Thus optimal computing unit $(\tilde{l}^{(N)},\widetilde{\varphi}_N)$ realizes the exact order ψ_N.

Here we note that in the relations (2) and (3), instead of the parameters $\alpha,\beta,\ldots$ the parameters of the class F and the space Y are taken.

Calculations for each function f from class F the value $l_N^{(1)}(f),\ldots,l_N^{(N)}(f)$ of functionals $l_N^{(1)} : F \to \mathbb{C},\ldots,l_N^{(N)} : F \to \mathbb{C}$, with rare exceptions, cannot be exact. Therefore, for the optimal computing unit $(\tilde{l}^{(N)},\widetilde{\varphi}_N)$, the problem arises of finding the error $\widetilde{\varepsilon}_N$ in calculating the values $\tilde{l}_N^{(1)}(f),\ldots,\tilde{l}_N^{(N)}(f)$ of the functionals $\tilde{l}_N^{(1)} : F \to \mathbb{C},\ldots,\tilde{l}_N^{(N)} : F \to \mathbb{C}$, which preserves the optimality of $(\tilde{l}^{(N)},\widetilde{\varphi}_N)$ and is the limiting in order. In [1] the error $\widetilde{\varepsilon}_N$ was called the limiting error of the optimal computing unit $(\tilde{l}^{(N)},\widetilde{\varphi}_N)$. Now we present definition of $\widetilde{\varepsilon}_N$, formulated in [2].

Definition 4. *A sequence* $\widetilde{\varepsilon}_N > 0$ *is called the limiting error of an optimal computing unit* $\left(\tilde{l}^{(N)},\widetilde{\varphi}_N\right)$, *if*

$$\triangle_N\left(\widetilde{\varepsilon}_N,(\tilde{l}^{(N)},\widetilde{\varphi}_N),T,F\right)_Y \underset{\alpha,\beta,\ldots}{\succ\prec} \delta_N(D_N,T,F)_Y \quad \textit{and} \tag{4}$$

$$\overline{\lim_{N\to\infty}} \frac{\triangle_N(\eta_N\widetilde{\varepsilon}_N,(\tilde{l}^{(N)},\widetilde{\varphi}_N),T,F)_Y}{\delta_N(D_N,T,F)_Y} = +\infty \tag{5}$$

for any positive sequence $\{\eta_N\}_{N\geq 1}$ *increasing arbitrarily slowly to* $+\infty$, *where*

$$\triangle_N(\varepsilon_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), T, F)_Y =$$

$$= \sup_{f\in F}\sup_{z_1,\ldots,z_N}\left\{\left\|(Tf)(\cdot)-\widetilde{\varphi}_N(z_1,\ldots,z_N;\cdot)\right\|_Y : \left|z_i-\tilde{l}_N^{(i)}(f)\right|\leq\varepsilon_N, i=1,\ldots,N\right\}\equiv$$

$$\equiv\sup_{f\in F}\sup_{|\gamma_N^{(1)}|\leq 1,\ldots,|\gamma_N^{(N)}|\leq 1}\left\|(Tf)(\cdot)-\widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f)+\gamma_N^{(1)}\varepsilon_N,\ldots,\tilde{l}_N^{(N)}(f)+\gamma_N^{(N)}\varepsilon_N;\cdot\right)\right\|_Y$$

for any positive sequence ε_N.

The relation (4) means that when calculating the values of the optimal computing unit $\widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f),\ldots,\tilde{l}_N^{(N)}(f);\cdot\right)$ each number $\tilde{l}_N^{(\tau)}(f)(\tau=1,\ldots,N=N(K))$ can be replaced with error $\widetilde{\varepsilon}_N$ by a number z_τ such that $|z_\tau-\tilde{l}_N^{(\tau)}(f)|\leq\varepsilon_N(\tau=1,\ldots,N=N(K))$, preserving the exact order of error of the optimal recovery.

According to equality (5), we can state that the error $\widetilde{\varepsilon}_N$ is of limiting error, because an arbitrarily slow infinite increase in the value of $\widetilde{\varepsilon}_N$ (i.e., replacement of $\widetilde{\varepsilon}_N$ by $\eta_N\widetilde{\varepsilon}_N$) violates the exact order of error of optimal recovery.

Many mathematicians have been and continue to be concerned with the problems of establishing the relation (1) and constructing optimal computing units for various F, Y, D_N and $T: F\mapsto Y$ (see, for example, [3-6] and the bibliography therein). The problem of finding limiting errors is a relatively new problem in approximation theory, computational mathematics and numerical analysis. Results on this problem can be found in the works [1], [2], [7] and [8]. In this article, when

$$Tf=f, F=W_2^{r;\alpha}[0,1]^s, Y=L^2[0,1]^s, D_N=L_N,$$

where $W_2^{r;\alpha}\equiv W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ is the anisotropic Sobolev class on a power –logarithmic scale (the definition of the class is given below), L_N is the set of computing units $(l^{(N)},\varphi_N)$ with linear functionals $l_N^{(1)}: W_2^{r;\alpha}\to\mathbb{C},\ldots,l_N^{(N)}: W_2^{r;\alpha}\to\mathbb{C}$, the limiting error of the optimal computing unit $\left(\bar{l}^{(N)},\overline{\varphi}_N\right)$ from [5] is found.

The importance of studying this work lies in the following: firstly, the anisotropic Sobolev class $W_2^{r;\alpha}\equiv W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ in the power-logarithmic scale is a finer scale of classifications of periodic functions in terms of the rate of decrease of their trigonometric Fourier coefficients than the usual anisotropic Sobolev class $W_2^{r_1,\ldots,r_s}[0,1]^s$ in the power scale; secondly, the recovery of functions from the class $W_2^{r;\alpha}[0,1]^s$ is carried out by computing units from a fairly wide set containing all partial sums of Fourier series over all possible orthonormal systems, all possible finite convolutions $\sum\limits_{i=1}^{N}f(\xi_i)K_N(x-\xi_i)$ with special kernels K_N, and all finite sums of approximation used in orthowidths, linear widths, and greedy algorithms; thirdly, previously, the problem of finding the limiting error of the optimal computing unit was not considered on a multidimensional functional class $W_2^{r;\alpha}$; fourthly, in the problem of finding the limiting error of the optimal computing unit on optimal recovery of functions from the class $W_2^{r;\alpha}$, unlike the classes Sobolev SW_2^r with a dominant mixed derivative, Korobov E_s^r and Sobolev W_p^r, and the exact order and limiting error does not depend on the number variable functions $f(x)=f(x_1,\ldots,x_s)$ (see, for example, [9] and [10]).

2 Main result

First, let's agree on the notation used. Everywhere below, the symbols $[a]$ and $|E|$ will denote the integer part of the number a and the amount elements of the finite set E. For each vector $r = (r_1, \ldots, r_s)$ with positive components, we assume $\lambda = 1/(1/r_1 + \ldots + 1/r_s)$. Instead of symbols

$$\|\cdot\|_{L^2}, \underset{s,r_1,\ldots,r_s,\alpha_1,\ldots,\alpha_s}{\ll}, \underset{s,r_1,\ldots,r_s,\alpha_1,\ldots,\alpha_s}{\gg} \quad \text{and} \quad \underset{s,r_1,\ldots,r_s,\alpha_1,\ldots,\alpha_s}{\succ\prec}$$

we will use the symbols $\|\cdot\|_2, \ll, \gg$ and $\succ\prec$ respectively. The symbol $\square$ will mean the end of the proofs.

Now we give a definition of the anisotropic Sobolev class $W_2^{r;\alpha}$ on a power – logarithmic scale. Let an integer number $s \geq 2$, vectors r $= (r_1, \ldots, r_s)$ and $\alpha = (\alpha_1, \ldots, \alpha_s)$ be given such that $r_i > 0$ and $\alpha_i \in \mathbb{R}$ for each $i = 2, 3, \ldots, s$. The class $W_2^{r;\alpha} \equiv W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ consist of all functions $f(x) = f(x_1, \ldots, x_s)$ that are summable on $[0,1]^s$ and 1 – periodic on each variable and whose trigonometric Fourier – Lebesgue coefficients $\hat{f}(m) = \int\limits_{[0,1]^s} f(x)e^{-2\pi i(m,x)}dx, m \in Z^s$ satisfy the condition

$$\sum_{m \in Z^s} |\hat{f}(m)|^2 (\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1 + 1) + \ldots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s + 1)) \leq 1,$$

where $\overline{m}_j = \max\{1; |m_j|\}$ for each $j = 1, \ldots, s$.

The main result of this article is the following

Theorem. *Let an integer number $s \geq 2$, vectors* r $= (r_1, \ldots, r_s), r_1 > 0, \ldots, r_s > 0$ *and* $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathrm{R}^s$ *be given such that* $r_i + \alpha_i > 0$ *for each* $i = 2, 3, \ldots, s$, *and let the inequality*

$$\left(\frac{1}{\min\{r_1, r_1 + \alpha_1\}} + \ldots + \frac{1}{\min\{r_s, r_s + \alpha_s\}}\right)^{-1} > \frac{1}{2} \tag{6}$$

hold. Then the quantity

$$\overline{\varepsilon}_N = \frac{1}{N^{\lambda + 1/2}(\ln N)^{\lambda(\alpha_1/r_1 + \ldots + \alpha_s/r_s)}}$$

is limiting error of the optimal computing unit

$$\left(\overline{l}^{(N)}, \overline{\varphi}_N\right) \equiv \overline{\varphi}_N\left(\overline{l}_N^{(1)}(f), \ldots, \overline{l}_N^{(N)}(f); x\right) = \sum_{\tau=1}^{N} \hat{f}(\overline{m}^{(\tau)}) e^{2\pi i(\overline{m}^{(\tau)}, x)},$$

where $N \equiv N(K) = \prod\limits_{i=1}^{s}(2N_i + 1)$,
$N_i \equiv N_i(K) = [K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1 + \ldots + \alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}], K \geq 2$ *for each* $i \in \{1, \ldots, s\}, \{\overline{m}^{(1)}, \overline{m}^{(2)}, \ldots, \overline{m}^{(N)}\}$ *is some ordering of the set* $A_K = \{m \in Z^s : |m_1| \leq N_1, \ldots, |m_s| \leq N_s\}$, *in the problem of optimal recovery of functions from the class* $W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ *in the metric of the space* $L^2[0,1]^s$.

In the case $\alpha_1 = \alpha_2 = \ldots = \alpha_s = 0$ from this theorem we obtain the following statement.

Corollary. *Let an integer number $s \geq 2$, vector $\mathrm{r} = (r_1, \ldots, r_s)$ be given such that $r_i > 0$ for each $i = 2, 3, \ldots, s$, and let the inequality $\left(\frac{1}{r_1} + \ldots + \frac{1}{r_s}\right)^{-1} > \frac{1}{2}$ hold. Then the quantity $\overline{\varepsilon}_N = \frac{1}{N^{\lambda+1/2}}$ is limiting error of the optimal computing unit*

$$\left(\bar{l}^{(N)}, \overline{\varphi}_N\right) \equiv \overline{\varphi}_N\left(\bar{l}_N^{(1)}(f), \ldots, \bar{l}_N^{(N)}(f); x\right) = \sum_{\tau=1}^{N} \hat{f}(\overline{m}^{(\tau)}) e^{2\pi i(\overline{m}^{(\tau)}, x)},$$

where $N \equiv N(K) = \prod_{i=1}^{s}(2N_i + 1), N_i \equiv N_i(K) = [K^{\lambda/r_i}], K \geq 2$ for each $i \in \{1, \ldots, s\}, \{\overline{m}^{(1)}, \overline{m}^{(2)}, \ldots, \overline{m}^{(N)}\}$ is some ordering of the set $A_K = \{m \in Z^s : |m_1| \leq N_1, \ldots, |m_s| \leq N_s\}$, in the problem of optimal recovery of functions from the class $W_2^{r_1,\ldots,r_s}[0,1]^s$ in the metric of the space $L^2[0,1]^s$.

3 Auxiliary statements

Lemma 1. *Let sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be given such that $\lim\limits_{n\to\infty} x_n = +\infty$ and $\lim\limits_{n\to\infty} y_n = +\infty$. Then for the sequence $z_n = \min\{x_n, y_n\}$ the equality $\lim\limits_{n\to\infty} z_n = +\infty$ holds.*

Proof. According to the equalities $\lim\limits_{n\to\infty} x_n = +\infty$ and $\lim\limits_{n\to\infty} y_n = +\infty$, for any positive number $\varepsilon > 0$ there is a number N_ε such that for all natural numbers $n \geq N_\varepsilon$ the inequalities $x_n > \varepsilon$ and $y_n > \varepsilon$ are satisfied. From these inequalities follows the inequality $\min\{x_n, y_n\} > \varepsilon$, which is true for each $n \geq N_\varepsilon$. Therefore, $\lim\limits_{n\to\infty} z_n = +\infty$. $\square$

Lemma 2. *For each $\gamma \in \mathrm{R}$ there exists a quantity $C_1(\gamma) \geq 2$ such that for all integers $K \geq C_1(\gamma)$ the relation*

$$\ln(K \ln^\gamma K) \underset{\gamma}{\succ\prec} \ln K \tag{7}$$

holds.

Proof. In case $\gamma \geq 0$ for all integers $K \geq 2$ the inequalities

$$K \leq K \ln^\gamma K \leq K^{\gamma+1} \Leftrightarrow$$
$$\Leftrightarrow \ln K \leq \ln(K \ln^\gamma K) \leq (\gamma + 1) \ln K \tag{8}$$

are satisfied.

Let $\gamma < 0$. Since $\lim\limits_{K\to\infty} \sqrt{K} \ln^\gamma K = +\infty$, there exists a number $K^{(0)} = K^{(0)}(\gamma)$ such that for all integers $K \geq K^{(0)}$ the inequalities

$$\frac{1}{2} \ln K \leq \ln(K \ln^\gamma K) \leq \ln K \tag{9}$$

hold. Therefore, taking $C_1(\gamma) = \begin{cases} 2, & \text{if} \quad \gamma \geq 0; \\ K^{(0)} + 1, & \text{if} \quad \gamma < 0, \end{cases}$ by virtue of the inequalities (8) and (9) for all integers $K \geq C_1(\gamma)$ we obtain (7). $\square$

Lemma 3. *For any function $f \in L^2$ the inequality*

$$\max\left\{\|f(\cdot) - \varphi_N(0, \ldots, 0; \cdot)\|_2, \|(-f)(\cdot) - \varphi_N(0, \ldots, 0; \cdot)\|_2\right\} \geq \|f\|_2$$

is satisfied.

Proof. Let us introduce the following notations:

$$a = \|f(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2 \quad \text{and} \quad b = \|(-f)(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2.$$

Then, according to the inequalities $\max\{a,b\} \geq (a+b)/2$ and $\|x\|_2 + \|y\|_2 \geq \|x-y\|_2$, we have

$$\max\Big\{\|f(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2, \|(-f)(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2\Big\}$$
$$\geq \frac{\|f(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2 + \|(-f)(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|}{2} \geq$$
$$\geq \frac{(\|f(\cdot) - \varphi_N(0,\ldots,0;\cdot)) - ((-f)(\cdot) - \varphi_N(0,\ldots,0;\cdot)\|_2}{2} \geq \|f\|_2. \square$$

4 Proof of the main result

We will begin the proof by checking the validity of the relations

$$\triangle_N(\overline{\varepsilon}_N, (\overline{l}^{(N)}, \overline{\varphi}_N), Tf = f, W_2^{r;\alpha})_{L^2} \succ\prec \delta_N(L_N, Tf = f, W_2^{r;\alpha})_{L^2}. \tag{10}$$

For arbitrarily given numbers $\gamma_N^{(\tau)}$ such that $|\gamma_N^{(\tau)}| \leq 1 (\tau = 1,\ldots,N)$ there is an inequality

$$\Big\|f(\cdot) - \overline{\varphi}_N(\overline{l}_N^{(1)}(f) + \gamma_N^{(1)}\overline{\varepsilon}_N, \ldots, \overline{l}_N^{(N)}(f) + \gamma_N^{(N)}\overline{\varepsilon}_N;\cdot)\Big\|_2 \leq$$
$$\leq \Big\|f(\cdot) - \overline{\varphi}_N(\overline{l}_N^{(1)}(f), \ldots, \overline{l}_N^{(N)}(f);\cdot)\Big\|_2 + \Big\|\sum_{\tau=1}^{N}(-\gamma_N^{(\tau)})\overline{\varepsilon}_N e^{2\pi i(\overline{m}^{(\tau)},\cdot)}\Big\|_2. \tag{11}$$

According to the theorem from [5], the relations

$$\delta_N(L_N, Tf = f, W_{L^2}^{r;\alpha})_{L^2} \succ\prec \delta_N\left((\overline{l}^{(N)}, \overline{\varphi}_N), Tf = f, W_2^{r;\alpha}\right)_{L^2} \succ\prec$$
$$\succ\prec \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \ldots + \alpha_s/r_s)}} \tag{12}$$

are valid. Using the Parseval equality, we find

$$\Big\|\sum_{\tau=1}^{N}(-\gamma_N^{(\tau)})\overline{\varepsilon}_N e^{2\pi i(\overline{m}^{(\tau)},\cdot)}\Big\|_2 \ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \ldots + \alpha_s/r_s)}}.$$

Further, due to inequalities (11) and (12), we obtain

$$\Big\|f(\cdot) - \overline{\varphi}_N\left(\overline{l}_N^{(1)}(f) + \gamma_N^{(1)}\overline{\varepsilon}_N, \ldots, \overline{l}_N^{(N)}(f) + \gamma_N^{(N)}\overline{\varepsilon}_N;\cdot\right)\Big\|_2 \ll$$
$$\ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \ldots + \alpha_s/r_s)}}.$$

From where, by virtue of the arbitrariness of the numbers $\gamma_N^{(\tau)} (\tau = 1,\ldots,N)$ and the function f , we have

$$\triangle_N \left(\overline{\varepsilon}_N, (\overline{l}^{(N)}, \overline{\varphi}_N), Tf=f, W_2^{r;\alpha}\right)_{L^2} \ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)}}. \qquad (13)$$

Since

$$\delta_N(L_N, Tf=f, W_2^{r;\alpha})_{L^2} \le \delta_N\left((\overline{l}^{(N)}, \overline{\varphi}_N), Tf=f, W_2^{r;\alpha}\right)_{L^2} \le$$

$$\le \triangle_N \left(\overline{\varepsilon}_N, (\overline{l}^{(N)}, \overline{\varphi}_N), Tf=f, W_2^{r;\alpha}\right)_{L^2},$$

then taking into account (12) and (13) we obtain (10).

Let the set of pairs $\left(l^{(N)}, \varphi_N\right)$ with functionals

$$l_N^{(1)}(f) = \hat{f}(m^{(1)}), \ldots, l_N^{(N)}(f) = \hat{f}(m^{(N)})$$

be denoted by Φ_N. Now let us verify that for all $(l^{(N)}, \varphi_N) \in \Phi_N$ and any arbitrarily slowly increasing to $+\infty$ sequence $\{\eta_{N(K)}\}_{K\ge1}$ the equality

$$\overline{\lim_{N\to\infty}} \frac{\triangle_N(\eta_N \overline{\varepsilon}_N, (\overline{l}^{(N)}, \overline{\varphi}_N), Tf=f, W_2^{r;\alpha})_{L^2}}{\delta_N(L_N, Tf=f, W_2^{r;\alpha})_{L^2}} = +\infty. \qquad (14)$$

holds. Next, for each integer $K > C(r,\alpha,s)$ we define the set

$$H_K^* = \{m \in Z^s : [M_1^*] \le |m_1| \le 2\cdot[M_1^*], \ldots, [M_s^*] \le |m_s| \le 2\cdot[M_s^*]\},$$

where $M_i^* = N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i}(\ln N)^{-\alpha_i/r_i}\beta_K^{-1/r_i}$ for all $i \in \{1,2,\ldots,s\}, N = N(K), \beta_K = \min\{\eta_N, \ln N\}$.

Since $\lim\limits_{K\to+\infty} \beta_K = +\infty$ (see Lemma 1), there exists a number $K_0 \ge C(r,\alpha,s)$ such that for all integers $K \ge K_0$ the inequality $\beta_K \ge 1$ holds.

Now for any component $m_i (i = 1,2,\ldots,s)$ of the vector $m \in H_K^*$ we will prove the inequality

$$\ln^{\alpha_i}(2\overline{m}_i) \ll \ln^{\alpha_i} N, \alpha_i \in \mathbb{R}. \qquad (15)$$

If $\alpha_i \ge 0$, then by virtue of inequalities $(\ln N)^{-\alpha_i/r_i} \le 1$ and $\beta_K^{-1/r_i} \le 1$ and Lemma 2, we have

$$\ln^{\alpha_i}(2\overline{m}_i) \ll \ln^{\alpha_i}\left(2N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i}(\ln N)^{-\alpha_i/r_i}\beta_K^{-1/r_i}\right) \ll$$

$$\ll \ln^{\alpha_i}\left(2N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i}\right) \ll \ln^{\alpha_i} N.$$

Comparing the beginning and the end of this chain of inequalities, we obtain (15). Let $\alpha_i < 0$. Since $\beta_K = \min\{\eta_N, \ln N\}$, then $\beta_K^{-1/r_i} \ge \ln^{-1/r_i} N$.

Therefore,

$$2\overline{m}_i \gg 2N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i}(\ln N)^{-\alpha_i/r_i}\beta_K^{-1/r_i} \gg$$

$$\gg N^{\lambda/r_i}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)/r_i-\alpha_i/r_i-1/r_i},$$

whence by virtue of Lemma 2 and the inequality $\alpha_i < 0$, we again obtain (15).

