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CONVEXITY AND CONCAVITY IN SUBMAJORIZATION INEQUALITIES FOR τ-MEASURABLE OPERATORS

We proved the following result. Consider a semi-finite von Neumann algebra equipped with a trace, and let there be several τ-measurable operators together with a nonnegative function defined on the nonnegative real line. Suppose also that we are given positive weights whose total sum equals one. If the function obtained by applying f to the square root of its argument is convex and if f vanishes at zero, then the weighted sum of the values of f applied to the absolute values of the operators is at least as large as a certain expression involving f evaluated at both the average of all the operators and their pairwise differences. If the same function of the square root is concave, the inequality reverses: the mentioned expression becomes no smaller than the weighted sum of the transformed absolute values. This theorem yields a significant generalization of Clarkson-type inequalities in the noncommutative setting and extends the result previously established by Alrimawi, Hirzallah, and Kittaneh.
Keywords: Clarkson inequality, $\tau-$measurable operator, von Neumann algebra, generalized singular value function, submajorization inequality, convex function, concave function.

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$\tau-$өлшемді операторлар үшін субмажорланған теңсіздіктердегі ойыс және дөңес функциялар

Біз келесі нәтижені дәлелдедік. Жартылай ақырлы фон Нейман алгебрасы қарастырылсын, сонымен қатар, мұнда бірнеше τ-өлшемді операторлар және нақты санның теріс емес мәндерінде анықталған функция бар болсын. Сондай-ақ қосындысы бірге тең болатын оң салмақтар берілсін. Егер нөлдегі мәні нөлге тең f функциясы өзінің аргументінің квадрат түбіріне қолданылған кезде ойыс болса, онда операторлардың модульдеріне осы f функцияны қолдану арқылы алынған мәндердің салмақталған қосындысы барлық операторлардың орташа мәніне және олардың айырмашылықтарына қолданылған f функциядан құралған белгілі бір өрнектен кем болмайды. Ал егер аталған функция квадрат түбірге қолданылғанда дөңес болса, онда бұл теңсіздік қарама-қарсы бағытта орындалады: көрсетілген өрнек салмақталған қосындыдан кем емес болады. Бұл нәтиже бейкоммутативтік ортадағы Кларксон типті теңсіздіктерді едәуір кеңейтеді және Alrimawi, Hirzallah және Kittaneh еңбегінде алынған нәтижені толықтырады.
Түйін сөздер: Кларксон теңсіздігі, τ-өлшемді оператор, фон Нейман алгебрасы, сингулярлы жалпылама функция, субмажоризация теңсіздігі, дөңес функция, ойыс функция.

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Выпуклость и вогнутость в субмажоризационных неравенствах для $\tau-$измеримых операторов

Мы доказали следующий результат. Рассматривается полу-конечная алгебра фон Неймана, несколько τ-измеримых операторов и неотрицательная функция, определённая на неотрицательной полуоси. Также заданы положительные веса, сумма которых равна единице. Если функция, которая получается при применении f к квадратному корню своего аргумента, является выпуклой и при этом значение f в нуле равно нулю, то взвешенная сумма значений функции, применённой к модулям операторов, не меньше определённого выражения, построенного из значений f, взятых на среднем операторе и на их попарных различиях. Если же указанная функция, применённая к квадратному корню аргумента, является вогнутой, то справедливо обратное неравенство: это выражение не меньше взвешенной суммы соответствующих значений. Полученный результат существенно расширяет неравенства типа Кларксона в некоммутативной среде и дополняет форму, ранее установленную Alrimawi, Hirzallah и Kittaneh.
Ключевые слова: неравенство Кларксона, τ-измеримый оператор, алгебра фон Неймана, обобщённая сингулярная функция, неравенство субмажоризации, выпуклая функция, вогнутая функция.

1 Introduction

Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$, where τ is a faithful normal semifinite trace on $\mathcal{M}$. The $*$-algebra $S(\tau)$ of all τ-measurable operators affiliated with $\mathcal{M}$ provides a natural framework for extending classical inequalities to the noncommutative setting.

An operator $x \in S(\tau)$ is said to be positive (denoted $x \geq 0$) if $\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in \mathrm{dom}(x)$. For two self-adjoint operators $x, y \in S(\tau)$, we write $x \geq y$ if $x - y \geq 0$.

In this framework, unitarily invariant norms play a central role. We say $|||\cdot|||$ is unitarily invariant if

$$|||uxv||| = |||x|||$$

for all $x \in S(\tau)$ and for all unitary operators $u, v \in \mathcal{M}$. Typical examples include the generalized Schatten p-norms given by

$$\|x\|_p = \tau\left(|x|^p\right)^{1/p}, \quad 0 < p < \infty.$$

A fundamental result in the theory of unitarily invariant norms is given by the classical *Clarkson inequalities* in Schatten classes, which provide bounds for operator combinations under p-norms. These inequalities have been studied and generalized in various contexts, particularly in the setting of bounded operators on Hilbert spaces.

We now consider $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. This algebra is a von Neumann algebra under the weak operator topology and forms one of the most important examples in functional analysis. Within $\mathcal{B}(\mathcal{H})$, the positive operators are those x for which

$$\langle x\xi, \xi \rangle \geq 0 \quad \text{for all } \xi \in \mathcal{H}.$$

A notable extension of Clarkson-type inequalities to this noncommutative setting was established by Alrimawi, Hirzallah, and Kittaneh in [1]. They proved the following result:

Let $|||\cdot|||$ be a unitarily invariant norm, $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ be positive operators, and let $\alpha_1, \ldots, \alpha_n$ be positive real numbers such that $\sum_{j=1}^{n} \alpha_j = 1$

Define the index set $S_\ell = \{1, \ldots, n\} \setminus \{\ell\}$. Then:

1. If f is a nonnegative function on $[0,\infty)$ with $f(0)=0$ and $g(t)=f\left(\sqrt{t}\right)$ is convex on $[0,\infty)$, then for every unitarily invariant norm,

$$\left|\left|\left| f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|A_\ell-\sum_{j=1}^n \alpha_j A_j\right|\right) + \sum_{j,k\in S_\ell} f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|A_j-A_k|\right) + f\left(\left|\sum_{j=1}^n \alpha_j A_j\right|\right)\right|\right|\right| \le \left|\left|\left|\sum_{j=1}^n \alpha_j f(|A_j|)\right|\right|\right|,$$

for each $\ell=1,\dots,n$.

2. If f is a nonnegative function on $[0,\infty)$ such that $g(t)=f\left(\sqrt{t}\right)$ is concave on $[0,\infty)$, then for every unitarily invariant norm,

$$\left|\left|\left|\sum_{j=1}^n \alpha_j f(|A_j|)\right|\right|\right| \le \left|\left|\left| f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|A_\ell-\sum_{j=1}^n \alpha_j A_j\right|\right) + \sum_{j,k\in S_\ell} f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|A_j-A_k|\right) + f\left(\left|\sum_{j=1}^n \alpha_j A_j\right|\right)\right|\right|\right|,$$

for each $\ell=1,\dots,n$.

This paper develops new norm inequalities for operators that generalize and refine the classical Clarkson inequalities. Prior results on non-commutative Clarkson-type inequalities for the symmetric norm of τ-measurable operators can be found in [7], while related submajorization inequalities were established in [4,5].

The structure of the paper is as follows. Section 2 develops several key operator identities on Hilbert spaces that form the foundation for our results. In Section 3, we establish the main norm inequalities for convex and concave functions of τ-measurable operators.

2 Preliminaries

Throughout this paper, we denote by $\mathcal{M}$ a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, equipped with a faithful normal semifinite trace τ. A densely defined, closed linear operator x on $\mathcal{H}$ with domain $D(x)$ is said to be *affiliated* with $\mathcal{M}$ if

$$u^*xu = x \quad \text{for all unitary } u\in\mathcal{M}',$$

where $\mathcal{M}'$ is the commutant of $\mathcal{M}$.

An operator x affiliated with $\mathcal{M}$ is said to be *τ-measurable* if, for every $\varepsilon>0$, there exists a projection $e\in\mathcal{M}$ such that $e(\mathcal{H})\subseteq D(x)$ and $\tau\left(e^\perp\right)<\varepsilon$, where $e^\perp := 1-e$.

The set of all τ-measurable operators is denoted by $S(\tau)$. This space forms a $*$-algebra under strong closures of algebraic operations.

Let $\mathcal{P}(\mathcal{M})$ denote the lattice of projections in $\mathcal{M}$. For $\varepsilon,\delta>0$, define the sets

$$\mathcal{N}(\varepsilon,\delta) := \left\{x\in S(\tau) : \exists\, e\in\mathcal{P}(\mathcal{M}) \text{ such that } \|xe\|<\varepsilon \text{ and } \tau\left(e^\perp\right)<\delta\right\}.$$

These sets form a neighborhood base at 0 for a metrizable Hausdorff topology on $S(\tau)$, called the *measure topology*. With this topology, $S(\tau)$ becomes a complete topological $*$ algebra (see [11]).

For $x \in S(\tau)$, *generalized singular value function* $\mu(t;x)$ is defined by

$$\mu(t;x) := \inf\left\{\|xe\| : e \in \mathcal{P}(\mathcal{M}),\ \tau\left(e^{\perp}\right) \le t\right\}, \quad t \ge 0,$$

(see [9] for more details about the generalized singular value function).

If $x, y \in S(\tau)$, then we say that x is *submajorized* by y and write $x \preceq y$ if and only if

$$\int_0^t \mu(s;x)ds \le \int_0^t \mu(s;y)ds, \quad t \in [0,\infty).$$

See [2,3] and the references therein.

3 Main results

The next result is a restatement of [6, Lemma 2].

Lemma 1 *Let $x_0, \cdots, x_{n-1}$ be τ-measurable operators and let $\alpha_0, \cdots, \alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1} \alpha_j x_j = 0$ and $\sum_{j=0}^{n-1} \alpha_j = 1$. For each $\ell \in \{0, \ldots, n-1\}$, we have*

$$\frac{\alpha_\ell}{(1-\alpha_\ell)}|x_\ell|^2 + \sum_{j,k\in S_\ell} \frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}|x_j - x_k|^2 = \sum_{j=0}^{n-1} \alpha_j |x_j|^2,$$

where $S_\ell = \{0, \ldots, n-1\} \setminus \{\ell\}$.

Lemma 2 *Let $\sum_{j=0}^{n-1} \alpha_j = 1$. Then for each $\ell = 0, \ldots, n-1$,*

$$\frac{\alpha_\ell}{(1-\alpha_\ell)}\left|x_\ell - \sum_{j=0}^{n-1}\alpha_j x_j\right|^2 + \sum_{j,k\in S_\ell} \frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}|x_j - x_k|^2 = \sum_{j=0}^{n-1} \alpha_j |x_j|^2 - \left|\sum_{j=0}^{n-1}\alpha_j x_j\right|^2.$$

In particular,

$$\frac{\alpha_\ell}{(1-\alpha_\ell)}\left|x_\ell - \sum_{j=0}^{n-1}\alpha_j x_j\right|^2 \le \sum_{j=0}^{n-1} \alpha_j |x_j|^2 - \left|\sum_{j=0}^{n-1}\alpha_j x_j\right|^2,$$

with equality if and only if $x_j = x_k$ for all $j, k \in S_\ell$.

Proof. Let

$$\bar{x} := \sum_{j=0}^{n-1} \alpha_j x_j.$$

Define new variables:

$$y_j := x_j - \bar{x}, \quad \text{so that} \quad \sum_{j=0}^{n-1} \alpha_j y_j = 0.$$

Applying Lemma 1 to the y_j (since they satisfy the same assumptions), we get:

$$\frac{\alpha_\ell}{1-\alpha_\ell}\left|y_\ell\right|^2+\sum_{j,k\in S_\ell}\frac{\alpha_j\alpha_k}{2\left(1-\alpha_\ell\right)}\left|y_j-y_k\right|^2=\sum_{j=0}^{n-1}\alpha_j\left|y_j\right|^2. \tag{1}$$

Note that:

$$y_\ell=x_\ell-\bar{x},\quad \left|y_j-y_k\right|^2=\left|(x_j-\bar{x})-(x_k-\bar{x})\right|^2=\left|x_j-x_k\right|^2,$$

and

$$\begin{aligned}\sum_{j=0}^{n-1}\alpha_j\left|y_j\right|^2&=\sum_{j=0}^{n-1}\alpha_j\left|x_j-\bar{x}\right|^2\\&=\sum_{j=0}^{n-1}\alpha_j\left(\left|x_j\right|^2-x_j^*\bar{x}-\bar{x}^*x_j+\left|\bar{x}\right|^2\right)\\&=\sum_{j=0}^{n-1}\alpha_j\left|x_j\right|^2-\left(\sum_{j=0}^{n-1}\alpha_jx_j^*\right)\bar{x}-\bar{x}^*\sum_{j=0}^{n-1}\alpha_jx_j+\left|\bar{x}\right|^2\sum_{j=0}^{n-1}\alpha_j\\&=\sum_{j=0}^{n-1}\alpha_j\left|x_j\right|^2-\bar{x}^*\bar{x}-\bar{x}^*\bar{x}+\left|\bar{x}\right|^2\\&=\sum_{j=0}^{n-1}\alpha_j\left|x_j\right|^2-\left|\bar{x}\right|^2.\end{aligned}$$

Substituting these into equation (1), we obtain:

$$\frac{\alpha_\ell}{1-\alpha_\ell}\left|x_\ell-\bar{x}\right|^2+\sum_{j,k\in S_\ell}\frac{\alpha_j\alpha_k}{2\left(1-\alpha_\ell\right)}\left|x_j-x_k\right|^2=\sum_{j=0}^{n-1}\alpha_j\left|x_j\right|^2-\left|\bar{x}\right|^2,$$

which is exactly the statement of the lemma.

The second part of the lemma follows from the nonnegativity of the second term on the left-hand side:

$$\sum_{j,k\in S_\ell}\frac{\alpha_j\alpha_k}{2\left(1-\alpha_\ell\right)}\left|x_j-x_k\right|^2\geq 0,$$

with equality if and only if $x_j=x_k$ for all $j,k\in S_\ell$.

We obtain the following result, similar to [9, Proposition 4.6] (see also [6, Lemma 3]).

Lemma 3 *Let $x_0,\ldots,x_{n-1}$ be positive τ-measurable operators and let $\alpha_0,\ldots,\alpha_{n-1}$ be positive real numbers such that $\sum_{j=0}^{n-1}\alpha_j=1$.*

(i) If $g:[0,\infty)\to[0,\infty)$ is a convex function with $g(0)=0$, then

$$\mu\left(t;g\left(\sum_{j=0}^{n-1}\alpha_jx_j\right)\right)\leq\mu\left(t;\sum_{j=0}^{n-1}\alpha_jg\left(x_j\right)\right).$$

(ii) If $h:[0,\infty)\to[0,\infty)$ is a concave function, then

$$\mu\left(t;\sum_{j=0}^{n-1}\alpha_j h\left(x_j\right)\right)\leq\mu\left(t;h\left(\sum_{j=0}^{n-1}\alpha_j x_j\right)\right).$$

We recall the following well-known result (see [8, Theorem 5.3]).

Lemma 4 *Let for all $x_0,\cdots,x_{n-1}$ be positive $\tau-$measurable operators.*

(i) If $g:[0,\infty)\longrightarrow[0,\infty)$ is convex function with $g(0)=0$, then

$$\sum_{j=0}^{n-1}g\left(x_j\right)\preceq g\left(\sum_{j=0}^{n-1}x_j\right)$$

(ii) If $h:[0,\infty)\longrightarrow[0,\infty)$ is concave function, then

$$h\left(\sum_{j=0}^{n-1}x_j\right)\preceq\sum_{j=0}^{n-1}h\left(x_j\right)$$

The following theorem offers a natural extension of [1, Theorem 3.8].

Theorem 1 *Let $x_0,\ldots,x_{n-1}$ be τ-measurable operators and $\alpha_0,\ldots,\alpha_{n-1}$ be positive real numbers such that*

$$\sum_{j=0}^{n-1}\alpha_j=1.$$

Then for each $\ell\in\{0,\ldots,n-1\}$:

(i) If $f:[0,\infty)\to[0,\infty)$ is a non-negative function with $f(0)=0$ and $g(t)=f(\sqrt{t})$ is convex, then

$$f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell-\sum_{j=0}^{n-1}\alpha_j x_j\right|\right)+\sum_{j,k\in S_\ell}f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j-x_k|\right)+f\left(\left|\sum_{j=0}^{n-1}\alpha_j x_j\right|\right)\preceq\sum_{j=0}^{n-1}\alpha_j f(|x_j|).$$

(ii) If $h(t)=f(\sqrt{t})$ is concave, then the reverse inequality holds:

$$\sum_{j=0}^{n-1}\alpha_j f(|x_j|)\preceq f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell-\sum_{j=0}^{n-1}\alpha_j x_j\right|\right)+\sum_{j,k\in S_\ell}f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j-x_k|\right)+f\left(\left|\sum_{j=0}^{n-1}\alpha_j x_j\right|\right).$$

Proof. Let $f : [0,\infty) \to [0,\infty)$ be a non-negative function. Define $g(t) = f(\sqrt{t})$ and $h(t) = f(\sqrt{t})$, depending on convexity or concavity of $f(\sqrt{t})$.

We prove each case separately.

(i) Assume that $g(t) = f(\sqrt{t})$ is convex and $f(0) = 0$.

Let

$$A_\ell = \sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell - \sum_{j=0}^{n-1}\alpha_j x_j\right|, \quad B_{\ell,j,k} = \sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j - x_k|, \quad C = \left|\sum_{j=0}^{n-1}\alpha_j x_j\right|.$$

Then,

$$f(A_\ell) + \sum_{j,k\in S_\ell} f\left(B_{\ell,j,k}\right) + f(C) = g(A_\ell^2) + \sum_{j,k\in S_\ell} g\left(B_{\ell,j,k}^2\right) + g(C^2).$$

By convexity of g and Lemma 4(i), we have

$$g(A_\ell^2) + \sum_{j,k\in S_\ell} g\left(B_{\ell,j,k}^2\right) + g(C^2) \preceq g\left(A_\ell^2 + \sum_{j,k\in S_\ell} B_{\ell,j,k}^2 + C^2\right).$$

Note that by Lemma 2, we can write:

$$A_\ell^2 + \sum_{j,k\in S_\ell} B_{\ell,j,k}^2 + C^2 = \sum_{j=0}^{n-1}\alpha_j|x_j|^2. \tag{2}$$

Hence,

$$f(A_\ell) + \sum_{j,k\in S_\ell} f\left(B_{\ell,j,k}\right) + f(C) \preceq g\left(\sum_{j=0}^{n-1}\alpha_j|x_j|^2\right).$$

Applying Lemma 3(i) (since g is convex and $g(0) = 0$), we conclude:

$$g\left(\sum_{j=0}^{n-1}\alpha_j|x_j|^2\right) \preceq \sum_{j=0}^{n-1}\alpha_j g(|x_j|^2) = \sum_{j=0}^{n-1}\alpha_j f(|x_j|).$$

Therefore,

$$f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell - \sum_{j=0}^{n-1}\alpha_j x_j\right|\right) + \sum_{j,k\in S_\ell} f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j - x_k|\right) + f\left(\left|\sum_{j=0}^{n-1}\alpha_j x_j\right|\right) \preceq \sum_{j=0}^{n-1}\alpha_j f(|x_j|).$$

(ii) Now assume $h(t) = f(\sqrt{t})$ is concave.

Then using Lemma 3(ii) and (2),

$$\sum_{j=0}^{n-1}\alpha_j f(|x_j|) = \sum_{j=0}^{n-1}\alpha_j h(|x_j|^2) \preceq h\left(\sum_{j=0}^{n-1}\alpha_j|x_j|^2\right) = h\left(A_\ell^2 + \sum_{j,k\in S_\ell} B_{\ell,j,k}^2 + C^2\right).$$

Then by Lemma 4(ii) and concavity of h, we get:

$$h\left(A_\ell^2+\sum_{j,k\in S_\ell} B_{\ell,j,k}^2+C^2\right)\preceq h(A_\ell^2)+\sum_{j,k\in S_\ell} h\left(B_{\ell,j,k}^2\right)+h(C^2)$$
$$=f(A_\ell)+\sum_{j,k\in S_\ell} f\left(B_{\ell,j,k}\right)+f(C).$$

Thus,

$$\sum_{j=0}^{n-1}\alpha_j f(|x_j|)\preceq f\left(\sqrt{\frac{\alpha_\ell}{1-\alpha_\ell}}\left|x_\ell-\sum_{j=0}^{n-1}\alpha_j x_j\right|\right)$$
$$+\sum_{j,k\in S_\ell} f\left(\sqrt{\frac{\alpha_j\alpha_k}{2(1-\alpha_\ell)}}|x_j-x_k|\right)+f\left(\left|\sum_{j=0}^{n-1}\alpha_j x_j\right|\right).$$

This completes the proof.

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REDUCTION THEOREMS FOR DISCRETE HARDY OPERATOR ON MONOTONE SEQUENCE CONES $(0 < p < 1)$

In this work, we investigate the discrete Hardy and Copson operators acting on the cone of nonnegative monotone sequences. It is established that the weighted inequalities of type $l_p \to l_q$ for these operators, in the case $0 < q < \infty$, $0 < p < 1$, can be reduced to the corresponding inequalities defined on the cone of general nonnegative sequences. The latter possess a broader basis for proof, which significantly extends the possibilities for their analysis. Weighted inequalities for the integral Hardy operator (in the continuous setting) on the cone of nonnegative nonincreasing functions have been studied previously by many authors. Reduction theorems for inequalities involving Hardy-type integral operators on the cone of nonincreasing functions to inequalities on the cone of nonnegative functions are well established. In this paper, we provide several theorems that demonstrate the equivalence between inequalities for discrete Hardy and Copson operators on the cone of nonnegative nonincreasing sequences and the corresponding inequalities on the cone of nonnegative sequences. Our proofs differ substantially from those in the continuous case. Methods applicable in the continuous setting do not always work in the discrete setting. For the case $p > 1$, analogous results were obtained by the authors earlier.
Key words: reduction theorems, discrete Hardy operator, monotone sequences, weighted inequalities, Copson operator.