From (15) follows the inequality

$$\overline{m}_i^{r_i}\ln^{\alpha_i}(2\overline{m}_i) \ll N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)}\beta_K^{-1}. \quad (16)$$

Consider the function $h_K(x) = \beta_K\overline{\varepsilon}_N \sum\limits_{m\in H_K^*} e^{2\pi i(m,x)}$. Then, using the relation

$$|H_K^*| \succ\prec N\cdot\beta_K^{-1/\lambda} \quad (17)$$

and the inequalities (16) and $\beta_K \geq 1 (K\geq K_0)$, we have

$$\sum_{m\in Z^s}|\hat{h}_K(m)|^2\left(\overline{m}_1^{2r_1}\ln^{2\alpha_1}(\overline{m}_1+1)+\ldots+\overline{m}_s^{2r_s}\ln^{2\alpha_s}(\overline{m}_s+1)\right) =$$

$$=\sum_{m\in H_K^*}|\hat{h}_k(m)|^2\left(\overline{m}_1^{2r_1}\ln^{2\alpha_1}(\overline{m}_1+1)+\ldots+\overline{m}_s^{2r_s}\ln^{2\alpha_s}(\overline{m}_s+1)\right) \ll$$

$$\ll \sum_{m\in H_K^*}|\hat{h}_K(m)|^2\left(\overline{m}_1^{2r_1}\ln^{2\alpha_1}(2\overline{m}_1)+\ldots+\overline{m}_s^{2r_s}\ln^{2\alpha_s}(2\overline{m}_s)\right) \ll$$

$$\ll \beta_K^2\overline{\varepsilon}_N^2\sum_{m\in H_K^*}N^{2\lambda}(\ln N)^{2\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)}\beta_K^{-2} \ll$$

$$\ll \frac{1}{N}\sum_{m\in H_K^*}1 \ll \frac{1}{\beta_K^{1/\lambda}} \ll 1.$$

Therefore, for some $C(r,\alpha,s)>0$ the function $t_K(x) = C(r,\alpha,s)\cdot h_K(x)$ belongs to the class $W_2^{r;\alpha}$. By virtue of Parseval's equality and relation (17) we have

$$\|t_K\|_2 \gg \frac{\beta_K^{1-1/(2\lambda)}}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\ldots+\alpha_s/r_s)}}. \quad (18)$$

Further, having fixed the values of $K\geq K_0$ for any $\tau = 1,\ldots,N\equiv N(K)$ we put

$$\tilde{\gamma}_N^{(\tau)} = -\frac{\hat{t}_K(m^{(\tau)})}{\overline{\varepsilon}_N\eta_N} \quad \text{and} \quad \widetilde{\omega}_N^{(\tau)} = -\frac{(-\hat{t}_K)(m^{(\tau)})}{\overline{\varepsilon}_N\eta_N}.$$

Since for each $\tau = 1,\ldots,N$ inequalities $|\tilde{\gamma}_N^{(\tau)}|\leq 1, |\widetilde{\omega}_N^{(\tau)}|\leq 1$ and equalities

$$\hat{t}_K(m^{(\tau)})+\eta_N\tilde{\gamma}_N^{(\tau)}\overline{\varepsilon}_N = 0, (-\hat{t}_K)(m^{(\tau)})+\eta_N\widetilde{\omega}_N^{(\tau)}\overline{\varepsilon}_N = 0$$

are true, then by virtue of Lemma 3 for each pair $(l^{(N)},\varphi_N)\in\Phi_N$ we have

$$\sup_{f\in W_2^{r;\alpha}}\ \sup_{|\gamma_N^{(1)}|\leq 1,\ldots,|\gamma_N^{(N)}|\leq 1}\Big\|f(\cdot)-\varphi_N\Big(\hat{f}(m^{(1)})+\gamma_N^{(1)}\eta_N\overline{\varepsilon}_N,\ldots,$$

$$\hat{f}(m^{(N)})+\gamma_N^{(N)}\eta_N\overline{\varepsilon}_N;\cdot\Big)\Big\|_2 \geq$$

$$\geq \max\Big\{\Big\|t_K(\cdot)-\varphi_N\Big(\hat{t}_K(m^{(1)})+\tilde{\gamma}_N^{(1)}\eta_N\overline{\varepsilon}_N,\ldots,\hat{t}_K(m^{(N)})+\tilde{\gamma}_N\eta_N\overline{\varepsilon}_N;\cdot\Big)\Big\|_2,$$

$$\left\|(-t_K)(\cdot)-\varphi_N\Big((-\hat{t}_K)(m^{(1)})+\widetilde{\omega}_N^{(1)}\eta_N\overline{\varepsilon}_N,\ldots,(-\hat{t}_K)(m^{(N)})+\widetilde{\omega}_N^{(N)}\eta_N\overline{\varepsilon}_N;\cdot\Big)\right\|_2\Big\}=$$

$$=\max\left\{\Big\|t_K(\cdot)-\varphi_N(0,\ldots,0;\cdot)\Big\|_2,\Big\|(-t_K)(\cdot)-\varphi_N(0,\ldots,0;\cdot)\Big\|_2\right\}\geq\|t_K\|_2.$$

Next, using inequality (18), we find

$$\triangle_N\left(\eta_N\overline{\varepsilon}_N,(l^{(N)},\varphi_N),Tf=f,W_2^{r;\alpha}\right)_{L^2}\gg\delta_N\left(L_N,Tf=f,W_2^{r;\alpha}\right)_{L^2}\beta_K^{1-1/(2\lambda)}. \tag{19}$$

Since

$$\left(\frac{1}{\min\{r_1,r_1+\alpha_1\}}+\ldots+\frac{1}{\min\{r_s,r_s+\alpha_s\}}\right)^{-1}>\frac{1}{2}$$

(see condition (6)), then $2\lambda>1$. Therefore, in view of the equality $\lim\limits_{K\to+\infty}\beta_K=+\infty$ and the inequalitiy (19) for each pair $(l^{(N)},\varphi_N)\in\Phi_N$ and any positive sequence $\{\eta_{N(K)}\}_{K\geq1}$ that increases arbitrarily slowly to $+\infty$, the inequality

$$\lim_{N\to\infty}\frac{\triangle_N(\eta_N\overline{\varepsilon}_N,(l^{(N)},\varphi_N),Tf=f,W_2^{r;\alpha})_{L^2}}{\delta_N(L_N,Tf=f,W_2^{r;\alpha})_{L^2}}=+\infty \tag{20}$$

holds. It is clear that $(\overline{l}^{(N)},\overline{\varphi}_N)\in\Phi_N$. Consequently, from (20) follows (14). □

Remark. *Since the equality (20) is proved for all pairs $(l^{(N)},\varphi_N)\in\Phi_N$, then any optimal computing unit $\varphi_N\left(\hat{f}(m^{(1)}),\ldots,\hat{f}(m^{(N)});\cdot\right),N=N(K)$ does not have a greater (in order) limiting error than $\overline{\varepsilon}_N$.*

5 Conclusion

The theorem proved above is a new result in approximation theory, numerical analysis, and computational mathematics. This study can be continued by replacing condition (6) with a weaker condition that ensures absolute convergence of the trigonometric Fourier series $\sum\limits_{m\in Z^s}\hat{f}(m)e^{2\pi i(m,x)}$ functions $f\in W_2^{r;\alpha}$.

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ON q-DEFORMATED HÖRMANDER MULTIPLIER THEOREM

Abstract. The main purposes of this work, we introduce the q-deformed Fourier multiplier A_g defined on the space $L^2_q(\mathbb{R}_q)$ through the framework of the q^2-Fourier transform, while also extending the functional setting of $L^p_q(\mathbb{R}_q)$ with $1 \leq p < \infty$. Our approach provides a natural extension of classical Fourier multiplier theory into the q-deformed setting, which is relevant in the context of quantum groups and noncommutative analysis. Furthermore, we establish several key q-analogues of classical harmonic analysis inequalities for the q^2-Fourier transform, including the Paley inequality, Hausdorff-Young inequality, Hausdorff-Young-Paley inequality, and Hardy-Littlewood inequality. These results not only generalize their classical counterparts but also open new avenues for analysis on q-deformed spaces. As a significant application, we prove a q-deformed version of the Hörmander multiplier theorem, which provides sufficient conditions for the boundedness of multipliers in the q-deformed setting. This work sets the stage for further developments in the field of q-deformed harmonic analysis.

Key words: q-Jackson integral, q-caclulus, Fourier multiplier, inequality, multiplier, Hausdorff-Young inequality.

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q-деформацияланған Хёрмандердің мультипликаторлар теоремасы туралы

Аннотация. Бұл жұмыстың негізгі мақсаттары: біз $L^2_q(\mathbb{R}_q)$ кеңістігінде анықталған q^2-Фурье түрлендіруі шеңберінде A_g q-деформацияланған Фурье көбейткішін енгіземіз және $1 \leq p < \infty$ үшін функционалды параметрді $L^p_q(\mathbb{R}_q)$ кеңістіктеріне кеңейтеміз. Біздің көзқарасымыз, әрине, Фурье көбейткіштерінің классикалық теориясын q-деформацияланған параметрге дейін кеңейтеді, бұл кванттық топтар мен коммутативті емес талдау контекстінде өте маңызды. Содан кейін біз q^2-Фурье түрлендіруі үшін гармоникалық талдаудың классикалық теңсіздіктерінің бірқатар негізгі q-аналогтарын белгілейміз, соның ішінде Палей, Хаусдорф–Янг, Хаусдорф–Янг–Пэйли және Харди-Литлвуд теңсіздіктері. Алынған нәтижелер олардың классикалық прототиптерін қорытып қана қоймай, q-деформацияланған кеңістіктерге талдаудың жаңа бағыттарын ашады. Маңызды қолданба ретінде біз көбейткіштер туралы Хёрмандер теоремасының q-деформацияланған нұсқасын дәлелдейміз, ол q-деформацияланған параметрде көбейткіштердің шектелгендігі үшін жеткілікті шарттарды береді. Бұл жұмыс q-деформацияланған гармоникалық талдауды одан әрі дамытуға негіз қалайды.
Түйін сөздер: q-Джексон интегралы, q-есептеу, Фурье көбейткіші, теңсіздік, мультипликатор, Хаусдорф–Янг теңсіздігі.

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О q-деформированной теореме Хёрмандера о мультипликаторах

Аннотация. Основные цели данной работы состоят в следующем: мы вводим q-деформированный мультипликатор Фурье A_g, определённый на пространстве $L^2_q(\mathbb{R}_q)$ в рамках q^2-преобразования Фурье, а также расширяем функциональную постановку до пространств $L^p_q(\mathbb{R}_q)$ при $1 \leq p < \infty$. Наш подход естественным образом продолжает классическую теорию фурье-мультипликаторов в q-деформированную постановку, что существенно в контексте квантовых групп и некоммутативного анализа. Далее мы устанавливаем ряд ключевых q-аналогов классических неравенств гармонического анализа для q^2-преобразования Фурье, включая неравенства Пэли, Хаусдорфа–Янга, Хаусдорфа–Янга–Пэли и Харди–Литтлвуда. Полученные результаты не только обобщают их классические прототипы, но и открывают новые направления анализа на q-деформированных пространствах. В качестве существенного приложения мы доказываем q-деформированный вариант теоремы Хёрмандера о мультипликаторах, дающий достаточные условия ограниченности мультипликаторов в q-деформированной постановке. Эта работа закладывает основу для дальнейшего развития q-деформированного гармонического анализа.
Ключевые слова: q-интеграл Джексона, q-исчисление, мультипликатор Фурье, неравенство, мультипликатор, неравенство Хаусдорфа–Янга.

1 Introduction

The history of quantum calculus (or q-deformation) started in the 18th century when L. Euler [9] investigated the infinite product in the following form:

$$(q;q)_\infty^{-1} = \prod_{k=0}^{\infty} \frac{1}{1-q^{k+1}}, \qquad |q| < 1.$$

It serves as a generating function for the partition function $p(n)$, which enumerates the number of distinct ways to express n as a sum of positive integers. In the early 20th century, F.H. Jackson introduced the q-derivative and the definite q-integral [6, 7], forming the basis of modern q-calculus. Over the past two decades, research on q-deformation has expanded rapidly. For instance, V. Kac and P. Cheung [8] studied its fundamental properties, while T. Ernst [10, 11] highlighted its importance in quantum computing models. Further developments include the work of N. Bettaibi and R.H. Bettaieb [4], who introduced a q-deformed Dunkl operator and analyzed its Fourier transform in [13, 14] (see also [16]). This operator is defined using Rubins q-differential operator ∂_q [17, 18]. For more details on the history and recent progress in q-calculus, see the monographs [1, 10–12, 15].

The q-difference calculus dates back to the early 20th century, with pioneering contributions by F. Jackson [6, 7] and R.D. Carmichael [5]. More recently, W. Al-Salam [3] and R.P. Agarwal [2] introduced the concept of fractional q-difference calculus. In recent years, fueled by the rapid growth of research in the q-partial dif equation, this theory has also undergone significant development (see, [25–27, 29–31]).

In this work, we establish some basic q-deformed integral inequalities for q^2-Fourier transform such as the Paley, Hausdorff-Young, Hausdorff-Young-Paley, and Hardy-Littlewood inequalities. The problem under consideration can be reformulated as proving the boundedness of an associated Fourier multiplier via an appropriate transformation. In this context, the *Hörmander multiplier theorem* is a fundamental result in Fourier analysis that provides conditions ensuring the boundedness of Fourier multiplier operators on L^p spaces. Specifically, it characterizes the regularity requirements for a multiplier function so that the associated operator, defined by multiplication in the Fourier domain, acts boundedly on

$L^p(\mathbb{R}^d$. Let σ be a function on $\mathbb{R}^d$, and define the Fourier multiplier operator A_σ by

$$A_\sigma f(x) = \mathcal{F}^{-1}\big[\sigma \cdot \hat{f}\big](x),$$

where $\mathcal{F}$ denotes the classical Fourier transform.

The theorem states that A_σ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \leq 2 \leq q < \infty$. if σ satisfies a condition, often expressed as

$$\sup_{\lambda>0} \lambda \Big(\int_{|\sigma(s)|\geq\lambda} d_q s \Big)^{\frac{1}{p}-\frac{1}{q}}.$$

This statement generalizes earlier results by Mikhlin and provides a powerful framework for analyzing multipliers. It has important applications in partial differential equations, signal processing, and control theory, among others. Comprehensive discussions of these results and their further developments are available in the works of L. Hörmander [23], E.M. Stein [32], as well as in more recent texts like L. Grafakos [24]. Our formulation of q-deformed Fourier multiplier is more intuitive and aligns closely with the classical, commutative framework, which allows many of the same properties to carry over. Similar to the classical case, the key part of the proof depends on the Paley inequality and the Hausdorff–Young–Paley inequality for the classical Fourier transform, both of which are obtained through the Hausdorff–Young inequality. In the course of our work, we also derive q-analogue of several important inequalities such as the Paley, Hausdorff–Young–Paley, Hardy–Littlewood. Moreovere, we present a simple proof of the $L^p - L^q$ boundedness of Fourier multipliers that avoids using the Paley and Hausdorff–Young–Paley inequalities, drawing on the method introduced in [33].

2 Preliminaries

2.1 Basic notations on $\mathbb{R}_q$ space

Throughout this paper, we assume $0 < q < 1$. In this section, we will fix some notations and recall some preliminary results. We put $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$ and $\tilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}$. For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), \quad n = 1, 2, \ldots; \quad (a;q)_\infty = \prod_{k=0}^{\infty} \left(1 - aq^k\right).$$

We denote also

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

The q-analogue differential operator $D_q f(x)$ is

$$D_q f(x) := \frac{f(x) - f(qx)}{x(1-q)}.$$

The q-Jackson integrals are defined by (see, [6, 7])

$$\int_0^a f(x)\mathrm{d}_q x = (1-q)a\sum_{n=0}^{+\infty} q^n f\left(aq^n\right) \qquad (1)$$

$$\int_a^b f(x)\mathrm{d}_q x = (1-q)\sum_{n=0}^{+\infty} q^n \left(bf\left(bq^n\right) - af\left(aq^n\right)\right) \qquad (2)$$

and

$$\int_{\mathbb{R}_q} f(x)\mathrm{d}_q x = (1-q)\sum_{n=-\infty}^{+\infty} q^n \left\{f\left(q^n\right) + f\left(-q^n\right)\right\},$$

provided the sums converge absolutely.

In the following we denote by

- $L_q^p\left(\mathbb{R}_q\right) = \left\{f : \|f\|_{L_q^p(\mathbb{R}_q)} = \left(\int_{\mathbb{R}_q} |f(x)|^p \, \mathrm{d}_q x\right)^{1/p} < \infty\right\}.$
- $L_q^\infty\left(\mathbb{R}_q\right) = \left\{f : \|f\|_{L_q^\infty(\mathbb{R}_q)} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty\right\}.$

2.2 Fourier transform and Fourier multiplier on $\mathbb{R}_q$

The q^2-exponentials (see [18] and [17])

$$e(x;q^2) = \cos(-ix;q^2) + i\sin(-ix;q^2),$$

where the q^2-trigonometric functions

$$\cos(x;q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} x^{2k}}{[2k]_q!}$$

and

$$\sin(x;q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} x^{2k+1}}{[2k+1]_q!}.$$

Definition 2.1 *Let $f \in \mathcal{D}_q\left(\mathbb{R}_q\right)$. Then the q^2-Fourier transform of f is defined as follows*

$$\mathcal{F}(\xi;q^2) := \widehat{f}(\xi) \;=\; K\int_{\mathbb{R}_q} f(x)e(-ix\xi;q^2)d_q x \qquad (3)$$

and its inverse

$$f(x) \;=\; K\int_{\mathbb{R}_q} e(ix\xi;q^2)\mathcal{F}(\xi;q^2)d_q\xi,$$

where $K = \frac{(1+q)^{1/2}}{2\Gamma_{q^2}(1/2)}$.

Moreover, we have the Plancherel (or Parseval) identity (see, [17])

$$\|f\|_{L^2_q(\mathbb{R}_q)} = \|\widehat{f}\|_{L^2_q(\mathbb{R}_q)}. \tag{4}$$

In [17], δ_y denote the weighted Dirac-measure at $y \in \mathbb{R}_q$ defined on $\mathbb{R}_q$ by

$$\delta_y(x) = \begin{cases} [(1-q)y]^{-1}, & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases}$$

and It satisfies the following properties:

1) for all $x, y \in \mathbb{R}_q$, we have the orthogonality relation

$$\delta_y(x) \;=\; K^2 \int\limits_{\mathbb{R}_q} e(ix\xi; q^2)e(-iy\xi; q^2)d_q\xi. \tag{5}$$

2) If $f \in L^1_q(\mathbb{R}_q)$, then we get that

$$f(y) \;=\; \int\limits_{\mathbb{R}_q} f(x)\delta_y(x)d_qx. \tag{6}$$

Definition 2.2 *We assume that the function $g : \mathbb{R}_q \to \mathbb{C}$ is bounded. Then, we introduce the q-deformated Fourier multiplier A_g on $L^2_q(\mathbb{R}_q)$ as follows*

$$A_g(f)(x) = K \int_{\mathbb{R}_q} g(\xi)\widehat{f}(\xi)e(ix\xi; q^2)d_q\xi. \tag{7}$$

Definition 2.3 *Let $1 \leq p, r \leq \infty$. Let $B : L^p_q(\mathbb{R}_q) \to L^r_q(\mathbb{R}_q)$ be a bounded linear operator. The, we define its adjoint operator $B^* : L^{r'}_q(\mathbb{R}_q) \to L^{p'}_q(\mathbb{R}_q)$ as follows*

$$(B(f_1), f_2) := \int_{\mathbb{R}_q} B(f_1)(\xi)\overline{f_2(\xi)}d_q\xi = \int_{\mathbb{R}_q} f_1(\xi)\overline{B(f_2)(\xi)}d_q\xi = (f_1, B^*(f_2)), \tag{8}$$

for all $f_1 \in L^p_q(\mathbb{R}_q)$ and $f_2 \in L^{r'}_q(\mathbb{R}_q)$.

2.3 The q-distribution function

In subsection, we state the distribution function $d_f(\lambda; q)$ on $\mathbb{R}_q$. Let Ω be a subset of $(0, \infty)$ and $z > 0$. Then, the definite q-integral with the function $\chi_\Omega(x)$ introduced as follows

$$\int\limits_{\mathbb{R}^+_q} \chi_{(0,z]}(x)f(x)\mathrm{d}_qx = (1-q)\sum_{q^n \leq z} q^n f(q^n) \tag{9}$$

and

$$\int\limits_{\mathbb{R}^+_q} \chi_{[z,\infty)}(x)f(x)\mathrm{d}_qx = (1-q)\sum_{z \leq q^n} q^n f(q^n), \tag{10}$$

where $\chi_\Omega(x)$ is the characteristic function of the set Ω (see, [20, formals 2.6-2.7] and [21]).

Definition 2.4 *(see, [28, Definition 2. p. 504]) The q-distribution function $d_f(\lambda;q)$ of $f:\mathbb{R}_q\to\mathbb{R}$ is a real-valued function, which expressed as*

$$d_f(\lambda;q)=\mu_q\{x\in\mathbb{R}_q:|f(x)|>\lambda\},\quad \lambda>0.$$

Moreover, we observe that

$$d_{f+g}(2\lambda;q)\le d_f(\lambda;q)+d_g(\lambda;q). \tag{11}$$

Using the distribution function, we present and demonstrate the following key characterization of the $L_q^p(\mathbb{R}_q)$ norm.

Proposition 2.5 *(see, [28, Proposition 4. p. 506])Let $0<p<\infty$ and $f\in L_q^p(\mathbb{R}_q)$. Then*

$$\|f\|^p_{L_q^p(\mathbb{R}_q)}=[p]_q\int_{\mathbb{R}_q^+}\lambda^{p-1}d_f(\lambda;q)d_q\lambda. \tag{12}$$

Lemma 2.6 *(see, [28, Lemma 1. p. 506]) Let $f\in L_q^p(\mathbb{R}_q)$ for $0\le p<\infty$. Then*
a) We assume that $E_\lambda=\{x\in\mathbb{R}_q:|f(x)|>\lambda\}$

$$d_f(\lambda;q)\le\frac{1}{\lambda}\int_{\mathbb{R}_q}\chi_{E_\lambda}(x)|f(x)|d_qx\le\frac{1}{\lambda}\int_{\mathbb{R}_q}|f(x)|d_qx;$$

b) (The q-Chebyshev inequality).

$$d_f(\lambda;q)\le\frac{1}{\lambda^p}\int_{\mathbb{R}_q}\chi_{E_\lambda}(x)|f(x)|^pd_qx,$$

3 A q-deformated interpolation theorem

In this section we establish a q-deformated interpolation theorem.