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Монотонды тізбектер конусындағы дискретті Харди операторы үшін редукциялық теоремалар $(0 < p < 1)$

Бұл жұмыста біз теріс емес монотонды тізбектер конусындағы дискретті Харди және Копсон операторларын қарастырамыз. $0 < q < \infty$, $0 < p < 1$ жағдайындағы монотонды тізбектер конусындағы дискретті Харди және Копсон операторлары үшін $l_p \to l_q$ түріндегі салмақты теңсіздіктерді теріс емес тізбектер конусындағы сәйкес теңсіздіктерге келтіруге болатыны көрсетілген. Соңғыларың дәлелдеу негізі кеңірек, бұл оларды талдау мүмкіндіктерін айтарлықтай кеңейтеді. Теріс емес өспейтін функциялар конусындағы интегралдық Харди операторы үшін (үзіліссіз жағдайда) салмақты теңсіздіктерді бұрын көптеген авторлар зерттеген. Бұл мақалада теріс емес өспейтін тізбектер конусындағы дискретті Харди және Копсон операторлары үшін теңсіздіктердің және теріс емес тізбектер конусындағы теңсіздіктерге эквиваленттілігіне қатысты әртүрлі теоремалар ұсынылған. ұсынылған дәлелдеулер үздіксіз жағдайдағылардан айтарлықтай ерекшеленеді. Үздіксіз параметрде қолданылатын әдістер дискретті параметрде әрқашан жұмыс істей бермейді. $p > 1$ жағдайына ұқсас нәтижелерді біз бұрын алғанбыз.
Түйін сөздер: редукция теоремалары, дискретті Харди операторы, монотонды тізбектер, салмақталған теңсіздіктер, Копсон операторы.

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Редукционные теоремы для дискретного оператора Харди на конусах монотонных последовательностей $(0 < p < 1)$

В данной статье мы рассматриваем дискретные операторы Харди и Копсона на конусе неотрицательных монотонных последовательностей. Показано, что весовые неравенства вида $l_p \to l_q$ для дискретных операторов Харди и Копсона на конусе монотонных последовательностей в случае $0 < q < \infty$, $0 < p < 1$ могут быть сведены к соответствующим неравенствам на конусе неотрицательных последовательностей. Последние обладают более широкой основой для доказательства, что существенно расширяет возможности их анализа. Весовые неравенства для интегрального оператора Харди (в непрерывном случае) на конусе неотрицательных невозрастающих функций ранее исследовались многими авторами. Хорошо известны также теоремы сведения неравенств для интегральных операторов типа Харди на конусе невозрастающих функций к неравенствам на конусе неотрицательных функций. В работе приводятся различные теоремы, касающиеся эквивалентности неравенств для дискретных операторов Харди и Копсона на конусе неотрицательных невозрастающих последовательностей и неравенств на конусе неотрицательных последовательностей. Представленные доказательства существенно отличаются от доказательств в непрерывном случае. Методы, применимые в непрерывной постановке, не всегда работают в дискретной. Для случая $p > 1$ аналогичные результаты были ранее получены нами.

Ключевые слова: редукционные теоремы, дискретный оператор Харди, монотонные последовательности, весовые неравенства, оператор Копсона.

1 Introduction

In this paper, we study weighted Hardy and Copson inequalities, focusing on their behavior on cones of nonnegative and monotone sequences. We establish conditions under which these inequalities hold and explore their relation to corresponding results in the continuous setting.

$$\left(\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}x(k)\right)^{q}a(m)\right)^{\frac{1}{q}}\leq C\left(\sum_{m=1}^{\infty}x(m)^{p}b(m)\right)^{\frac{1}{p}}, \tag{1}$$

$$\left(\sum_{m=1}^{\infty}\left(\sum_{k=m}^{\infty}x(k)\right)^{q}a(m)\right)^{\frac{1}{q}}\leq C\left(\sum_{m=1}^{\infty}x(m)^{p}b(m)\right)^{\frac{1}{p}} \tag{2}$$

for non-negative, non-increasing sequences $x = \{x(m)\}$. Here $\{a(m)\}$ and $\{b(m)\}$ are given non-negative weight sequences, $p \in (0,1)$ and $q \in (0,\infty)$ are fixed parameters and the constant $C > 0$ is independent of x. The case $1 < p < \infty$ and $0 < q < \infty$ was considered in the recent paper [1], for discrete cases, Sawyer's duality theorem was obtained by R.Oinarov and S.Kh.Shalginbaeva [2], and using an effective method based on Sawyer's duality principle, which reduces inequalities (1) for non-negative, non-increasing sequences to modified inequalities for non-negative sequences, they give a characterization of the inequality of (1), in the cases $1 < p, q < \infty$, Sawyer's duality theorem cannot possible to use in the case when $q \in (0,1)$. The corresponding problem for unrestricted, non-negative sequences $\{x\}$ was solved by the authors in [3], [4] and ([5], Theorem 9.2). In the continuous setting, this problem has been extensively studied over the last twenty years (see [6], [7], [8], [11], and the references therein). Under the conditions $1 \le q < \infty$, and $0 < p < \infty$, an effective approach is provided by Sawyer's duality principle [12], which reduces inequalities (1) for nonnegative non increasing sequences to modified inequalities for the same class of sequences.

Our main results are the discrete analogues of the theorems of A.Gogatishvili and V.D.Stepanov [6].

The structure of the paper is as follows. In the next section, we formulate the main results of the study. Section 3 is devoted to preliminary results, while the final section contains the proofs of the principal statements concerning supremum operators with kernels. Throughout the paper, we shall adhere to the following notation. The set of all natural numbers will be denoted by $\mathbb{N}$. We write $A \lesssim B$ if there exists a positive constant C such that $A \le CB$. Furthermore, the notation $A \approx B$ ill indicate that both $A \lesssim B$ and $B \lesssim A$ hold simultaneously.

2 Main Results

We assume that $\{a(m)\}$ and $\{b(m)\}$ are sequences of non-negative terms throughout. Their partial sums are denoted by

$$A(m) = \sum_{k=1}^{m} a(k), \quad B(m) = \sum_{k=1}^{m} b(k), \quad m \in \mathbb{N}.$$

Theorem 1 *Let $0 < q \le \infty$, $0 < p < 1$. Suppose that $\{a(m)\}$ and $\{b(m)\}$ are given non-negative weight sequences.Then, the following six conditions are equivalent.*

(i) *Inequality (1) holds for every non-negative, non-increasing sequence $\{x(m)\}$.*

(ii) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty} \left(\sum_{i=1}^{m} \left(\sum_{k=i}^{\infty} y(k) \right)^{\frac{1}{p}} \right)^{q} a(m) \right)^{\frac{1}{q}} \le C \left(\sum_{m=1}^{\infty} y(m) B(m) \right)^{\frac{1}{p}} \tag{3}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iii) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty} \left(\sum_{i=1}^{m} i^{p-1} \left(\sum_{k=i}^{\infty} y(k) \right) \right)^{\frac{q}{p}} a(m) \right)^{\frac{1}{q}} \le C \left(\sum_{m=1}^{\infty} y(m) B(m) \right)^{\frac{1}{p}} \tag{4}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iv) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty} \left(\sup_{1 \le i \le m} i^{p} \left(\sum_{i=k}^{\infty} y(k) \right) \right)^{\frac{q}{p}} a(m) \right)^{\frac{1}{q}} \le C \left(\sum_{m=1}^{\infty} y(m) B(m) \right)^{\frac{1}{p}} \tag{5}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(v) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\left(\sup_{i\le k<\infty} y(k)\right)^{\frac{1}{p}}\right)^{q} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{6}$$

for every sequence $\{y(m)\}$ *consisting of non-negative terms,*

(vi) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{1\le i\le m} i^{p}\left(\sup_{i\le k\le\infty} y(k)\right)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{7}$$

for every sequence $\{y(m)\}$ *consisting of non-negative terms.*

Theorem 2 *Let* $0 < q \le \infty$, $0 < p < 1$. *Suppose that* $\{a(m)\}$ *and* $\{b(m)\}$ *are given non-negative weight sequences. Then the following six conditions are equivalent:*

(i) *Inequality (1) holds for every non-negative, non-increasing sequence* $\{x(m)\}$

(ii) *For any* $\alpha > 0$, *the following inequality is satisfied:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i} B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{q} a(i)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{8}$$

for every sequence $\{y(m)\}$ *consisting of non-negative terms,*

(iii) *For any* $\alpha > 0$ *the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sup_{1\le k<i} B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{q} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{9}$$

for every sequence $\{y(m)\}$ *consisting of non-negative terms,*

(iv) *For any* $\alpha > 0$ *the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p} B(i)^{\alpha+1}y(i)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{10}$$

for every sequence $\{y(m)\}$ *consisting of non-negative terms,*

(v) *For any* $\alpha > 0$ *the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{1\le i\le m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sum_{k=1}^{i} B(k)^{\alpha+1}y(k)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}} \tag{11}$$

for every sequence $\{y(m)\}$ *consisting of non-negative terms,*

(vi) *For any* $\alpha > 0$ *the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{1\le i\le m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\le k\le i} B(k)^{\alpha+1}y(k)\right)^{\frac{q}{p}} a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}}, \tag{12}$$

for every sequence $\{y(m)\}$ *consisting of non-negative terms.*

Theorem 3 *Let $0 < q \le \infty$, $0 < p < 1$. Suppose that $\{a(m)\}$ and $\{b(m)\}$ are prescribed non-negative weight sequences. Then the six conditions listed below are mutually equivalent:*
(i) *Inequality (2) is satisfied for every non-negative, non-increasing sequence $\{x(m)\}$.*
(ii) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{k=m}^{\infty}\left(\sum_{i=k}^{\infty}y(i)\right)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{13}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,
(iii) *The inequality stated below holds true:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty}(i-m)^{p}y(i)\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{14}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,
(iv) *The following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{m\le i\le\infty}(i-m)^{p}\sum_{k=i}^{\infty}y(k)\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{15}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,
(v) *The following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{k=m}^{\infty}\sup_{k\le i\le\infty}y(i)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{16}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,
(vi) *The following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{m\le i\le\infty}(i-m)^{p}\sup_{i\le k\le\infty}y(k)\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{17}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms.

Theorem 4 *Let $0 < q \le \infty$, $0 < p < 1$. Suppose that $\{a(m)\}$ and $\{b(m)\}$ are given non-negative weight sequences. Then the following six conditions are equivalent:*
(i) *Inequality (2) holds for every non-negative, non-increasing sequence $\{x(m)\}$.*
(ii) *For any $\alpha > 0$, the inequality below is satisfied:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty}B(i)^{-\frac{\alpha+1}{p}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{18}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iii) *For any $\alpha > 0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty}B(i)^{-\frac{\alpha+1}{p}}\left(\sup_{1\le k<i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}}\le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{19}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(iv) *For any $\alpha > 0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty}\left(\sum_{k=i}^{\infty}B(k)^{-\frac{\alpha+1}{p}}\right)^{p-1}B(i)^{-\frac{\alpha+1}{p}}\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{\alpha}{p}}a(m)\right)\le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{20}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(v) *For any $\alpha > 0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{m\le i<\infty}B(i)^{-\frac{\alpha+1}{p}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}}\le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{21}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms,

(vi) *For any $\alpha > 0$ the following inequality is valid:*

$$\left(\sum_{m=1}^{\infty}\left(\sup_{m\le i\le\infty}\left(\sum_{k=i}^{\infty}B(k)^{\frac{\alpha+1}{p}}\right)^{p}\left(\sup_{1\le k\le i}B(k)^{\alpha+1}y(k)\right)\right)^{\frac{q}{p}}a(m)\right)^{\frac{1}{q}}\le C\left(\sum_{m=1}^{\infty}y(m)B(m)\right)^{\frac{1}{p}} \tag{22}$$

for every sequence $\{y(m)\}$ consisting of non-negative terms.

3 Preliminaries

Lemma 1 *(Fubini, see [4])*

$$\sum_{m=1}^{\infty}a(m)\sum_{k=1}^{m}b(k)<\infty \quad \text{if and only if} \quad \sum_{m=1}^{\infty}b(m)\sum_{k=m}^{\infty}a(k)<\infty. \tag{23}$$

Moreover,

$$\sum_{m=1}^{\infty}a(m)\sum_{k=1}^{m}b(k)=\sum_{m=1}^{\infty}b(m)\sum_{k=m}^{\infty}a(k). \tag{24}$$

Lemma 2 *(Power Rule, see [15]) If $0<p<\infty$, then*

$$\min(p,1)\sum_{k=1}^{m} b(k)\left(\sum_{j=1}^{k} b(j)\right)^{p-1} \leq \left(\sum_{k=1}^{m} b(k)\right)^{p} \leq \max(p,1)\sum_{k=1}^{m} b(k)\left(\sum_{j=1}^{k} b(j)\right)^{p-1}. \quad (25)$$

Lemma 3 *(Power Rule for Tails, see [15]) If $0<p<\infty$, then*

$$\min(p,1)\sum_{k=1}^{m} b(k)\left(\sum_{j=k}^{m} b(j)\right)^{p-1} \leq \left(\sum_{k=1}^{m} b(k)\right)^{p} \leq \max(p,1)\sum_{k=1}^{m} b(k)\left(\sum_{j=k}^{m} b(j)\right)^{p-1}. \quad (26)$$

Lemma 4 *(Generalized Partial Sums lemma, see [13]) Let $N\in\mathbb{N}$ and $0<p\leq 1$.*
(a) *If*

$$\left(\sum_{k=1}^{m} a(k)\right)^{p} \lesssim \sum_{k=1}^{m} b(k) \quad (m=1,2,\cdots,N)$$

then

$$\left(\sum_{k=1}^{N} a(k)x(k)\right)^{p} \lesssim \sup_{1\leq m\leq N}\frac{(\sum_{k=1}^{m} a(k))^{p}}{\sum_{k=1}^{m} b(k)}\sum_{k=1}^{N} b(k)x(k)^{p} \quad (m=1,2,\cdots,N)$$

and all non-negative non -ieasing sequences $(x(1),x(2),\cdots,x(N))$.
(b) *If*

$$\left(\sum_{k=m}^{N} a(k)\right)^{p} \lesssim \sum_{k=m}^{N} b(k) \quad (m=1,2,\cdots,N)$$

then

$$\left(\sum_{k=1}^{N} a(k)x(k)\right)^{p} \lesssim \sup_{1\leq m\leq N}\frac{\left(\sum_{k=m}^{N} a(k)\right)^{p}}{\sum_{k=m}^{N} b(k)}\sum_{k=1}^{N} b(k)x(k)^{p} \quad (m=1,2,\cdots,N)$$

and for all non-negative, non-increasing sequences $(x(1),x(2),\cdots,x(N))$;

4 Proofs of Main Results

Proof 1 Proof of Theorem 1. *We have the following estimates*

$$\begin{aligned}\left(\sup_{1\leq i\leq m} i^{p}\left(\sup_{i\leq k\leq\infty} y(k)\right)\right)^{\frac{1}{p}} &\leq \sup_{1\leq i\leq m} i\left(\sum_{k=i}^{\infty} y(k)\right)^{\frac{1}{p}} \leq \sum_{i=1}^{m}\left(\sum_{k=i}^{\infty} y(k)\right)^{\frac{1}{p}}\\ &\leq \sup_{1<k<m} k\left(\sum_{i=1}^{k} i^{p-1}\right)^{-\frac{1}{p}}\left(\sum_{i=1}^{m} i^{p-1}\left(\sum_{k=i}^{\infty} y(k)\right)\right)^{\frac{1}{p}}. \end{aligned} \quad (27)$$

The first two inequalities are trivial, and the last inequality follows from Lemma 4.

We also need the following trivial inequalities,

$$\left(\sup_{1\le i\le m} i^p\left(\sup_{i\le k\le\infty} y(k)\right)\right)^{\frac{1}{p}} \le \sum_{i=1}^{m}\left(\sup_{i\le k<\infty} y(k)\right)^{\frac{1}{p}} \le \sum_{i=1}^{m}\left(\sum_{k=i}^{\infty} y(k)\right)^{\frac{1}{p}}. \tag{28}$$

The equivalence

$$(i) \Leftrightarrow (ii)$$

easily follows by taking $x(k) = (\sum_{i=k}^{\infty} y(i))^{\frac{1}{p}}$ *in (1) and use Fubini Lemma 1.*

From the estimates (27) follow the following implications

$$(iii) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (vi),$$

and from the estimates (28)we have

$$(ii) \Rightarrow (v) \Rightarrow (vi).$$

The implication

$$(vi) \Rightarrow (iii),$$

follows from [[14], Theorem 2.3 and Theorem 2.4]. Therefore, we have all the implications.

Proof 2 Proof of Theorem 2.

$(i) \Rightarrow (ii)$.

In (1) consider decreasing sequence:

$$x(k) = \left(\frac{1}{\sum_{i=1}^{k} b(i)B(i)^{\frac{\alpha+1}{p}-1}}\sum_{i=1}^{k} B(i)^{\frac{\alpha+1}{p}-1} b(i)\sum_{j=i}^{\infty} y(j)\right)^{\frac{1}{p}}$$

. Using the following estimate,

$$x(k) \ge \left(\frac{1}{\sum_{i=1}^{k} b(i)B(i)^{\frac{\alpha+1}{p}-1}}\sum_{j=1}^{k}\left(\sum_{i=1}^{j} B(i)^{\frac{\alpha+1}{p}-1} b(i)\right) y(j)\right)^{\frac{1}{p}} \approx \left(\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\sum_{j=1}^{k} B(j)^{\frac{\alpha+1}{p}} y(j)\right)^{\frac{1}{p}}$$

and (25) and (26) we obtain

$$\begin{aligned}
\left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m}\left(\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\sum_{k=1}^{i} B(k)^{\frac{\alpha+1}{p}} y(k)\right)\right)^{q} a(m)\right)^{\frac{1}{q}} &\lesssim \left(\sum_{m=1}^{\infty}\left(\sum_{i=1}^{m} x(i)\right)^{q} a(m)\right)^{\frac{1}{q}} \\
&\le C\left(\sum_{m=1}^{\infty} x(m)^p b(m)\right)^{\frac{1}{p}} \\
&= C\left(\sum_{m=1}^{\infty}\left(\frac{1}{\sum_{i=1}^{k} b(i)B(i)^{\frac{\alpha+1}{p}-1}}\sum_{i=1}^{k} B(i)^{\frac{\alpha+1}{p}-1} b(i)\sum_{j=i}^{\infty} y(j)\right) b(m)\right)^{\frac{1}{p}} \\
&\lesssim C\left(\sum_{m=1}^{\infty}\left(\sum_{j=m}^{\infty} y(j)\right) b(m)\right)^{\frac{1}{p}} \le C\left(\sum_{m=1}^{\infty} y(m)B(m)\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence (8) follows.
(ii) ⇒ (i).

$$x(k)=\frac{B(k)^{\frac{\alpha+1}{p}}}{B(k)^{\frac{\alpha+1}{p}}}x(k)\le(\alpha+1)^{\frac{1}{p}}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\left(\sum_{j=1}^{k}B(j)^{\alpha}b(j)x(j)\right)^{\frac{1}{p}}.$$

Using this estimate, (8) for seguence $y(j)=B(j)^{-1}b(j)$, we get

$$\begin{aligned}&\left(\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}x(k)\right)^{q}a(m)\right)^{\frac{1}{q}}\\&\le(\alpha+1)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\left(\frac{1}{B(k)^{\alpha+1}}\sum_{j=1}^{k}B(j)^{\alpha}b(j)x(j)\right)^{\frac{1}{p}}\right)^{q}a(m)\right)^{\frac{1}{q}}\\&\le C(\alpha+1)^{\frac{1}{p}}\left(\sum_{i=1}^{\infty}b(m)x(m)^{p}\right)^{\frac{1}{p}}.\end{aligned}$$

We have the following estimates:

$$\begin{aligned}&\left(\sup_{1\le i\le m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\le k\le i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\\&\le\left(\sup_{1\le i\le m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}?\\&\le\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\\&\le\left(\sum_{i=1}^{m}\left(\sum_{k-i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}B(i)^{\alpha+1}y(i)\right)^{\frac{1}{p}}.\end{aligned}\tag{29}$$

The first two inequalities are trivial, and the last inequality follows from Lemma 4(b). We also have the following trivial estimates:

$$\begin{aligned}&\left(\sup_{1\le i\le m}\left(\sum_{k=i}^{m}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\le k\le i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\\&\le\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sup_{1\le k\le i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}\le\sum_{i=1}^{m}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}.\end{aligned}\tag{30}$$

From the estimates (29) follows following implications

$(iv) \Rightarrow (ii) \Rightarrow (v) \Rightarrow (vi).$

From the estimates (30) we have

$(ii) \Rightarrow (iii) \Rightarrow (vi),$

The implication

$(vi) \Rightarrow (iv).$

Follows from [[14], Theorem 3.2 and Theorem 2.4]. Therefore, we have all the implications.

Proof 3 Proof of Theorem 3. *We have the following estimates:*

$$\left(\sup_{m\le i\le\infty}(i-m)^p\left(\sup_{i\le k\le\infty}y(k)\right)\right)^{\frac{1}{p}} \le \left(\sup_{m\le i\le\infty}(i-m)^p\left(\sum_{k=i}^{\infty}y(k)\right)\right)^{\frac{1}{p}}$$
$$\le \sum_{i=m}^{\infty}\left(\sum_{k=i}^{\infty}y(k)\right)^{\frac{1}{p}} \le C_p\left(\sum_{i=m}^{\infty}(i-m)^{p-1}\left(\sum_{k=i}^{\infty}y(k)\right)\right)^{\frac{1}{p}}$$
$$\approx C_p\left(\sum_{i=m}^{\infty}(i-m)^p y(k)\right)^{\frac{1}{p}}. \tag{31}$$

The first two inequalities are trivial, and the third inequality follows from Lemma 4(a), and the last equivalent follows by using (24).

We also need the following trivial inequalities,

$$\left(\sup_{m\le i\le\infty}\left(\sup_{i\le k\le\infty}y(k)\right)\right)^{\frac{1}{p}} \le \sum_{i=m}^{\infty}\left(\sup_{i\le k<\infty}y(k)\right)^{\frac{1}{p}} \le \sum_{i=m}^{\infty}\left(\sum_{k=i}^{\infty}y(k)\right)^{\frac{1}{p}}. \tag{32}$$

The equivalence

$(i) \Leftrightarrow (ii)$

easily follows by taking $x(k) = \left(\sum_{i=k}^{\infty} y(i)\right)^{\frac{1}{p}}$ *in (2).*

From the estimates (31) follows the implications

$(iii) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (vi).$

From the estimates (32) follows the implications

$(ii) \Rightarrow (v) \Rightarrow (vi).$

The implication

$(vi) \Rightarrow (iii),$

follows from [[14], Theorem 3.2 and Theorem 2.4]. Therefore, we have all the implications.