3.1 The q-deformated Marcinkiewicz Interpolation theorem

Definition 3.1 *(see, [28, Definition4. p. 507]) Assume that $0<p<\infty$. Then, we defined the space weak $L_q^{p,\infty}(\mathbb{R}_q)$ as follows*

$$\|f\|_{L_q^{p,\infty}(\mathbb{R}_q)}\;:=\;\left\{\inf_{\lambda>0}\left\{C_q>0:d_f(\lambda;q)\le\frac{C_q}{\lambda^p}\right\}=\sup_{\lambda>0}\left\{\lambda d_f^{1/p}(\lambda;q)\right\}<\infty\right\}. \tag{13}$$

The weak $L_q^{p,\infty}(\mathbb{R}_q)$ spaces are larger than the usual $L_q^p(\mathbb{R}_q)$ spaces.
For any $0<p<\infty$ and any f in $L_q^\infty(\mathbb{R}_q)$ we have

$$\|f\|_{L_q^{p,\infty}(\mathbb{R}_q)}\le\|f\|_{L_q^p(\mathbb{R}_q)}, \tag{14}$$

Hence, the embedding $L_q^{p,\infty}(\mathbb{R}_q)\hookrightarrow L_q^p(\mathbb{R}_q)$ holds.
Indeed, by (13) and the q-Chebyshev's inequality (see, Lemma 2.6 (b)), and we have

$$\|f\|_{L_q^{p,\infty}(\mathbb{R}_q)} = \sup_{\lambda>0}\{\lambda d_f^{1/p}(\lambda;q)\} = \sup_{\lambda>0}\Big\{\big(\int_{E_\lambda}\chi_{E_\lambda}(x)|f(x)|^p d_qx\big)^{1/p}\Big\} \le \|f\|^p_{L_q^p(\mathbb{R}_q)},$$

which implies that (14) holds.

Now, we can prove the following interpolation theorem, which will let us deduce $L_q^p(\mathbb{R}_q)$ boundedness from weak inequalities, since they measure the size of the distribution function.

Theorem 3.2 *(q-deformated Marcinkiewicz interpolation) Let* $0 < p < s \le \infty$ *and* T *is a sublinear operator defined on* $L_q^{p,\infty}(\mathbb{R}_q) + L_q^{s,\infty}(\mathbb{R}_q) := \left\{f_0 + f_1 : f_0 \in L_q^{p,\infty}(\mathbb{R}_q), f_1 \in L_q^{s,\infty}(\mathbb{R}_q)\right\}$. *Assume that*

$$\|T(f)\|_{L_q^{p,\infty}(\mathbb{R}_q)} \le C_0\|f\|_{L_q^{p,\infty}(\mathbb{R}_q)}, \quad \forall f \in L_q^{p,\infty}(\mathbb{R}_q), \tag{15}$$

$$\|T(f)\|_{L_q^{s,\infty}(\mathbb{R}_q)} \le C_1\|f\|_{L_q^{s,\infty}(\mathbb{R}_q)}, \quad \forall f \in L_q^{s,\infty}(\mathbb{R}_q), \tag{16}$$

Then $\forall r \in (p,s)$ *and* $\forall f \in L_q^{r,\infty}(\mathbb{R})$ *the following estimate holds*

$$\|T(f)\|_{L_q^{r,\infty}(\mathbb{R}_q)} \le C\|f\|_{L_q^{r,\infty}(\mathbb{R}_q)}, \tag{17}$$

where $C := 2[r]_q^{1/r}\left(\frac{1}{[r-p]_q} + \frac{1}{[s-r]_q}\right)^{1/r} C_0^\theta C_1^{1-\theta}$ *and* $\theta := \frac{1/r-1/s}{1/p-1/s}$.

Proof. For a fixed $\lambda > 0$ we suppose that the functions f_0 and f_1 by

$$f_0(x) = \begin{cases} f(x), & \text{if } |f(x)| \le C\lambda, \\ 0, & \text{if } |f(x)| > C\lambda, \end{cases} \qquad f_1(x) = \begin{cases} 0, & \text{if } |f(x)| \le C\lambda, \\ f(x), & \text{if } |f(x)| > C\lambda, \end{cases}$$

for some $C > 0$ to be determined later.

Let $0 < p < r < s < \infty$. We assume that $E_0 := \{x : |f(x)| \le C\lambda\}$ and $E_1 := \{x : |f(x)| > C\lambda\}$. Then it can then be easily verified that f_1 (the unbounded part of f) is an L_q^p function for $p < r$:

$$\begin{aligned}\int_{\mathbb{R}_q^+}\lambda^{r-p-1}\|f_0\|^p_{L_q^p(\mathbb{R}_q)}d_q\lambda &= \int_{\mathbb{R}_q^+}\lambda^{r-p-1}\int_{\mathbb{R}_q}\chi_{E_0}(x)|f(x)|^p d_qx d_q\lambda \\ &= \int_{\mathbb{R}_q}|f(x)|^p\int_{\mathbb{R}_q^+}\lambda^{r-p-1}\chi_{E_1}(x)d_q\lambda d_qx \le \frac{[C\lambda]^{p-r}\|f\|^r_{r,q}}{[r-p]_q},\end{aligned} \tag{18}$$

and that f_0 (the bounded part of f) is an $L^s_{s,q}(\mathbb{R}_q)$ function for $r < s$:

$$\begin{aligned}\int_{\mathbb{R}_q^+}\lambda^{r-s-1}\|f_1\|^s_{s,q}d_q\lambda &= \int_{\mathbb{R}_q^+}\lambda^{r-s-1}\int_{\mathbb{R}_q}\chi_{E_1}(x)|f(x)|^s d_qx d_q\lambda \\ &= \int_{\mathbb{R}_q}|f(x)|^s\int_{\mathbb{R}_q^+}\lambda^{r-s-1}\chi_{E_0}(x)d_q\lambda d_qx \le \frac{[C\lambda]^{s-r}\|f\|^r_{r,q}}{[s-r]_q}.\end{aligned} \tag{19}$$

the subadditivity property of T and Hypotheses (15) and (16) together with (11) now give

$$d_{Tf}(2\lambda;q) \overset{(11)}{\leq} d_{Tf_0}(\lambda;q)+d_{Tf_1}(\lambda;q) \overset{(15)(16)}{\leq} \frac{C_0^p}{\lambda^p}\|f_0\|_{p,q}^p+\frac{C_1^s}{\lambda^s}\|f_1\|_{s,q}^s \tag{20}$$

In view of the last estimates (18)-(20) and (12), we conclude that

$$\begin{aligned}\|Af\|_{r,q}^r &\overset{(12)}{=} [r]_q\int_{\mathbb{R}_q^+}[2\lambda]^{r-1}d_{Tf}(2\lambda;q)d_q2\lambda \\ &\overset{(20)}{=} 2^r[r]_q\left[C_0^p\int_{\mathbb{R}_q^+}\lambda^{r-p-1}\|f_0\|_{p,q}^p d_q\lambda+C_1^s\int_{\mathbb{R}_q^+}\lambda^{r-s-1}\|f_1\|_{s,q}^s d_q\lambda\right] \\ &\overset{(18)(19)}{\leq} 2^r[r]_q\left[\frac{C_0^pC^{r-p}}{[r-p]_q}+\frac{C_1^sC^{s-r}}{[s-r]_q}\right]\|f\|_{r,q}^r.\end{aligned}$$

We assume that $C_0^pC^{r-p}=C_1^sC^{s-r}$, we get that

$$C=C_0^{\frac{p}{s-p}}C_1^{\frac{s}{s-p}} \quad\Rightarrow\quad C_0^pC^{r-p}=C_0^pC_0^{\frac{p(r-p)}{s-p}}C_1^{\frac{s(r-p)}{s-p}}=C_0^{\frac{p(r-p)}{s-p}}C_1^{\frac{s(r-p)}{s-p}}.$$

Therefore, we have shown (17), where

$$C^r=2^r[r]_qC_0^{\frac{p(r-p)}{s-p}}C_1^{\frac{s(r-p)}{s-p}}\left\{\frac{1}{r-p}+\frac{1}{s-r}\right\}.$$

This completes the proof.

We say that $\mathcal{A}\lesssim\mathcal{B}$ if there exists a positive constant $c>0$, which depends only on certain parameters of the spaces involved, such that $\mathcal{A}\leq c\mathcal{B}$. Similarly, we write $\mathcal{A}\asymp\mathcal{B}$ to indicate that both inequalities $\mathcal{A}\lesssim\mathcal{B}$ and $\mathcal{A}\gtrsim\mathcal{B}$ are satisfied, possibly with different constants in each inequality. In other words, $\mathcal{A}$ and $\mathcal{B}$ are equivalent up to multiplicative constants depending only on the space parameters.

4 The q-deformated Hausdorff-Young-Paley Inequality

Now, we start to prove q-deformated Hausdorff-Young-Paley inequality and its inverse inequality.

Theorem 4.1 *Let $1\leq p\leq 2$ and $\frac{1}{p}+\frac{1}{p'}=1$. Then for any $f\in L_q^p(\mathbb{R}_q)$ we have*

$$\|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)}\leq\|f\|_{L_q^p(\mathbb{R}_q)}. \tag{21}$$

Proof. Let A is a linear operator such that $A(f)(\xi)=\widehat{f}(\xi)$ for $f\in L_q^p(\mathbb{R}_q)$, $1\leq p\leq 2$. Then, by using the Hölder inequality (see [19, Proposition 37.2]), we have

$$\begin{aligned}\|A(f)\|_{L_q^\infty(\mathbb{R}_q)} &= \|\widehat{f}\|_{L_q^\infty(\mathbb{R}_q)}=\sup_{\xi\in\mathbb{R}_q}|\widehat{f}(\xi)| \\ &\leq \sup_{\xi\in\mathbb{R}_q}\|e(-i\cdot\xi;q^2)\|_{L_q^\infty(\mathbb{R}_q)}\|f\|_{L_q^1(\mathbb{R}_q)}\leq\|f\|_{L_q^1(\mathbb{R}_q)},\end{aligned}$$

where $\sup_{\xi\in\mathbb{R}_q} \|e(-i\cdot\xi;q^2)\|_{L_q^\infty(\mathbb{R}_q)} \leq 1$. Moreover, by Plancherel's identity (4), we have

$$\|A(f)\|_{L_q^2(\mathbb{R}_q)} = \|\widehat{f}\|_{L_q^2(\mathbb{R}_q)} \overset{(4)}{=} \|f\|_{L_q^2(\mathbb{R}_q)}, \quad f\in L_q^2(\mathbb{R}_q).$$

Therefore, we derive that $A: L_q^1(\mathbb{R}_q)\to L_q^\infty(\mathbb{R}_q)$ and $A: L_q^2(\mathbb{R}_q)\to L_q^2(\mathbb{R}_q)$, with the operator norms at most 1. In the case, $\theta = 2/p'$, then $0\leq\theta\leq 1$. Moreover, we have $\frac{1}{p}=\frac{1-\theta}{1}+\frac{\theta}{2}$ and $\frac{1}{p'}=\frac{1-\theta}{\infty}+\frac{\theta}{2}$. Hence, It is follows from the Theorem 3.2 that the inequality (21) holds.

We now derive the reverse form of inequality (21) in the range $2\leq p\leq\infty$.

Theorem 4.2 *Suppose that* $2\leq p\leq\infty$ *and* $\widehat{f}\in L_q^{p'}(\mathbb{R}_q)$. *Then*

$$\|f\|_{L_q^p(\mathbb{R}_q)} \leq \|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)}, \tag{22}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. Let $f\in L_q^p(\mathbb{R}_q)$. then, from duality of $L_q^p(\mathbb{R}_q)$ we find that

$$\|f\|_{L_q^p(\mathbb{R}_q)} = \sup\left\{|(f,\overline{\varphi})| : \varphi\in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1\right\}.$$

and using the Plancherel identity (4), we get

$$(f,\overline{\varphi}) = \int_{\mathbb{R}_q} \widehat{x}(s)\overline{\widehat{y}(s)}d_qs, \quad x,y\in\mathcal{S}(\mathbb{R}_q).$$

Therefore,

$$\begin{aligned}
\|f\|_{L_q^p(\mathbb{R}_q)} &= \sup\{|(f,\overline{\varphi})| : \varphi\in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1\} \\
&= \sup\left\{\left|\int_{\mathbb{R}_q}\widehat{f}(s)\overline{\widehat{\varphi}(s)}d_qs\right| : \varphi\in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1\right\} \\
&\leq \sup\left\{\int_{\mathbb{R}_q}|\widehat{f}(s)\overline{\widehat{\varphi}(s)}|d_qs : \varphi\in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1\right\} \\
&\leq \sup\left\{\int_{\mathbb{R}_q}|\widehat{f}(s)||\widehat{\varphi}(s)|d_qs : \varphi\in L_q^{p'}(\mathbb{R}_q), \quad \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1\right\} \\
&\leq \sup_{\substack{\varphi\in L_q^{p'}(\mathbb{R}_q)\\ \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1}} \left\{\left(\int_{\mathbb{R}_q}|\widehat{f}(s)|^{p'}d_qs\right)^{1/p'}\cdot\left(\int_{\mathbb{R}_q}|\widehat{\varphi}(s)|^{p}d_qs\right)^{1/p}\right\} \\
&= \sup_{\substack{\varphi\in L^{p'}(\mathbb{R}_q)\\ \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1}} \left\{\|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)}\cdot\|\widehat{\varphi}\|_{L_q^{p}(\mathbb{R}_q)}\right\}.
\end{aligned}$$

Here we used the inequality $|\widehat{f}(\xi)\overline{\widehat{\varphi}(\xi)}| \leq |\widehat{f}(\xi)||\overline{\widehat{\varphi}(\xi)}|$ for any $\xi\in\mathbb{R}_q$, applying the Hölder inequality (see [19, Proposition 37.2]) with respect to Fourier transforms of f and $\varphi\in L_q^{p'}(\mathbb{R}_q)$

with $\|\varphi\|_{L_q^{p'}(\mathbb{R}_q)} = 1$. by using inequality (21) with respect to φ, we write that

$$\|f\|_{L_q^p(\mathbb{R}_q)} \overset{(21)}{\leq} \sup_{\substack{\varphi \in L_q^{p'}(\mathbb{R}_q) \\ \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)}=1}} \left\{ \|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)} \cdot \|\varphi\|_{L_q^{p'}(\mathbb{R}_q)} \right\} = \|\widehat{f}\|_{L_q^{p'}(\mathbb{R}_q)},$$

thereby completing the proof.

Next, we establish the q-deformated Hausdorff-Young-Paley inequality.

Theorem 4.3 *Assume that $1 < p \leq 2$ and let $\varphi : \mathbb{R}_q \to \mathbb{R}_+$ be a strictly positive function satisfyingthe following condition*

$$M_\varphi := \sup_{t>0} t \int\limits_{\varphi(\xi) \geq t} d_q\xi < \infty. \tag{23}$$

Then, we have the following inequality

$$\left(\int\limits_{\mathbb{R}_q} |\widehat{f}(\xi)|^p \varphi^{2-p}(\xi) d_q\xi \right)^{\frac{1}{p}} \leq c_p M_\varphi^{\frac{2-p}{p}} \|f\|_{L_q^p(\mathbb{R}_q)} \quad \textit{for} \quad f \in L_q^p(\mathbb{R}_q), \tag{24}$$

where $c_p > 0$ is a constant independent of f.

Proof. We assume that ν be a measure on $\mathbb{R}_q$ by $\nu(\xi) := \varphi^2(\xi) d_q\xi > 0$. Detone a space $L_q^p(\mathbb{R}_q, \nu)$ as follows

$$\|f\|_{L_q^p(\mathbb{R}_q,\nu)} := \left\{ f : \left(\int\limits_{\mathbb{R}_q} |f(\xi)|^p \varphi^2(\xi) d_q\xi \right)^{\frac{1}{p}} < \infty \right\}.$$

One can readily verify that, endowed with the above norm, this space is Banach. We then introduce the operator $A : L_q^p(\mathbb{R}_q) \to L_q^p(\mathbb{R}_q, \nu)$ via the formula

$$(Af)(\xi) = \frac{\widehat{f(\xi)}}{\varphi(\xi)}.$$

It follows from $\widehat{f+\varphi}(\xi) \overset{(3)}{=} \widehat{f}(\xi) + \widehat{\varphi}(\xi), \quad f, \varphi \in L_q^2(\mathbb{R}_q)$, that A is a sub-liner (or quasi-linear) operator. Now, we will prove that A $A : L_q^p(\mathbb{R}_q) \to L_q^p(\mathbb{R}_q, \nu)$ is well-defined and bounded with $1 \leq p \leq 2$. Equivalently, we claim that (24) is valid under condition (23). We first verify that A is of weak types $(2,2)$ and $(1,1)$. The distribution function $d_{A(f)}(t)$, $t > 0$, with respect to $\varphi^2(s) > 0$, is defined by

$$d_{A(f)}(t) := \nu\{s > 0 : |A(f)| > t\} = \int\limits_{|A(f)|>t} \varphi^2(\xi) d_q\xi.$$

The next step is to show that

$$d_{A(f)}(t) \leq \left(\frac{c_2\|f\|_{L^2_q(\mathbb{R}_q)}}{t}\right)^2 \quad \text{with} \quad c_2 = 1, \tag{25}$$

and

$$d_{A(f)}(t) \leq \frac{c_1\|f|_{L^1_q(\mathbb{R}_q)}}{t} \quad \text{with} \quad c_1 = 2M_\varphi. \tag{26}$$

To begin with, we prove inequality (25). Using the q-Chebyshev inequality (see Lemma 2.6 (b)) together with (3), we obtain

$$td_{A(f)}(t) \leq \|A(f)\|^2_{L^2_q(\mathbb{R}_q,\nu)} = \int\limits_{\mathbb{R}_q} |\widehat{f}(s)|^2 d_q s = \|\widehat{f}\|^2_{L^2_q(\mathbb{R}_q)} \overset{(3)}{=} \|f\|^2_{L^2_q(\mathbb{R}_q)}.$$

Therefore, the operator A is of weak type $(2,2)$ with its norm bounded above by $c_2 = 1$. Next, we proceed to prove inequality (26). Using Hцlder's inequality (cf. [19, Proposition 37.2]) for the exponents $p = 1$ and $p' = \infty$, we obtain

$$\begin{aligned}\frac{|\widehat{f}(\xi)|}{\varphi(\xi)} &\overset{(3)}{\leq} K\frac{\left|\int_{\mathbb{R}_q} f(x)e(-ix\xi;q^2)d_q x\right|}{\varphi(\xi)} \\ &\leq \frac{\|e(-i\cdot\xi;q^2)\|_{L^\infty_q(\mathbb{R}_q)}\|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)} \leq \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)}, \quad \xi \in \mathbb{R}_q.\end{aligned}$$

Therefore, we have

$$\{\xi \in \mathbb{R}_q : \frac{|\widehat{f}(\xi)|}{\varphi(\xi)} > t\} \subset \{\xi \in \mathbb{R}_q : \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)} > t\}$$

for any $t > 0$. Consequently,

$$\nu\{\xi \in \mathbb{R}_q : \frac{|\widehat{f}(\xi)|}{\varphi(\xi)} > t\} \leq \nu\{\xi \in \mathbb{R}_q : \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)} > t\}$$

for any $t > 0$. Setting $v := \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{t}$, we obtain

$$\nu\{\xi \in \mathbb{R}_q : \frac{|\widehat{f}(\xi)|}{\varphi(\xi)} > t\} \leq \nu\{\xi \in \mathbb{R}_q : \frac{\|f\|_{L^1_q(\mathbb{R}_q)}}{\varphi(\xi)} > t\} = \int\limits_{\varphi(\xi)\leq v} \varphi^2(\xi) d_q\xi. \tag{27}$$

Let us estimate the right hand side. Now we claim that

$$\int\limits_{\varphi(\xi)\leq v} \varphi^2(\xi) d_q\xi \leq (1 + q^{1/2})v \cdot M_\varphi. \tag{28}$$

Indeed, from this equality $\varphi^2(\xi) = (1-q) \sum\limits_{q^i \le \varphi^2(\xi)} q^i$, and (9)-(10) first we have

$$\begin{aligned}
\int\limits_{\varphi(\xi)\le v} \varphi^2(\xi) d_q\xi &= (1-q) \int\limits_{\varphi(\xi)\le v} \sum_{q^i \le \varphi^2(\xi)} q^i d_q\xi = (1-q)^2 \sum_{\varphi(q^k)\le v} q^k \sum_{q^{i/2}\le \varphi(q^k)} q^i \\
&= (1-q)^2 \sum_{q^{i/2}\le v} q^i \sum_{q^{i/2}\le \varphi(q^k)\le v} q^k \\
&\le (1+q^{1/2})(1-q^{1/2}) \sum_{q^{i/2}\le v} q^i (1-q) \sum_{q^{i/2}\le \varphi(q^k)} q^k \\
&\le (1+q^{1/2})(1-q^{1/2}) \sum_{q^{i/2}\le v} q^{i/2} q^{i/2} \int\limits_{q^{i/2}\le \varphi(\xi)} d_q\xi \\
&= (1+q^{1/2}) \int\limits_0^v \left(t \int\limits_{t\le\varphi(\xi)} d_q\xi \right) d_{q^{1/2}}t. \qquad (29)
\end{aligned}$$

Since

$$t \int\limits_{t\le\varphi(\xi)} d_q\xi \le \sup_{t>0} t \int\limits_{t\le\varphi(\xi)} d_q\xi = M_\varphi$$

and $M_\varphi < \infty$ by assumption and (29), it follows that

$$\int\limits_{\varphi(\xi)\le v} \varphi^2(\xi) d_q\xi \overset{(29)}{\le} (1+q^{1/2}) M_\varphi \int\limits_0^v d_{q^{1/2}}t \le \ (1+q^{1/2}) v \cdot M_\varphi.$$

This establishes the claim (28). By combining (27) and (28), we derive (26), which confirms that A is of weak type $(1,1)$ with operator norm at most $c_1 = 2M_\varphi$. Applying Theorem 3.2 with parameters $p_1 = 1$, $p_2 = 2$, and $\frac{1}{p} = \frac{1-\eta}{1} + \frac{\eta}{2}$, we consequently obtain inequality (24). This completes the proof.