Proof 4 Proof of Theorem 4. *The equivalence (i) ⇔ (ii) follows same way as in the proof of Theorem 3. As in the proof of the Theorem 3, we have*

$$\left(\sup_{m\le i\le\infty}\left(\sum_{k=i}^{\infty}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\le k\le i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}$$

$$\le\left(\sup_{m\le i\le\infty}\left(\sum_{k=i}^{\infty}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p}\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}$$

$$\le\sum_{i=m}^{\infty}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}$$

$$\le\left(\sum_{i=m}^{\infty}\left(\sum_{k=i}^{\infty}\frac{1}{B(k)^{\frac{\alpha+1}{p}}}\right)^{p-1}\frac{1}{B(i)^{\frac{\alpha+1}{p}}}\sum_{k=1}^{i}B(k)^{\alpha+1}y(k)\right)^{\frac{1}{p}}$$

which gives the implications

$$(iv)\Rightarrow(ii)\Rightarrow(v)\Rightarrow(vi).$$

Using the trivial estimates

$$\left(\sup_{m\le i\le\infty}\left(\sum_{k=i}^{\infty}\frac{1}{B_k^{\frac{\alpha+1}{p}}}\right)^{p}\sup_{1\le k\le i}B_k^{\alpha+1}y_k\right)^{\frac{1}{p}}\le\sum_{i=m}^{\infty}\frac{1}{B_i^{\frac{\alpha+1}{p}}}\left(\sup_{1\le k\le i}B_k^{\alpha+1}y_k\right)^{\frac{1}{p}}$$

$$\le\sum_{i=m}^{\infty}\frac{1}{B_i^{\frac{\alpha+1}{p}}}\left(\sum_{k=1}^{i}B_k^{\alpha+1}y_k\right)^{\frac{1}{p}}.$$

We are obtaining the implications:

$$(ii)\Rightarrow(iii)\Rightarrow(vi).$$

The implication

$$(vi)\Rightarrow(iv),$$

follows from [14, Theorem 2.3 and Theorem 2.4].

5 Conclusion

In this paper, we establish that weighted inequalities of the type $l_p\to l_q$ for discrete Hardy and Copson operators on the cone of non-negative monotone sequences can, in the case $0<q<\infty$, $0<p<1$, be reduced to the corresponding inequalities on the cone of non-negative sequences. This approach makes it possible to employ a broader range of proof techniques and significantly simplifies the analysis of the inequalities under consideration. The equivalence theorems relating inequalities for discrete Hardy and Copson operators on the cone of non-increasing sequences to those on the cone of non-negative sequences emphasize the intrinsic differences between the discrete and continuous cases. It is shown that methods successfully applied in the continuous case do not always prove effective in the discrete one. The results obtained complement previously known statements for the case $p > 1$ and broaden the theoretical framework for studying discrete Hardy and Copson operators in weighted spaces.

6 Declaration

The authors have no competing interests to declare. Relevant to the content of this article

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ILL-POSEDNESS OF A MIXED PROBLEM IN A CYLINDRICAL DOMAIN FOR THE MULTIDIMENSIONAL LAVRENTIEV-BITSADZE EQUATION

Studies of well-posed and ill-posed problems in mathematical physics, including inverse problems and their practical applications, are of considerable interest, where the key issue is the correct formulation of the direct problem. Hyperbolic and elliptic equations are widely used in biomedical modeling, including to describe tumor growth and deformations of biological tissues. Analogies between membrane oscillations and tissue dynamics are widely used in biomechanics and mathematical medicine. For example, the spatial oscillations of elastic membranes are described by partial differential equations. When the membrane deflection is specified by a function $u(x, t)$, $x \in R^m$, $m \geq 2$, application of Hamilton's principle leads to a multidimensional wave equation, and in the case of equilibrium, to the Laplace equation. Consequently, the dynamics of elastic membranes can be described by the multidimensional Lavrentiev-Bitsadze equation. The problems considered in the article are ill-posed problems. The proof of non-unique solvability and the construction of an explicit solution is in fact a regularization of an ill-posed problem through the spectral method and integral representations, etc. In this article, the ambiguity of the solution is proven and an explicit form of the classical solution of a mixed problem for the multidimensional Lavrentiev-Bitsadze equation, is presented.
Key words: Ill-posedness, mixed problem, cylindrical domain, Bessel function, boundary conditions.

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Көп өлшемді Лаврентьев-Бицадзе теңдеуі үшін цилиндрлік облыстағы аралас есептің қисынды қойылмауы

Математикалық физиканың қисынды және қисынды емес қойылған есептерді, соның ішінде кері есептерді және олардың практикалық қолданылуын зерттеу айтарлықтай қызығушылық тудырады, мұндағы негізгі мәселе тура есептің қисынды қойылуы болып табылады. Гиперболалық және эллиптикалық теңдеулер биомедициналық модельдеуде, соның ішінде ісіктің ісуі мен биологиялық тіндердің деформацияларын сипаттау үшін кеңінен қолданылады. Мембрана тербелістері мен тіннің өзгеріс динамикасы биомеханика мен математикалық медицинада кеңінен қолданылады. Мысалы, серпімді мембраналардың кеңістіктік тербелістері дербес дифференциалдық теңдеулермен сипатталады. Мембрананың ауытқуы $u(x, t)$, $x \in R^m$, $m \geq 2$, функциясымен анықталған кезде, Гамильтон принципін қолдану көп өлшемді толқындық теңдеуге, ал тепе-теңдік жағдайында - Лаплас теңдеуіне әкеледі. Демек, серпімді мембраналардың динамикасын көп өлшемді Лаврентьев-Бицадзе теңдеуімен сипаттауға болады. Бұл мақалада қарастырылатын есептер қисынды емес есептер болып табылады.

Бірегей шешілімділікті дәлелдеу және айқын шешімді құруда спектрлік әдіс және интегралдық бейнелеулер арқылы қисынды емес есепті регуляризациялау болып табылады. Мақалада теориялық нәтижелер келтірілген – көп өлшемді Лаврентьев-Бицадзе теңдеуі үшін аралас есептің бірегей шешілетіндігін дәлелдеп және классикалық шешімі үшін айқын түрін алған.
Түйін сөздер: қисынды, аралас есеп, цилиндрлік облыс, Бессель функциясы, шекаралық шарттар.

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Некорректность смешанной задачи в цилиндрической области для многомерного уравнения Лаврентьева-Бицадзе

Исследования корректных и некорректных задач математической физики, включая обратные задачи и их практические применения, представляют значительный интерес, где ключевым моментом выступает корректная постановка прямой задачи. Гиперболические и эллиптические уравнения широко используются в биомедицинском моделировании, в том числе для описания роста опухолевых образований и деформаций биологических тканей. Аналогии между колебаниями мембран и динамикой поведения тканей широко применяются в биомеханике и математической медицине. Так колебания упругих мембран в пространстве описываются уравнениями в частных производных. При задании прогиба мембраны функцией $u(x,t)$, $x \in R^m$, $m \geq 2$, применение принципа Гамильтона приводит к многомерному волновому уравнению, а в случае равновесного состояния - к уравнению Лапласа. Следовательно, динамика упругих мембран может быть описана многомерным уравнением Лаврентьева-Бицадзе. Рассматриваемые в статье задачи являются некорректными. Доказательство неоднозначности решения и построение явного решения фактически представляет собой регуляризацию некорректной задачи с помощью спектрального метода, интегральных представлений и т.д. В статье доказана неоднозначность решения и представлен явный вид классического решения смешанной задачи для многомерного уравнения Лаврентьева-Бицадзе.
Ключевые слова: корректность, смешанная задача, цилиндрическая область, функция Бесселя, граничные условия.

1 Introduction

The team of authors has maintained scientific cooperation for many years, based on the results of research the field of well-posedness and ill-posed of problems in mathematical physics and related inverse problems, as well as their applications. When studying inverse and ill-posed problems, the formulation of the direct problem and its well-posedness conditions are of great importance, such a connection is presented [1–3]. As is known, hyperbolic and elliptic equations are used in biomedical models, including tumor growth and deformation of biological tissues. In [4], a linear stability analysis of growing tissues is discussed. The equations for small perturbations are reduced to mixed-type systems. In particular, wave-like and diffusion-like regimes (hyperbolic and elliptic, respectively) are described, which may coexist due to heterogeneous growth. Indeed, analogies between membrane vibrations and tissue dynamics are actively used in biomechanics and mathematical medicine. For example, this includes predicting brain tumor growth, where mechanical pressure on the surrounding tissues is taken into account, as well as analyzing tissue deformation during tumor invasion (such as glioblastoma), where the deformation follows wave-type equations with viscoelastic terms. The above problems have been studied in detail in [5–8], but for multidimensional hyperbolic-elliptic equations these problems have not yet been studied.

2 Problem statement and result

Let $\Omega_{\alpha\beta}$ be a finite domain of the Euclidean space E_{m+1} of points $(x_1, \dots, x_m, t)$, bounded at $t > 0$ by the cylinder $\Gamma_\beta = \{(x,t) : |x| = 1\}$ and the plane $t = \beta > 0$, and for $t < 0$ the cylinder $\Gamma_\alpha = \{(x,t) : |x| = 1\}$ and the plane $t = \alpha < 0$, where $|x|$ is the length of the vector $x = (x_1, \dots, x_m)$, $m \geq 2$. Denote by Ω_β^+ and Ω_α^- parts of the domain $\Omega_{\alpha\beta}$, lying in the half-spaces $t > 0$ and $t < 0$; σ_β – the upper base of the domain Ω_β^+,a σ_α – the lower base of the domain Ω_α^-.

Let S be the common part of the boundaries of the domains Ω_β^+ and Ω_α^-, representing the set of $\{t = 0, 0 < |x| < 1\}$ points from E_m [9].

In the domain of $\Omega_{\alpha\beta}$ we consider

$$(\mathrm{sgn}t)\Delta_x u - u_{tt} = 0, \tag{1}$$

where Δ_x is the Laplace operator for variables $x_1, \dots, x_m$ [9].

Next, it is convenient [9] for us to move from Cartesian coordinates $x_1, \dots, x_m, t$ to spherical $r, \theta_1, \dots, \theta_{m-1}, t$, $r \geq 0$, $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_i \leq \pi$, $i = \overline{2, m-1}$.

As a multidimensional mixed problem, consider the following problem

Problem 1 *Find the solution of the equation (1) in the domain $\Omega_{\alpha\beta}$ for $t \neq 0$ from the class $C(\overline{\Omega}_{\alpha\beta}) \cap C^1(\Omega_{\alpha\beta}) \cap C^2(\Omega_\beta^+ \cup \Omega_\alpha^-)$ satisfying the boundary conditions*

$$u\Big|_{\Gamma_\alpha} = \psi_1(t,\theta), \tag{2}$$

$$u\Big|_{\Gamma_\alpha} = \psi_2(t,\theta), \quad u\Big|_{\sigma_\alpha} = \varphi(r,\theta), \tag{3}$$

at the same time, $\psi_1(0,\theta) = \psi_2(0,\theta)$, $\psi_2(\alpha,\theta) = \varphi(r,\theta)$.

Let $\{Y_{n,m}^k(\theta)\}$ be a system of linearly independent spherical functions of the order n, $1 \leq k \leq k_n$, $(m-2)!n!k_n = (n+m-3)!(2n+m-2)$ and $W_2^l(S)$, $l = 0, 1, \dots$ are Sobolev spaces [9].

It takes place ([10], p. 142-144)

Lemma 1 *Let $f(r,\theta) \in W_2^l(S)$. If $l \geq m-1$, then the series*

$$f(r,\theta) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} f_n^k(r) Y_{n,m}^k(\theta), \tag{4}$$

and also the series obtained from it by differentiation of order $p \leq l-m+1$ converge absolutely and uniformly [9, 10].

Lemma 2 *In order for $f(r,\theta) \in W_2^l(S)$ it is necessary and sufficient that the coefficients of the series (4) satisfy the inequalities [9, 10]*

$$|f_0^1(r)| \leq c_1, \quad \sum_{n=1}^{\infty} \sum_{k=1}^{k_n} n^{2l} |f_n^k(r)|^2 \leq c_2, \quad c_1, c_2 = const.$$

By $\psi_{2n}^k(t)$, $\overline{\varphi}_n^k(r)$, we denote the expansion coefficients of the series (4), respectively, of the functions $\psi_2(t,\theta)$, $\varphi(r,\theta)$ [9].

Then the following is true

Theorem 1 *If $\psi_1(t,\theta) \in W_2^l(\Gamma_\beta)$, $\psi_2(t,\theta) \in W_2^l(\Gamma_\alpha)$, $\varphi(r,\theta) \in W_2^l(\sigma_\alpha)$, $l > \frac{3m}{2}$, then Problem 1 is solvable and not unique.*

Proof. In spherical coordinates the equation (1) in the domain Ω_α^- has the form [10,11]

$$u_{rr} + \frac{m-1}{r}u_r - \frac{1}{r^2}\delta u + u_{tt} = 0, \tag{5}$$

where is

$$\delta \equiv -\sum_{j=1}^{m-1} \frac{1}{g_j \sin^{m-j-1}\theta_j} \frac{\partial}{\partial\theta_j}\left(\sin^{m-j-1}\theta_j \frac{\partial}{\partial\theta_j}\right),\ g_1 = 1,\ g_j = (\sin\theta_1 \dots \sin\theta_{j-1})^2,\ j > 1.$$

It is known ([9], [10], p. 239) that the spectrum of the operator δ consists of eigenvalues $\lambda_n = n(n+m-2)$, $n = 0,1,\dots$, each of which corresponds to $k_n$orthonormal functions $Y_{n,m}^k(\theta)$.

Since the desired solution to Problem 1 in the domain $\overline{\Omega}_\alpha$ belongs to the class $C(\overline{\Omega}_\alpha^-) \cap C^2(\Omega_\alpha^-)$, it can be sought in the form as

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \overline{u}_n^k(r,t) Y_{n,m}^k(\theta), \tag{6}$$

where $\overline{u}_n^k(r,t)$ are the functions to be defined.

Substituting (6) into (5), using the orthogonality of spherical functions [9,10], we arrive at the equation

$$\overline{u}_{nrr}^k + \frac{m-1}{r}\overline{u}_{nr}^k + \overline{u}_{ntt}^k - \frac{\lambda_n}{r^2}\overline{u}_n^k = 0, \quad k = \overline{1,k_n},\ n = 0,1,..., \tag{7}$$

Moreover, from the boundary condition (3), taking into account Lemma 1, we have

$$\overline{u}_n^k(1,t) = \psi_{2n}^k(t), \quad \overline{u}_n^k(r,\alpha) = \overline{\varphi}_n^k(r), \quad k = \overline{1,k_n}, \quad n = 0,1,\dots. \tag{8}$$

In (7), (8), replacing $\overline{v}_n^k(r,t) = \overline{u}_n^k(r,t) - \psi_n^k(t)$, we get

$$\overline{v}_{nrr}^k + \frac{m-1}{r}\overline{v}_{nr}^k - \frac{\lambda_n}{r^2}\overline{v}_n^k + \overline{v}_{ntt}^k = \overline{f}_n^k(r,t). \tag{9}$$

$$\begin{gathered}\overline{v}_n^k(1,t) = 0, \quad \overline{v}_n^k(r,\alpha) = \varphi_n^k(r), \quad k = \overline{1,k_n}, \quad n = 0,1,\dots. \\ \overline{f}_n^k(r,t) = \frac{\lambda_n}{r^2}\psi_{2n}^k(t) - \psi_{2ntt}^k, \quad \varphi_n^k(r) = \overline{\varphi}_n^k(r) - \psi_n^k(\alpha).\end{gathered} \tag{10}$$

By replacing $\overline{v}_n^k(r,t) = r^{(1-m)/2} v_n^k(r,t)$, the problem (9), (10) is reduced to the following problem

$$Lv_n^k \equiv v_{nrr}^k + \frac{\lambda_n}{r^2} v_n^k + v_{ntt}^k = \widetilde{f}_n^k(r,t), \tag{11}$$

$$v_n^k(1,t) = 0, \quad v_n^k(r,\alpha) = \widetilde{\varphi}_n^k(r), \quad k = \overline{1, k_n}, \quad n = 0, 1, \ldots,$$
$$\lambda_n = \frac{[(m-1)(3-m) - 4\lambda_n]}{2}, \quad \widetilde{f}_n^k(r,t) = r^{(m-1)/2} \overline{f}_n^k(r,t), \quad \widetilde{\varphi}_n^k(r) = r^{(m-1)/2} \varphi_n^k(r). \tag{12}$$

We seek the solution to problem (11), (12) in the form $v_n^k(r,t) = v_{1n}^k(r,t) + v_{2n}^k(r)$, where $v_{1n}^k(r,t)$ – is the solution to the problem

$$Lv_{1n}^k = \widetilde{f}_n^k(r,t), \quad v_{1n}^k(1,t) = 0, \quad v_{1n}^k(r,\alpha) = 0, \tag{13}$$

and $v_{2n}^k(r,t)$ – solving the problem

$$Lv_{2n}^k = 0, \quad v_{2n}^k(1,t) = 0, \quad v_{2n}^k(r,\alpha) = \widetilde{\varphi}_n^k(r). \tag{14}$$

We will consider the solution of the problems in the form

$$v_n^k(r,t) = \sum_{s=1}^{\infty} R_s(r) T_s(t). \tag{15}$$

At the same time, let

$$\widetilde{f}_n^k(r,t) = \sum_{s=1}^{\infty} a_{s,n}^k(t) R_s(r), \quad \widetilde{\varphi}_{2n}^k(r) = \sum_{s=1}^{\infty} b_{s,n}^k R_s(r). \tag{16}$$

Substituting (15) into (13), taking into account (16), we get

$$R_{srr} + \left(\frac{\overline{\lambda}_n}{r^2} + \mu \right) R_s = 0, \quad 0 < r < 1, \quad R_s(1) = 0, \quad |R_s(0)| < \infty, \tag{17}$$

$$T_{stt} - \mu T_s(t) = a_{s,n}^k(t), \quad \alpha < t < 0, \tag{18}$$

$$T_s(\alpha) = 0. \tag{19}$$

A limited solution to the (17) problem is [12]

$$R_s(r) = \sqrt{r} J_\nu(\mu_{s,n} r), \quad \mu = \mu_{s,n}^2. \tag{20}$$

$\nu = n + (m-2)/2, \mu_{s,n}$ – zeros of Bessel functions of the first kind $J_\nu(z), \mu = \mu_{s,n}^2$.

The general solution of the equation (18) is represented as [12]

$$T_{s,n}(t) = c_{1s}\cosh\mu_{s,n}t + c_{2s}\sinh\mu_{s,n}t - \frac{\cosh\mu_{s,n}t}{\mu_{s,n}}\int\limits_t^0 a^k_{s,n}(\xi)\sinh\mu_{s,n}\xi d\xi +$$

$$+\frac{\sinh\mu_{s,n}t}{\mu_{s,n}}\int\limits_t^0 a^k_{s,n}(\xi)\cosh\mu_{s,n}\xi d\xi,$$

c_{1s}, c_{2s} – arbitrary constants, satisfying the condition (19), we will have

$$\begin{gathered}\mu_{s,n}T_{s,n}(t) = c_{1s}\mu_{s,n}[\cosh\mu_{s,n}t - (\coth\mu_{s,n}\alpha)\sinh\mu_{s,n}t] + \\ + \left[(\coth\mu_{s,n}\alpha)\int\limits_\alpha^0 a^k_{s,n}(\xi)(\sinh\mu_{s,n}\xi)d\xi - \int\limits_\alpha^0 a^k_{s,n}(\xi)(\cosh\mu_{s,n}\xi)d\xi\right]\sinh\mu_{s,n}t - \\ -(\cosh\mu_{s,n}t)\int\limits_t^0 a^k_{s,n}(\xi)\sinh\mu_{s,n}\xi d\xi + (\sinh\mu_{s,n}t)\int\limits_t^0 a^k_{s,n}(\xi)\cosh\mu_{s,n}\xi d\xi.\end{gathered} \tag{21}$$

Substituting (20) into (16), we get

$$r^{-\frac{1}{2}}\widetilde{f}^k_n(r,t) = \sum_{s=1}^{\infty} a^k_{s,n}(t)J_\nu(\mu_{s,n}r), \quad r^{-\frac{1}{2}}\widetilde{\varphi}^k_n(r) = \sum_{s=1}^{\infty} b^k_{s,n}J_\nu(\mu_{s,n}r), \quad 0 < r < 1. \tag{22}$$

Series (22) – expansions into Fourier-Bessel series [12], if

$$a^k_{s,n}(t) = 2[J_{\nu+1}(\mu_{s,n})]^{-2}\int\limits_0^1 \sqrt{\xi}\widetilde{f}^k_n(\xi,t)J_\nu(\mu_{s,n}\xi)d\xi, \tag{23}$$

$$b^k_{s,n} = 2[J_{\nu+1}(\mu_{s,n})]^{-2}\int\limits_0^1 \sqrt{\xi}\widetilde{\varphi}^k_n(\xi)J_\nu(\mu_{s,n}\xi)d\xi, \tag{24}$$

where $\mu_{s,n}$, $s = 1, 2, \ldots$ are the positive zeros of the Bessel functions $J_\nu(z)$, arranged in order of increasing magnitude.

From (20), (21) we obtain the solution to the problem (13) in the form

$$v^k_{1n}(r,t) = \sum_{s=1}^{\infty}\sqrt{r}T_{s,n}(t)J_\nu(\mu_{s,n}r), \tag{25}$$

where $T_{s,n}(t)$ – are determined from (21), and $a^k_{s,n}(t)$ – from (23).

Next, substituting (15) into (14), taking into account (16) we will have

$$V_{stt} - \mu^2_{s,n}V_s = 0, \quad \alpha < t < 0, \tag{26}$$

$$V_s(\alpha) = b^k_{s,n}. \tag{27}$$

The general solution of the equation (26) has the form

$$V_{s,n}(t) = c'_{1s} \cosh \mu_{s,n} t + c'_{2s} \sinh \mu_{s,n} t,$$

where c'_{1s}, c'_{2s} – arbitrary constants, satisfying which condition (27) we get

$$V_{s,n}(t) = c'_{1s}[\cosh \mu_{s,n} t - (\coth \mu_{s,n}\alpha) \sinh \mu_{s,n} t] + \frac{b^k_{s,n} \sinh \mu_{s,n} t}{\sinh \mu_{s,n}\alpha}. \tag{28}$$

From (20), (28) we get the solution to the problem (14) by the formula

$$v^k_{2n}(r,t) = \sum_{s=1}^{\infty} \sqrt{r} V_{s,n}(t) J_\nu(\mu_{s,n} r), \tag{29}$$

where $V_{s,n}(t)$ are from (28), and $b^k_{s,n}$ – are from (24).

Thus, the boundary value problem for the equation (5) with data

$$u\Big|_{\Gamma_\alpha} = \psi_2(t,\theta), u\Big|_{\sigma_\alpha} = \varphi_2(r,\theta)$$

in the domain of Ω^-_α has countless solutions of the type

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{k_n} \{\psi^k_n(t) + r^{(1-m)/2}[v^k_{1n}(r,t) + v^k_{2n}(r,t)]\} Y^k_{n,m}(\theta), \tag{30}$$

where $v^k_{1n}(r,t)$, $v^k_{2n}(r,t)$ are defined from (25), (29).