From the q-deformated Paley-type inequality stated in Theorem 4.3, we derive the following q-deformated Hardy–Littlewood inequality.

Theorem 4.4 *Assume that $1 < p \le 2$ and $\varphi : \mathbb{R}_q \to \mathbb{R}_+$ be a strictly positive function satisfying the following condition*

$$\int\limits_{\mathbb{R}_q} \frac{1}{\varphi^\beta(s)} d_q s < \infty \quad \text{for} \quad \text{some} \quad \beta > 0. \qquad (30)$$

Then, we have q-deformated Hardy–Littlewood inequality as follows

$$\left(\int\limits_{\mathbb{R}_q} |\widehat{f}(s)|^p \varphi^{\beta(p-2)}(s) d_q s \right)^{\frac{1}{p}} \le C_p \|f\|_{L^p_q(\mathbb{R}_q)} \quad \text{for} \quad f \in L^p_q(\mathbb{R}_q),$$

where $C_p > 0$ is a constant independent of x.

Proof. It follows from the assumption 30 that

$$\begin{aligned} C_q &= (1-q)\sum_{k\in\mathbb{Z}} q^k\varphi^{-\beta}(q^k) = (1-q)\sum_{\varphi^\beta(q^k)\le\frac{1}{t}} q^k\varphi^{-\beta}(q^k) \\ &\ge (1-q)t\sum_{\varphi^\beta(q^k)\le\frac{1}{t}} q^k = t\int\limits_{\varphi^\beta(s)\le\frac{1}{t}} d_qs = t\int\limits_{t\le\frac{1}{\varphi^\beta(s)}} d_qs, \quad t>0. \end{aligned}$$

Therefore, taking the supremum over all positive t, we obtain the bound

$$\sup_{t>0}\, t\int_{\{s\in\mathbb{R}_q : t\le\frac{1}{\varphi^\beta(s)}\}} d_qs \le C_q < \infty.$$

This shows that the integral expression is uniformly controlled by the constant C_q. Then, by applying Theorem 4.3 to the function defined by

$$h(s) = \frac{1}{\varphi^\beta(s)}, \quad s\in\mathbb{R}_q.$$

we derive the desired inequality.

Theorem 4.5 *Suppose that* $2\le p<\infty$ *with* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\varphi:\mathbb{R}_q\to\mathbb{R}_+$ *be a strictly positive function satisfying the following condition*

$$\int\limits_{\mathbb{R}^d}\frac{1}{\varphi^\beta(s)}d_qs<\infty \quad \textit{for} \quad \textit{some} \quad \beta>0.$$

If

$$\int\limits_{\mathbb{R}^d}|\widehat{f}(s)|^p\varphi^{\frac{\beta p(2-p')}{p'}}(s)d_qs<\infty,$$

then

$$\|f\|^p_{L^p_q(\mathbb{R}_q)}\le C_{p,q}\int\limits_{\mathbb{R}_q}|\widehat{f}(s)|^p\varphi^{\frac{\beta p(2-p')}{p'}}(s)d_qs, \quad f\in L^p_q(\mathbb{R}_q),$$

where $C_{p,q}>0$ *is a constant independent of* x.

Proof. For $L_p(\mathbb{R}_q)$ we have

$$\|f\|_{L^p_q(\mathbb{R}_q)} = \sup\Big\{\big|\langle f,g\rangle_{L^2_q(\mathbb{R}_q)}\big| : g\in L^{p'}(\mathbb{R}_q), \quad \|g\|_{L^{p'}(\mathbb{R}_q)}=1\Big\}.$$

It follows from (4) that

$$\langle f,g\rangle_{L^2_q(\mathbb{R}_q)} = \int\limits_{\mathbb{R}^d}\widehat{f}(s)\widehat{g}(s)d_qs, \quad f,g\in L^2_q(\mathbb{R}_q). \tag{31}$$

Using the Hölder inequality for any function $g \in L_q^{p'}(\mathbb{R}_q)$ with $\|g\|_{L_q^{p'}(\mathbb{R}_q)} = 1$, we deduce that

$$\begin{aligned}
\|f\|_{L_q^p(\mathbb{R}_q)} &= \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \{|\langle f, g\rangle_{L_q^2(\mathbb{R}_q)}| : g \in L_q^{p'}(\mathbb{R}_q)\} \\
&= \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\Big|\int_{\mathbb{R}_q} \widehat{f}(s)\widehat{g}(s) d_q s\Big| : g \in L_q^{p'}(\mathbb{R}_q)\Big\} \\
&\leq \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\int_{\mathbb{R}_q} |\widehat{f}(s)\widehat{g}(s)| d_q s : g \in L_q^{p'}(\mathbb{R}_q)\Big\} \\
&\leq \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\int_{\mathbb{R}_q} |\widehat{f}(s)||\widehat{g}(s)| d_q s : g \in L_q^{p'}(\mathbb{R}_q)\Big\} \\
&\leq \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\int_{\mathbb{R}_q} \varphi^{\frac{\beta(2-p')}{p'}}(s)|\widehat{f}(s)| \cdot \varphi^{\frac{\beta(p'-2)}{p'}}(s)|\widehat{g}(s)| : g \in L_q^{p'}(\mathbb{R}_q)\Big\} \\
&\leq \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\Big(\int_{\mathbb{R}_q} \varphi^{\frac{\beta p(2-p')}{p'}}(s)|\widehat{f}(s)|^p d_q s\Big)^{1/p} \cdot \Big(\int_{\mathbb{R}_q} \varphi^{\beta(p'-2)}(s)|\widehat{g}(s)|^{p'} d_q s\Big)^{1/p'}\Big\}.
\end{aligned}$$

Now applying Theorem 4.4 with respect to p', we get

$$\begin{aligned}
\|f\|_{L_q^p(\mathbb{R}_q)} &\leq \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \Big\{\Big(\int_{\mathbb{R}_q} \varphi^{\frac{\beta p(2-p')}{p'}}(s)|\widehat{f}(s)|^p d_q s\Big)^{1/p} \cdot \Big(\int_{\mathbb{R}^d} \varphi^{\beta(p'-2)}(s)|\widehat{g}(s)|^{p'} d_q s\Big)^{1/p'}\Big\} \\
&\lesssim \Big(\int_{\mathbb{R}_q} \varphi^{\frac{\beta p(2-p')}{p'}}(s)|\widehat{f}(s)|^p d_q s\Big)^{1/p} \cdot \sup_{\|g\|_{L_q^{p'}(\mathbb{R}_q)}=1} \|g\|_{L_q^{p'}(\mathbb{R}_q)}.
\end{aligned}$$

Since $\|g\|_{L_q^{p'}(\mathbb{R}_q)} = 1$, taking $C_{p,q} = c_{p',q}$, we complete the proof.

Remark 4.6 *Suppose $p = 2$, then the inequalities stated in Theorems 4.1 and 4.2 both simplify to the identity given by (4).*

The following result can be inferred from [22, Corollary 5.5.2, p. 120].

Proposition 4.7 *Let $d_q\nu_1(\xi) = \omega_1(\xi)d_q\xi$, $d\nu_2(\xi) = \omega_2(\xi)d_q\xi$, $\xi \in \mathbb{R}_q$. Suppose that $1 \leq p, r_0, r_1 < \infty$. If a continuous linear operator A admits bounded extensions $A : L_q^p(\mathbb{R}_q) \to L_q^{r_0}(\mathbb{R}_q, \nu_1)$ and $A : L_q^p(\mathbb{R}_q) \to L_q^{r_1}(\mathbb{R}_q, \nu_2)$, then there exists a bounded extension $A : L_q^p(\mathbb{R}_q) \to L_q^r(\mathbb{R}_q, \nu)$ where $0 < \theta < 1$, $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$ and $d_q\nu(\xi) = \omega(\xi)d_q\xi$, $\omega = \omega_1^{\frac{r}{r_0}(1-\theta)} \cdot \omega_2^{\frac{r}{r_1}\theta}$.*

Now, we obtain the q-deformated Hausdorff-Young-Paley inequality.

Theorem 4.8 *Suppose that $1 < p \leq r \leq p' < \infty$ for $\frac{1}{p} + \frac{1}{p'} = 1$. Let φ is given as in Theorem 4.3. Then*

$$\Big(\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^r \varphi(\xi)^{r(\frac{1}{r}-\frac{1}{p'})} d_q\xi\Big)^{\frac{1}{r}} \leq c_{p,r,p'} M_\varphi^{\frac{1}{r}-\frac{1}{p'}} \|f\|_{L_q^p(\mathbb{R}_\theta^d)}.$$

where $c_{q,p,r,p'} > 0$ is a constant independent of f.

Proof. Let $A(x) := \widehat{f}$ be a linear operator acting on the space $L^p_q(\mathbb{R}_q)$. By using the inequality stated in (24) for $1 < p \leq 2$, we then deduce that

$$\Big(\int\limits_{\mathbb{R}_q} |\widehat{f}(\xi)|^p \varphi^{2-p}(\xi) d_q\xi\Big)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p_q(\mathbb{R}_q)}.$$

In other words, $A : L^p_q(\mathbb{R}_q) \to L^p_q(\mathbb{R}_q, \nu_1)$ is a bounded map, where the weight is given by ?1(?):=f2-p(?)>0 $\omega_1(\xi) := \varphi^{2-p}(\xi) > 0$ with $\xi \in \mathbb{R}_q$. moreover, for $1 \leq p \leq 2$ with $\frac{1}{p} + \frac{1}{p'} = 1$, by applying the inequality (21), we obtain that

$$\Big(\int\limits_{\mathbb{R}_q} |\widehat{f}(\xi)|^{p'} d_q\xi\Big)^{1/p'} = \|\widehat{f}\|_{L^{p'}_q(\mathbb{R}_q)} \leq \|f\|_{L^p_q(\mathbb{R}_q)},$$

which implyis that $A : L^p_q(\mathbb{R}_q) \to L^{p'}_q(\mathbb{R}_q, \nu_2)$, where $\nu_2(\xi) := 1 d_q\xi$ for all $\xi \in \mathbb{R}_q$. It follows from Proposition 4.7 that $A : L^p_q(\mathbb{R}_q) \to L^r_q(\mathbb{R}_q, \nu)$ with $d_q\nu = \omega(\xi) d_q\xi$, is bounded for any η such that $p \leq \eta \leq p'$, where the space $L^\eta_q(\mathbb{R}_q, \nu)$ is defined as

$$\|f\|_{L^\eta_q(\mathbb{R}_q,\nu)} := \left\{ f : \mathbb{R}_q \to \mathbb{R} : \left(\int\limits_{\mathbb{R}_q} |f(\xi)|^\eta \omega(\xi) d_q\xi\right)^{\frac{1}{\eta}} < \infty \right\},$$

where $\omega : \mathbb{R}_q \to \mathbb{R}$ is a positive function and will be defined later. Let us find the explicit form of ω. For fix $\theta \in (0,1)$ such that $\frac{1}{\eta} = \frac{1-\theta}{p} + \frac{\theta}{p'}$, we derive $\theta = \frac{p-\eta}{\eta(p-2)}$ and from Proposition 4.7 with respect to $r = \eta$, $r_0 = p$, and $r_1 = p'$, we have

$$\omega(\xi) = (\omega_1(\xi))^{\frac{\eta(1-\eta)}{r}} \cdot (\omega_2(\xi))^{\frac{\eta\eta}{r'}} = (\varphi^{2-r}(\xi))^{\frac{\eta(1-\eta)}{r}} \cdot 1^{\frac{\eta\eta}{r'}} = \varphi^{1-\frac{\eta}{r'}}(\xi) = \varphi^{\eta\left(\frac{1}{\eta}-\frac{1}{r'}\right)}(\xi)$$

for all $\xi \in \mathbb{R}_q$ and $\frac{2-r}{r} \cdot (1-\eta) = \frac{1}{\eta} - \frac{1}{r'}$. Hence, for $d_q\nu = \varphi^{\eta\left(\frac{1}{\eta}-\frac{1}{p'}\right)}(\xi) d_q\xi$ we obtain

$$\|A(x)\|_{L^\eta_q(\mathbb{R}_q,\nu)} \lesssim (M_\varphi^{\frac{2-r}{r}})^{1-\eta} \|x\|_{L^r_q(\mathbb{R}_q)} = M_\varphi^{\frac{1}{\eta}-\frac{1}{p'}} \|x\|_{L^p_q(\mathbb{R}_q)}, \quad x \in L^p_q(\mathbb{R}_q).$$

This completes the proof.

5 the q-deformated Hörmander multiplier theorem

First, we obtain the q^2-Fourier transform of the Fourier multiplier (7).

Lemma 5.1 *Let $g : \mathbb{R}_q \to \mathbb{C}$ be a bounded function. Then, we have*

$$\widehat{A_g(f)} = g \cdot \widehat{f}, \quad f \in L^p_q(\mathbb{R}_q). \tag{32}$$

for $f \in L^p_q(\mathbb{R}_q)$.

Proof. Let $f \in L^p_q(\mathbb{R}_q)$. Then, by (3), (5)-(6) and (7) we have

$$\begin{aligned}
\widehat{(A_g f)}(y) &\overset{(3)(7)}{=} K^2 \int\limits_{\mathbb{R}_q} \Big[\int\limits_{\mathbb{R}_q} g(\xi)\widehat{f}(\xi) e(ix\xi; q^2) d_q\xi \Big] e(-ixy; q^2) d_q x \\
&= \int\limits_{\mathbb{R}_q} g(\xi)\widehat{f}(\xi) \Big[K^2 \int\limits_{\mathbb{R}_q} e(ix\xi; q^2) e(-ixy; q^2) d_q x \Big] d_q\xi \\
&\overset{(5)}{=} \int\limits_{\mathbb{R}_q} g(\xi)\widehat{f}(\xi)\delta_y(\xi) d_q\xi \\
&\overset{(6)}{=} g(y)\widehat{f}(y),
\end{aligned}$$

for all $y \in L^p_q(\mathbb{R}_q)$.

Let us denote by $\overline{g}$ the complex conjugate of the function g, in Definition 2.3.

Lemma 5.2 *Suppose that $1 < p, q < \infty$. Let $A_g : L^p_q(\mathbb{R}^d_\theta) \to L^q_q(\mathbb{R}^d_\theta)$ be the Fourier multiplier defined by (7) with the symbol g. Then its adjoint $A^*_g = A_{\overline{g}}$ and $A_{\overline{g}} : L^{q'}_q(\mathbb{R}^d_\theta) \to L^{p'}_q(\mathbb{A}^d_\theta)$.*

Proof. For $h, f \in L^p_q(\mathbb{R}_q)$. Then, It follows frmm (8), (31), and (32) that

$$\begin{aligned}
(A_g f, h) &\overset{(31)}{=} \int\limits_{\mathbb{R}_q} \widehat{A_g f}(s)\widehat{h}(s) d_q s \\
&\overset{(32)}{=} \int\limits_{\mathbb{R}_q} g(s)\widehat{f}(s)\widehat{h}(s) d_q s = \int\limits_{\mathbb{R}_q} \widehat{f}(s) g(s)\widehat{h}(s) d_q s \\
&\overset{(32)}{=} \int\limits_{\mathbb{R}_q} \widehat{f}(s)\widehat{A_g(h)}(s) d_q s \overset{(31)}{=} (f, A_g h).
\end{aligned}$$

Since $L^\eta_q(\mathbb{R}_q)$ is dense in $L^p_q(\mathbb{R}_q)$, we have $A^*_g = A_g$.

Finally, we state the q-deformated Hörmander multiplier theorem.

Theorem 5.3 *Suppose that $1 < p \le 2 \le \eta < \infty$ and $g : \mathbb{R}_q \to \mathbb{R}$ be a bounded function. Then, the Fourier multiplier defined in (7) can be extended to act as a bounded linear operator from the space $L^p_q(\mathbb{R}_q)$ to the space $L^\eta_q(\mathbb{R}_q)$. Moreover, the following estimate holds*

$$\|A_g\|_{L^p_q(\mathbb{R}_q) \to L^\eta_q(\mathbb{R}_q)} \lesssim \sup_{\lambda > 0} \lambda \Big(\int\limits_{|g(s)| \ge \lambda} d_q s \Big)^{\frac{1}{p} - \frac{1}{q}}.$$

Proof. By duality it is sufficient to study two cases: $1 < p \le \eta' \le 2$ and $1 < \eta' \le p \le 2$, where $1 = \frac{1}{\eta} + \frac{1}{\eta'}$.

First, we consider the case $1 < p \le \eta' \le 2$, where $1 = \frac{1}{\eta} + \frac{1}{\eta'}$. By (32) we have

$$\widehat{A_g f} = g \cdot \widehat{f}, \quad f \in L^p_q(\mathbb{R}_q). \tag{33}$$

Then, it follows from it follows from Proposition 4.2 and (33) that

$$\|A_g f\|_{L^\eta_q(\mathbb{R}_q)} \overset{(22)}{\leq} \|\widehat{A_g f}\|_{L^{\eta'}_q(\mathbb{R}_q)} \overset{(33)}{=} \|g\widehat{f}\|_{L^{\eta'}_q(\mathbb{R}_q)}, \tag{34}$$

for all $f \in L^\eta_q(\mathbb{R}_q)$.

Thus, we denote $\eta' := r$ and $\frac{1}{s} := \frac{1}{p} - \frac{1}{\eta} = \frac{1}{\eta'} - \frac{1}{p'}$, then for $h(\xi) := |g(\xi)|^s, \xi \in \mathbb{R}_q$, then, by using the inequality in Theorem 4.8. In other words, we derive

$$\left(\int\limits_{\mathbb{R}_q} \left(|\widehat{f}(\xi)| \cdot |g(\xi)| \right)^{\eta'} d_q\xi \right)^{\frac{1}{\eta'}} \lesssim M^{\frac{1}{s}}_{|g|^s} \|x\|_{L^p_q(\mathbb{R}_q)} \tag{35}$$

for any $f \in L^p_q(\mathbb{R}_q)$. Let us study $M^{\frac{1}{s}}_{|g|^s}$ separately. Indeed, by definition

$$M^{\frac{1}{s}}_{|g|^s} := \left(\sup_{\lambda>0} \lambda \int\limits_{|g(\xi)|^s \geq \lambda} d_q\xi \right)^{\frac{1}{s}} = \left(\sup_{\lambda>0} \lambda \int\limits_{|g(\xi)| \geq \lambda^{\frac{1}{s}}} d_q\xi \right)^{\frac{1}{s}} = \left(\sup_{\lambda>0} \lambda^s \int\limits_{|g(\xi)| \geq \lambda} d_q\xi \right)^{\frac{1}{s}}.$$

Since $\frac{1}{s} := \frac{1}{p} - \frac{1}{\eta}$, it follows that

$$\begin{aligned} M^{\frac{1}{s}}_{|g|^s} &= \left(\sup_{\lambda>0} \lambda^s \int\limits_{|g(\xi)| \geq \lambda} d_q\xi \right)^{\frac{1}{p} - \frac{1}{\eta}} = \sup_{\lambda>0} \lambda^{s\left(\frac{1}{p} - \frac{1}{\eta}\right)} \left(\int\limits_{|g(\xi)| \geq \lambda} d_q\xi \right)^{\frac{1}{p} - \frac{1}{\eta}} \\ &= \sup_{\lambda>0} \lambda \left(\int\limits_{|g(\xi)| \geq \lambda} d_q\xi \right)^{\frac{1}{p} - \frac{1}{\eta}}. \end{aligned} \tag{36}$$

Hence, combining (34), (35), and (36) we obtain

$$\begin{aligned} \|A_g f\|_{L^\eta_q(\mathbb{R}^d_\theta)} &\overset{(34)}{\lesssim} \left(\int\limits_{\mathbb{R}_q} \left(|\widehat{x}(\xi)| \cdot |g(\xi)| \right)^{\eta'} d_q\xi \right)^{\frac{1}{\eta'}} \\ &\overset{(35)}{\lesssim} M^{\frac{1}{s}}_{|g|^s} \|x\|_{L^p_q(\mathbb{R}_q)} \overset{(36)}{=} \sup_{\lambda>0} \lambda \left(\int\limits_{|g(\xi)| \geq \lambda} d_q\xi \right)^{\frac{1}{p} - \frac{1}{\eta}} \|x\|_{L^\eta_q(\mathbb{R}_q)}, \end{aligned}$$

for $1 < p \leq \eta' \leq 2$ and $x \in L^p_q(\mathbb{R}^d_\theta)$.

Next, we consider the case $\eta' \leq p \leq 2$ so that $p' \leq (\eta')' = \eta$, where $1 = \frac{1}{\eta} + \frac{1}{\eta'}$ and $1 = \frac{1}{p} + \frac{1}{p'}$. Thus, the L^p_q-duality (see Lemma 5.2) yields that $A^*_g = A_g$ and

$$\|A_g\|_{L^p_q(\mathbb{R}_q) \to L^\eta_q(\mathbb{R}_q)} = \|A_g\|_{L^{\eta'}_q(\mathbb{R}_q) \to L^{p'}_q(\mathbb{R}_q)}.$$

Set $\frac{1}{p} - \frac{1}{\eta} = \frac{1}{s} = \frac{1}{\eta'} - \frac{1}{p'}$. Hence, by repeating the argument in the previous case we have

$$\begin{aligned} \|A_g(x)\|_{L^{p'}_q(\mathbb{R}_q)} &\lesssim \sup_{\lambda>0} \lambda \left(\int\limits_{\overline{|g(\xi)|} \geq \lambda} d_q\xi \right)^{\frac{1}{q'} - \frac{1}{\eta'}} \|x\|_{L^{\eta'}_q(\mathbb{R}_q)} \\ &= \sup_{\lambda>0} \lambda \left(\int\limits_{|g(\xi)| \geq \lambda} d_q\xi \right)^{\frac{1}{p} - \frac{1}{\eta}} \|x\|_{L^{\eta'}_q(\mathbb{R}_q)}. \end{aligned}$$

In other words, we have

$$\|A_g\|_{L_q^{\eta'}(\mathbb{R}_q)\to L_q^{p'}(\mathbb{R}_q)} \lesssim \sup_{\lambda>0} \lambda \left(\int\limits_{|g(\xi)|\geq\lambda} d_q\xi \right)^{\frac{1}{p}-\frac{1}{\eta}}.$$

Combining both cases, we obtain

$$\|A_g\|_{L_q^{p}(\mathbb{R}_q)\to L_q^{\eta}(\mathbb{R}_q)} \lesssim \sup_{\lambda>0} \lambda \left(\int\limits_{|g(\xi)|\geq\lambda} d_q\xi \right)^{\frac{1}{p}-\frac{1}{\eta}}$$

for all $1 < p \leq 2 \leq \eta < \infty$. This concludes the proof.