Using the formula [13] $2J'_\nu(z) = J_{\nu-1}(z) - J_{\nu+1}(z)$, estimates [10, 13]

$$|J_\nu(z)| \le \frac{1}{\Gamma(1+\nu)}\left(\frac{z}{2}\right)^\nu, \quad |k_n| \le c_1 n^{m-2},$$

$$\left|\frac{\partial^q}{\partial\theta^q_j} Y^k_{n,m}(\theta)\right| \le c_2 n^{\frac{m}{2}-1+q}, \quad c_1, c_2 = const, \quad j = \overline{1, m-1}, \quad q = 0, 1, \ldots,$$

$\Gamma(z)$ – gamma function, as well as lemmas, constraints on given functions $\psi_2(t,\theta)$, $\varphi(r,\theta)$, as in [14], [15] it can be shown that the resulting solution (30) belongs to the class $C(\overline{\Omega^-_\alpha}) \cap C(\Omega_\alpha)$.

Next, from (30) at $t \to -0$ it will have

$$u(r,\theta,0) = \tau(r,\theta) = \sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\sum_{s=1}^{\infty} \{\psi^k_{2n}(0) + r^{\frac{(2-m)}{2}}(c_{1s} + c'_{1s})\} J_{n+\frac{(m-2)}{2}}(\mu_{s,n} r) Y^k_{n,m}(\theta),$$

$$u_t(r,\theta,0) = \nu(r,\theta) = \sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\sum_{s=1}^{\infty} \Big\{\psi^k_{2nt}(0) + r^{\frac{(2-m)}{2}}\Big[-(c_{1s} + c'_{1s})\mu_{s,n} \coth \mu_{s,n}\alpha +$$
$$+(\coth \mu_{s,n}\alpha) \int_\alpha^0 a^k_{s,n}(\xi)(\sinh \mu_{s,n}\xi) d\xi - \int_\alpha^0 a^k_{s,n}(\xi)(\cosh \mu_{s,n}\xi) d\xi + \tag{31}$$
$$+\frac{\mu_{s,n} b^k_{s,n}}{\cosh \mu_{s,n}\alpha}\Big]\Big\} J_{n+\frac{(m-2)}{2}}(\mu_{s,n} r) Y^k_{n,m}(\theta)$$

with $\tau(r,\theta)$,$\nu(r,\theta) \in W_2^l(S)$, $l > \frac{3m}{2}$.

Now we will study Problem 1 in the domain of Ω_β^+, which, by virtue of (2) and (31) is reduced to a mixed problem for the multidimensional wave equation [9]

$$u_{rr} + \frac{(m-1)}{r}u_r - \frac{1}{r^2}\delta u - u_{tt} = 0 \tag{32}$$

with conditions

$$u\Big|_S = \tau(r,\theta), \quad u_t\Big|_S = \nu(r,\theta), \quad u\Big|_{\Gamma_\beta} = \psi_1(t,\theta). \tag{33}$$

The following is shown in [7]

Theorem 2 *The problem (32), (33) is uniquely solvable in the class* $C(\overline{\Omega}_\beta^+) \cap C^2(\Omega_\beta^+)$.

From representation (31), and also from Theorem 2 it follows that Problem 1 has countless classical solutions.

Theorem 1 has been proven.

Since in [7,9] an explicit form of solutions to problem (32), (33) was obtained, then it is possible to write an explicit representation of the solution for Problem 1.

3 Conclusion and discussion

It has been established that the mixed problem for the multidimensional Lavrentiev-Bitsadze equation admits an ambiguous solution, and its explicit classical form has been obtained. This ill-posedness, manifested in the solution's high sensitivity to small data changes, is directly related to the problems of tumor modeling, where parameter instability leads to significant variability in growth predictions and treatment response.It has been established that the mixed problem for the multidimensional Lavrentiev-Bitsadze equation admits an ambiguous solution, and its explicit classical form has been obtained. This ill-posedness, manifested in the solution's high sensitivity to small data changes, is directly related to the problems of tumor modeling, where parameter instability leads to significant variability in growth predictions and treatment response.

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ON THE UNIQUE SOLVABILITY OF NONLOCAL IN TIME PROBLEMS CONTAINING THE IONKIN OPERATOR IN THE SPATIAL VARIABLE

This paper studies a differential equation representing the differences of two operators. One of the operators is generated by linear differential expressions that depend on time. The second operator is the Ionkin operator with respect to the spatial variable. In this paper, the differential operator with respect to time is generated by two-point Birkhoff regular boundary conditions. At the same time, the elliptic operator with respect to the spatial variable does not satisfy the so-called Agmon conditions. Moreover, the operator with respect to the spatial variable is not self-adjoint. In the beginning, the solvability of the problem is proved. In the final part, the uniqueness of the solution is proved. Direct application of the methods of the authors' previous works to prove the uniqueness of the solution to the problem is quite problematic. However, the authors managed to modify the reasoning of previous works to prove the uniqueness of the solution to the problem.

Key words: elliptic operators, differential-operator equations, initial-boundary value problem, solvability of the problem, existence of a solution, uniqueness of a solution, operator eigenvalues, complete orthonormal systems.

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Кеңістік айнымалылы Ионкин операторын қамтитын уақыт бойынша жергілікті емес есептерің бірегей шешімі туралы

Бұл жұмыста екі оператордың айырмасы болып табылатын дифференциалдық теңдеу зерттеледі. Операторлардың біріншісі уақытқа тәуелді сызықтық дифференциалдық өрнектер арқылы туындайды. Операторлардың екіншісі кеңістік айнымалыға тәуелді Ионкин операторын сипаттайды. Бұл жұмыста уақытқа қатысты дифференциалдық оператор екі нүктелік регулярлы Биркгоф шекаралық шарттары арқылы құрылған. Бұл жағыдайда кеңістік айнымалыға тәуелді оператор Агмон шарттарын қанағаттандырмайды. Сонымен қатар, кеңістік айнымалылы оператор түйіндес болмайды. Жұмыстың кіріспесіде есептің шешілетіндігі дәлелденеді. Қорытынды бөлімде шешімінің жалғыздығы дәлелденеді. Есептің шешімінің жалғыздығын дәлелдеу үшін авторлардың бұрынғы еңбектеріндегі әдістерді тікелей қолдану ыңғайсыз. Дегенмен, есептің шешімінің жалғыздығын дәлелдеу үшін бұрынғы еңбектеріндегі пайымдауларды қолдана алды.
Түйін сөздер: эллиптикалық операторлар, дифференциалдық-операторлық теңдеулер, бастапқы-шектік есеп, есептің шешімі, шешімінің болуы, шешімінің жалғыздығы, оператордың меншікті мәндері, толық ортонормаланған жүйелер.

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Об однозначной разрешимости нелокальных по времени задач, содержащих оператор Ионкина по пространственной переменной

В данной работе исследуется дифференциальное уравнение, представляющее разно- сти двух операторов. Один из операторов, порождается линейными дифференциаль- ными выражениями, зависящими от времени. Второй из операторов представляет опе- ратор Ионкина по пространственной переменной. В настоящей работе дифференциальный оператор по времени порождается двухточечными регулярными по Биркгофу граничными условиями. В то же время эллиптический оператор по пространственной переменной не удовлетворяет так называемым условиям Агмона. Более того оператор по пространственной переменной не является самосопряженным. В начале доказывается разрешимость поставленной задачи. В заключительной части доказывается единственности решения. Непосредственное применение методов предыдущих работ авторов для доказательства единственности решения задачи достаточно проблемно. Однако авторам для доказательства единственности решения задачи удалось модифицировать рассуждения предыдущих работ.

Ключевые слова: эллиптические операторы, дифференциально-операторные уравнения, начально-краевая задача, разрешимость задачи, существование решения, единственность решения, собственные значения оператора, полные ортонормированные системы

1 Introduction

Let $0 < T < \infty$. We introduce the differential expression

$$l(t, \frac{d}{dt}) \equiv \frac{d^{2p}}{dt^{2p}} + \sum_{k=0}^{2p-2} p_k(t) \frac{d^k}{dt^k}, \quad t \in (0,T),$$

where $p_k(t) \in C^k[0,T]$, $k = 0, 1, \dots, 2p-2$.

Let us consider in the domain $Q_T = (0,1) \times (0,T)$ the differential equation

$$l(t, \frac{\partial}{\partial t}) u(x,t) - u(x,t) + \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

with boundary conditions on x for fixed $t \in (0,T)$

$$u(0,t) = 0, \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x}, \tag{2}$$

with conditions on t for fixed $x \in \Omega$

$$U_{2\xi-1}(u(x,\cdot)) \equiv \frac{\partial^{j_\xi} u(x,t)}{\partial t^{j_\xi}}\Big|_{t=0} + \sum_{s=0}^{j_\xi - 1} \left(\alpha_{2\xi-1,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=0} + \beta_{2\xi-1,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=T} \right) = 0,$$

$$U_{2\xi}(u(x,\cdot)) \equiv \frac{\partial^{j_\xi} u(x,t)}{\partial t^{j_\xi}}\Big|_{t=T} + \sum_{s=0}^{j_\xi - 1} \left(\alpha_{2\xi,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=0} + \beta_{2\xi,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=T} \right) = 0, \ \xi = 1, \dots m,$$

$$U_{2m+\xi}(u(x,\cdot)) \equiv \alpha_{2m+\xi,\nu_\xi} \frac{\partial^{\nu_\xi} u(x,t)}{\partial t^{\nu_\xi}}\Big|_{t=0} + \beta_{2m+\xi,\nu_\xi} \frac{\partial^{\nu_\xi} u(x,t)}{\partial t^{\nu_\xi}}\Big|_{t=T} +$$

$$\sum_{s=0}^{\nu_\xi - 1} \left(\alpha_{2m+\xi,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=0} + \beta_{2m+\xi,s} \frac{\partial^s u(x,t)}{\partial t^s}\Big|_{t=T} \right) = 0, \quad \xi = 1, \dots r. \tag{3}$$

Here $2m + r = 2p$, and also in the next line $(0 \le j_1 < ... < j_m \le 2p-1,\ 0 \le \nu_1 < ... < \nu_r \le 2p-1)$ there are no identical natural numbers. For the coefficients of the condition on t the inequalities are satisfied

$$|\alpha_{2m+q,\nu_q}| + |\beta_{2m+q,\nu_q}| \neq 0, \quad q = 1, ..., r.$$

The right-hand side $f(x,t)$ is a given function. Note that conditions (3) are non-decomposable boundary conditions for $j = 1, ..., 2p$, i.e. they have the form

$$U_j(u(x,\cdot)) \equiv U_{j0}(u(x,\cdot)) + U_{jT}(u(x,\cdot)), \tag{4}$$

where for $a = 0, T$ the linear form $U_{ja}(u(x,\cdot))$ represents a differential expression depending on

$$u(x,a), \frac{\partial u(x,a)}{\partial t}, ..., \frac{\partial^{2p-1} u(x,a)}{\partial t^{2p-1}}.$$

According to the monograph by M.A. Naimark [1], boundary conditions of the form (3) are regular boundary conditions. In the work by G.M. Kesel'man [2] it is shown that regular in the sense of Birkhoff boundary conditions of the form (4) can be normalized and reduced to the form (3).

Boundary conditions (2) with respect to the variable x were first introduced and studied in the work of N.I. Ionkin [3]. An important distinctive feature of Ionkin's conditions is that the corresponding eigenvalue problem

$$-v''(x) = \lambda v(x), \quad 0 < x < 1$$

with boundary conditions (2) has infinitely many associated functions. The latter fact significantly influences the spectral expansions with respect to the system of eigen and associated functions of the Ionkin problem.

The purpose of this work is to find out what requirements the right-hand side of $f(x,t)$ must satisfy so that problem (1)–(2)–(3) is uniquely solvable?

We define the functional space of solutions $V_2^{1,2p}(Q_T)$ of problem (1)–(2)–(3) as the linear space of functions $u(x,t)$ belonging to the space $L_2(Q_T)$ and having a generalized derivative with respect to the spatial variable x and with respect to the variable t up to the order $2p$ inclusive, belonging to the same space, with a finite norm

$$\|u\|^2_{V_2^{1,2p}(Q_T)} = \int_0^T \Big[\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} + |\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle|\Big]\, dt,$$

where

$$\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle = \int_0^1 l(t,\frac{\partial}{\partial t})u(x,t)\,\overline{u(x,t)}\, dx.$$

It is obvious that the space $V_2^{1,2p}(Q_T)$ is a Banach space.

In the work of N.I. Ionkin [3] the eigenvalue problem was considered:

$$v''(x) + \lambda\, v(x) = 0, \quad 0 < x < 1 \tag{5}$$

$$v(0) = 0, \tag{6}$$

$$v'(0) = v'(1) \tag{7}$$

which is not self-adjoint. The adjoint to problem (5)–(7) has the form

$$v^{*\prime\prime}(x) + \lambda\, v^*(x) = 0, \quad 0 < x < 1, \tag{8}$$

$$v^{*\prime}(1) = 0, \tag{9}$$

$$v^*(0) = v^*(1). \tag{10}$$

It is known [3] that problem (4)–(6) has eigenvalues

$$\lambda_k = (2\pi\, k)^2, \quad k = 0, 1, ... \tag{11}$$

The system of eigenfunctions of problem (5)–(7) is calculated in the work [3] and it has the form:

$$v_0(x) = x,\ v_{2k-1}(x) = x\, \cos(2\pi\, k\, x),\ v_{2k}(x) = \sin(2\pi\, k\, x),\ k = 1, 2, ... \tag{12}$$

Note that each eigenvalue λ_k for $k > 0$ corresponds to an eigenfunction $v_{2k}(x)$ and an associated $v_{2k-1}(x)$. At the same time, for $k = 0$ the eigenvalue $\lambda_0 = 0$ is simple.

The system of eigen and associated functions of the adjoint problem to problem (4)–(6) is denoted by [3]:

$$v_0^*(x) = 2,\ v_{2k-1}^*(x) = 4\cos(2\pi\, k\, x),\ v_{2k}^*(x) = 4(1-x)\sin(2\pi\, k\, x),\ k = 1, 2, ... \tag{13}$$

In this case, each $\lambda_k = (2\pi\, k)^2$ with $k > 0$ corresponds to an eigenfunction $v_{2k-1}^*(x)$ and an associated $v_{2k}^*(x)$.

In the work [3] the following lemmas are proved.

Lemma 1 *[3]. Sequences of functions (12) and (13) form biorthogonal on the interval* $(0, 1)$ *systems of functions, so that for any numbers* i *and* j *the following relation holds*

$$\langle v_i, v_j^* \rangle = \int_0^1 v_i(x)\, \overline{v_j^*(x)}\, dx = \delta_{ij},$$

here δ_{ij} *is the Kronecker delta.*

Lemma 2 *[3]. The sequence* $v_0(x) = x$, $v_{2k-1}(x) = x\, \cos(2\pi\, k\, x)$, $v_{2k}(x) = \sin(2\pi\, k\, x)$, $k = 1, 2, ...$ *forms a basis in the space of functions* $L_2(0, 1)$*, and for any function* $\varphi(x) \in L_2(0, 1)$ *the inequalities of F. Riesz [5] with some constants are valid* r_1, r_2, R_1, R_2, *:*

$$r_1\, \|\varphi\|_{L_2(0,1)} \leq \sum_{k=0}^{\infty} |\varphi_k|^2 \leq R_1\, \|\varphi\|_{L_2(0,1)}, \tag{14}$$

$$r_2\, \|\varphi\|_{L_2(0,1)} \leq \sum_{k=0}^{\infty} |\varphi_k^*|^2 \leq R_2\, \|\varphi\|_{L_2(0,1)}, \tag{15}$$

where $\varphi_k = \int_0^1 \varphi(x)\, \overline{v_k^*(x)} dx$, $\quad \varphi_k^* = \int_0^1 \varphi(x)\, \overline{v_k(x)} dx$.

In what follows we denote by $\rho_0 = 1$, $\rho_{2k-1} = \rho_{2k} = \sqrt[2p]{1 + (2\pi k)^2}$ for $k \geq 1$. Sequence of numbers We denote $\{\rho_k,\ k \geq 0\}$ by ρ.

If the right-hand side of the equation $f(x,t)$ belongs to the space $L_2(Q_T)$, then we introduce the sequences

$$f_k(t) = \int_0^1 f(x,t)\,\overline{v_k^*(x)}dx, \quad k \geq 0, \tag{16}$$

$$f_k^*(t) = \int_0^1 f(x,t)\,\overline{v_k(x)}dx, \quad k \geq 0. \tag{17}$$

Let $W_2^{0,1}(Q_T)$ denote the space of functions $f(x,t) \in L_2(Q_T)$ such that

$$\frac{\partial f(x,t)}{\partial t} \in L_2(Q_T),$$

$$\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,T)\|^2_{L_2(0,1)} + \int_0^T \|f(\cdot,t)\|^2_{L_2(0,1)}\,dt + \int_0^T \|\frac{\partial f(\cdot,t)}{\partial t}\|^2_{L_2(0,1)}\,dt < \infty.$$

The main result of this article is formulated in the following statement.

Theorem 1 *Let p be an arbitrary natural number. Let the condition $\Delta(\rho_k) \neq 0$ be satisfied for all $k \geq 0$ (the characteristic determinant $\Delta(\rho_k)$ is introduced by formula (23)).*

If the right-hand side $f(x,t)$ belongs to the space $W_2^{0,1}(Q_T)$, then there exists a unique solution $u(x,t) \in V_2^{1,2p}(Q_T)$ of problem (1)-(2)-(3), and the estimate

$$\int_0^T \left[\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} + |\langle l(t,\frac{\partial}{\partial t})u(\cdot,t),u(\cdot,t)\rangle|\right]dt \leq$$

$$M\left[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,T)\|^2_{L_2(0,1)} + \int_0^T \|f(\cdot,t)\|^2_{L_2(0,1)}\,dt + \int_0^T \|\frac{\partial f(\cdot,t)}{\partial t}\|^2_{L_2(0,1)}dt\right], \tag{18}$$

where M is some constant independent of $f(x,t)$.

In the works of N.I. Ionkin [3,4] a mixed problem for the heat equation was investigated. Theorem 1 generalizes the results of N.I. Ionkin [3,4] when the differential part of the heat conductivity operator with respect to the variable t is replaced by a differential expression with respect to the variable t of order $2p$.

At the same time, in the works of A.I. Kozhanov [6,7], R.R. Ashurov [8], K.B. Sabitov [9,10] similar problems were studied, when the differential part of the heat conductivity operator with respect to the spatial variable x is replaced by more complex differential expressions with respect to the variable x.

Note that an equation of the form (1), according to the terminology of A.A. Dezin [11], refers to differential-operator equations. The issues of solvability of differential-operator equations were studied in the works [12–17]. V.V. Sheluchin [18,19] studied the problem of forecasting ocean temperature based on average data for the previous period of time, which also belongs to the class of differential-operator equations.

There are various methods of proving uniqueness. Usually, an effective means of proving uniqueness is the maximum principle [20] and its various generalizations such as the Hopf [21] and Zaremba-Giraud [22] principles. For problem (1)–(2)–(3), the above principles are not satisfied. Therefore, when proving the uniqueness of a solution, we needed a toolkit other than the extremum principle.

In the work of V.A. Ilyin [23] a fairly universal method for proving the uniqueness of a solution for hyperbolic and parabolic equations is proposed. Under fairly general restrictions on the domain Ω, in the work [23] a theorem of uniqueness of a solution for hyperbolic and parabolic equations is proved. The meaning of the requirements of V.A. Ilyin's theorem [23] is that the elliptic part of a hyperbolic and parabolic operator has a complete system of eigenfunctions in the corresponding functional space.

We also note the work of I.V. Tikhonov [24], devoted to uniqueness theorems for linear nonlocal problems for abstract differential equations. This work is interesting because I.V. Tikhonov proposed a new method for proving uniqueness theorems. I.V. Tikhonov's method of proving uniqueness is based on the "method of quotients"for entire functions of exponential type. In the work of A.Yu. Popov, I.V. Tikhonov [25], the class of unique solvability of the heat equation with a nonlocal condition expressed by an integral over time on a fixed interval was determined. They managed to give a complete description of uniqueness classes in terms of the behavior of solutions for $|x| \to \infty$.

The differential equation (1) is the sum of two operators. One of the operators is generated by linear differential expressions depending on time. The second operator is the Ionkin operator with respect to the spatial variable. In this paper, the differential operator with respect to time is generated by two-point Birkhoff regular boundary conditions. At the same time, the elliptic operator with respect to the spatial variable does not satisfy the so-called Agmon conditions [26].

Moreover, the operator with respect to the spatial variable is not self-adjoint. Therefore, direct application of the methods of works [28–34] to prove the uniqueness of the solution to problem (1)–(2)–(3) is quite problematic. However, the authors managed to modify the reasoning of works [28–30] to prove the uniqueness of the solution to problem (1)–(2)–(3).

Recall that for unique solvability, the mutual arrangement of the spectra of the two operators indicated above plays an essential role. In the articles of the authors [28–30], the method for proving the uniqueness theorem is a hybrid of the method of guiding functionals of M.G. Crane [1, 27] and the method of V.A. Ilyin [23].

2 Formal representation of the solution to problem (1)–(2)–(3)

We seek the solution to problem (1)–(2)–(3) in the form

$$u(x,t) = y_0(t)\, v_0(x) + \sum_{k=1}^{\infty} \big(y_{2k-1}(t)\, v_{2k-1}(x) + y_{2k}(t)\, v_{2k}(x)\big), \tag{19}$$

Using the results of the monograph [1], we find the coefficients $y_k(t)$ using the formula

$$y_k(t) = \int_0^T \frac{H(t,\tau,\lambda_k)}{\Delta(\lambda_k)}\, f_k(\tau)\, d\tau, \tag{20}$$

where $f_k(\tau)$ are calculated using formula (16).
The formulas for the expressions $H(t,\tau,\rho_k^{2p})$ and $\Delta(\rho_k^{2p})$ are taken from the monograph [1] and will be given below.

For further calculations, it is convenient to denote by $\{\omega_\mu\}$ the roots of (-1) of degree $2p$. In this section, we assume that p is an even number. The results formulated for even p remain valid for odd p. In this case, minor changes are required in the course of proving the results.

If p is an even number, then the numbering of the numbers $\{\omega_1,...,\omega_{2p}\}$ can be subordinated to the inequalities

$$\mathrm{Re}\omega_1 \le \cdots \le \mathrm{Re}\omega_p < 0 < \mathrm{Re}\omega_{p+1} \le \cdots \le \mathrm{Re}\omega_{2p}. \tag{21}$$

Let $\rho_k = \sqrt[2p]{1+4\pi^2k^2}$. According to the monograph [1], we introduce a system of solutions $\{y_\mu(t,\rho_k)\}$ of the homogeneous equation $l(t,\frac{d}{dt})y_\mu(t)+\rho_k^{2p}\,y_\mu(t)=0$, which has an asymptotic representation

$$y_\mu(t,\rho_k) = e^{\omega_\mu\rho_k t}\cdot[1], \tag{22}$$

where $[1]=1+O(1/\rho_k)$ при $\rho_k\to\infty$.

Now we can enter the characteristic determinant

$$\Delta(\rho_k) = \det[U_j(y_\mu)]; \quad j,\, \mu = 1,\ldots,2p. \tag{23}$$

From inequalities (21) and asymptotic representations (22) we obtain an asymptotic representation of the characteristic determinant for $\rho_k\to\infty$

$$\Delta(\rho_k) = \rho_k^{2(j_1+\ldots+j_m)+\nu_1+\ldots+\nu_r}\cdot e^{\rho_k(\omega_{p+1}+\ldots+\omega_{2p})T}\cdot\Delta_0\cdot[1], \tag{24}$$

where

$$\Delta_0 = \begin{vmatrix} \omega_1^{j_1} & \ldots & \omega_p^{j_1} & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \omega_{p+1}^{j_1} & \ldots & \omega_{2p}^{j_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \omega_1^{j_m} & \ldots & \omega_p^{j_m} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \omega_{p+1}^{j_m} & \ldots & \omega_{2p}^{j_m} \\ \alpha_{2m+1,\nu_1}\omega_1^{\nu_1} & \ldots & \alpha_{2m+1,\nu_1}\omega_1^{\nu_1} & \beta_{2m+1,\nu_1}\omega_{p+1}^{\nu_1} & \ldots & \beta_{2m+1,\nu_1}\omega_{2p}^{\nu_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{2m+r,\nu_r}\omega_1^{\nu_r} & \ldots & \alpha_{2m+1,\nu_r}\omega_1^{\nu_r} & \beta_{2m+1,\nu_r}\omega_{p+1}^{\nu_r} & \ldots & \beta_{2m+1,\nu_r}\omega_{2p}^{\nu_r} \end{vmatrix} \neq 0.$$

Let W_μ denote the algebraic complement of the element $y_\mu^{(2p-1)}(\tau,\rho_k)$ in the determinant

$$W(\tau) = \begin{vmatrix} y_1^{(2p-1)}(\tau,\rho_k) & \ldots & y_{2p}^{(2p-1)}(\tau,\rho_k) \\ y_1^{(2p-2)}(\tau,\rho_k) & \ldots & y_{2p}^{(2p-2)}(\tau,\rho_k) \\ \cdot & \ldots & \cdot \\ y_1(\tau,\rho_k) & \ldots & y_{2p}(\tau,\rho_k) \end{vmatrix}.$$

Let's define the function

$$g(t,\tau,\rho_k) = \pm\frac{1}{2}\sum_{\mu=1}^{2p} y_\mu(t,\rho_k)\,z_\mu(\tau,\rho_k),$$

where

$$z_\mu(\tau,\rho_k)=\frac{W_\mu(\tau)}{W(\tau)}.$$

Moreover, the sign $\ll + \gg$ is taken when $t>\tau$, and the sign $\ll - \gg$ when $t<\tau$.

From inequalities (21) and representations (22) it follows that

$$z_\mu(\tau,\rho_k)=\frac{e^{-\rho_k\omega_\mu\tau}}{2p\cdot\rho_k^{2p-1}}\cdot[1].$$

According to the monograph [1] we introduce the function

$$H(t,\tau,\rho_k)=\begin{vmatrix} y_1(t,\rho_k) & \dots & y_{2p}(t,\rho_k) & g(t,\tau,\rho_k)\\ U_1(y_1) & \dots & U_1(y_{2p}) & U_1(g)(\tau)\\ \cdot & \dots & \cdot & \cdot\\ U_{2p}(y_1) & \dots & U_{2p}(y_{2p}) & U_{2p}(g)(\tau)\end{vmatrix},$$

where

$$U_j(g)=-\frac{1}{2}\sum_{j=1}^{2p}U_{j0}(y_\mu)\,z_\mu(\tau)+\frac{1}{2}\sum_{j=1}^{2p}U_{jT}(y_\mu)\,z_\mu(\tau),\quad j=1,\dots,2p.$$

Let $t>\tau$. In the determinant $H(t,\tau,\rho_k)$ we multiply the columns with numbers $1,\dots,p$ by the functions $\frac{1}{2}z_1(\tau),\dots,\frac{1}{2}z_p(\tau)$, and the columns $p+1,\dots,2p$ by the functions $-\frac{1}{2}z_{p+1}(\tau),\dots,-\frac{1}{2}z_{2p}(\tau)$ and add them with the last column. As a result, we obtain the relation

$$H(t,\tau,\rho_k)=\begin{vmatrix} y_1(t,\rho_k) & \dots & y_{2p}(t,\rho_k) & H_0(t,\tau)\\ U_1(y_1) & \dots & U_1(y_{2p}) & H_1(\tau,\rho_k)\\ \cdot & \dots & \cdot & \cdot\\ U_{2p}(y_1) & \dots & U_{2p}(y_{2p}) & H_{2p}(\tau,\rho_k)\end{vmatrix},$$

where

$$H_0(t,\tau,\rho_k)=\sum_{\mu=1}^{p}y_\mu(t)\,z_\mu(\tau),$$

$$H_j(\tau,\rho_k)=\sum_{\mu=1}^{p}U_{jT}(y_\mu)\,z_\mu(\tau)-\sum_{\mu=p+1}^{2p}U_{j0}(y_\mu)\,z_\mu(\tau),\quad j=1,\dots,2p.$$

From inequalities (21) and asymptotic representations (22) we derive the asymptotic formula

$$H(t,\tau,\rho_k)=\frac{\rho_k^{2(j_1+\ldots+j_m)+\nu_1+\ldots+\nu_r}\cdot e^{\rho_k(\omega_{p+1}+\ldots+\omega_{2p})T}}{2p\cdot\rho_k^{2p-1}}\cdot$$

$$\begin{vmatrix} e^{\rho_k\omega_1 t} & \dots & e^{\rho_k\omega_1 t} & \widetilde{H}_0(t,\tau,\rho_k)\\ \cdot & \dots & \cdot & \widetilde{H}_1(\tau,\rho_k)\\ \cdot & \Delta_0 & \cdot & \cdot\\ \cdot & \dots & \cdot & \widetilde{H}_{2p}(\tau,\rho_k)\end{vmatrix}[1],$$

where

$$\widetilde{H}_0(t,\tau,\rho_k) = -\sum_{\mu=1}^{p}\omega_\mu e^{\rho_k\omega_\mu(t-\tau)}[1],$$

$$\widetilde{H}_{2\xi-1}(\tau,\rho_k) = -\sum_{\mu=1}^{p}\beta_{2\xi-1,j_\xi-1}\omega_\mu^{j_\xi+1}\rho_k^{-1}e^{\rho_k\omega_\mu(T-\tau)}[1] + \sum_{\mu=p+1}^{2p}\omega_\mu^{j_\xi+1}e^{-\rho_k\omega_\mu\tau}[1],$$

$$\widetilde{H}_{2\xi}(\tau,\rho_k) = -\sum_{\mu=1}^{p}\omega_\mu^{j_\xi+1}e^{\rho_k\omega_\mu(T-\tau)}[1] + \sum_{\mu=p+1}^{2p}\alpha_{2\xi,j_\xi-1}\omega_\mu^{j_\xi}\rho_k^{-1}e^{-\rho_k\omega_\mu\tau}[1], \xi=1,\ldots,m,$$

$$\widetilde{H}_{2m+\xi}(\tau,\rho_k) = -\sum_{\mu=1}^{p}\beta_{2m+\xi,\nu_\xi}\omega_\mu^{\nu_\xi+1}e^{\rho_k\omega_\mu(T-\tau)}[1] + \sum_{\mu=p+1}^{2p}\alpha_{2m+\xi,\nu_\xi}\omega_\mu^{\nu_\xi+1}e^{-\rho_k\omega_\mu\tau}[1], \xi=1,\ldots,r.$$

Therefore, for $0<\tau<t<T$ the representation is correct

$$\frac{H(t,\tau,\rho_k)}{\Delta(\lambda_k)} = \frac{1}{2p\,\rho_k^{2p-1}}\cdot\widetilde{H}_0(t,\tau,\rho_k) - \sum_{s=1}^{2p}\frac{1}{\Delta_0\,2p\,\rho_k^{2p-1}}h_s(t,\rho_k)\cdot\widetilde{H}_s(\tau,\rho_k), \tag{25}$$

where $h_s(t,\rho_k)$ is the determinant obtained from the characteristic determinant $\Delta(\lambda_k)$ by replacing its s-th row with the row

$$||e^{\omega_1\rho_k t}[1],...,e^{\omega_p\rho_k t}[1],\ e^{\omega_{p+1}\rho_k(t-T)}[1],...,e^{\omega_{2p}\rho_k(t-T)}[1]||.$$

The expansion of the determinant $h_s(t,\rho_k)$ over the s-th row has the form

$$h_s(t,\rho_k) = \sum_{\mu=1}^{p}h_{s,\mu}e^{\rho_k\omega_\mu t}[1] + \sum_{\mu=p+1}^{2p}h_{s,\mu}e^{\rho_k\omega_\mu(t-T)}[1], s=1,...,2p.$$

In this case, $h_{s,\mu}, s=1,\ldots,2p$ are numbers representing the corresponding algebraic complements.
The asymptotic relation (25) remains valid for $0<t<\tau<T$. As a result, equality (19), taking into account (20) and (25), will take the form

$$u(x,t) = \frac{1}{2p}v_0(x)\Big[-\int_0^T\widetilde{H}_0(t,\tau,\rho_0)\,f_0(\tau)d\tau - \frac{1}{\Delta_0}\sum_{s=1}^{2p}h_s(t,\rho_0)\int_0^T\widetilde{H}_s(\tau,\rho_0)\,f_0(\tau)d\tau\Big]+$$

$$\frac{1}{2p}\sum_{k=1}^{\infty}\frac{v_{2k-1}(x)}{\rho_{2k-1}^{2p-1}}\Big[-\int_0^T\widetilde{H}_0(t,\tau,\rho_{2k-1})\,f_{2k-1}(\tau)d\tau - \frac{1}{\Delta_0}\sum_{s=1}^{2p}h_s(t,\rho_{2k-1})\int_0^T\widetilde{H}_s(\tau,\rho_{2k-1})\,f_{2k-1}(\tau)d\tau\Big]+$$

$$\frac{1}{2p}\sum_{k=1}^{\infty}\frac{v_{2k}(x)}{\rho_{2k}^{2p-1}}\Big[-\int_0^T\widetilde{H}_0(t,\tau,\rho_{2k})\,f_{2k}(\tau)d\tau - \frac{1}{\Delta_0}\sum_{s=1}^{2p}h_s(t,\rho_{2k})\int_0^T\widetilde{H}_s(\tau,\rho_{2k})\,f_{2k}(\tau)d\tau\Big] =$$

$$\frac{1}{2p}\sum_{k=0}^{\infty}\frac{v_k(x)}{\rho_k^{2p-1}}\Big[-\int_0^T\widetilde{H}_0(t,\tau,\rho_k)\,f_k(\tau)d\tau - \frac{1}{\Delta_0}\sum_{s=1}^{2p}h_s(t,\rho_k)\int_0^T\widetilde{H}_s(\tau,\rho_k)\,f_k(\tau)d\tau\Big]. \tag{26}$$

Let us introduce the following notations

$$I^{(1)}_{k\mu}(t)=\rho_k\omega_\mu\int_0^t e^{\rho_k\omega_\mu(t-\tau)}\,f_k(\tau)d\tau,\quad I^{(2)}_{k\mu}(t)=\rho_k\omega_\mu\int_t^T e^{\rho_k\omega_\mu(t-\tau)}\,f_k(\tau)d\tau,$$

$$I^{(3)}_{k\mu s}(t)=e^{\rho_k\omega_\mu t}\rho_k\omega_s\int_0^T e^{\rho_k\omega_s(T-\tau)}\,f_k(\tau)d\tau,\quad I^{(4)}_{k\mu s}(t)=e^{\rho_k\omega_\mu t}\rho_k\omega_s\int_0^T e^{-\rho_k\omega_s\tau}\,f_k(\tau)d\tau,$$

$$I^{(5)}_{k\mu s}(t)=e^{\rho_k\omega_\mu(t-T)}\rho_k\omega_s\int_0^T e^{\rho_k\omega_s(T-\tau)}\,f_k(\tau)d\tau,\quad I^{(6)}_{k\mu s}(t)=e^{\rho_k\omega_\mu(t-T)}\rho_k\omega_s\int_0^T e^{-\rho_k\omega_s\tau}\,f_k(\tau)d\tau. \tag{27}$$

Now let us take into account the asymptotic representations of the functions $\widetilde{H}_0(t,\tau,\rho_k)$, $\widetilde{H}_s(\tau,\rho_k)$, $s=1,\dots,2p$. As a result, from (26), using the notation (27), we have the following representation

$$u(x,t)=\frac{1}{2p}\sum_{k=0}^{\infty}\frac{v_k(x)}{\rho_k^{2p}}\cdot A_k(t), \tag{28}$$

where

$$A_k(t)=\Big[-\sum_{\mu=1}^{p}I^{(1)}_{k\mu}(t)-\sum_{\mu=p+1}^{2p}I^{(2)}_{k\mu}(t)+$$

$$\frac{1}{\Delta_0}\sum_{\xi=1}^{m}\Big(-\sum_{\mu=1}^{p}\sum_{s=1}^{p}h_{2\xi-1,\mu}\beta_{2\xi-1,j_\xi-1}\frac{\omega_s^{j_\xi}}{\rho_k}I^{(3)}_{k\mu s}(t)+\sum_{\mu=1}^{p}\sum_{s=p+1}^{2p}h_{2\xi-1,\mu}\omega_s^{j_\xi}I^{(4)}_{k\mu s}(t)-$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p}h_{2\xi-1,\mu}\beta_{2\xi-1,j_\xi-1}\frac{\omega_s^{j_\xi}}{\rho_k}I^{(5)}_{k\mu s}(t)+\sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p}h_{2\xi-1,\mu}\omega_s^{j_\xi}I^{(6)}_{k\mu s}(t)-$$

$$\sum_{\mu=1}^{p}\sum_{s=1}^{p}h_{2\xi,\mu}\omega_s^{j_\xi}I^{(3)}_{k\mu s}(t)+\sum_{\mu=1}^{p}\sum_{s=p+1}^{2p}h_{2\xi,\mu}\alpha_{2\xi,j_\xi-1}\frac{\omega_s^{j_\xi-1}}{\rho_k}I^{(4)}_{k\mu s}(t)-$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p}h_{2\xi,\mu}\omega_s^{j_\xi}I^{(5)}_{k\mu s}(t)+\sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p}h_{2\xi,\mu}\alpha_{2\xi,j_\xi-1}\frac{\omega_s^{j_\xi-1}}{\rho_k}I^{(6)}_{k\mu s}(t)\Big)-$$

$$\sum_{\xi=1}^{r}\Big(-\sum_{\mu=1}^{p}\sum_{s=1}^{p}h_{2m+\xi,\mu}\beta_{2m+\xi,\nu_\xi}\omega_s^{j_\xi}I^{(3)}_{k\mu s}(t)+\sum_{\mu=1}^{p}\sum_{s=p+1}^{2p}h_{2m+\xi,\mu}\alpha_{2m+\xi,\nu_\xi}\omega_s^{\nu_\xi}I^{(4)}_{k\mu s}(t)-$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p}h_{2m+\xi,\mu}\beta_{2m+\xi,\nu_\xi}\omega_s^{\nu_\xi}I^{(5)}_{k\mu s}(t)+\sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p}h_{2m+\xi,\mu}\alpha_{2m+\xi,\nu_\xi}\omega_s^{j_\xi}I^{(6)}_{k\mu s}(t)\Big)\Big].$$

Remark 1 *If, when solving problem (1)-(2)-(3), we use the expansion in the biorthogonal system $v_k^*(x)$, then we would obtain a representation of the solution $u(x,t)$, similar to the representation (28)*

$$u(x,t) = \frac{1}{2p}\sum_{k=0}^{\infty}\frac{v_k^*(x)}{\rho_k^{2p}}\cdot A_k^*(t), \tag{29}$$

where

$$A_k^*(t) = \Big[-\sum_{\mu=1}^{p} J_{k\mu}^{(1)}(t) - \sum_{\mu=p+1}^{2p} J_{k\mu}^{(2)}(t) +$$

$$\frac{1}{\Delta_0}\sum_{\xi=1}^{m}\Big(-\sum_{\mu=1}^{p}\sum_{s=1}^{p} h_{2\xi-1,\mu}\beta_{2\xi-1,j_\xi-1}\frac{\omega_s^{j_\xi}}{\rho_k}J_{k\mu s}^{(3)}(t) + \sum_{\mu=1}^{p}\sum_{s=p+1}^{2p} h_{2\xi-1,\mu}\omega_s^{j_\xi}J_{k\mu s}^{(4)}(t) -$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p} h_{2\xi-1,\mu}\beta_{2\xi-1,j_\xi-1}\frac{\omega_s^{j_\xi}}{\rho_k}J_{k\mu s}^{(5)}(t) + \sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p} h_{2\xi-1,\mu}\omega_s^{j_\xi}J_{k\mu s}^{(6)}(t) -$$

$$\sum_{\mu=1}^{p}\sum_{s=1}^{p} h_{2\xi,\mu}\omega_s^{j_\xi}J_{k\mu s}^{(3)}(t) + \sum_{\mu=1}^{p}\sum_{s=p+1}^{2p} h_{2\xi,\mu}\alpha_{2\xi,j_\xi-1}\frac{\omega_s^{j_\xi-1}}{\rho_k}J_{k\mu s}^{(4)}(t) -$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p} h_{2\xi,\mu}\omega_s^{j_\xi}J_{k\mu s}^{(5)}(t) + \sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p} h_{2\xi,\mu}\alpha_{2\xi,j_\xi-1}\frac{\omega_s^{j_\xi-1}}{\rho_k}J_{k\mu s}^{(6)}(t)\Big) -$$

$$\sum_{\xi=1}^{r}\Big(-\sum_{\mu=1}^{p}\sum_{s=1}^{p} h_{2m+\xi,\mu}\beta_{2m+\xi,\nu_\xi}\omega_s^{j_\xi}J_{k\mu s}^{(3)}(t) + \sum_{\mu=1}^{p}\sum_{s=p+1}^{2p} h_{2m+\xi,\mu}\alpha_{2m+\xi,\nu_\xi}\omega_s^{\nu_\xi}J_{k\mu s}^{(4)}(t) -$$

$$\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p} h_{2m+\xi,\mu}\beta_{2m+\xi,\nu_\xi}\omega_s^{\nu_\xi}J_{k\mu s}^{(5)}(t) + \sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p} h_{2m+\xi,\mu}\alpha_{2m+\xi,\nu_\xi}\omega_s^{j_\xi}J_{k\mu s}^{(6)}(t)\Big)\Big][1].$$

Here we introduced the functions $J_{k\mu}^{(1)}(t), ..., J_{k\mu s}^{(6)}(t)$, which were obtained from $I_{k\mu}^{(1)}(t), ..., I_{k\mu s}^{(6)}(t)$ as a result of replacing $f_k(\tau)$ with $f_k^*(\tau)$, i.e.

$$J_{k\mu}^{(1)}(t) = \rho_k\omega_\mu\int_0^t e^{\rho_k\omega_\mu(t-\tau)}\, f_k^*(\tau)d\tau, \quad J_{k\mu}^{(2)}(t) = \rho_k\omega_\mu\int_t^T e^{\rho_k\omega_\mu(t-\tau)}\, f_k^*(\tau)d\tau,$$

$$J_{k\mu s}^{(3)}(t) = e^{\rho_k\omega_\mu t}\rho_k\omega_s\int_0^T e^{\rho_k\omega_s(T-\tau)}\, f_k^*(\tau)d\tau, \quad J_{k\mu s}^{(4)}(t) = e^{\rho_k\omega_\mu t}\rho_k\omega_s\int_0^T e^{-\rho_k\omega_s\tau}\, f_k^*(\tau)d\tau,$$

$$J_{k\mu s}^{(5)}(t) = e^{\rho_k\omega_\mu(t-T)}\rho_k\omega_s\int_0^T e^{\rho_k\omega_s(T-\tau)}\, f_k^*(\tau)d\tau, \quad J_{k\mu s}^{(6)}(t) = e^{\rho_k\omega_\mu(t-T)}\rho_k\omega_s\int_0^T e^{-\rho_k\omega_s\tau}\, f_k^*(\tau)d\tau.$$

3 Some auxiliary statements

In this section we will prove the following lemma.

Lemma 3 *Let $\gamma = \sin\frac{\pi}{2p}$ and p be an even number. If $f(x,t)$ is differentiable with respect to t, then for all $k \geq 1$ and $t \in (0,T)$ the following estimates hold:*

$$|I^{(1)}_{k\mu}(t)| \leq |f_k(t)| + |f_k(0)|e^{-\gamma\rho_k t} + \int_0^t |f'_k(\tau)|\, e^{-\gamma\rho_k(t-\tau)}\, d\tau \quad for \quad \mu = 1,\dots,p;$$

$$|I^{(2)}_{k\mu}(t)| \leq |f_k(t)| + |f_k(T)|e^{-\gamma\rho_k(T-t)} + \int_t^T |f'_k(\tau)|\, e^{\gamma\rho_k(t-\tau)}\, d\tau \quad for \quad \mu = p+1,\dots,2p;$$

$$|I^{(3)}_{k\mu s}(t)| \leq |f_k(T)|e^{-\gamma\rho_k t} + |f_k(0)|e^{-\gamma\rho_k(t+T)} + e^{-\gamma\rho_k t}\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k(T-\tau)}\, d\tau,$$

$$for \quad s,\mu = 1,\dots,p;$$

$$|I^{(4)}_{k\mu s}(t)| \leq |f_k(T)|e^{-\gamma\rho_k(t+T)} + |f_k(0)|e^{-\gamma\rho_k t} + e^{-\gamma\rho_k t}\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau,$$

for $s = p+1,\dots,2p, \quad \mu = 1,\dots,p$;

$$|I^{(5)}_{k\mu s}(t)| \leq |f_k(T)|e^{\gamma\rho_k(t-T)} + |f_k(0)|e^{\gamma\rho_k(t-2T)} + e^{\gamma\rho_k(t-T)}\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k(T-\tau)}\, d\tau,$$

$$for \quad s = 1,\dots,p, \quad \mu = p+1,\dots,2p;$$

$$|I^{(6)}_{k\mu s}(t)| \leq |f_k(T)|e^{\gamma\rho_k(t-2T)} + |f_k(0)|e^{\gamma\rho_k(t-T)} + e^{\gamma\rho_k(t-T)}\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau,$$

$$for \quad s,\mu = p+1,\dots,2p; \qquad (30)$$

Similarly, similar estimates also hold for integrals $J^{(1)}_{k\mu}(t), J^{(2)}_{k\mu}(t), J^{(i)}_{k\mu s}(t), i = 3,4,5,6.$

The proof of Lemma 3 follows from the fact that for p – even the following inequalities hold:

$$\mathrm{Re}\omega_1 \leq \dots \leq \mathrm{Re}\omega_p = -\gamma < 0 < \gamma = \mathrm{Re}\omega_{p+1} \leq \dots \leq \mathrm{Re}\omega_{2p}.$$

For example, let us prove an estimate for $|I^{(1)}_{k\mu}(t)|$. The following identity holds

$$I^{(1)}_{k\mu}(t) = \rho_k\omega_\mu \int_0^t e^{\rho_k\omega_\mu(t-\tau)}\, f_k(\tau)d\tau = -\int_0^t f_k(\tau)\,\frac{d}{d\tau}e^{\rho_k\omega_\mu(t-\tau)}\, d\tau =$$

$$-f_k(t) + f_k(0)e^{\rho_k\omega_\mu t} + \int_0^t f'_k(\tau)\, e^{\rho_k\omega_\mu(t-\tau)}\, d\tau.$$

This implies the required estimate for $|I^{(1)}_{k\mu}(t)|$. The other statements in Lemma 3 are proved similarly.