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Inverse Source Recovery in a Class of Singular Diffusion Equations via Optimal Control

This paper addresses the inverse problem of identifying a space-dependent source term in a singular parabolic equation involving an inverse-square potential, knowing final time measurement data. The problem is reformulated within an optimal control framework, minimizing a Tikhonov-regularized functional to ensure stability. Theoretical contributions include existence and uniqueness of weak solutions for the direct problem, along with a stability estimate for the inverse problem under a first-order optimality condition. A Landweber-type iterative algorithm is designed for numerical reconstruction, validated through synthetic examples with both exact and noisy data.

Key words: Inverse problem; singular parabolic equation; stability; regularization; Landweber method.

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Сингулярлық диффузиялық теңдеулер класы үшін көзді оңтайлы басқару әдісімен қалпына келтіру

Бұл жұмыста кері квадратты потенциалы бар сингулярлық параболалық теңдеудегі кеңістік-тік тәуелді көзді анықтаудың кері есебі қарастырылады, ол ақырлы уақыт мезетіндегі өлшеу деректерін пайдаланады. Есеп тиімді басқарудағы орнықтылықты қамтамасыз ету үшін Тихоновтың регуляризацияланған функционалын минимизациялауға негізделіп тұжырымдалған. Теориялық нәтижелері ретінде тура есеп үшін әлсіз шешімнің бар және жалғыздығы дәлелденуін, сонымен қатар, бірінші реттік оптималдық шарты орындалған жағдайда кері есептің орнықтылығының бағалауын айтуға болады. Сандық нәтижелері ретінде дәл және шулы деректермен синтетикалық мысалдарда тексерілген Ландвебер типіндегі итерациялық алгоритм әзірленді.

Түйінді сөздер: Кері есеп; сингулярлы параболалық теңдеу; тұрақтылық; тұрақтандыру; Ландвебер әдісі.

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Восстановление источника в классе сингулярных уравнений диффузии с использованием метода оптимального управления

В данной работе рассматривается обратная задача идентификации пространственно-зависимого источника в сингулярном параболическом уравнении с обратно-квадратичным потенциалом на основе данных измерений в конечный момент времени. Задача переформулируется в рамках оптимального управления путём минимизации регуляризованного функционала Тихонова, что обеспечивает устойчивость решения. Теоретические результаты включают доказательство существования и единственности слабого решения для прямой задачи, а также оценку устойчивости для обратной задачи, основанную на условии оптимальности первого порядка. Для численной реконструкции разработан итерационный алгоритм типа Ландвебера, эффективность которого подтверждена на синтетических примерах с точными и зашумлёнными данными.
Түйінді сөздер: Кері есеп; сингулярлы параболалық теңдеу; тұрақтылық; тұрақтандыру; Ландвебер әдісі.

1 introduction

Inverse problems are concerned with the identification of unknown inputs or sources from partial or indirect observations of the system's response, in contrast to forward problems, where the output is computed from given inputs. It is well known that inverse problems are often ill-posed in the sense of Hadamard; that is, the solution may not exist, may not be unique, or may not depend continuously on the data. Consequently, small perturbations in the measurements—such as those due to noise—can lead to significant errors in the solution

In the present work, we investigate the inverse problem of identifying a spatially dependent source term in a singular parabolic equation from measurements of the solution at a fixed final time. More precisely, we consider the following initial-boundary value problem

$$\begin{cases} \partial_t\theta(x,t) - \theta_{xx}(x,t) - \dfrac{\mu}{|x|^2}\theta(x,t) = f(x), \quad (x,t) \in Q_T := \Omega \times (0,T), \\ \theta(0,t) = \theta(1,t) = 0, \quad t \in (0,T), \\ \theta(x,0) = \theta_0(x), x \in \Omega, \end{cases} \tag{1}$$

where $\Omega := (0,1)$, $0 < T < \infty$ is an arbitrary final fixed time, θ_0 is a given smooth function describe the initial state, $f(x)$ represents the unknown source term which is assumed to be kept independent of time variable t.

We are particularly concerned with the inverse problem of recovering the spatially dependent source term $f(x)$ appearing in the governing parabolic equation. To this end, we assume that the solution $u(x,t)$ is observed at the final time $t = T$ over the spatial domain Ω, that is

$$u(x,T) = \omega(x), \quad x \in \Omega, \tag{2}$$

where $\omega \in L^2(\Omega)$ denotes the final-time observation. When the source term $f(x)$ is known, the associated initial-boundary value problem (1) defines the so-called direct (or forward) problem. In the present study, however, $f(x)$ is unknown and must be identified from the final observation (2). Accordingly, we formulate the inverse problem as the determination of $f(x)$ from a prescribed admissible class such that the corresponding solution to (1) satisfies the final-time constraint (2).

Singular inverse-square potentials have attracted considerable attention in recent years due to their relevance in modeling various physical phenomena across multiple disciplines, including quantum cosmology [5], combustion theory [6], electron capture processes [8], and quantum mechanics [7]. Moreover, such potentials naturally arise in the linearization of certain reaction–diffusion systems governed by the heat equation involving supercritical source terms [1].

In the context of inverse problems for parabolic equations, a substantial body of literature has addressed issues related to stability and well-posedness for various classes of equations using a range of analytical and numerical techniques [12, 17–21].

Concerning inverse problems for singular parabolic equations, we mention, among other works, the study in [15], where the inverse source problem for the model (1) was investigated in a multidimensional setting. In [11], the author addressed the inverse problem of identifying a source term in degenerate singular parabolic equations, with degeneracy and singularity occurring in the interior of the spatial domain. More recently, in [14], the inverse source problem for a heat equation involving multipolar inverse-square potentials was considered.

From a numerical perspective, it is worth noting that only a limited number of works have been devoted to the identification of source terms or coefficients in parabolic equations with inverse-square potentials, despite the fact that such models arise naturally in both theoretical studies and applied contexts.

In contrast to the aforementioned studies, which commonly rely on techniques based on Carleman estimates [17], our approach is framed within the context of optimal control theory—a widely used methodology for addressing inverse source problems in a broad class of evolution equations [1, 10, 13, 22]. Specifically, we recast the inverse problem as an optimal control problem, where the unknown source term is treated as a control variable. The objective is then to minimize a suitably defined cost functional, which yields a quasi-solution to the original inverse problem.

By deriving and analyzing the first-order necessary optimality conditions, we establish both the local stability and uniqueness of the quasi-solution. More precisely, our main stability result can be stated as follows: let (U, f) and $(\tilde{U}, \tilde{f})$ be two solutions to the inverse problem (1)–(2) corresponding to final-time observations ω and $\tilde{\omega}$, respectively. Then, there exists a constant $C > 0$, independent of the final time T, such that

$$\|f - \tilde{f}\|^2_{L^2(\Omega)} \leq C\|\omega - \tilde{\omega}\|^2_{L^2(\Omega)}.$$

The second main contribution of this work concerns the numerical reconstruction of the unknown source term in the problem (1), based on the final-time observation (2). To this end, we develop a numerical scheme built upon the well-known Landweber iterative method. This approach has proven to be both reliable and efficient, as demonstrated through a series of numerical experiments.

The remainder of the paper is organized as follows. In Section 2, we establish the well-posedness of the direct problem (1). Section 3 is devoted to the analysis of the inverse problem within an optimal control framework; in particular, we prove the existence of a minimizer for the cost functional and derive the associated first-order necessary optimality condition. In Section 4, using the optimality condition, we establish a stability result for the inverse problem. Section 5 is concerned with the numerical reconstruction of the unknown source term. To this end, we implement a Landweber-type iterative method to compute an approximate solution to the inverse problem based on the final-time data.

2 Analysis of the direct problem

2.1 Functional framework

As is well known in the analysis of parabolic equations involving singular inverse-square potentials, the constant μ plays a crucial role in determining the well-posedness of the associated problem. Specifically, there exists a critical threshold $\mu^* > 0$ beyond which the problem becomes ill-posed. This upper bound is given by the optimal constant in the Hardy inequality, which ensures that for any function $z \in H_0^1(\Omega)$, the weighted function $\frac{z}{x} \in L^2(\Omega)$, and the following inequality holds:

$$\mu^* \int_\Omega \frac{z^2(x)}{x^2}\, dx \leq \int_\Omega |z_x(x)|^2\, dx. \tag{3}$$

In the one-dimensional setting $\Omega = (0,1)$, it is known that the critical constant is $\mu^* = \frac{1}{4}$. For fixed $\mu \in (0, \mu^*]$, we define the following functional space:

$$H^1_{\mu,0}(\Omega) := \left\{ z \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega) \,:\, z(0) = z(1) = 0, \quad \int_\Omega \left(z_x^2(x) - \mu \frac{z^2(x)}{x^2} \right) dx < +\infty \right\}.$$

This space is a Hilbert space when equipped with the inner product

$$(z_1, z_2)_\mu := \int_\Omega \left(z_{1,x}(x) z_{2,x}(x) - \mu \frac{z_1(x) z_2(x)}{x^2} \right) dx,$$

and the corresponding norm

$$\|z\|_\mu := \left(\int_\Omega \left(z_x^2(x) - \mu \frac{z^2(x)}{x^2} \right) dx \right)^{1/2}.$$

By standard arguments, one can show that there exist positive constants $C_1, C_2 > 0$, depending on μ, such that

$$(1 - 4\mu) \int_\Omega z_x^2\, dx + C_1 \int_\Omega z^2\, dx \leq \|z\|_\mu^2 \leq (1 + 4\mu) \int_\Omega z_x^2\, dx + C_2 \int_\Omega z^2\, dx.$$

This implies that for the subcritical case $\mu < \mu^*$, the spaces $H^1_{\mu,0}(\Omega)$ and $H^1_0(\Omega)$ are topologically equivalent with respect to their norms. However, in the critical case $\mu = \mu^*$, the space $H^1_{\mu,0}(\Omega)$ strictly contains $H^1_0(\Omega)$, that is,

$$H^1_0(\Omega) \subsetneq H^1_{\mu,0}(\Omega).$$

In this work, we restrict our attention to the subcritical case $0 < \mu < \mu^*$. Now, le us define the space $H^1_\mu(\Omega)$ as the completion of $H^1(\Omega)$ with respect to the norm

$$\|z\|_{H^1_\mu(\Omega)} := \left(\|z\|^2_{L^2(\Omega)} + \|z\|^2_\mu\right)^{1/2}.$$

Accordingly, we may write

$$H^1_{\mu,0}(\Omega) = \left\{z \in H^1_\mu(\Omega) \,:\, z(0) = z(1) = 0\right\}.$$

Under the assumption $\mu < \frac{1}{4}$, it is known that $H^1_\mu(\Omega)$ embeds continuously into the Sobolev space $W^{1,q}_0(\Omega)$ for all $1 \le q < 2$, and also into the fractional Sobolev spaces $H^s_0(\Omega)$ for all $0 \le s < 1$. Moreover, due to the compact embedding $W^{1,q}_0(\Omega) \hookrightarrow H^s_0(\Omega)$ for suitable $q = q(s)$ sufficiently close to 2, and the compactness of $H^s_0(\Omega) \hookrightarrow L^2(\Omega)$, we conclude that

$$H^1_\mu(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega),$$

where the embedding is compact. For more details on the properties of $H^1_\mu(\Omega)$, we refer the reader to [2] and [15].

2.2 Well-posedness of the Direct Problem

In order to analyze the inverse problem associated with the differential equation under consideration, a thorough understanding of the corresponding direct problem is essential. Therefore, we begin by establishing the well-posedness of the direct problem, with a detailed analysis of the existence, uniqueness, and regularity of its solutions.

To define a weak solution, we multiply equation (1) by a test function $\phi \in H^1_{\mu,0}(\Omega)$, integrate over Ω, and use integration by parts. This leads to the following variational formulation.

Definition 1. *Let $\theta_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$. A function θ is said to be a* weak solution *to problem (1) if*

$$\theta \in L^2(0,T; H^1_{\mu,0}(\Omega)), \quad \theta_t \in L^2(0,T; H^{-1}_\mu(\Omega)),$$

and for all test functions $\phi \in L^2(0,T; H^1_{\mu,0}(\Omega))$, the following variational identity holds:

$$\iint_{Q_T} \theta_t \phi \, dx\, dt + \iint_{Q_T} \theta_x \phi_x \, dx\, dt - \mu \iint_{Q_T} \frac{\theta\phi}{x^2} \, dx\, dt = \iint_{Q_T} f\phi \, dx\, dt, \tag{4}$$

with the initial condition $\theta(0) = \theta_0$ satisfied in $L^2(\Omega)$.

Remark 1. *The use of the weighted Sobolev space $H^1_{\mu,0}(\Omega)$ is crucial due to the singularity of the potential term $\mu x^{-2}\theta$, which renders the classical space $H^1_0(\Omega)$ inadequate when $\mu > 0$. For $\mu < \mu^*$, the Hardy inequality ensures that the bilinear form associated with the operator is coercive on $H^1_{\mu,0}(\Omega)$.*

Before formulating the inverse problem, it is necessary to establish that the associated direct problem is well posed.
This ensures that for any admissible source term, the governing singular parabolic equation admits a unique weak solution that depends continuously on the data.
Such a result guarantees that the forward operator is mathematically well defined, which is a fundamental prerequisite for the subsequent optimal control framework.

Theorem 1. *Let $\theta_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$. Then, problem (1) admits a unique weak solution θ in the sense of Definition 1, satisfying*

$$\theta \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_{\mu,0}(\Omega)), \quad \theta_t \in L^2(0,T; H^{-1}_{\mu}(\Omega)).$$

Moreover, the following a priori energy estimate *holds:*

$$\sup_{t\in[0,T]} \|\theta(t)\|^2_{L^2(\Omega)} + \int_0^T \|\theta(t)\|^2_{\mu}\, dt + \int_0^T \|\theta_t(t)\|^2_{H^{-1}_{\mu}(\Omega)}\, dt \leq C\left(\|\theta_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(Q_T)}\right), \quad (5)$$

where the constant $C > 0$ depends only on μ, Ω, and T.

3 Optimal control

The inverse problem under consideration is ill-posed in the sense of Hadamard, meaning that uniqueness and stability of solutions cannot be guaranteed without introducing additional constraints.
A widely used strategy in such cases is to recast the inverse problem as an optimal control problem, where the unknown source term is treated as a control variable.
This approach allows us to incorporate a regularization mechanism that stabilizes the inversion procedure.

More precisely, the inverse problem is reformulated as the minimization of a Tikhonov-type cost functional, consisting of two terms: a data misfit term that enforces consistency with the final-time observation, and a penalty term that ensures stability by controlling the norm of the source.
The admissible set of controls is restricted to bounded functions in $L^2(\Omega)$, which reflects a priori physical knowledge about the source.

This optimal control formulation serves as the foundation for the subsequent analysis. In particular, it allows us to establish the existence of minimizers (**Th**2), to derive necessary optimality conditions (**Th**3), and to prove stability estimates for the reconstructed source (**Th**4). Hence, Section 3 plays a crucial role in bridging the direct analysis of the forward problem with the theoretical and numerical treatment of the inverse problem.

4 Formulation of the Inverse Problem

The inverse problem addressed in this work can be stated as follows: given an initial condition $\theta_0(x) \in L^2(\Omega)$ and a final-time observation $\omega(x) \in L^2(\Omega)$, determine the spatially dependent source term $f(x)$ such that the corresponding solution θ to the initial-boundary value problem (1) satisfies the over-specified final condition

$$\theta(x,T) = \omega(x), \quad \text{for all } x \in \Omega. \tag{6}$$

To tackle this ill-posed problem, we adopt an optimal control framework. The inverse problem is reformulated as the following constrained optimization problem: find $f^* \in \mathcal{A}$ such that

$$\min_{f\in\mathcal{A}} \mathcal{J}(f) = \mathcal{J}(f^*), \quad \text{subject to } \theta[f] \text{ solving (1)}, \tag{7}$$

where the cost functional $\mathcal{J} : L^2(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{J}(f) := \frac{1}{2}\,\|\theta[f](\cdot,T) - \omega\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|f\|^2_{L^2(\Omega)}, \tag{8}$$

and $\gamma > 0$ is a regularization parameter. The admissible set $\mathcal{A} \subset L^2(\Omega)$ is given by

$$\mathcal{A} := \left\{ f \in L^2(\Omega) \,:\, c_0 \le f(x) \le c_1 \text{ a.e. in } \Omega \right\}, \tag{9}$$

for some constants $0 < c_0 < c_1$. The regularization term in (8) ensures the stability of the minimization problem and reflects a priori bounds on the unknown source.

Next, we establish the existence of an optimal solution to the minimization problem (7) by means of the following result.

Theorem 2. *Let $\theta_0 \in L^2(\Omega)$, $\omega \in L^2(\Omega)$, and assume that the direct problem (1) admits a unique weak solution $\theta[f]$ for every $f \in \mathcal{A}$, as guaranteed by Theorem 1. Then, the optimal control problem (7) admits at least one solution; that is, there exists $f^* \in \mathcal{A}$ such that*

$$\mathcal{J}(f^*) = \min_{f\in\mathcal{A}} \mathcal{J}(f).$$

Proof 1. *Since $\mathcal{J}(f) \ge 0$ for all $f \in \mathcal{A}$, the cost functional $\mathcal{J}$ admits an infimum over the admissible set $\mathcal{A}$, denoted by*

$$d := \inf_{f\in\mathcal{A}} \mathcal{J}(f).$$

Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence such that

$$d < \mathcal{J}(f_n) \le d + \frac{1}{n}, \quad \text{for all } n \in \mathbb{N}^*. \tag{10}$$

Since $\mathcal{A} \subset L^2(\Omega)$ is closed, convex, and bounded, there exists a subsequence (still denoted f_n) and a limit $f^* \in \mathcal{A}$ such that

$$f_n \rightharpoonup f^* \quad \text{weakly in } L^2(\Omega). \tag{11}$$

Let $\theta_n := \theta[f_n]$ denote the unique weak solution to problem (1) with source term f_n. By Theorem 1, the sequence (θ_n) is uniformly bounded in the spaces

$$L^2(0,T;H^1_{\mu,0}(\Omega)), \quad L^\infty(0,T;L^2(\Omega)), \quad \text{and} \quad L^2(0,T;H^{-1}_\mu(\Omega)).$$

Hence, up to a subsequence, there exists $\theta^* \in L^2(0,T;H^1_{\mu,0}(\Omega))$ *such that*

$$\begin{aligned} &\theta_n \rightharpoonup \theta^* \quad \text{weakly in } L^2(0,T;H^1_{\mu,0}(\Omega)),\\ &\theta_n \overset{*}{\rightharpoonup} \theta^* \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)),\\ &\partial_t\theta_n \rightharpoonup \partial_t\theta^* \quad \text{weakly in } L^2(0,T;H^{-1}_\mu(\Omega)). \end{aligned} \tag{12}$$

Furthermore, by the Aubin–Lions lemma and the compact embedding $H^1_{\mu,0}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, *we also obtain the strong convergence*

$$\theta_n \to \theta^* \quad \text{strongly in } L^2(Q_T). \tag{13}$$

Now, subtracting the weak formulations satisfied by $\theta^* = \theta[f^*]$ *and* $\theta_n = \theta[f_n]$, *and testing the resulting equation by* $\phi = \theta^* - \theta_n$, *we obtain the energy inequality:*

$$\frac{1}{2}\frac{d}{dt}\|\theta^*(t) - \theta_n(t)\|^2_{L^2(\Omega)} \le h(t)\int_\Omega (f^*(x) - f_n(x))(\theta^*(x,t) - \theta_n(x,t))\,dx. \tag{14}$$

Integrating both sides over $(0,T)$, *we get*

$$\|\theta^*(T) - \theta_n(T)\|^2_{L^2(\Omega)} \le \iint_{Q_T} h(t)(f^*(x) - f_n(x))(\theta^*(x,t) - \theta_n(x,t))\,dx\,dt.$$

Using the weak convergence $f_n \rightharpoonup f^*$ *in* $L^2(\Omega)$ *and strong convergence* $\theta_n \to \theta^*$ *in* $L^2(Q_T)$, *we deduce that the right-hand side vanishes as* $n \to \infty$, *hence:*

$$\|\theta^*(T) - \theta_n(T)\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty. \tag{15}$$

To conclude, we analyze the convergence of the misfit term. Define:

$$\begin{aligned} I_n &:= \left| \|\theta^*(T) - \omega\|^2_{L^2(\Omega)} - \|\theta_n(T) - \omega\|^2_{L^2(\Omega)} \right| \\ &\le \|\theta^*(T) - \theta_n(T)\|_{L^2(\Omega)} \cdot \|\theta^*(T) + \theta_n(T) - 2\omega\|_{L^2(\Omega)}. \end{aligned}$$

Due to (15), *we conclude:*

$$\lim_{n\to\infty} \|\theta_n(T) - \omega\|^2_{L^2(\Omega)} = \|\theta^*(T) - \omega\|^2_{L^2(\Omega)}. \tag{16}$$

Finally, applying weak lower semi-continuity of the L^2*-norm to* (11), *and using* (16), *we obtain:*

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{J}(f_n) &= \liminf_{n\to\infty} \left(\frac{1}{2}\|\theta_n(T) - \omega\|^2 + \frac{\gamma}{2}\|f_n\|^2 \right) \\ &\ge \frac{1}{2}\|\theta^*(T) - \omega\|^2 + \frac{\gamma}{2}\|f^*\|^2 = \mathcal{J}(f^*). \end{aligned}$$

Combining this with the minimality of the sequence (10), *we conclude that* f^* *is indeed a minimizer of the functional* $\mathcal{J}$, *i.e.,* $\mathcal{J}(f^*) = d$. *This completes the proof.*

Theorem 3 provides the first-order necessary condition characterizing the optimal control. This condition links the unknown source term with the adjoint state and plays a central role both in the theoretical analysis of stability and in the numerical implementation of the Landweber-type method.