Lemma 4 *There is an estimate for the integral*

$$|I^{(6)}_{k\mu s}(t)|^2 \leq |f_k(T)|^2 + |f_k(0)|^2 + \frac{1}{\gamma}\int_0^T |f'_k(\tau)|^2 d\tau. \tag{31}$$

For other quantities $|I^{(j)}_{k\mu s}(t)|^2$, $j = \overline{1,5}$, *similar estimates hold. Similar estimates also hold for quantities* $|J^{(j)}_{k\mu s}(t)|^2$, $j = \overline{1,6}$.

Proof 1 *For* $s > p$ *and* $\mu > p$ *we write out the value using relations* (30)

$$|I^{(6)}_{k\mu s}(t)|^2 \leq \Big(|f_k(T)|e^{\gamma\rho_k(t-2T)} + |f_k(0)|e^{\gamma\rho_k(t-T)} + e^{\gamma\rho_k(t-T)}\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau\Big)^2 \leq$$

$$\leq 3\Big(|f_k(T)|^2 e^{2\gamma\rho_k(t-2T)} + |f_k(0)|^2 e^{2\gamma\rho_k(t-T)} + e^{2\gamma\rho_k(t-T)}(\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau)^2\Big).$$

Since $0 < t < T$, *then* $(t-2T) < (t-T) < 0$. *Therefore, the following estimates are valid*

$$|f_k(T)|^2 e^{2\gamma\rho_k(t-2T)} \leq |f_k(T)|^2,$$

$$|f_k(0)|^2 e^{2\gamma\rho_k(t-T)} \leq |f_k(0)|^2,$$

$$e^{2\gamma\rho_k(t-T)}(\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau)^2 \leq (\int_0^T |f'_k(\tau)|\, e^{-\gamma\rho_k\tau}\, d\tau)^2 \leq$$

$$\int_0^T |f'_k(\tau)|^2 d\tau \cdot \int_0^T e^{-2\gamma\rho_k\tau}\, d\tau = \frac{1-e^{-2\gamma\rho_k T}}{2\gamma\rho_k}\cdot\int_0^T |f'_k(\tau)|^2 d\tau \leq \frac{1}{\gamma}\int_0^T |f'_k(\tau)|^2 d\tau.$$

This implies (31). *Lemma 4 is proved. Other quantities* $|I^{(j)}_{k\mu s}(t)|^2$, $j = \overline{1,5}$ *are estimated similarly.*

4 Proof of solvability of problem (1)–(2)–(3)

In Section 3, formal representations of the solution to problem (1)–(2)–(3) are obtained. The solutions to problem (1)–(2)–(3) have representations (28) and (29). Now we will justify the convergence of these representations.

From Lemma 3 and representation (28) follow the necessary estimates for $\|u(\cdot,t)\|^2_{L_2(0,1)}$, $\|\frac{\partial u(\cdot,t)}{\partial x})\|^2_{L_2(0,1)}$.

First, let us evaluate the expression $\|u(\cdot,t)\|^2_{L_2(Q)} + \|\frac{\partial u(\cdot,t)}{\partial x})\|^2_{L_2(0,1)}$. To do this, let's consider

$$\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} = \int_0^1 (u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2})\,\overline{u(x,t)}\, dx.$$

From (28) it follows that

$$u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{2p}\sum_{k=0}^{\infty} v_k(x)\cdot\, A_k(t).$$

We multiply the last expression by (29) as a scalar. As a result, we have

$$\|u(\cdot,t)\|^2_{L_2(0,1)}+\|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)}=\int_0^1\frac{1}{2p}\Big(\sum_{k=0}^{\infty}v_k(x)\cdot A_k(t)\Big)\frac{1}{2p}\Big(\sum_{j=0}^{\infty}\frac{1}{\rho_j^{2p}}\cdot\overline{v_j^*(x)\,A_j^*(t)}\Big)\,dx=$$

$$\frac{1}{4p^2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{\rho_j^{2p}}A_k(t)\cdot\overline{A_j^*(t)}\cdot\int_0^1 v_k(x)\cdot\overline{v_j^*(x)}\,dx=\frac{1}{4p^2}\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}A_k(t)\cdot\overline{A_k^*(t)}.$$

Thus, we get

$$\|u(\cdot,t)\|^2_{L_2(0,1)}+\|\frac{\partial u(\cdot,t)}{\partial x})\|^2_{L_2(0,1)}=\frac{1}{4p^2}\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}A_k(t)\cdot\overline{A_k^*(t)}. \tag{32}$$

From (32) we obtain the following upper bound

$$\frac{1}{4p^2}\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}|A_k(t)\cdot\overline{A_k^*(t)}|\le\frac{1}{4p^2}\sum_{k=0}^{\infty}\frac{1}{2}\Big(|\frac{A_k(t)}{\rho_k^p}|^2+|\frac{\overline{A_k^*(t)}}{\rho_k^p}|^2\Big)\le$$

$$\frac{1}{8p^2}\sum_{k=0}^{\infty}\Big(|A_k(t)|^2+|\overline{A_k^*(t)}|^2\Big). \tag{33}$$

Taking into account (28), we estimate the expression from above

$$|A_k(t)|^2\le M\cdot\Big[\sum_{\mu=1}^{p}|I^{(1)}_{k\mu}(t)|^2+\sum_{\mu=p+1}^{2p}|I^{(2)}_{k\mu}(t)|^2+\sum_{\mu=1}^{p}\sum_{s=1}^{p}|I^{(3)}_{k\mu s}(t)|^2+$$

$$\sum_{\mu=1}^{p}\sum_{s=p+1}^{2p}|I^{(4)}_{k\mu s}(t)|^2+\sum_{\mu=p+1}^{2p}\sum_{s=1}^{p}|I^{(5)}_{k\mu s}(t)|^2+\sum_{\mu=p+1}^{2p}\sum_{s=p+1}^{2p}|I^{(6)}_{k\mu s}(t)|^2\Big]. \tag{34}$$

From Lemma 3, as well as from Lemma 4, the following inequality follows

$$|A_k(t)|^2\le M\Big[|f_k(0)|^2+|f_k(t)|^2+|f_k(T)|^2+\frac{1}{\gamma}\int_0^T|f_k'(\tau)|^2d\tau\Big]. \tag{35}$$

Summing both parts of inequality (35) over k, we have

$$\sum_{k=0}^{\infty}|A_k(t)|^2\le M\,p^2\Big[\|f(\cdot,0)\|^2_{L_2(0,1)}+\|f(\cdot,t)\|^2_{L_2(0,1)}+$$

$$\|f(\cdot,T)\|^2_{L_2(0,1)}+\frac{1}{\gamma}\int_0^T\|\frac{\partial f(\cdot,\tau)}{\partial\tau}\|^2_{L_2(0,1)}d\tau\Big]. \tag{36}$$

Here we use the inequalities from Lemma 2.
Using the statements of the second part of Lemma 3, we have a similar estimate for the expression

$$\sum_{k=0}^{\infty} |\overline{A_k^*(t)}|^2 \leq M\,p^2\Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(0,1)}+$$

$$\|f(\cdot,T)\|^2_{L_2(0,1)} + \frac{1}{\gamma}\int_0^T \|\frac{\partial f(\cdot,\tau)}{\partial \tau}\|^2_{L_2(0,1)} d\tau\Big]. \tag{37}$$

As a result, from inequalities (32), (34)–(35)–(36)–(37) we obtain the estimate

$$\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x})\|^2_{L_2(0,1)} \leq M_1\cdot\Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(0,1)}+$$

$$\|f(\cdot,T)\|^2_{L_2(0,1)} + \frac{1}{\gamma}\int_0^T \|\frac{\partial f(\cdot,\tau)}{\partial \tau}\|^2_{L_2(0,1)} d\tau\Big]. \tag{38}$$

Scalar multiplication of the expression $l(t,\frac{\partial}{\partial t})u(\cdot,t)$ by $u(\cdot,t)$ we get

$$\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle = \int_0^1 \Big[f(x,t) - \frac{1}{2p}\sum_{k=0}^{\infty} v_k(x)\cdot A_k(t),\ \overline{u(x,t)}\Big]\,dx =$$

$$\int_0^1 f(x,t)\,\overline{u(x,t)}\,dx - \frac{1}{2p}\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}\,A_k(t)\cdot\overline{A_k^*(t)}. \tag{39}$$

Next, using the above estimates, we obtain

$$|\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle| \leq \|u(\cdot,t)\|_{L_2(0,1)}\cdot\|f(\cdot,t)\|_{L_2(0,1)} + \frac{1}{2p}\sum_{k=0}^{\infty}\frac{1}{\rho_k^{2p}}\,|A_k(t)\cdot\overline{A_k^*(t)}| \leq$$

$$\frac{1}{2}\Big(\|u(\cdot,t)\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(0,1)}\Big) + M\cdot\sum_{k=0}^{\infty}\Big(|A_k(t)|^2 + |\overline{A_k^*(t)}|^2\Big). \tag{40}$$

Summing up inequalities (38) and (40), we get as a result

$$\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} + |\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle| \leq$$

$$M_1\Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(0,1)} + \|f(\cdot,T)\|^2_{L_2(0,1)} + \frac{1}{\gamma}\int_0^T \|\frac{\partial f(\cdot,\tau)}{\partial \tau}\|^2_{L_2(0,1)} d\tau\Big]. \tag{41}$$

Now it remains to integrate both parts of inequality (41) over t from 0 to T. As a result, we have

$$\int_0^T \Big[\|u(\cdot,t)\|^2_{L_2(0,1)} + \|\frac{\partial u(\cdot,t)}{\partial x}\|^2_{L_2(0,1)} + |\langle l(t,\frac{\partial}{\partial t})u(\cdot,t), u(\cdot,t)\rangle|\Big]\,dt \leq$$

$$M_2\Big[\|f(\cdot,0)\|^2_{L_2(0,1)} + \|f(\cdot,T)\|^2_{L_2(0,1)} + \int_0^T \|f(\cdot,t)\|^2_{L_2(0,1)}\,dt + \int_0^T \|\frac{\partial f(\cdot,t)}{\partial t}\|^2_{L_2(0,1)} dt\Big]. \tag{42}$$

As a result, for the solution $u(x,t)$ we have the required estimate.

5 Proof of the uniqueness of the solution of problem (1)–(2)–(3)

In the previous section, sufficient conditions for solvability on the right-hand side $f(x,t)$ of equation (1) were found.

Now we will prove the uniqueness of the solution $u(x,t)$ of problem (1)–(2)–(3). To do this, we denote by A the operator corresponding to Ionkin's problem (5)–(6)–(7). We also introduce the operator B, defined by the differential expression $l(t,\frac{d}{dt})$ according to the formula

$$Bw(t) = l(t,\frac{d}{dt})w(t), \quad 0 < t < T$$

on the domain of definition

$$D(B) = \{w(t) \in W_1^{2p}(0,T) : U_j(w) = 0, j = 1,2,...,2p\}.$$

Let us introduce for $s = 1,\dots,2p$ solutions $\kappa_s(t,\mu)$ of the homogeneous equation

$$l^+(t,\frac{d}{dt})\,æ_s(t,\mu) = \mu \cdot \kappa_s(t,\mu), \quad 0 < t < T$$

with inhomogeneous conditions

$$U_j^*(æ_s(\cdot,\mu)) = \delta_{j,s-1} \cdot \Delta^*(\mu), \quad j = 1,\dots,2p.$$

Here the expression $l^+(t,\frac{d}{dt})$ is the adjoint differential expression to the expression $l(t,\frac{d}{dt})$. The set of linear forms $U_1^*, U_2^*, ..., U_{2p}^*$ defines the domain of the adjoint operator B^*, that is

$$D(B^*) = \{w(t) \in W_1^{2p}(0,T) : \ U_j^*(w) = 0, j = 1,2,...,2p\}.$$

Here $\Delta^*(\mu)$ is the characteristic determinant of the operator B^*. Note that all solutions $æ_s(t,\mu)$ represent entire functions of μ.

Let μ_0 be the zero of the characteristic determinant $\Delta^*(\mu)$ and its multiplicity be m_0. Then for any $s = 1,\dots,2p$ in the ordered row

$$\left[\kappa_s(t,\mu_0), \frac{1}{1!}\frac{\partial}{\partial\mu}\kappa_s(t,\mu_0), \cdots, \frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial\mu^{m_0-1}}\kappa_s(t,\mu_0)\right] \tag{43}$$

the first non-zero function represents the eigenfunction of the operator B^*, and the subsequent members of the row give a chain of associated functions generated by it.

Finally, for the sake of completeness, we present the Lagrange formula [1]. For any two functions $w(t)$ and $R(t)$ from $W_2^{2p}(0,T)$, the identity holds

$$\int_0^T l(t,\frac{d}{dt})w(t)\overline{R(t)}dt - \int_0^T w(t)\overline{l^+(t,\frac{d}{dt})R(t)}dt =$$

$$\sum_{j=1}^{2p}\left[U_j(w)\cdot\overline{U_{4p-j+1}^*(R)} - U_{j+2p}(w)\cdot\overline{U_j^*(R)}\right], \tag{44}$$

where the linear forms $U_{2p+1}(\cdot), ..., U_{4p}(\cdot)$ are chosen so that the system of $4p$ forms $U_1(\cdot), ..., U_{4p}(\cdot)$ is a linearly independent system of linear forms. According to the results of the monograph [1], the set of linear forms $U_1^*(\cdot), ..., U_{4p}^*(\cdot)$ is determined uniquely by the system of forms $U_1(\cdot), ..., U_{4p}(\cdot)$.

Now we proceed to the proof of the uniqueness of the solution to problem (1)–(2)–(3). Consider $u(x,t)$ the solution to the homogeneous boundary value problem (1)–(2)–(3) for $f \equiv 0$ and show that $u(x,t) \equiv 0$ for $(x,t) \in Q_T$. For a fixed $x \in (0,1)$ we introduce the function

$$F_s(x,\mu) = \int_0^T u(x,t)\,\overline{\kappa_s(t,\overline{\mu})}dt. \tag{45}$$

t is not difficult to see that

$$F_s(x,\mu) - \frac{d^2}{dx^2}F_s(x,\mu) = \int_0^T (u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2})\,\overline{\kappa_s(t,\overline{\mu})}dt =$$

$$\int_0^T l(t,\frac{\partial}{\partial t})u(x,t)\cdot\overline{\kappa_s(t,\overline{\mu})}dt. \tag{46}$$

According to the Lagrange formula (44) we have

$$F_s(x,\mu) - \frac{d^2}{dx^2}F_s(x,\mu) = \int_0^T u(x,t)\,\overline{l^+(t,\frac{d}{dt})\kappa_s(t,\overline{\mu})}dt - U_{s+2p}(u(x,\cdot))\,\overline{\Delta^*(\overline{\mu})}$$

$$= \mu\int_0^T u(x,t)\,\overline{\kappa_s(t,\overline{\mu})}dt - U_{s+2p}(u(x,\cdot))\,\overline{\Delta^*(\overline{\mu})}$$

$$= \mu F_s(x,\mu) - U_{s+2p}(u(x,\cdot))\,\overline{\Delta^*(\overline{\mu})}. \tag{47}$$

Note the connection between $\Delta(\lambda)$ and $\Delta^*(\lambda)$. For all complex λ the identity holds

$$\Delta(\lambda) = \overline{\Delta^*(\overline{\lambda})}.$$

Therefore, for $s = 1, \ldots, 2p$, equality (47) can be rewritten as

$$F_s(x,\mu) - \frac{d^2}{dx^2}F_s(x,\mu) = \mu F_s(x,\mu) - U_{s+2p}(u(x,\cdot))\,\Delta^*(\mu). \tag{48}$$

which is valid for all $x \in (0,1)$ and all complex μ.

If μ_0 is zero of the characteristic determinant $\Delta(\lambda)$ of multiplicity m_0, then for all $s = 1, \ldots, 2p$ and all $x \in (0,1)$ from (48) the equalities follow

$$F_s(x,\mu_0) - \frac{d^2}{dx^2}F_s(x,\mu_0) = \mu_0 F_s(x,\mu_0),$$

$$\frac{1}{1!}\frac{\partial}{\partial\mu}\big(F_s(x,\mu) - \frac{\partial^2}{\partial x^2}F_s(x,\mu)\big)\big|_{\mu=\mu_0} = \mu_0\frac{1}{1!}\frac{\partial}{\partial\mu}(F_s(x,\mu)\big|_{\mu=\mu_0} + F_s(x,\mu_0),$$

$$\dots$$

$$\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}}{\partial\mu^{m_0-1}}\big(F_s(x,\mu)-\frac{\partial^2}{\partial x^2}F_s(x,\mu)\big)\big|_{\mu=\mu_0}$$

$$=\mu_0\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}F_s(x,\mu)}{\partial\mu^{m_0-1}}\big|_{\mu=\mu_0}+\frac{1}{(m_0-2)!}\frac{\partial^{m_0-2}F_s(x,\mu)}{\partial\mu^{m_0-2}}\big|_{\mu=\mu_0}. \tag{49}$$

Since, according to the condition of the theorem, $\Delta(\lambda_k)\neq 0$ for all $k\geq 0$, then no λ_k can coincide with the eigenvalue μ_0. Therefore, from relations (49) it follows that for all $x\in(0,1)$ the equalities are satisfied

$$F_s(x,\mu_0)=0,\ \frac{1}{1!}\frac{\partial F_s(x,\mu)}{\partial\mu}\big|_{\mu=\mu_0}=0,\dots,\frac{1}{(m_0-1)!}\frac{\partial^{m_0-1}F_s(x,\mu)}{\partial\mu^{m_0-1}}\big|_{\mu=\mu_0}=0. \tag{50}$$

Thus, for all $x\in(0,1)$, the complex number μ_0 is a zero of the function $F_s(x,\mu)$ of multiplicity no less than m_0.

Since $F_s(x,\mu)$ is an entire function of μ and each zero μ_0 of the characteristic determinant $\Delta(\mu)$ of multiplicity m_0 is also a zero of $F_s(x,\mu)$ of multiplicity not less than m_0, then the ratio $\frac{F_s(x,\mu)}{\Delta(\mu)}$ is also an entire function of μ. According to the methodology of V. A. Ilyin [23] we multiply the function $F_s(x,\mu)$ scalarly by the root function $v_k(x)$, $k\geq 0$ of the operator A and denote them by

$$G_{sk}(\mu)=\int_0^1 F_s(x,\mu)\overline{v_k(x)}\,dx,\quad k\geq 0,\quad 1\leq s\leq 2p. \tag{51}$$

The multiplicities of zeros in μ of the functional $G_{sk}(\mu)$ are not less than the multiplicities of zeros of the functions $F_s(x,\mu)$. We also introduce the functions

$$Q_{sk}(\mu)\equiv\frac{G_{sk}(\mu)}{\Delta(\mu)}=\int_0^1\overline{v_k(x)}\int_0^T u(x,t)\,\frac{\overline{æ_s(t,\overline{\mu})}}{\Delta(\mu)}\,dt\,dx, \tag{52}$$

which also represent entire functions of μ.

Further analysis of entire functions $Q_{sk}(\mu)$ is based on the technique of estimating the orders of growth and types of entire functions. Note that the entire function $Q_{sk}(\mu)$ does not depend on the choice of the fundamental system of solutions of the homogeneous equation

$$l^+(t,\frac{d}{dt})\,R(t)=\mu\cdot R(t),\quad 0<t<T.$$

Let $\mu=\rho^{2p}$. Let ρ be an arbitrary complex number from the sector $S_0=\{\rho\in\mathbb{C}|\,0<\arg\rho<\frac{\pi}{2p}\}$. Let us enumerate the numbers $\omega_1,\omega_2,\dots,\omega_{2p}$ in the following order

$$\mathrm{Re}(\rho\omega_1)\leq\mathrm{Re}(\rho\omega_2)\leq\dots\leq\mathrm{Re}(\rho\omega_p)<0<\mathrm{Re}(\rho\omega_{p+1})\leq\dots\leq\mathrm{Re}(\rho\omega_{2p}), \tag{53}$$

when ρ lies strictly inside the sector S_0.

Let us choose a fundamental system of solutions of the homogeneous adjoint equation

$$l^+(t,\frac{d}{dt})\,h(t)=-\rho^{2p}\cdot h(t),\quad 0<t<T,$$

so that the asymptotic relations are satisfied

$$h_1(t,\rho) = e^{\rho\omega_1 t}[1 + O(1/\rho)], \dots, h_n(t,\rho) = e^{\rho\omega_{2p} t}[1 + O(1/\rho)],\ \rho \in S_0,\ \rho \to \infty. \tag{54}$$

As a result [1] for any ρ from the sector S_0 we have an asymptotic representation of the characteristic determinant $\widetilde{\Delta}(\rho)$ for $\rho \to \infty$, written through the fundamental system of solutions $\{h_1(t,\rho), \dots, h_{2p}(t,\rho)\}$.

In the work [2] the conjugate linear forms $U^*_{2p}(\cdot), \dots, U^*_1(\cdot)$ are written out explicitly. Taking into account their representation for $\rho \in S_0$, $\rho \to \infty$ we have for $j \le p$

$$U^*_{2p}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_1)}[1],$$

$$U^*_{2p-1}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_1)}[0],$$

$$\dots$$

$$U^*_{2p-2m+2}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_m)}[1],$$

$$U^*_{2p-2m+1}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_m)}[0],$$

$$U^*_{r}(h_j) = (\rho\omega_j)^{(2p-1-\nu_1)}[\overline{\alpha}_1],$$

$$\dots$$

$$U^*_{1}(h_j) = (\rho\omega_j)^{(2p-1-\nu_r)}[\overline{\alpha}_r].$$

Similarly, when $j > p$ for $\rho \in S_0$, $\rho \to \infty$ we have

$$U^*_{2p}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_1)}e^{\rho\omega_j T}[0],$$

$$U^*_{2p-1}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_1)}e^{\rho\omega_j T}[1],$$

$$\dots$$

$$U^*_{2p-2m+2}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_m)}e^{\rho\omega_j T}[0],$$

$$U^*_{2p-2m+1}(h_j) = (\rho\omega_j)^{(2p-1-\gamma_m)}e^{\rho\omega_j T}[1],$$

$$U^*_{r}(h_j) = (\rho\omega_j)^{(2p-1-\nu_1)}e^{\rho\omega_j T}[\overline{\beta}_1],$$

$$\dots$$

$$U^*_{1}(h_j) = (\rho\omega_j)^{(2p-1-\nu_r)}e^{\rho\omega_j T}[\overline{\beta}_r].$$

Here it is designated for brevity $[a] = a + O(1/\rho)$.

We substitute all these expressions into the characteristic determinant

$$\widetilde{\Delta}^*(\overline{\mu}) = \det(U^*_\nu(h_j)) = \rho^{\widehat{\alpha}}\, e^{\rho(\omega_{p+1}+\ldots+\omega_{2p})T}\, \Delta^*_0, \tag{55}$$

where

$$\widehat{\alpha} = 2[2p-1-\gamma_1 + \ldots + 2p-1-\gamma_m] + (2p-1-\nu_1) + \ldots + (2p-1-\nu_r),$$

$$\Delta^*_0 = [\theta^*_0].$$

The number θ_0^* is nonzero, since according to the work of [2] the conjugate linear forms $U_{2p}^*(\cdot),\ldots,U_1^*(\cdot)$ are also Birchhoff regular.

For any ρ from the sector S_0 the asymptotic representation of $\overline{\widetilde{\kappa_1}}(t,\rho)$ for $\rho\to\infty$ has the following form, written in terms of the fundamental system of solutions

$$\overline{\widetilde{\kappa_1}}(t,\rho)=\frac{1}{(\rho\omega_p)^{2p-1-\gamma_1}}\rho^{\widehat{\alpha}}e^{\rho(\omega_{p+1}+\cdots+\omega_{2p})T}[\xi_0^*], \tag{56}$$

where ξ_0^* is some numerical determinant.

We obtain similar asymptotic representations for $\overline{\widetilde{\kappa_s}}(t,\rho)$ for $s>1$.

From this it follows that

$$Q_{1k}(\mu)=\int_0^1\left(\int_0^T\frac{\overline{\widetilde{\kappa_1}}(t,\overline{\mu})}{\widetilde{\Delta^*}(\rho)}u(x,t)dt\right)v_k(x)\,dx=$$

$$\int_0^1\int_0^T\frac{[\xi_0^*]}{(\rho\omega_p)^{2p-1-\gamma_1}[\theta_0^*]}\,u(x,t)\,v_k(x)\,dt\,dx. \tag{57}$$

Using Riemann's lemma ([19], p. 496), we easily obtain

$$\lim_{|\rho|\to\infty}Q_{1k}(\mu)=0,\quad \rho\in S_0.$$

It immediately follows from this

$$\lim_{\rho\to\infty}Q_{1k}(\mu)=0,\quad \rho\in S_0.$$

Thus, along all rays $\rho\in S_0$ and $\rho\to\infty$ we have the limit equality

$$\lim_{\rho\to\infty}Q_{1k}(\mu)=0.$$

We obtain similar asymptotic representations for $Q_{sk}(\mu)$ for $s>1$ and for all $k=0,1,2,\ldots$.

Exactly the same analysis can be carried out for the sector $\rho\in S_1$, where $S_1=\{\rho\in\mathbb{C}|\ \frac{\pi}{2p}<\arg\rho<\frac{\pi}{p}\}$.

Therefore, according to the Phragmen-Lindelof and Liouville theorems ([20], p. 203) for functions of finite order we obtain that

$$Q_{sk}(\mu)\equiv 0\quad \text{при всех}\quad \mu\in\mathbb{C}.$$

From here for any $k=0,1,2,\ldots$ and for any $s=1,\ldots,2p$ we have

$$\int_0^1 v_k(x)F_s(x,\mu)\,dx\equiv 0,\quad \forall\mu\in\mathbb{C}.$$

Then from the completeness of the system $\{v_k(x),\,k=0,1,2,\ldots\}$ in $L_2(0,1)$ it follows that

$$F_s(x,\mu)\equiv 0,\quad \forall x\in(0,1),\quad \forall\mu\in\mathbb{C},\quad s=1,\ldots,2p.$$

Therefore, we have

$$\int_0^T \overline{\widetilde{æ}_s(t,\overline{\mu})}\, u(x,t)\, dt \equiv 0, \quad \forall x \in (0,1), \quad \forall \mu \in \mathbb{C}, \quad s = 1,\dots,2p.$$

It follows from this

$$\frac{1}{\nu!}\frac{\partial^\nu}{\partial \mu^\nu}\int_0^T \overline{\widetilde{\kappa}_s(t,\overline{\mu})}\, u(x,t)\, dt \equiv 0, \quad \forall x \in (0,1), \quad \forall \mu \in \mathbb{C}, \quad s = 1,\dots,2p, \quad \forall \nu \geq 0. \tag{58}$$

Now, instead of μ in equality (58), we substitute μ_τ – an arbitrary eigenvalue of the operator B. The multiplicity of the eigenvalue μ_τ is considered equal to m_τ. Let the parameter ν in formula (58) take the values $1, 2, \dots, m_\tau - 1$. Then, by virtue of (43), from (58) we obtain that for any fixed $x \in (0,1)$ the function $u(x,t)$ is orthogonal to all eigenfunctions of the operator B^*. Since the system of eigenfunctions of the operator B^* is a complete system in $L_2(0,T)$, it follows from this that

$$u(x,t) \equiv 0, \quad \forall t \in (0,T), \quad x \in (0,1).$$

Thus, the uniqueness of the solution to problem (1)–(2)–(3) is completely proven.

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ON THE OPTIMAL RECOVERY OF FUNCTIONS FROM THE CLASS $W_2^{\mathrm{r},\alpha}$

In this paper, the problem of optimal recovery of functions from the anisotropic Sobolev class $W_2^{\mathrm{r},\alpha}$ in the power-logarithmic scale by the values of linear functionals is solved in the Hilbert metric, and the limiting error of the optimal computing unit is found. Thus, the following results are obtained here: 1) The exact order of error of optimal recovery of functions $f \in W_2^{\mathrm{r},\alpha}$ by computing units constructed based on the values of linear functionals defined on the class under consideration has been established;2) The computing unit that realizes the established exact order is written out in explicit form; 3) The limiting error of the specified optimal computing unit is found, which preserves its optimality and can not be improved in order. The actuality of the problem studied here is that, firstly, the class $W_2^{\mathrm{r},\alpha}$ is a finer scale of classifications of periodic functions by the rate of decrease of their trigonometric Fourier coefficients than the anisotropic Sobolev class in the power scale W_2^{r}, secondly, the set of computing units $(l^{(N)}, \varphi_N)$ with linear functionals is a fairly wide set containing all partial sums of Fourier series over all possible orthonormal systems, all possible finite convolutions with special kernels, as well as all finite sums of approximation used in orthowidths, linear widths and greedy algorithms.
Key words: optimal recovery, optimal computing unit, limiting error, exact order, anisotropic Sobolev class, trigonometric Fourier coefficients, linear functionals

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$W_2^{\mathrm{r};\alpha}$ класы функцияларының оптималды қалыптастыруы туралы

Бұл жұмыста гильберттік метрикада дәреже – логарифмдік шкаладағы анизотропты Соболев $W_2^{\mathrm{r},\alpha}$ класы функцияларын оптималды қалыптастыру есебі шешілген және оптималды есептеу агрегатының шектік қателігі табылған. Сонымен, мұнда келесі нәтижелер алынған: 1) $f \in W_2^{\mathrm{r},\alpha}$ функцияларын қарастырып отырған класта анықталған сызыќтық функционалдар мәндері арқылы ќұрылған есептеу агрегаттарымен оптималды қалыптастырудағы қателіктің дәл реті тағайындалған; 2) Тағайындалған дәл реті жүзеге асыратын есептеу агрегаты айқын түрде жазылып келтірілген; 3) Көрсетілген оптималды есептеу агрегатының оның оптималдығын сақтайтын және реті бойынша жақсармайтын шектік қателігі табылған. Осында зерттеліп отырылған есептің өзектілігі мына жайттар арқылы түсіндіріледі: біріншіден, периодты функцияларды олардың тригонометриялық Фурье коэффициенттерінің кему жылдамдығы бойынша классификациялап сипаттауда $W_2^{\mathrm{r},\alpha}$ класы дәрежелік шкаладағы анизотропты W_2^{r} класымен салыстырғанда дәл әрі терең шкаладағы класс болады, екіншіден, сызықтық функционалдарға сәйкес $(l^{(N)}, \varphi_N)$ есептеу агрегаттарының жиыны жеткілікті кең жиын болып табылады, өйткені, бұл жиын құрамына барлық мүмкін ортонормаланған жүйелерге сәйкес Фурье қатарларының барлық дербес қосындылары, арнайы өзегі бар ақырлы конволюциялар, сонымен бірге, ортодиаметрлерде, сызықтық диаметрлерде және гриди алгоритмдердегі жуықтауларда қолданылатын барлық ақырлы қосындылар кіреді.

Түйін сөздер: оптималды қалыптастыру, оптималды есептеу агрегаты, шектік қателік, дәл рет, анизотропты Соболев класы, тригонометриялық Фурье коэффициенттері, сызықтық функционал

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Об оптимальном восстановлении функций из класса $W_2^{\mathrm{r};\alpha}$

В данной работе в гильбертовой метрике решена задача оптимального восстановления функций из анизотропного класса Соболева в степенно – логарифмической шкале $W_2^{\mathrm{r},\alpha}$ по значениям линейных функционалов и найдена предельная погрешность оптимального вычислительного агрегата. Тем самым, здесь получены следующие результаты: 1) Установлен точный порядок погрешности оптимального восстановления функций $f \in W_2^{\mathrm{r},\alpha}$ вычислительными агрегатами, построенными по значениям линейных функционалов, определенных на рассматриваемом классе; 2) В явном виде выписан вычислительный агрегат, реализующий установленный точный порядок; 3) Найдена предельная погрешность указанного оптимального вычислительного агрегата, сохраняющая его оптимальность и неулучшаемая по порядку. Актуальность изучаемой здесь задачи заключается в том, что во – первых, класс $W_2^{\mathrm{r},\alpha}$ является более тонкой шкалой классификаций периодических функций по скорости убывания их тригонометрических коэффициентов Фурье, чем анизотропный класс Соболева W_2^{r} в степенной шкале, во – вторых, множество вычислительных агрегатов $(l^{(N)}, \varphi_N)$ с линейными функционалами является достаточно широким множеством, содержащим все частичные суммы рядов Фурье по всевозможным ортонормированным системам, всевозможные конечные свертки со специальными ядрами, а также все конечные суммы приближения, использующиеся в ортопоперечниках, линейных поперечниках и жадных алгоритмах.

Ключевые слова: оптимальное восстановление, оптимальный вычислительный агрегат, предельная погрешность, точный порядок, анизотропный класс Соболева, тригонометрические коэффициенты Фурье, линейный функционал

1 Introduction

The problem of optimal recovery of functions of class $F = \{f(x) : x \in \Omega\}$ by the values $l_N^{(1)}(f), \ldots, l_N^{(N)}(f)$ of linear functionals $l_N^{(1)} : F \to \mathbb{C}, \ldots, l_N^{(N)} : F \to \mathbb{C}$ consist in establishing the exact order of quantity

$$\delta_N(L_N, F, L^q) = \inf_{(l^{(N)}, \varphi_N) \in L_N} \delta_N((l^{(N)}, \varphi_N), F, L^q), \tag{1}$$

where $L^q \equiv L^q(\Omega)$ is a set of measurable functions on Ω with finite norm

$$\|f\|_{L^q} = \begin{cases} \left(\int\limits_{\Omega} |f(x)|^q dx \right)^{1/q}, & \text{if} \quad 1 \le q < \infty; \\ \sup\limits_{x \in \Omega} \mathrm{vrai}|f(x)|, & \text{if} \quad q = \infty, \end{cases}$$

$$\delta_N((l^{(N)}, \varphi_N), F, L^q) = \sup_{f \in F} \left\| f(\cdot) - \varphi_N(l_N^{(1)}(f), \ldots, l_N^{(N)}(f); \cdot) \right\|_{L^q},$$

φ_N – an arbitrary function such that $\varphi_N(\tau_1,\ldots,\tau_N;x) \in L^q$ as a function of x for any $\tau_1,\ldots,\tau_N, l^{(N)} \equiv l^{(N)}(f) = (l_N^{(1)}(f),\ldots,l_N^{(N)}(f)), L_N = \{l^{(N)}\} \times \{\varphi_N\}$.

Here we note that by 'establishing the exact order of quantity (1)' we should understand finding a positive sequence $\{\psi_N\}_{N\geq 1}$ such that for some constants $A_1 > 0$ and $A_2 > 0$ independent of N, the inequalities $A_1\psi_N \leq \delta_N(L_N, F, L^q) \leq A_2\psi_N$ are satisfied for all N.

Everywhere below, the pair $(l^{(N)}, \varphi_N) \in L_N$ will be called a computing unit, and the pair $(\tilde{l}^{(N)}, \widetilde{\varphi}_N) \in L_N$ that realizes the exact order of quantity (1), an optimal computing unit.

The quantity (1) was first considered in [1] when recovering analytical functions in the L^2 metric. Then the research of the paper [1] was continued in [2]. In [3], the relationships of the quantity (1) with the known quantities - the Kolmogorov width, the Gelfand width, and the linear width - were studied. Further, in [4] - [8] the exact orders of (1) were established with an indication of the optimal computing units. Moreover, in [6] and [7], in addition to establishing the exact order of (1), the limiting errors of the optimal computing units were found (the definition of the limiting error is given below).

In [8], the Sobolev class $W_2^{r;\alpha}$ in the power-logarithmic scale was considered for the first time, and under certain conditions on the vector $\alpha = (\alpha_1,\ldots,\alpha_s)$, a theorem on the optimal recovery of functions from the considered class in the metric of the space $L^q, 2 \leq q \leq \infty$ was formulated. Here we managed to remove these conditions on the optimal recovery in the metric of L^2 and to find the limiting error of the optimal computing unit.

2 The definitions of the class $W_2^{\mathrm{r},\alpha}$ and of the limiting error

First, we agree on the notation used. As usual, $[a]$ denotes the integer part of a number $a, m = (m_1,\ldots,m_s) \in Z^s, |J|$ denote the amount of elements of a finite set J. Here and throughout the text, for each vector $\mathrm{r} = (r_1,\ldots,r_s)$ with positive components , we set

$$\lambda \equiv \lambda(r_1,\ldots,r_s) = (1/r_1 + \cdots + 1/r_s)^{-1}.$$

For positive functions $f(x), x \geq 1$ and $g(x), x \geq 1$ the notation $f(x) \ll g(x)$ will mean the existence of some quantity $C > 0$, independent of the variable x, such that $f(x) \leq Cg(x)$ holds for all $x \geq 1$. And the simultaneous fulfillment of inequalities $f(x) \leq Cg(x)$ and $g(x) \leq Cf(x)$ is written as $f(x) \succ\prec g(x)$. Everywhere below, the symbol $\square$ will denote the end of the proof.

Let be given an integer number $s \geq 2$, vectors $r = (r_1,\ldots,r_s)$ and $\alpha = (\alpha_1,\ldots,\alpha_s)$ such that $r_i > 0$ and $\alpha_i \in \mathbb{R}$ for each $i = 2,3,\ldots,s$. The anisotropic Sobolev class $W_2^{r;\alpha} \equiv W_2^{r_1,\ldots,r_s;\alpha_1,\ldots,\alpha_s}[0,1]^s$ in a power – logarithmic scale, by definition, consists of all summable on a cube $[0,1]^s$ and 1 – periodic by each variable functions $f(x) = f(x_1,\ldots,x_s)$, that satisfy the condition

$$\sum_{m\in Z^s} \left|\hat{f}(m)\right|^2 (\overline{m}_1^{2r_1}\ln^{2\alpha_1}(\overline{m}_1+1) + \cdots + \overline{m}_s^{2r_s}\ln^{2\alpha_s}(\overline{m}_s+1)) \leq 1,$$

where $\hat{f}(m) = \int\limits_{[0,1]^s} f(x)e^{-2\pi i(m,x)}dx, m \in Z^s$ are the trigonometric Fourier – Lebesgue coefficients of the function $f, \overline{m}_j = \max\{1; |m_j|\}$ for each $j = 1,\ldots,s$.

For each $f \in F$ calculating the values $l_N^{(1)}(f), \dots, l_N^{(N)}(f)$ of the functionals $l_N^{(1)} : F \to \mathbb{C}, \dots, l_N^{(N)} : F \to \mathbb{C}$, with few exceptions, cannot be exact. Therefore, for the optimal computing unit $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$, the problem arises of finding the error $\widetilde{\varepsilon}_N$ in calculating the values $\tilde{l}_N^{(1)}(f), \dots, \tilde{l}_N^{(N)}(f)$ functionals $\tilde{l}_N^{(1)} : F \to \mathbb{C}, \dots, \tilde{l}_N^{(N)} : F \to \mathbb{C}$, that preserves optimality $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ and cannot be improved in order. The definition of $\widetilde{\varepsilon}_N$ was first given in [9] within the framework of the study «Computational(Numerical) Diameter». There, the error $\widetilde{\varepsilon}_N$ was called the limiting error of the optimal computing unit $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$. Further, in [10], an equivalent definition of $\widetilde{\varepsilon}_N$ was formulated within the general formulation of the problem of recovering the operator $T : F \to Y$, where F is a functional class, Y is a normed space. Now we present from [7] the definition of the limiting error, stated at $Tf = f$ and $Y = L^q, q \in [2, +\infty]$. The limiting error of an optimal computing $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ is defined as a sequence $\widetilde{\varepsilon}_N > 0$ such that

$$\triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), F, L^q) \succ\prec \delta_N(D_N, F, L^q) \quad \text{and}$$

$$\overline{\lim_{N\to\infty}} \frac{\triangle_N(\eta_N\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), F, L^q)}{\delta_N(D_N, F, L^q)} = +\infty$$

for any arbitrary slowly increasing to $+\infty$ positive sequence $\{\eta_N\}_{N\geq 1}$, where

$$\triangle_N(\varepsilon_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), F, L^q) =$$

$$= \sup_{f\in F} \sup_{|\gamma_N^{(1)}|\leq 1,\dots,|\gamma_N^{(N)}|\leq 1} \left\| f(\cdot) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f) + \gamma_N^{(1)}\varepsilon_N, \dots, \tilde{l}_N^{(N)}(f) + \gamma_N^{(N)}\varepsilon_N; \cdot) \right\|_{L^q}$$

for each positive sequence ε_N.

3 Formulations of the obtained results

Further, for the sake of brevity, we will use the following notations

$$\delta_N(L_N) \equiv \delta_N(L_N, W_2^{\mathrm{r};\alpha}, L^2), \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \equiv \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N), W_2^{\mathrm{r};\alpha}, L^2),$$

$$\triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \equiv \triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N), W_2^{\mathrm{r};\alpha}, L^2).$$

We have proved the following two theorems.

Theorem 1. *Let the number* $s \in \mathbb{N}\backslash 1$, *vectors* $\mathrm{r} = (r_1, \dots, r_s), r_1 > 0, \dots, r_s > 0$ *and* $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ *be given, and let* $\lambda \equiv (1/r_1 + \dots + 1/r_s)^{-1} > 1/2$,

$$N_i \equiv N_i(K) = [K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}], K \geq 2$$

for each $i \in 1, \dots, s, N \equiv N(K) = \prod_{i=1}^{s}(2N_i + 1)$. *Then there exists a quantity* $C_0 > 0$ *such that for all integers* $K \geq C_0$ *the relations*

$$\delta_N(L_N) \succ\prec \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \succ\prec \frac{1}{N^\lambda(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}}, \tag{2}$$

hold, here, the computing unit $(\tilde{l}^{(N)}, \widetilde{\varphi}_N)$ *is defined by the equalities*

$$\tilde{l}_N^{(1)}(f) = \hat{f}(\widetilde{m}^{(1)}), \dots, \tilde{l}_N^{(N)}(f) = \hat{f}(\widetilde{m}^{(N)}), \widetilde{\varphi}_N(z_1, \dots, z_N; x) = \sum_{\tau=1}^{N} z_\tau e^{2\pi i(\widetilde{m}^{(\tau)},x)},$$

where $\{\widetilde{m}^{(1)}, \widetilde{m}^{(2)}, \dots, \widetilde{m}^{(N)}\}$ *is some ordering of the set*

$$A_K = \{m \in Z^s : |m_1| \le N_1, \dots, |m_s| \le N_s\}.$$

Theorem 2. *The quantity* $\widetilde{\varepsilon}_N = \frac{1}{N^{\lambda+1/2}(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}}$ *is the limiting error of the optimal computing unit*

$$(\tilde{l}^{(N)}, \widetilde{\varphi}_N) \equiv \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \dots, \tilde{l}_N^{(N)}(f); x) = \sum_{\tau=1}^{N} f(\widetilde{m}^{(\tau)}) e^{2\pi i(\widetilde{m}^{(\tau)}, x)},$$

i.e.

$$\triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \succ\prec \delta_N(L_N) \tag{3}$$

and for any arbitrarily slowly increasing to $+\infty$ *positive sequence* $\{\eta_{N(K)}\}_{K\ge 1}$ *the equality*

$$\overline{\lim_{K\to\infty}} \frac{\triangle_N(\eta_N\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N))}{\delta_N(L_N)} = +\infty \tag{4}$$

holds.

Remark 1. *Relations (2) under the conditions* $1/\min\{r_1, r_1+\alpha_1\}+\dots+\min\{r_s, r_s+\alpha_s\} < 2$ *and* $r_i + \alpha_i > 0 (i = 1, \dots, s)$ *on the vector* $\alpha = (\alpha_1, \dots, \alpha_s)$ *obtained in [8].*

Remark 2. *In the case* $\alpha_1 = \alpha_2 = \dots = \alpha_s = 0$ *the main results of the article [6] follow from Teorems 1 and 2.*

4 Auxiliary statements

Lemma 1 (see, Lemma 1.2.4 from [11]). *For each* $\gamma \in \mathbb{R}$ *there exists a quantity* $C_1(\gamma) \ge 2$ *such that for all integers* $K \ge C_1(\gamma)$ *the relation* $\ln(K \ln^\gamma K) \succ\prec \ln K$ *holds.*

Lemma 2. *Let a number* $B \ge 2$, *vectors* $\mathrm{r} = (r_1, \dots, r_s), r_1 > 0, \dots, r_s > 0$ *and* $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ *be given, and let* $\gamma_i = \alpha_1/r_1 + \dots + \alpha_s/r_s - \alpha_i/\lambda$ *for each* $i \in \{1, \dots, s\}$. *Then there is an integer* $K_0 \ge 2$ *such that for all integers* $K \ge \max\{C_1(\gamma_i), K_0\}$ *the relations*

$$\ln(BN_i) \succ\prec \ln K \succ\prec \ln N (i = 1, \dots, s), \tag{5}$$

hold, where $N_i \equiv N_i(K) = [K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}]$ *for each* $i = 1, \dots, s, N \equiv N(K) = \prod_{i=1}^{s}(2N_i + 1)$.