Theorem 3. *Let $f^* \in \mathcal{A}$ be an optimal solution to the control problem (7), and let $\theta^* := \theta[f^*]$ denote the corresponding solution to the state equation (1). Then, the following variational inequality holds:*

$$\int_\Omega [\theta^*(x,T) - \omega(x)]\, \xi(x,T)\, dx + \gamma \int_\Omega f^*(x)\, (h(x) - f^*(x))\, dx \geq 0, \quad \forall h \in \mathcal{A}, \tag{17}$$

where $\xi \in L^2(0,T; H^1_{\mu,0}(\Omega)) \cap C([0,T]; L^2(\Omega))$ is the unique weak solution to the following adjoint problem:

$$\begin{cases} \partial_t \xi(x,t) - \xi_{xx}(x,t) - \dfrac{\mu}{x^2}\xi(x,t) = h(x) - f^*(x), & \text{in } Q_T := \Omega \times (0,T), \\ \xi(0,t) = \xi(1,t) = 0, & \text{for } t \in (0,T), \\ \xi(x,0) = 0, & \text{for } x \in \Omega := (0,1). \end{cases} \tag{18}$$

Proof 2. *Let $h \in \mathcal{A}$ and $\delta \in [0,1]$, and define a convex perturbation of the optimal control f^* by*

$$f_\delta := f^* + \delta(h - f^*).$$

Since $\mathcal{A}$ is convex, it follows that $f_\delta \in \mathcal{A}$ for all $\delta \in [0,1]$. Let $\theta_\delta := \theta[f_\delta]$ denote the unique weak solution to problem (1) associated with the control f_δ.

We define the perturbed cost functional

$$\mathcal{J}_\delta := \mathcal{J}(f_\delta) = \frac{1}{2}\int_\Omega |\theta_\delta(x,T) - \omega(x)|^2 dx + \frac{\gamma}{2}\int_\Omega |f_\delta(x)|^2 dx. \tag{19}$$

Since f^* is an optimal control, the function $\delta \mapsto \mathcal{J}(f_\delta)$ attains its minimum at $\delta = 0$. Therefore, the derivative of $\mathcal{J}_\delta$ with respect to δ satisfies

$$\left.\frac{d}{d\delta}\mathcal{J}(f_\delta)\right|_{\delta=0} \geq 0. \tag{20}$$

We now compute this derivative. By differentiating under the integral sign and using the chain rule, we obtain:

$$\frac{d}{d\delta}\mathcal{J}(f_\delta) = \int_\Omega [\theta_\delta(x,T) - \omega(x)]\, \frac{\partial \theta_\delta}{\partial \delta}(x,T)\, dx + \gamma \int_\Omega f_\delta(x)(h(x) - f^*(x))\, dx. \tag{21}$$

Evaluating (21) at $\delta = 0$, we define $\xi := \frac{\partial \theta_\delta}{\partial \delta}\big|_{\delta=0}$. Then inequality (20) becomes:

$$\int_\Omega [\theta^*(x,T) - \omega(x)]\, \xi(x,T)\, dx + \gamma \int_\Omega f^*(x)(h(x) - f^*(x))\, dx \geq 0, \tag{22}$$

which is precisely the desired variational inequality (17).

It remains to characterize ξ. Differentiating the state equation with respect to δ, we find that ξ satisfies the following linearized problem:

$$\begin{cases} \partial_t \xi(x,t) - \xi_{xx}(x,t) - \dfrac{\mu}{x^2}\xi(x,t) = h(x) - f^*(x), & \text{in } Q_T, \\ \xi(0,t) = \xi(1,t) = 0, & t \in (0,T), \\ \xi(x,0) = 0, & x \in \Omega, \end{cases}$$

which coincides with problem (18). This concludes the proof.

5 Stability Results

In this section, we investigate the stability of the inverse problem with respect to perturbations in the final-time observation data. Stability plays a central role in inverse problems, especially due to their inherent ill-posedness in the sense of Hadamard. In our context, the goal is to assess how the optimal solution f^* depends continuously on the measured data $\omega \in L^2(\Omega)$.

We consider two final-time observations $\omega, \tilde{\omega} \in L^2(\Omega)$, and analyze the corresponding solutions $f^*, \tilde{f}^* \in \mathcal{A}$ obtained by minimizing the cost functional (7). Under appropriate assumptions, we prove that small perturbations in the data lead to small changes in the recovered source, thereby establishing a Lipschitz-type stability estimate for the inverse problem.

Inverse problems are typically unstable with respect to perturbations in the data.

Theorem 4 demonstrates that, under the proposed optimal control formulation, the recovered source satisfies a Lipschitz-type stability estimate.
This result ensures robustness of the reconstruction and provides a rigorous theoretical justification for the numerical performance observed in Section 6.

Theorem 4. *Let $f, \tilde{f} \in \mathcal{A}$ be two optimal controls corresponding to the final observations $\omega, \tilde{\omega} \in L^2(\Omega)$, respectively, and let $\theta := \theta[f]$, $\tilde{\theta} := \theta[\tilde{f}]$ be the associated solutions to the state equation (1). Then, the following Lipschitz-type stability estimate holds:*

$$\|f - \tilde{f}\|^2_{L^2(\Omega)} \leq \frac{1}{2\gamma}\|\omega - \tilde{\omega}\|^2_{L^2(\Omega)}. \tag{23}$$

Proof 3. *Let $f, \tilde{f} \in \mathcal{A}$ be two optimal controls corresponding to the final-time data $\omega, \tilde{\omega} \in L^2(\Omega)$, and let $\theta := \theta[f]$, $\tilde{\theta} := \theta[\tilde{f}]$ be the associated solutions to the state problem (1).*

We apply the first-order optimality condition (17) with $f^* = f$ and $h = \tilde{f}$, yielding:

$$\int_\Omega [\theta(x,T) - \omega(x)]\,\xi(x,T)\,dx + \gamma\int_\Omega f(x)(\tilde{f}(x) - f(x))\,dx \geq 0, \tag{24}$$

where ξ solves the adjoint problem:

$$\begin{cases} \partial_t \xi - \xi_{xx} - \dfrac{\mu}{x^2}\xi = \tilde{f} - f, & \text{in } Q_T, \\ \xi(0,t) = \xi(1,t) = 0, & t \in (0,T), \\ \xi(x,0) = 0, & x \in \Omega. \end{cases} \tag{25}$$

Similarly, applying (17) with $f^* = \tilde{f}$ and $h = f$, we obtain:

$$\int_\Omega \left[\tilde{\theta}(x,T) - \tilde{\omega}(x)\right] \tilde{\xi}(x,T)\,dx + \gamma \int_\Omega \tilde{f}(x)(f(x) - \tilde{f}(x))\,dx \geq 0, \tag{26}$$

where $\tilde{\xi}$ solves:

$$\begin{cases} \partial_t \tilde{\xi} - \tilde{\xi}_{xx} - \dfrac{\mu}{x^2}\tilde{\xi} = f - \tilde{f}, & in\ Q_T, \\ \tilde{\xi}(0,t) = \tilde{\xi}(1,t) = 0, & t \in (0,T), \\ \tilde{\xi}(x,0) = 0, & x \in \Omega. \end{cases} \tag{27}$$

Adding inequalities (24) and (26) yields:

$$\gamma \|f - \tilde{f}\|^2_{L^2(\Omega)} \leq \int_\Omega \left[\theta(T) - \omega\right] \xi(T)\,dx + \int_\Omega \left[\tilde{\theta}(T) - \tilde{\omega}\right] \tilde{\xi}(T)\,dx. \tag{28}$$

Now, define the error functions $E := \theta - \tilde{\theta}$, and $X := \xi - \tilde{\xi}$. Then, E solves:

$$\begin{cases} \partial_t E - E_{xx} - \dfrac{\mu}{x^2}E = f - \tilde{f}, & in\ Q_T, \\ E(0,t) = E(1,t) = 0, & t \in (0,T), \\ E(x,0) = 0, & x \in \Omega, \end{cases} \tag{29}$$

and X solves the homogeneous problem:

$$\begin{cases} \partial_t X - X_{xx} - \dfrac{\mu}{x^2}X = 0, & in\ Q_T, \\ X(0,t) = X(1,t) = 0, & t \in (0,T), \\ X(x,0) = 0, & x \in \Omega. \end{cases} \tag{30}$$

Hence, by uniqueness of weak solutions, we conclude $X = 0$, i.e., $\xi = \tilde{\xi}$, and similarly $E = -\xi$ from comparing (29) and (25).

Substituting into (28) and using $E = -\xi$, we obtain:

$$\begin{aligned} \gamma \|f - \tilde{f}\|^2 &\leq \int_\Omega E(x,T)\xi(x,T)\,dx + \int_\Omega (\omega(x) - \tilde{\omega}(x))\xi(x,T)\,dx \\ &= -\|\xi(\cdot,T)\|^2_{L^2(\Omega)} + \int_\Omega (\omega - \tilde{\omega})\xi(\cdot,T)\,dx. \end{aligned}$$

Applying the Cauchy–Schwarz and Young inequalities, we get:

$$\begin{aligned} \gamma \|f - \tilde{f}\|^2 &\leq -\|\xi(T)\|^2 + \|\omega - \tilde{\omega}\| \cdot \|\xi(T)\| \\ &\leq -\|\xi(T)\|^2 + \frac{1}{2}\|\omega - \tilde{\omega}\|^2 + \frac{1}{2}\|\xi(T)\|^2 \\ &= -\frac{1}{2}\|\xi(T)\|^2 + \frac{1}{2}\|\omega - \tilde{\omega}\|^2 \\ &\leq \frac{1}{2}\|\omega - \tilde{\omega}\|^2. \end{aligned}$$

Dividing both sides by $\gamma > 0$, we conclude:

$$\|f - \tilde{f}\|^2_{L^2(\Omega)} \leq \frac{1}{2\gamma}\|\omega - \tilde{\omega}\|^2_{L^2(\Omega)},$$

which completes the proof.

Corollary 1. *Assume that assumptions of Theorem (4) hold. Furthermore, suppose that ω matches $\tilde{\omega}$ over Ω then $f = \tilde{f}$*

6 Numerical identification

In this section, we present a numerical strategy for identifying the unknown source term $f(x)$ in the singular parabolic problem (1), based on the final-time observation $\omega(x)$. Due to the ill-posedness of the inverse problem, direct inversion is highly unstable, and regularization techniques are essential to obtain stable and meaningful numerical approximations.

To this end, we implement an iterative regularization scheme based on the classical Landweber method, which is widely used in inverse problems due to its simplicity and robustness. The approach consists of iteratively updating the source term by moving along the negative gradient direction of the cost functional (7), evaluated via the solution of the associated forward and adjoint problems.

6.1 Landweber iteration method

Let us define the input-output operator $\mathcal{T}$ associated with the parabolic problem (1), which maps a source term to the final-time state of the corresponding solution. For simplicity of computation, we assume the initial condition is homogeneous, i.e., $\theta_0 = 0$. Then, the operator $\mathcal{T}$ is given by:

$$\begin{aligned}\mathcal{T}\colon\ & L^2(\Omega) \longrightarrow H^1_{\mu,0}(\Omega),\\ & f \mapsto \mathcal{T}f := \theta[f](\cdot, T),\end{aligned}$$

where $\theta[f]$ denotes the weak solution to problem (1) with source term $f \in L^2(\Omega)$, and $\theta_0 = 0$ as initial data. In this framework, $\mathcal{T}f$ represents the output measurement at the final time $t = T$.

In view of the above considerations, our inverse problem can be equivalently reformulated as the operator equation

$$\text{Find } f^\dagger \in \mathcal{A} \text{ such that } \mathcal{T}f^\dagger = \omega,$$

where $\mathcal{T}\colon L^2(\Omega) \to H^1_{\mu,0}(\Omega)$ is the input-output operator defined in the previous subsection, and $\omega \in L^2(\Omega)$ denotes the measured final-time data. Formally, the exact solution $f^\dagger$ satisfies the associated normal equation

$$\mathcal{T}^*\mathcal{T}f^\dagger = \mathcal{T}^*\omega,$$

where $\mathcal{T}^*$ denotes the adjoint of the operator $\mathcal{T}$. This normal equation can be interpreted as a fixed-point problem of the form

$$f^\dagger = f^\dagger - \beta\,\mathcal{T}^*\left(\mathcal{T}f^\dagger - \omega\right),$$

where $\beta > 0$ is a relaxation parameter. Based on this formulation, we construct an iterative Landweber-type method to approximate $f^\dagger$. Starting from an initial guess $f_0 \in L^2(\Omega)$, the iteration proceeds as:

$$\begin{aligned} f_{m+1} &= f_m - \beta\,\mathcal{T}^*\left(\mathcal{T}(f_m) - \omega\right) \\ &= f_m - \beta\,\mathcal{T}^*\left(\theta_m(\cdot, T) - \omega\right), \end{aligned} \tag{31}$$

where $\theta_m := \theta[f_m]$ is the solution of the forward problem (1) associated with the current iterate f_m.

It is well known (see, e.g., [9]) that the Landweber iteration (31) converges strongly to the minimum-norm solution $f^\dagger$, provided that $0 < \beta < 1/\|\mathcal{T}\|^2$ and the initial guess $f_0 \in \mathcal{D}(\mathcal{T})$. In practice, the iteration is terminated according to a suitable discrepancy principle or tolerance-based stopping rule.

For the numerical implementation of the Landweber algorithm, it is essential to compute the adjoint of the input–output operator.

Lemma 1 provides an explicit characterization of this adjoint in terms of the solution of an auxiliary boundary value problem.
This result enables the efficient numerical realization of the iterative reconstruction scheme.

Lemma 1. *Let $\psi \in L^2(\Omega)$, and let $\eta \in L^2(0, T; H^1_{\mu,0}(\Omega))$ be the unique weak solution of the following initial-boundary value problem:*

$$\begin{cases} \partial_t \eta(x,t) - \partial_{xx}\eta(x,t) + \dfrac{\mu}{x^2}\eta(x,t) = \psi(x), & \text{in } Q_T := \Omega \times (0,T), \\ \eta(x,0) = 0, & x \in \Omega, \\ \eta(0,t) = \eta(1,t) = 0, & t \in (0,T). \end{cases} \tag{32}$$

Then, the adjoint operator $\mathcal{T}^*\colon L^2(\Omega) \to L^2(\Omega)$, corresponding to the input-output operator $\mathcal{T}f = \theta[f](\cdot, T)$, is given by

$$\mathcal{T}^*\psi = \eta(\cdot, T),$$

Proof 4. *Let $f \in L^2(\Omega)$, and denote by $\theta = \theta[f] \in L^2(0, T; H^1_{\mu,0}(\Omega))$ the unique weak solution of the forward problem:*

$$\begin{cases} \partial_t \theta(x,t) - \partial_{xx}\theta(x,t) + \dfrac{\mu}{x^2}\theta(x,t) = f(x), & \text{in } Q_T, \\ \theta(x,0) = 0, & x \in \Omega, \\ \theta(0,t) = \theta(1,t) = 0, & t \in (0,T). \end{cases} \tag{33}$$

Then the input-output operator $\mathcal{T}\colon L^2(\Omega) \to L^2(\Omega)$ is defined by

$$\mathcal{T}f = \theta(\cdot, T).$$

Let $\psi \in L^2(\Omega)$, and let $\eta \in L^2(0,T;H^1_{\mu,0}(\Omega))$ be the solution of the following adjoint problem:

$$\begin{cases} \partial_t \eta(x,t) - \partial_{xx}\eta(x,t) + \dfrac{\mu}{x^2}\eta(x,t) = \psi(x), & in\ Q_T, \\ \eta(x,0) = 0, & x \in \Omega, \\ \eta(0,t) = \eta(1,t) = 0, & t \in (0,T). \end{cases} \tag{34}$$

We want to compute $\mathcal{T}^*\psi$ using the definition of the adjoint. By definition, $\mathcal{T}^*$ is the operator such that

$$\langle \mathcal{T} f, \psi \rangle_{L^2(\Omega)} = \langle f, \mathcal{T}^*\psi \rangle_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega).$$

Now, compute the left-hand side:

$$\langle \mathcal{T} f, \psi \rangle_{L^2(\Omega)} = \int_\Omega \theta(x,T)\psi(x)\,dx.$$

We aim to express this quantity in terms of f and η, and thereby identify $\mathcal{T}^*\psi$. To this end, we define the auxiliary function $v(x,t) := \eta(x,T-t)$. It is easy to verify (by direct substitution) that v satisfies the backward parabolic problem:

$$\begin{cases} -\partial_t v(x,t) - \partial_{xx} v(x,t) + \dfrac{\mu}{x^2} v(x,t) = \psi(x), & in\ Q_T, \\ v(x,T) = 0, & x \in \Omega, \\ v(0,t) = v(1,t) = 0, & t \in (0,T). \end{cases} \tag{35}$$

We now multiply the equation for θ by v, integrate over Q_T, and use integration by parts in time and space. We obtain:

$$\begin{aligned} \iint_{Q_T} f(x)v(x,t)\,dxdt &= \iint_{Q_T} \left(\partial_t\theta \cdot v + \partial_x\theta \cdot \partial_x v + \frac{\mu}{x^2}\theta v\right) dxdt \\ &= \iint_{Q_T} \left(-\partial_t v \cdot \theta + \partial_x\theta \cdot \partial_x v + \frac{\mu}{x^2}\theta v\right) dxdt, \end{aligned}$$

where we have used the fact that $\theta(x,0) = v(x,T) = 0$.

Since v satisfies (35), the right-hand side becomes:

$$\iint_{Q_T} \psi(x)\theta(x,t)\,dxdt.$$

Thus, we have established the identity:

$$\iint_{Q_T} f(x)v(x,t)\,dxdt = \iint_{Q_T} \psi(x)\theta(x,t)\,dxdt.$$

Now, reversing the change of variables $v(x,t) = \eta(x,T-t)$, we have:

$$\int_0^T v(x,t)\,dt = \int_0^T \eta(x,s)\,ds.$$

Similarly,

$$\iint_{Q_T} f(x)v(x,t)\,dxdt = \int_\Omega f(x) \int_0^T \eta(x,s)\,ds\,dx,$$

and

$$\iint_{Q_T} \psi(x)\theta(x,t)\,dxdt = \int_\Omega \psi(x) \int_0^T \theta(x,t)\,dt\,dx.$$

Assuming that this identity holds for all $T > 0$, we formally differentiate both sides with respect to T, obtaining:

$$\int_\Omega \psi(x)\theta(x,T)\,dx = \int_\Omega f(x)\eta(x,T)\,dx.$$

Therefore, we have:

$$(\mathcal{T}f,\psi)_{L^2(\Omega)} = (\theta(\cdot,T),\psi)_{L^2(\Omega)} = (f,\eta(\cdot,T))_{L^2(\Omega)},$$

and since this holds for all $f \in L^2(\Omega)$, we conclude:

$$\mathcal{T}^*\psi = \eta(\cdot,T).$$

To summarize, we now outline the main steps of the iterative procedure used to numerically reconstruct the unknown source term f in problem (1), based on the Landweber method.

Algorithm 1 Iterative Landweber Method for Source Identification

Require: Relaxation parameter $\beta > 0$, tolerance $\varepsilon > 0$, final-time data $\omega \in L^2(\Omega)$
Ensure: Approximate solution $f^\dagger$ and corresponding state $\theta^\dagger$ to the inverse problem

```
Initialization: Choose an initial guess f_0 ∈ A, and set k = 0
Solve Forward Problem: Compute θ_0 := θ[f_0] by solving (1)
Solve Adjoint Problem: Compute η_0 by solving (32) with source ψ = θ_0(·,T) − ω
Update Control: Set

      f_1 := f_0 − β η_0(·,T)

for k = 1, 2, ... until convergence do
   Solve θ_k := θ[f_k] from (1)
   if ‖θ_k(·,T) − ω‖_{L²(Ω)} < ε then
      Set f† := f_k, θ† := θ_k, and stop
   else
      Solve η_k from (32) with ψ = θ_k(·,T) − ω
      Update f_{k+1} := f_k − β η_k(·,T)
   end if
end for
```

6.2 Numerical results and discussions

In this subsection, we present numerical experiments that illustrate the performance of the proposed Landweber algorithm for reconstructing the space-dependent source term. The experiments are designed to validate both the accuracy and stability of the method under noise-free and noisy final-time data.

We begin with Example 1, where the exact solution of the forward problem is available in closed form. This allows for a direct comparison between the reconstructed and exact source profiles. In Example 2, the forward solution is generated numerically, thereby testing the algorithm in a more realistic setting. In both cases, the reconstructions confirm the theoretical predictions: the Landweber method converges towards the true source when noise-free data are used, while in the presence of noisy data, the algorithm still yields stable and accurate approximations, as shown in Figures 6.1–6.3.

The relative error $E_2(k)$ is also monitored as a function of the iteration index k. The error curves demonstrate a rapid initial decrease followed by saturation, which is consistent with the discrepancy principle and the finite accuracy of the numerical discretization. Overall, these results validate the effectiveness and robustness of the proposed method.

6.3 Numerical Implementation and Discretization

This subsection is devoted to numerical examples that illustrate the performance of the proposed Landweber algorithm for reconstructing the space-dependent source term $f(x)$ in the inverse problem (1). The solutions to both the direct and adjoint problems are approximated using finite-difference methods.

We fix the final time $T = 1$, so that the spatio-temporal domain is $Q_T = (0,1) \times (0,1)$. Let $M, N \in \mathbb{N}^*$ denote the number of spatial and temporal subdivisions, respectively. Define the mesh sizes

$$\Delta x = \frac{1}{M}, \quad \Delta t = \frac{1}{N}.$$

The spatial and temporal grid points are given by:

$$x_i = i\Delta x, \quad \text{for } i = 0, 1, \ldots, M, \qquad t_j = j\Delta t, \quad \text{for } j = 0, 1, \ldots, N.$$

The functions $\theta(x,t)$ (solution of the forward problem) and $\eta(x,t)$ (solution of the adjoint problem) are evaluated at these grid points. The numerical schemes employed for the discretization are based on finite-difference approximations of second-order spatial derivatives and backward or Crank–Nicolson schemes in time, ensuring stability in the presence of the singular potential μ/x^2. Boundary conditions are imposed explicitly at $x = 0$ and $x = 1$.

In the numerical tests, we measure the accuracy of the reconstructed source using the relative error at iteration k, defined by

$$E_2(k) := \|f^k - f\|^2_{L^2(\Omega)} = \frac{1}{M+1} \sum_{i=0}^{M} \left(f(x_i) - f^k(x_i) \right)^2,$$

where f is the exact source function and f^k is the reconstructed approximation at the k-th iteration, evaluated on the discrete grid $\{x_i\}_{i=0}^{M}$.

To test robustness against measurement errors, we also consider noisy data. The perturbed observation $\omega_\varepsilon(x)$ is generated from the exact final state $\omega(x) = \theta(x,T)$ by injecting a multiplicative random noise:

$$\omega_\varepsilon(x) = \omega(x) + \varepsilon \cdot \omega(x) \cdot \mathrm{rand}(x), \quad x \in \Omega, \tag{36}$$

where $\varepsilon \in (0,1)$ denotes the noise level and $\mathrm{rand}(x) \in (0,1)$ is a uniformly distributed random function over the spatial domain. This simulates realistic data perturbations encountered in practice.

Example 1. *In this first test case, we consider the inverse problem (1)–(2) with singularity parameter $\mu = \frac{1}{5}$, and a source term given by*

$$f(x,t) = -5\sin(\pi t)\left((x^2 - \pi^2 x^2 + \mu)\sin(\pi x)\right), \quad (x,t) \in Q_T.$$

It is easy to verify that the corresponding exact solution of the forward problem (1) is

$$\theta(x,t) = x^2 \sin(\pi x)(1 - e^{-t}), \quad x \in \Omega, \quad t \in [0,T].$$

Consequently, the final-time observation used in the inverse problem is computed as $\omega(x) = u(x,T)$. This example allows for direct comparison between the reconstructed and exact source terms.

Example 2. *In this second test, we consider a synthetic example in which the exact source term is prescribed as*

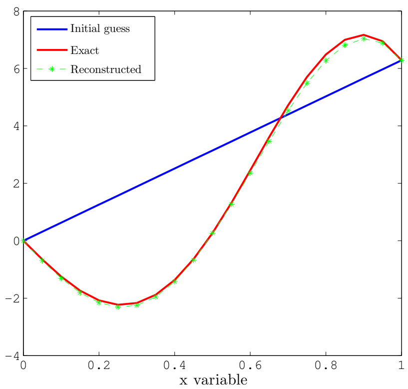
$$f(x) = \sin(\pi x), \quad x \in \Omega.$$

We set the singularity parameter to $\mu = \frac{1}{6}$. The final-time data $\omega(x) = u(x,T)$ is generated by solving the direct problem (1) using this exact source. This test serves to validate the reconstruction algorithm when the forward solution is numerically simulated, without using an explicit expression for $u(x,t)$.

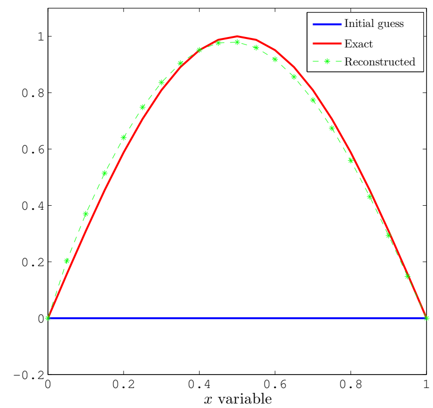
Discussion on Example 1

For the inversion process, we employ moderate discretization parameters, setting $\Delta t = 10^{-3}$ and $\Delta x = 5 \times 10^{-2}$. The Landweber iteration is initialized with the admissible guess $f_0(x) = x^2$. Figure 6.1 (a) shows a comparison between the exact source $f^\dagger$ and the reconstructed profile f_k in Example 1 after $k = 8000$ iterations. The agreement is notably close, confirming the convergence of the algorithm in the noise-free setting.

To assess the robustness of the method under measurement perturbations, we conduct additional experiments using noisy final-time data ω, generated according to the perturbation model (36). The reconstruction is evaluated after $k = 400$ iterations. As shown in Figure 6.2-(a), the reconstructions remain satisfactory under moderate noise levels, and the computed state $\theta_k(\cdot,T)$ matches the perturbed observations ω with high accuracy. However, for higher noise levels, the reconstruction quality deteriorates significantly. The evolution of $E_2(k)$ is shown in Figure 6.3-(a). We observe a monotonic decay of the error up to around $k = 400$, after which the reduction halts due to accumulated discretization errors in the numerical solution of the direct and adjoint problems.

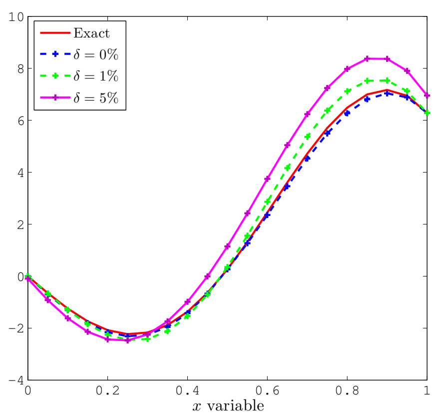


(a) Example 1 with $k = 800$

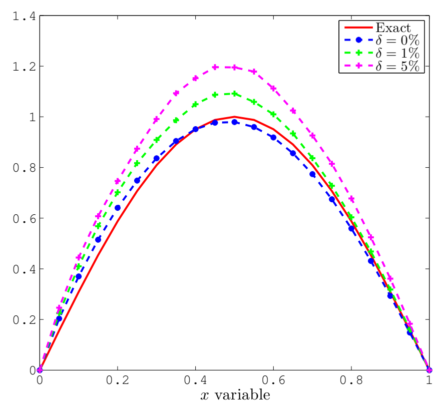


(b) Example 2 with $k = 400$

Figure 6.1: Numerical reconstruction.

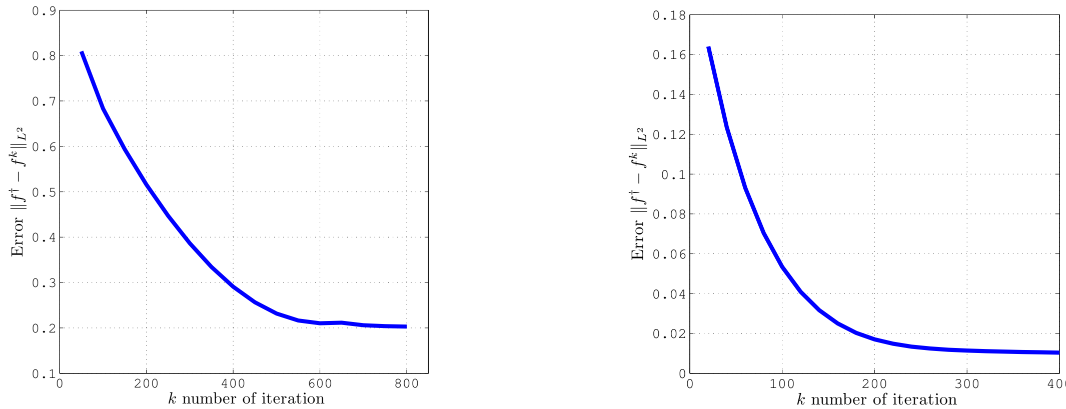


(a) Example 1



(b) Example 2

Figure 6.2: The numerical results with noise



(a) Example 1 (b) Example 2

Figure 6.3: Behaviour of reconstruction error $E_2(k)$ as a function of k.

Discussion on Example 2

For the second example, we consider a synthetic source term $f^\dagger(x) = \sin(\pi x)$ with singularity parameter $\mu = \frac{1}{6}$. The final-time observation $\omega(x)$ is generated by numerically solving the direct problem (1). The Landweber iteration is initialized with the same admissible guess $f_0(x) = 0$, and discretization parameters are set to $\Delta t = 10^{-3}$ and $\Delta x = 5 \times 10^{-2}$, as in the previous example. Figure 6.1 (b) displays the comparison between the exact source $f^\dagger$ and the reconstructed solution f_k after $k = 400$ iterations. The reconstruction achieves high accuracy with significantly fewer iterations than in Example 1, which is attributed to the simpler spectral content of the source.

To evaluate stability with respect to data perturbations, we introduce noisy observations based on the same noise model (36). The reconstruction after $k = 400$ iterations is reported in Figure 6.2 (b). The results indicate that the reconstructed state $\theta_k(\cdot, T)$ approximates the noisy data ω well for low to moderate noise levels. However, as the noise amplitude increases, the reconstruction degrades, consistent with the sensitivity of the inverse problem to measurement errors.

The convergence history of the relative error $E_2(k)$ is depicted in Figure 6.3 (b). Similar to the first example, we observe a rapid decay of the error up to $k \approx 300$, followed by stagnation. The early saturation is again due to the discretization effects and the finite resolution of the spatial grid, which limit further improvements in accuracy despite continued iteration.

Conclusion

In this work, we have addressed an inverse problem concerned with the identification of a space-dependent source term in a diffusion equation governed by a singular inverse-square potential. The proposed approach is based on an optimal control framework.

We began by establishing the existence and uniqueness of weak solutions to the direct problem. The inverse problem was then reformulated as a constrained optimization problem, for which we proved the existence of a minimizer and derived a first-order necessary optimality condition. This condition was further employed to demonstrate a Lipschitz-type stability result with respect to perturbations in the final-time data.

On the numerical side, we developed an iterative Landweber-type algorithm to reconstruct the unknown source term from noisy final measurements. A series of numerical experiments were carried out, confirming the effectiveness, stability, and robustness of the proposed reconstruction method, even in the presence of data perturbations.

As directions for future work, we plan to extend the current methodology to more complex models, including systems of coupled singular parabolic equations and fractional-order singular diffusion problems.

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2-бөлім Раздел 2 Section 2

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MATHEMATICAL MODELING OF RADIATION DEFECT FORMATION PROCESSES ON LIGHT TARGETS

The article analyzes the problem of studying the mechanisms of radiation defect generation in materials under ion irradiation. During the research, algorithms were developed to calculate the cascade-probability function (CPF) and the concentration of cascade regions as a function of the depth of the irradiated material, which allowed for an increase in the accuracy of modeling defect formation processes. The calculations of the CPF and the concentration of cascade regions revealed patterns in the behavior of radiation defects depending on the physical parameters of irradiation. The comparison of the obtained calculated data with experimental results confirmed the validity of the developed algorithms and models. A distinctive feature of the proposed method is the application of an analytical cascade-probability approach, which allows tracking the dynamics of defect formation at any depth of the target, unlike traditional numerical methods that require significant computational resources.

These results can be explained by the fact that the process of particle interaction with matter and the formation of radiation defects is probabilistic, allowing for the determination of the probabilities of ion interactions with materials (CPF) at any depth of the irradiated material, which enables more accurate modeling of defect formation processes and their dependence on physical parameters such as energy and depth. The developed models and algorithms can be applied in materials science, micro- and nanoelectronics, and in predicting the radiation resistance of structural materials.

Key words: ion, algorithm, ion implantation, cascade-probabilistic function, concentration of radiation defects.

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Математическое моделирование процессов радиационного дефектообразования на легких мишенях

В данной статье анализируется проблема изучения механизмов генерирования радиационных дефектов в материалах при ионном облучении. В процессе исследования были созданы алгоритмы для расчета каскадно-вероятностной функции (КВФ) и концентрации каскадных областей в зависимости от глубины облучаемого материала, что дало возможность повысить точность моделирования процессов дефектообразования. Выполненные расчеты КВФ и концентрации каскадных областей позволили выявить закономерности поведения радиационных дефектов в зависимости от физических параметров облучения. Сопоставление полученных расчетных данных с экспериментальными результатами подтвердило достоверность разработанных алгоритмов и моделей. Отличительной чертой предложенного метода является применение аналитического каскадно-вероятностного подхода, который позволяет отслеживать динамику дефектообразования на любой глубине мишени, в отличие от традиционных численных методов, требующих значительных вычислительных ресурсов.

Эти результаты объясняются тем, что процесс взаимодействия частиц с веществом и образования радиационных дефектов является вероятностным и позволяет получить вероятности взаимодействия ионов с материалами (КВФ), на любой глубине облучаемого материала, что позволяет более точно моделировать процессы дефектообразования и их зависимость от физических параметров, таких как энергия, глубина. Разработанные модели и алгоритмы могут быть применены в материаловедении, микро- и наноэлектронике, при прогнозировании радиационной стойкости конструкционных материалов.
Ключевые слова: ион, алгоритм, ионная имплантация, каскадно-вероятностная функция, концентрация радиационных дефектов.

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Жеңіл нысталарда радиациялық дектіктердің қалыптасу процестерін математикалық модельдеу

Мақала ион сәулеленуі кезінде материалдардағы радиациялық ақауларды генерациялау механизмдерін зерттеу мәселесін талдайды. Зерттеу барысында сәулеленген материалдың тереңдігіне байланысты каскадты ықтималдық функциясын (КЫФ) және каскадтық аймақтардың концентрациясын есептеу үшін алгоритмдер әзірленді, бұл ақау түзілу процестерін модельдеудің дәлдігін арттыруға мүмкіндік берді. КЫФ және каскадтық аймақтардың концентрациясы бойынша жүргізілген есептеулер радиациялық ақаулардың физикалық сәулелену параметрлеріне байланысты мінез-құлқын анықтауға мүмкіндік берді. Алынған есептік деректерді эксперименттік нәтижелермен салыстыру әзірленген алгоритмдер мен модельдердің дұрыстығын растады. Ұсынылған әдістің ерекшелігі - ақау түзілу динамикасын мақсаттың кез келген тереңдігінде бақылауға мүмкіндік беретін аналитикалық каскадты ықтималдық тәсілін қолдану, дәстүрлі сандық әдістердің едәуір есептеу ресурстарын талап ететіндігімен салыстырғанда.
Бұл нәтижелер бөлшектердің затпен өзара әрекеттесу және радиациялық ақаулардың түзілу процесі ықтималдықты болып табылатындығымен түсіндіріледі, бұл иондардың материалдармен (КЫФ) өзара әрекеттесу ықтималдықтарын сәулеленген материалдың кез келген тереңдігінде анықтауға мүмкіндік береді, бұл ақау түзілу процестерін және олардың энергия, тереңдік сияқты физикалық параметрлерге тәуелділігін дәл модельдеуге мүмкіндік береді. Дамытылған модельдер мен алгоритмдер материалтану, микро- және наноэлектроникада, конструкциялық материалдардың радиациялық төзімділігін болжауда қолданылуы мүмкін.
Түйін сөздер: ион, алгоритм, иондық имплантация, каскадты-ықтималдық функция, радиациялық ақаулардың концентрациясы.

1 Introduction

Research in the field of ion implantation and radiation-induced defect formation has been conducted and continues to be an important topic for the scientific community to this day.

This area is particularly relevant for the advancement of science in Kazakhstan, as many organizations are engaged in experimental studies on the effects of various types of radiation, including electron (1–10 MeV), proton and alpha (1–50 MeV), and ion (100–1000 keV) irradiation. There is a need for further explanation and analysis of experiments related to ion irradiation.

The relevance of the topic is confirmed by a number of factors. First, with the increasing consumption of materials and the growing demands for their properties, it is essential to develop new methods for their production and processing. It is expected that in the coming years, the demand for structural materials will significantly increase, highlighting the need to optimize ion irradiation processes to enhance radiation resistance and other key properties of materials. Second, the results of research in this area can have a substantial impact on their practical applications. The development of algorithms for calculating defect distribution will allow for more accurate predictions of material behavior under various operating conditions. This, in turn, could lead to the creation of more efficient and reliable structural materials, contributing to the advancement of technologies and increasing their competitiveness in the market. For example, the Institute of Nuclear Physics in Almaty has a proton accelerator and an alpha-particle accelerator (light ions), a cyclotron, and a nuclear reactor. The Eurasian University in Astana has an ion accelerator, and work is being conducted at the National Nuclear Center in the city of Kurchatov. Similar research is being carried out in countries near and far abroad.

Previously, mathematical models were developed to describe the processes of radiation defect formation within the framework of an analytical CP-method using the simple CPF (probability of transition in n steps) that did not account for energy losses due to ionization and excitation. Mathematical models have been developed taking into account energy losses for alpha particles, protons, electrons and ions. Unlike electrons, protons, and alpha particles, for ions it is necessary to find the actual result area for calculating transition probabilities and the concentration of cascade regions.

The object of the study is a solid body. The subject of the research is the CPFs depending on the number of interactions and the depth of particle penetration, the concentration of cascade regions during ion irradiation. The aim of the research is to mathematically model the processes of radiation defect formation in materials irradiated with ions, taking into account energy losses. Accordingly, the following tasks have been formulated:

- to develop algorithms for calculating the CPFs and the concentration of cascade regions as a function of the depth of the material irradiated with ions, and to create a software package (SP) for performing the calculations of these characteristics;
- to carry out calculations of the CPFs and the concentration of cascade regions;
- to verify the developed algorithm through a comparison of the simulation results with experimental data..

2 Literature review and problem statement

The paper [1] presents the results of research aimed at assessing the suitability of glassy carbon as a material for packaging nuclear waste. It is shown, that ion bombardment with xenon leads to the amorphization of the glassy carbon structure, which is confirmed by Raman spectroscopy analysis. However, unresolved questions remain regarding the influence of defects and radiation damage on the microstructure and surface of glassy carbon. This may be due to objective difficulties related to the lack of data on the behavior of glassy carbon under radiation exposure. A way to overcome these difficulties could be the use of computer modeling methods to predict material behavior. This approach was used in article [2], but the results indicated that a broader range of factors affecting the microstructure needs to be considered. All of this suggests that it is advisable to conduct research for a deeper understanding of the impact of radiation defect formation.

In article [3], the problem of understanding how the energy transferred to electronic and atomic subsystems can affect defect dynamics in materials is addressed. The interaction of displacement and ionization cascades induced by irradiation in silicon carbide (SiC) is investigated. It is shown that under ion irradiation, a delay in damage accumulation is observed, which linearly depends on both the increase in ionization and the energy transferred to the material. However, unresolved questions remain regarding the evolution of defects and their influence on material properties. This may be due to the limitations of existing models, making the investigation of this issue relevant. A way to overcome these difficulties could be the use of more complex models, such as Monte Carlo methods, which are classified as statistical trial methods and are numerical approaches to solving mathematical problems by predicting random variables. This method began to be widely applied in the 1970s for statistical modeling of particle trajectories and calculating the energy distribution transferred from ions to the atoms of the material. This approach was used in article [4], allowing for the calculation of particle penetration depth and the determination of radiation defect concentrations, such as vacancy clusters and interstitial atoms, which became the basis for quantitative analysis of radiation damage to materials. However, the results indicated that the dynamics of defect interactions need to be considered, as the algorithm only allows for the calculation of the distribution and concentration of primary defects, without accounting for their subsequent evolution.

In article [5], the method of pulsed ion bombardment was used to investigate the interaction of noble gas ions with potassium tantalate (KTaO3) and its influence on damage formation and amorphization. It was shown that the mechanism of amorphization is primarily due to defects caused by ion irradiation. However, the results indicated that additional factors influencing defect dynamics need to be considered.

Article [6] presents the results of a study dedicated to the formation of nanostructured TiAlN coatings on AISI 304 stainless steel substrates using reactive magnetron sputtering. It is shown that irradiation of the coatings with argon ions at an energy of 200 keV leads to changes in their mechanical properties, including hardness and Young's modulus. However, the calculations only considered the distribution of implanted ions, not the defects generated by them.

Article [7] employs a more detailed analytical method, such as scanning electron microscopy. However, the results indicated that the influence of various irradiation conditions on mechanical properties needs to be taken into account.

Thus, existing research highlights the need for further investigation into the effects of ion implantation and irradiation on material properties, opening new horizons for scientific research.

One of the key issues of ion implantation is the formation of radiation defects. First and foremost, it is essential to know the distributions of defects generated in atomic collision cascades. Despite the well-known numerical methods and models, analytical methods have undeniable advantages over them, even if they can only approximate certain phenomena. In this regard, a cascade-probabilistic method has been developed using a CPF, which allows for the creation of mathematical models in analytical form and, consequently, provides the opportunity to track the entire defect formation process at any depth of the irradiated material dynamically.

Previously, a simple CPF was used [8], which did not account for the actual changes in the range and angle of ejection of particles after each collision. This is not always justified, especially if the interaction range depends on energy. Such an approach can at best be used only for estimating results. Therefore, work has been conducted in this direction, resulting in mathematical models of the CPF that consider energy losses, the dependence of range and cross-section on energy for electrons, protons, alpha particles, and ions [9,10].

This research aims to address the specified problems, which will allow for the management of defect generation and evolution, ultimately leading to the production of materials with desired properties.

3 Materials and methods

The interaction cross-section for ions is calculated using the Rutherford formula [11]. The observation depths are based on data from tables [12]. The obtained interaction cross-section values are approximated by the following expression:

$$\sigma(h) = \frac{1}{\lambda_0}\left(\frac{1}{a(E_0 - kh)} - 1\right), \tag{1}$$

where λ_0, a, k, E_0 – approximation parameters.

It is not possible to use the provided formula (1) from [13] for the calculations of the CPF, as it leads to overflow issues when λ_0 is small or when n takes on large values (which can reach several million). By modifying this formula, we obtain:

$$\psi_n(h', h, E_0) = exp\left(-\left(\frac{h-h'}{\lambda_0}\right) + \frac{1}{\lambda_0 ak} ln\left(\frac{E_0 - kh'}{E_0 - kh}\right)\right) * \prod_{i=1}^{n}\left(\frac{h - h' - \frac{ln\frac{(E_0-kh')}{(E_0-kh)}}{ak}}{\lambda_0 i}\right), \tag{2}$$

where n – number of interactions; h', h – depths of ion generation and registration, $l = \frac{1}{\lambda_0 ak}$

In order to optimize the algorithms for calculating the CPF as a function of n and h, as well as the concentration of cascade regions, Stirling's formulas (5) and (6) from [14] are applied. To automate the determination of the CPF result area based on n, h, and the concentration of cascade regions, Binary [15] and Ternary [16] search algorithms are used. When ions interact with matter, defects are formed in the form of cascade regions, which consist of vacancy clusters and interstitial atom aggregates.

To calculate the concentration of cascade regions, the following formula from [9] is used:

$$C_k(E_0, h) = \int_{E_c}^{E_{2max}} W(E_0, E_2, h)\, dE_2, \tag{3}$$

$$E_{2max} = \frac{4(m_1c^2 m_2c^2)}{(m_1c^2 + m_2c^2)^2} E_1,$$

E_1 – the energy of the particle after energy losses at h, E_0 – the initial energy of the ion, $C_k(E_0, h)$ is defined considering that the energy of the particle at depth h is $E_1(h)$, E_{2max} –

the maximum possible energy gained by an atom, E_c – the threshold energy, E_2 – the energy of the primary knocked-out atom, m_1c_2 – the rest energy of the ion, m_2c_2 – the rest energy of the atom.

The spectrum of primary knocked-out atoms (PKA) is calculated using the (4.26) from [9]. Modifying equation (3) , we obtain:

$$C_k(E_0,h)=\frac{E_d}{\lambda_2 E_c}\frac{(E_{2max}-E_c)}{(E_{2max}-E_d)}\sum_{n=n_0}^{n_1}\int_{h-k\lambda_2}^{h} exp\left(-\frac{h-h'}{\lambda_2}\right)\psi_n(h')\frac{dh'}{\lambda_1(h')}, \tag{4}$$

where $\psi_n(h')$ is used as (2), E_d – the average displacement energy, n_0, n_1 –the initial and final values of the number of collisions from the area of the CPF, k – is an integer greater than one.

$$\lambda_1(h')=\frac{1}{\sigma_0 n_0(\frac{1}{a(E_0-kh')}-1)}*10^{24}\ (cm),$$

$$\lambda_2=\frac{1}{\sigma_2 n_0}*10^{24}\ (cm).$$

The cross-section σ_2 is calculated using the Rutherford formula; λ_1, λ_2 – are the mean free paths for ion-atomic and atomic-atomic collisions, respectively, $\sigma_0 = 1/\lambda_0$.

4 Results and discussions

When approximating curves, difficulties arise in specifying the initial data λ_0, a, E_0, and k in the approximation formula. The approximation expression best describes the cross-section values, as the theoretical correlation coefficient is sufficiently close to 1. The approximating curves of the dependence of σ on h are shown in Fig. 4.1. Table 1 presents the approximation parameters and the theoretical correlation coefficients for boron in silicon at various initial energy values. The targets are metal - aluminum and semiconductor - silicon.

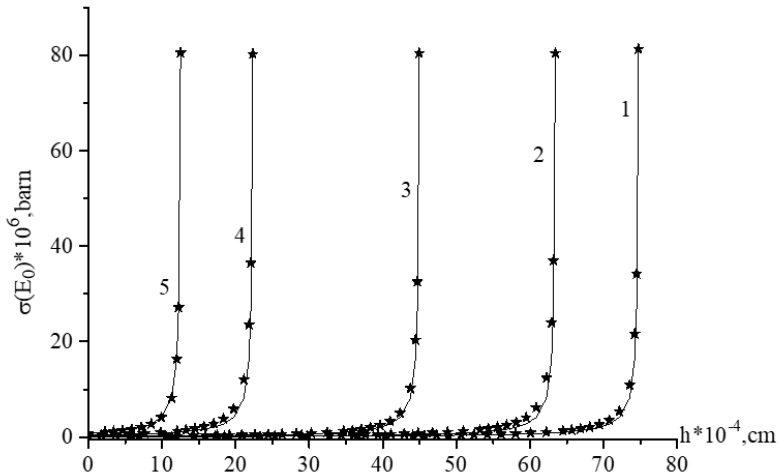

Figure 4.1: Approximation of the modified cross-section of the CPF for boron in silicon: $E_0 =$ 1000, 800, 500, 200, 100 (1-5) keV. Solid lines – approximation values, stars – calculated data for the dependence of the cross-section on h

Table 1: Approximation parameters for boron in silicon

E_0	$\sigma_0 * 10^6$	a	k	E_0'	η
1000	1,96808	0,2161	584,57	4,3801	0,9961
800	1,96898	1,199	91,07	0,57988	0,9954
500	1,61508	2,01	55,807	0,2519	0,9887
200	4,16808	0,254	921,908	2,0801	0,978
100	3101242	0,32	2342,4	3,041	0,9811

The CPF represents the probability that a particle generated at a certain depth h' will reach a specific depth h(registration depth) after n collisions.

Let's conduct a study of the CPF and examine its main properties:

1. Domain of the function: $E_0/ak(k-1) < h < E_0/k$.
2. $lim_{h \to h'} \psi_n(h', h, E_0) = 0$, $lim_{h \to h'} \psi_0(h', h, E_0) = 1$.
3. $lim_{k \to 0} \psi_n(h', h, E_0) = \frac{1}{n!}(\frac{h-h'}{\lambda})^n exp(-\frac{h-h'}{\lambda})$, that is the probability of transitioning over n steps, taking energy losses into account, reduces to the simplest CP-function without considering energy losses.
4. The sum of the CPF over all interactions is equal to 1, i.e., $K_\infty = \sum_{n=0}^{\infty} \psi_n(h', h, E_0) = 1$.
5. $lim_{n \to 0} \psi_n(h', h, E_0) = (\frac{E_0 - kh'}{E_0 - kh})^{-l} exp(\frac{h-h'}{\lambda_0}) = \psi_0(h', h, E_0)$
6. $lim_{n \to \infty} \psi_n(h', h, E_0) = 0$, that is the probability of a particle experiencing an infinite number of collisions while traversing a depth from h' to h is undoubtedly equal to zero.
7. $\int_{h'}^{h} \frac{\psi_n(h', h, E_0) dh}{\lambda(h)} = 1$, where $\lambda(h) = 1/(\sigma(h) n_0)$.

The results of the CPF as a function of n are presented in Figs. 4.2 and 4.3.

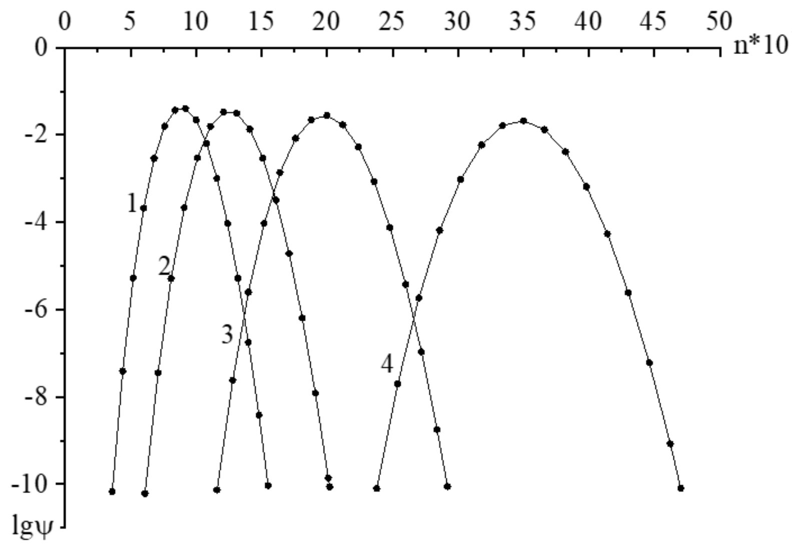

Figure 4.2: Dependence CPF on n for boron in silicon at $E_0 = 800\ keV$; $h = 5,0 \times 10^{-3}; 5,5 \times 10^{-3}; 6,0 \times 10^{-3}; 6,3 \times 10^{-3}$ (cm.) (1–4)

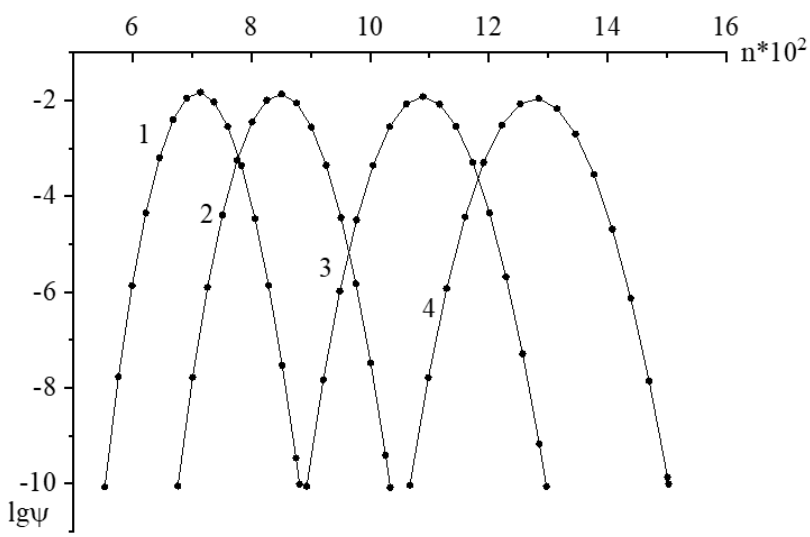

Figure 4.3: Dependence CPF on n for selenium in aluminum at $E_0 = 200\ keV$; $h = 1,5 \times 10^{-4}; 1,7 \times 10^{-4}; 2,0 \times 10^{-4}; 2,2 \times 10^{-4}$ (cm.) (1–4)

The results of the CPF as a function of h are presented in Figs. 4.4 and 4.5.

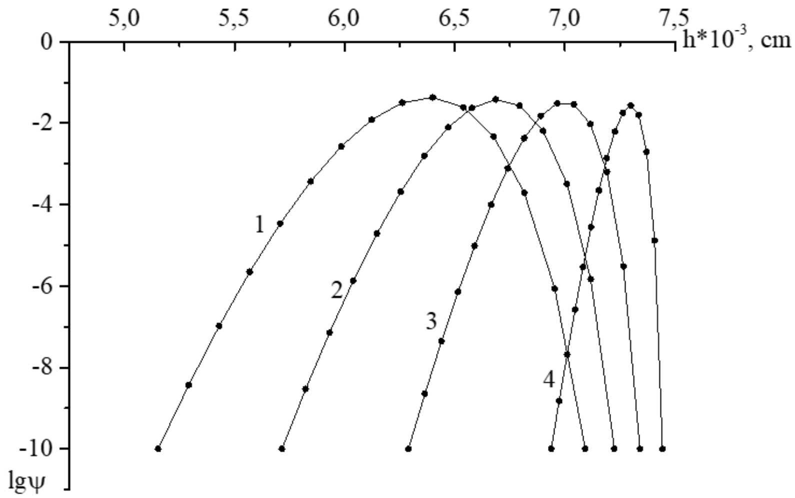

Figure 4.4: Dependence CPF on h for boron in silicon at $E_0 = 1000\ keV$; $n = 116611; 306622; 651245; 1421513$ (cm.) (1–4)

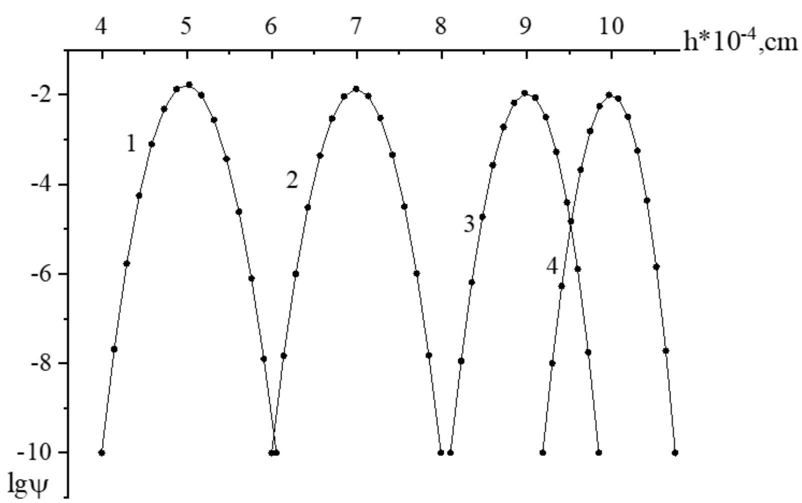

Figure 4.5: Dependence CPF on h for selenium in aluminum at $E_0 = 800\ keV$; $n = 549; 866; 1288; 1564$ (1–4)

The results of the calculations of the concentration of cascade regions for boron in silicon are presented in Fig. 4.6 and Table 2 and for selenium in aluminum in Fig. 4.7 and Table 3.

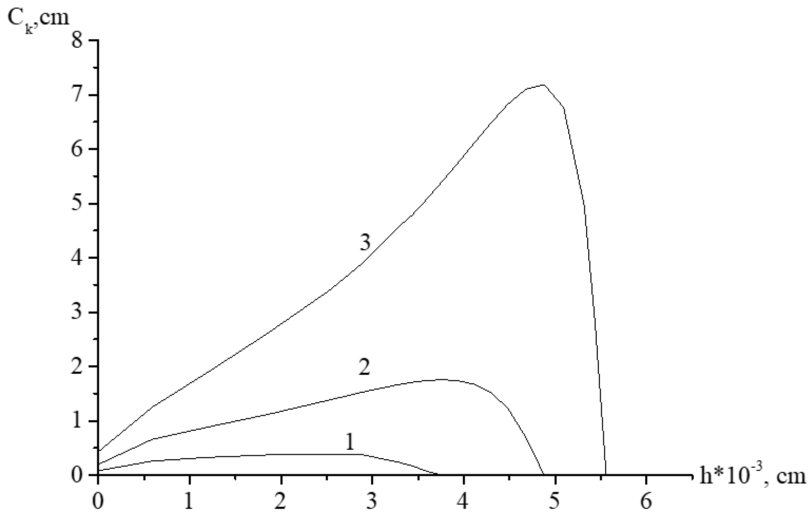

Figure 4.6: Dependence of concentration of cascade regions on h during the irradiation of silicon with boron ions for: $E_0 = 800\ keV$, $E_c = 200\ keV$ (1), $100\ keV$ (2), $50\ keV$ (3)

Table 2: The boundaries of the region for determining the concentration of cascade regions for boron in silicon at $E_c = 50\ keV$, $E_0 = 1000\ keV$

$h * 10^{-3}$, cm	C_k, cm	E_0, keV	n_0	n_1
0,1000	0,011332	1000	1	9
0,5000	0,1567	900	1	14
1,1286	0,52027	800	1	20
1,7200	0,946	700	1	27
2,3330	1,3808	600	1	34
2,9770	1,91	500	1	44
3,6600	2,61075	400	1	57
4,0270	3,0578	350	1	65
4,4070	3,5944	300	1	75
4,5860	3,6484	280	1	77
4,7270	4,1036	260	1	85
4,8930	4,3868	240	1	90
5,0640	4,536	220	1	96
5,2390	5,012	200	2	103
5,4204	5,3427	180	4	110
5,6080	5,63	160	6	119
5,8030	5,9128	140	9	129
6,0070	5,9962	120	13	141
6,2215	5,7033	100	18	156
6,4470	4,1819	80	25	175
6,5640	2,40099	70	30	187
6,6860	0	60	36	200

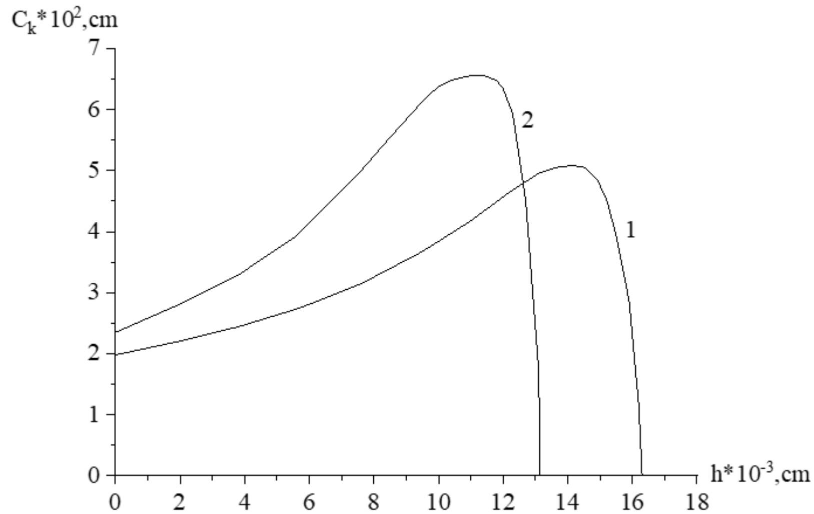

Figure 4.7: Dependence of concentration of cascade regions on h during the irradiation of aluminum with selenium ions for: $E_c = 100\ keV$, $E_0 = 1000\ keV$ (1), 800 keV (2)

Table 3: The boundaries of the region for determining the concentration of cascade regions for selenium in aluminum at $E_c = 50\ keV$ $E_0 = 1000\ keV$

$h * 10^{-4}$, cm	C_k, cm	E_0, keV	n_0	n_1
1,15	451,7	1000	21	173
2,21	477,7	900	74	288
4,33	544,8	800	214	521
6,71	417,84	700	320	678
8,49	412,58	600	429	831
10,23	406,59	500	671	1154
27,6	408,17	400	1713	2441
28,1	397,25	350	1749	2483
29,3	383,21	300	1835	2584
30,6	376,91	280	1925	2695
32,1	369,6	260	2036	2822
33,1	360,18	240	2109	2907
34,2	348,9	220	2189	3001
35,3	335,06	200	2270	3095
36,2	317,6	180	2336	3172
37,6	295,92	160	2440	3292
38,1	266,74	140	2477	3335
39,4	228,11	120	2573	3447
40,9	173,37	100	2686	3577
41,8	89,97	80	2754	3655
42,4	30,11	70	2799	3707

For comparison with experimental data, Fig. 8 shows the distributions of implanted boron ions with depth in irradiated silicon as a function of depth at an energy of 50 keV.

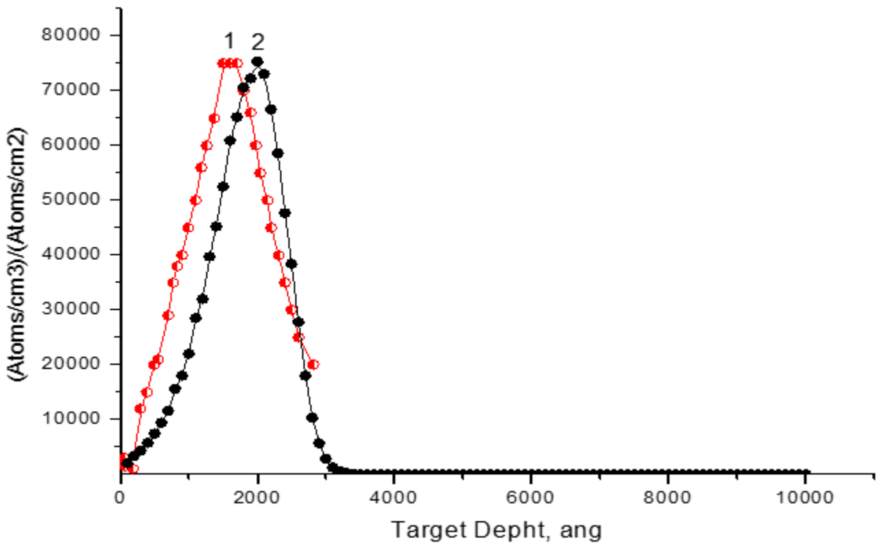

Figure 4.8: Distribution of implanted boron atoms with depth in Si: 1 – Experiment (50 keV); 2 – SRIM (50 keV)

The distribution of implanted boron atoms in Si exhibits clear maxima, and their concentration is unevenly distributed with depth. Comparing the calculated distributions of implanted boron ions (50 keV) in silicon with experimental data shows a good agreement. The slight discrepancies between the calculations and experimental data for boron are attributed to the incomplete consideration of the influence of the ambient temperature.

Unlike previously developed mathematical models of radiation defect formation that used simple CPF [8], this work proposes improved models that:

– take into account energy losses due to ionization and excitation of the medium's atoms, as well as the dependence of interaction range and cross-section on energy, achieving a closer agreement of the obtained results with physical experimental data (within 15%);

– they provide the ability to observe the entire process of ion interaction with the substance as a function of h.

Unlike electrons [17], protons [18], and alpha particles [19], modeling the interaction process of ions with matter is more complex [10], [13], [20–23]. In the proposed approach:

– it is possible to perform calculations for various incoming particles and targets from the periodic table;

– patterns of cascade regions distribution are identified based on threshold energy, penetration depth, and initial ion energy;

– the actual region of the result is found when calculating the transition probabilities and the concentration of cascade regions.

5 Conclusion

Algorithms and SP have been developed to calculate transition probabilities as a function of the number of collisions, the penetration depth of particles, and the concentration of cascade regions for ions. This enables the identification of patterns in the behavior of radiation defects based on the physical parameters of irradiation. All CPF calculations were performed using (2) , and concentrations were calculated using (4) in C#, with MS SQL Server 2022 used as the database management system.

The developed SP enables the calculation of interaction cross-sections, ionization losses, observation depths, and the determination of approximation coefficients. The created algorithms have enabled the automation of the area of result finding and the identification of patterns in the behavior of this area.

An analysis of the CPF has been conducted, and the properties that these functions should possess have been outlined. A comparison of the calculation results of the distributions of implanted particles for boron (50 keV) in silicon has been made with experimental data.

The study of the distribution of implanted ions and energy losses is crucial for understanding the processes occurring during ion implantation. The application of the obtained results can significantly enhance the understanding of radiation processes related to defect formation in materials irradiated by charged particles.

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