Proof. Since for each $i \in \{1, \dots, s\}$ the equality

$$\lim_{K\to\infty} K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i} = +\infty$$

is true, there exists some integer $K_0 \ge 2$ such for all integers $K \ge K_0$ the inequality $K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i} \ge B$ holds. Therefore,

$$\ln(BN_i) \le \ln\left(BK^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}\right) \le$$

$$\leq 2\ln\left(K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}\right) \leq \frac{2\lambda}{r_i}\ln(K\ln^{\gamma_i}K) \tag{6}$$

It is clear that

$$\ln(BN_i) \geq \ln\left(K^{\lambda/r_i}(\ln K)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)/r_i}(\ln K)^{-\alpha_i/r_i}\right) \geq \frac{\lambda}{r_i}\ln(K\ln^{\gamma_i}K). \tag{7}$$

According to Lemma 1, for all integers $K \geq C_1(\gamma_i)$ the relation $\ln(K\ln^{\gamma_i}K) \succ\prec \ln K$ is valid. Therefore, due to (6) and (7), for all integers $K \geq \max\{C_1(\gamma_i), K_0\}$ the relation

$$\ln(BN_i) \succ\prec \ln K, i \in \{1,\ldots,s\} \tag{8}$$

holds. From the definition of number N_i and the equality $N = \prod\limits_{i=1}^{s}(2N_i+1)$ the inequalities $K \leq N \leq 2^{2s}K$ easily follow, hence, taking $K \geq 2$ into account, we obtain $\ln K \leq \ln N \leq (2s+1)\ln K$, which together with (8) will lead to the relations (5). $\square$

Lemma 3. *Let be given numbers $r > 0, \beta \neq 0$ and a positive, strictly increasing on the interval $(C_2, +\infty)$ function $\varphi(\delta)$, where $C_2 \geq 1$. If for some $C_3 > 0$ and $C_4 > 0$ the inequalities*

$$C_3\delta^r\ln^\beta(2\delta) \leq \varphi(\delta) \leq C_4\delta^r\ln^\beta(2\delta), \tag{9}$$

are true for each $\delta \in (C_2, +\infty)$, then there exist positive quantities $C_5 \geq 1, C_6 > 0$ and $C_7 > 0$ such that the inequalities

$$C_6\delta^{1/r}\ln^{-\beta/r}(2\delta) \leq \varphi^*(\delta) \leq C_7\delta^{1/r}\ln^{-\beta/r}(2\delta), \tag{10}$$

are satisfied for all $\delta \in (C_5, +\infty)$, where φ^* is the function inverse to φ.

Proof. Let us prove the lemma for case $\beta > 0$ (in the case $\beta < 0$, analogous arguments are carried out). Let

$$C_6 = (r^\beta C_4^{-1})^{1/r} \quad \text{and} \quad C_7 = (r^\beta C_3^{-1}2^\beta)^{1/r}. \tag{11}$$

Since

$$\lim_{\delta\to+\infty}\delta^{1/r}\ln^{-\beta/r}(2\delta) = +\infty, \tag{12}$$

there exists a positive quantity $C_8 > C_2$ such that the inequality $C_6\delta^{1/r}\ln^{-\beta/r}(2\delta) > C_2$ is satisfied for all $\delta \in (C_8, +\infty)$. Therefore, by virtue of the right-hand side of (9), we obtain

$$\varphi(C_6\delta^{1/r}\ln^{-\beta/r}(2\delta)) \leq \frac{C_4C_6^r\delta\ln^{-\beta}(2\delta)}{r^\beta}\ln^\beta(C_6^r\delta\ln^{-\beta}(2\delta)). \tag{13}$$

Since $\lim\limits_{\delta\to+\infty}\ln^{-\beta}(2\delta) = 0$, then for some positive $C_9 \geq C_8$ the inequality $\ln^{-\beta}(2\delta) \leq C_6^{-r}$ is satisfied for all $\delta \in (C_9, +\infty)$. Consequently, continuing (13), taking into account the first equality from (11), for all $\delta \in (C_9, +\infty)$ we have

$$C_6\delta^{1/r}\ln^{-\beta/r}(2\delta) \leq \varphi^*(\delta). \tag{14}$$

According to (12), there is a positive quantity $C_{10} > C_2$ such that the inequality $C_7\delta^{1/r}\ln^{-\beta/r}(2\delta) > C_2$ holds for all $\delta \in (C_{10}, +\infty)$. Therefore, by virtue of the left part of (9), we obtain

$$\varphi_N(C_7\delta^{1/r}\ln^{-\beta/r}(2\delta)) \geq \frac{C_3C_7\delta\ln^{-\beta}(2\delta)}{r^\beta}\ln^\beta(C_7^r\delta\ln^{-\beta}(2\delta)). \tag{15}$$

Since $\lim\limits_{\delta\to+\infty} C_7^r\sqrt{\delta}\ln^{-\beta}(2\delta) = +\infty$, then there is a quantity $C_{11} > C_{10}$ such that for all $\delta \in (C_{11}, +\infty)$ the inequality $C_7^r\sqrt{\delta}\ln^{-\beta}\delta \geq 1$ is satisfied. Therefore, using (15) and second equality from (11) we arrive to the inequality

$$C_7\delta^{1/r}\ln^{-\beta/r}(2\delta) \geq \varphi^*(\delta). \tag{16}$$

Now, if we take $C_5 = \max\{C_9, C_{11}\}$, then by virtue of inequalities (14) and (16), the inequalities (10) are satisfied for all $\delta \in (C_5, +\infty)$. □

Lemma 4. *If $\lambda > 1/2$, then the trigonometric Fourier series of each function $f \in W_2^{r;\alpha}$ converges absolutely.*

Proof. It is clear that for all $\delta \in [1, +\infty)$ the relations

$$\ln(1+\delta) \succ\prec \ln(2\delta) \succ\prec \ln(A\delta) \tag{17}$$

hold, where $A > \max\{2, e^{-\alpha_1/r_1}, \dots, e^{-\alpha_s/r_s}\}$. Using Hölder inequality and the definition of class $W_2^{r;\alpha}$ for every $x \in [0,1]^s$ we have

$$\sum_{m\in Z^s}|\hat{f}(m)e^{2\pi i(m,x)}| \leq \sum_{m\in Z^s}|\hat{f}(m)| \leq$$

$$\leq \left(\sum_{m\in Z^s}\frac{1}{\overline{m}_1^{2r_1}\ln^{2\alpha_1}(\overline{m}_1+1)+\dots+\overline{m}_s^{2r_s}\ln^{2\alpha_s}(\overline{m}_s+1)}\right)^{1/2}. \tag{18}$$

Let $\omega_1(x) = (1/\ln^{\alpha_1}A)x^{r_1}\ln^{\alpha_1}(Ax), \dots, \omega_s(x) = (1/\ln^{\alpha_s}A)x^{r_s}\ln^{\alpha_s}(Ax)$ for all $x \geq 1$. Then by virtue of (17) and (18) we obtain

$$\sum_{m\in Z^s}|\hat{f}(m)| \ll \left(\sum_{m\in Z^s}\frac{1}{\omega_1^2(\overline{m}_1)+\dots+\omega_s^2(\overline{m}_s)}\right)^{1/2}. \tag{19}$$

The functions $\omega_1, \dots, \omega_s$ strictly increase on the interval $[1, +\infty)$. Indeed, for each $i \in \{1, \dots, s\}$ we have

$$\omega_i'(x) = (1/\ln^{\alpha_i}A)(r_i\ln(Ax)+\alpha_i)x^{r_i-1}\ln^{\alpha_i-1}(Ax) >$$

$$> (1/\ln^{\alpha_i}A)(r_i\ln e^{-\alpha_i/r_i}+\alpha_i)x^{r_i-1}\ln^{\alpha_i-1}(Ax) = 0.$$

Further, denoting by ω_i^* the inverse of the function ω_i, for each $p = 1, 2, \dots,$ we define the set $X_p = \{m \in Z^s : |m_1| \leq \omega_1^*(\omega(2p)), \dots, |m_s| \leq \omega_s^*(\omega(2p))\}$, where ω is the inverse of the function $\omega^*(x) = \omega_1^*(x) \times \dots \times \omega_s^*(x)$ strictly increasing on $[1, +\infty)$.

For $m = (m_1, \ldots, m_s) \in X_{p+1} \backslash X_p$ there is an index $\theta \in \{1, \ldots, s\}$ such that

$$|m_\theta| \geq \omega_\theta^*(\omega(2^p)) \Leftrightarrow \omega_\theta(|m_\theta|) \geq \omega(2^p).$$

Therefore, taking into account the inequality $\omega_1^2(\overline{m}_1)+\cdots+\omega_s^2(\overline{m}_s) \geq \omega_\theta^2(\overline{m}_\theta)$ and the relation $|X_{p+1}\backslash X_p| \succ\prec 2^p$, for all $p = 1, 2, \ldots$ we obtain

$$\sum_{m\in X_{p+1}\backslash X_p} \frac{1}{\omega_1^2(\overline{m}_1) + \cdots + \omega_s^2(\overline{m}_s)} \ll \frac{2^p}{\omega^2(2^p)}. \tag{20}$$

If we put $X_0 = \emptyset$, then according to equality $X_p = \bigcup\limits_{K=0}^{p-1} (X_{K+1}\backslash X_K)$ we have

$$\sum_{m\in Z^s} \frac{1}{\omega_1^2(\overline{m}_1) + \cdots + \omega_s^2(\overline{m}_s)} =$$

$$= \lim_{p\to\infty} \sum_{K=0}^{p-1} \sum_{m\in X_{K+1}\backslash X_K} \frac{1}{\omega_1^2(\overline{m}_1) + \cdots + \omega_s^2(\overline{m}_s)}.$$

Therefore , by virtue of (20) the inequality

$$\sum_{m\in Z^s} \frac{1}{\omega_1^2(\overline{m}_1) + \cdots + \omega_s^2(\overline{m}_s)} \ll \sum_{K=0}^{\infty} \frac{2^K}{\omega^2(2^K)} \tag{21}$$

is true. According to Lemma 3, there exists a quantity $C_{12} > 0$ such that for all integers $K > C_{12}$ the inequalities

$$\omega(2^K) \gg 2^{\lambda K} \ln^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}(3\cdot 2^K) \geq 2^{\lambda K}(K+1)^{\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}$$

are satisfied, whence,

$$\sum_{\{K:K>C_{12}\}} \frac{2^K}{\omega^2(2^K)} \ll \sum_{\{K:K>C_{12}\}} \frac{1}{2^{(2\lambda-1)K}(K+1)^{2\lambda(\alpha_1/r_1+\cdots+\alpha_s/r_s)}} < \infty, \tag{22}$$

since for $\lambda > 1/2$ the last series converges according to D'Alembert's criterion. As a result of (22), the series $\sum\limits_{K=0}^{\infty} \frac{2^K}{\omega^2(2^K)}$ converges. Therefore, due to (21) and (19) for $\lambda > 1/2$ the trigonometric Fourier series of each function $f \in W_2^{\mathrm{r};\alpha}$ converges absolutely. $\square$

Lemma 5 (see, Lemma A from [12]). *Suppose we are given an integer $s \geq 1$. Then, for each integer $N \geq 1$, the following assertion holds: for any set $G \equiv \{m^{(1)}, \ldots, m^{(N')}\} \subset Z^s$ such that $N' = |G| \geq 2N$ and $|G| \succ\prec N$, and for arbitrary linear functionals $l_1, \ldots, l_N$ defined at least on the set of all trigonometric polynomials with spectrum in G, there exist complex numbers $\{c_k\}_{k=1}^{N'}$ satisfying the conditions $\sum\limits_{k=1}^{N'} |c_k| \geq N$, $\sum\limits_{k=1}^{N'} |c_k|^2 = N$, moreover, if $\chi(x) = \sum\limits_{k=1}^{N'} c_k e^{2\pi i(m^{(k)},x)}$, then $l_1(\chi) = 0, \ldots, l_N(\chi) = 0$ and $\|\chi\|_{L^\infty} \geq N, \|\chi\|_{L^2} = \sqrt{N}$.*

5 Proof of theorem 1

Next, we will consider integers $K > C_0 = \max\{C_1(\gamma_1), \ldots, C_1(\gamma_s), K_0\}$. Since

$$\widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f); x\right) = \sum_{\tau=1}^{N} \hat{f}(\overline{m}^{(\tau)}) e^{2\pi i(\widetilde{m}^{(\tau)}, x)} = \sum_{m \in A_K} \hat{f}(m) e^{2\pi i(m,x)},$$

then according to Lemma 4 the equality

$$f(x) - \widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f); x\right) = \sum_{m \in Z^s \setminus A_K} \hat{f}(m) e^{2\pi i(m,x)} \tag{23}$$

holds. For any $m \in Z^s \setminus A_K$ there is a number $\nu \in \{1, \ldots, s\}$ such that $|m_\nu| > N_\nu$. Therefore, taking into account the monotony of the function ω_ν, we have

$$\frac{1}{\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1 + 1) + \cdots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s + 1)} \ll$$

$$\ll \frac{1}{\omega_1^2(\overline{m}_1) + \cdots + \omega_s^2(\overline{m}_s)} \ll \frac{1}{\omega_\nu^2(\overline{m}_\nu)} \ll \frac{1}{N_\nu^{2r_\nu} \ln^{2\alpha_\nu}(AN_\nu)}. \tag{24}$$

Due to the definition of the number N_ν and the relation (5) the inequality

$$N_\nu^{r_\nu} \ln^{\alpha_\nu}(AN_\nu) \gg N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \cdots + \alpha_s/r_s)}$$

is true. Therefore, continuing the inequality (24) we obtain

$$\frac{1}{\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1 + 1) + \cdots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s + 1)} \ll$$

$$\ll \frac{1}{N^{2\lambda} (\ln N)^{2\lambda(\alpha_1/r_1 + \cdots + \alpha_s/r_s)}}. \tag{25}$$

From (23), by virtue Parseval equality, the definition of the class under consideration and the equality (25), follows

$$\left\| f(\cdot) - \widetilde{\varphi}_N\left(\tilde{l}_N^{(1)}(f), \ldots, \tilde{l}_N^{(N)}(f); \cdot\right) \right\|_{L^2} \ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \cdots + \alpha_s/r_s)}}, \tag{26}$$

wherefrom we have

$$\delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \cdots + \alpha_s/r_s)}}. \tag{27}$$

Let the function $\varphi_N(z_1, \ldots, z_N; y) : \mathbb{C}^N \times [0,1]^s \mapsto \mathbb{C}$ and linear functionals $l_N^{(1)}, \ldots, l_N^{(N)}$ defined on class $W_2^{\mathrm{r};\alpha}$ be given and let $G_N = \{m \in Z^s : |m_1| \le [M_1], \ldots, |m_s| \le [M_s]\}$, where $M_i = N^{\lambda/r_i} (\ln N)^{\lambda(\alpha_1/r_1 + \cdots + \alpha_s/r_s)/r_i} (\ln N)^{-\alpha_i/r_i}$ for each $i = 1, 2, \ldots, s$. Since the inequality

$|G_N| > 2N$ and the relation $|G_N| \succ\prec N$ are valid for G_N, then by virtue of Lemma 5 for linear functionals $l_N^{(1)}, \dots, l_N^{(N)}$ there are complex numbers $c_m, m \in G_N$ such that

$$\sum_{m \in G_N} |c_m|^2 = N, \tag{28}$$

moreover, if $\prod_N(x) = \sum\limits_{m \in G_N} c_m e^{2\pi i(m,x)}$, then $l_N^{(1)}(\prod_N) = 0, \dots, l_N^{(N)}(\prod_N) = 0$ and

$$\left\| \prod\nolimits_N \right\|_{L^2} = \sqrt{N}. \tag{29}$$

Let's consider the function

$$g_N(x) = \frac{1}{N^{\lambda+1/2}(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}} \prod\nolimits_N(x).$$

Taking into account the definitions of the numbers $M_i, i \in \{1, \dots, s\}$ and equality (28) we obtain

$$\sum_{m \in G_N} |\hat{g}_N(m)|^2 \left(\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1 + 1) + \dots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s + 1) \right) \ll$$

$$\ll \frac{1}{N} \sum_{m \in G_N} |c_m|^2 \ll 1.$$

Therefore, for some $C_{13} > 0$ the function $f_N(x) = C_{13} g_N(x)$ belongs to class $W_2^{\mathrm{r};\alpha}$. Further, using the equality (29), we find

$$\|f_N\|_{L^2} \gg \frac{1}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}}. \tag{30}$$

Since $f_N \in W_2^{\mathrm{r};\alpha}$ and $l_N^{(1)}(f_N) = 0, \dots, l_N^{(N)}(f_N) = 0$, then

$$\sup_{f \in W_2^{\mathrm{r};\alpha}} \left\| f(\cdot) - \varphi_N(l_N^{(1)}(f), \dots, l_N^{(N)}(f); \cdot) \right\|_{L^2} \geq$$

$$\geq \frac{1}{2}(\|f_N(\cdot) - \varphi_N(0, \dots, 0; \cdot)\|_{L^2} + \|(-f_N)(\cdot) - \varphi_N(0, \dots, 0; \cdot)\|_{L^2}) \geq \|f_N\|_{L^2},$$

wherefrom, taking into account (30), we obtain

$$\delta_N(L_N) \gg \frac{1}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}}. \tag{31}$$

Consequently, by virtue of the inequalities $\delta_N(L_N) \leq \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N))$ and (27) the relations (2) hold. □

6 Proof of theorem 2

For arbitrarily given numbers $|\gamma_N^{(\tau)}| \leq 1 (\tau = 1, \dots, N)$ the inequality

$$\left\| f(\cdot) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f) + \gamma_N^{(1)}\widetilde{\varepsilon}_N, \dots, \tilde{l}_N^{(N)}(f) + \gamma_N^{(N)}\widetilde{\varepsilon}_N; \cdot) \right\|_{L^2} \leq$$

$$\leq \left\| f(\cdot) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f), \dots, \tilde{l}_N^{(N)}(f); \cdot) \right\|_{L^2} + \left\| \sum_{\tau=1}^{N} (-\gamma_N^{(\tau)})\widetilde{\varepsilon}_N e^{2\pi i(\widetilde{m}^{(\tau)}, \cdot)} \right\|_{L^2} \tag{32}$$

holds. It is clear, that

$$\left\| \sum_{\tau=1}^{N} (-\gamma_N^{(\tau)})\widetilde{\varepsilon}_N e^{2\pi i(\widetilde{m}^{(\tau)}, \cdot)} \right\|_{L^2} \ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \dots + \alpha_s/r_s)}}.$$

Therefore, according to inequalities (26) and (32), we have

$$\left\| f(\cdot) - \widetilde{\varphi}_N(\tilde{l}_N^{(1)}(f) + \gamma_N^{(1)}\widetilde{\varepsilon}_N, \dots, \tilde{l}_N^{(N)}(f) + \gamma_N^{(N)}\widetilde{\varepsilon}_N; \cdot) \right\|_{L^2} \ll$$

$$\ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \dots + \alpha_s/r_s)}},$$

wherefrom, due to the arbitrariness of the numbers $\gamma_N^{(\tau)} (\tau = 1, \dots, N)$ and the function f follows

$$\triangle_N(\widetilde{\varepsilon}_N, (\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \ll \frac{1}{N^\lambda (\ln N)^{\lambda(\alpha_1/r_1 + \dots + \alpha_s/r_s)}}. \tag{33}$$

Since $\delta_N(L_N) \leq \delta_N((\tilde{l}^{(N)}, \widetilde{\varphi}_N)) \leq \triangle_N(\widetilde{\varepsilon}_N, (\bar{l}^{(N)}, \overline{\varphi}_N))$, then using (31) and (33) we arrive to (3).

Further, for each integer $K > C_0$ we take $\beta_K = \min\{\eta_N, \ln N\}$ and define the set

$$H_K = \{m \in Z^s : [J_1] \leq |m_1| \leq 2[J_1], \dots, [J_s] \leq |m_s| \leq 2[J_s]\},$$

where $N = N(K), \eta_N \equiv \{\eta_{N(K)}\}_{K \geq 1}$ is a positive sequence arbitrarily slowly increasing to $+\infty$,

$$J_i = N^{\lambda/r_i} (\ln N)^{\lambda(\alpha_1/r_1 + \dots + \alpha_s/r_s)/r_i} (\ln N)^{-\alpha_i/r_i} \beta_K^{-1/r_i} (i = 1, \dots, s).$$

Since $\lim\limits_{K \to +\infty} \beta_K = +\infty$, there exist some quantity $C_{14} > C_0$ such that for all integers $K > C_{14}$ the inequality

$$\beta_K \geq 1 \tag{34}$$

holds. For an arbitrary vector $m = (m_1, \dots, m_s) \in H_K$ the inequality

$$\ln^{\alpha_i}(A\overline{m}_i) \ll \ln^{\alpha_i} N, \tag{35}$$

is true, where $\alpha_i \in \mathbb{R}$ and $\overline{m}_i = \max\{1; |m_i|\}$ for each $i \in \{1, 2, \dots, s\}$. The validity of the last inequality is established using Lemma 1 and inequalities $\ln^{-1/r_i} N \le \beta_K^{-1/r_i} \le 1$. Using (35) for any vector $m = (m_1, \dots, m_s) \in H_K$, we easily obtain

$$\overline{m}_i^{r_i} \ln^{\alpha_i}(A\overline{m}_i) \ll N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}\beta_K^{-1}. \tag{36}$$

Now, for each $K > C_{14}$ we consider the function $h_K(x) = \beta_K \widetilde{\varepsilon}_N \sum\limits_{m\in H_K} e^{2\pi i(m,x)}$. Due to inequalities $|H_K| \ll N \cdot \beta_K^{-1/\lambda}, 1/\lambda > 0$, (36) and (34) we have

$$\sum_{m\in H_K} |\hat{h}_K(m)|^2(\overline{m}_1^{2r_1} \ln^{2\alpha_1}(\overline{m}_1 + 1) + \dots + \overline{m}_s^{2r_s} \ln^{2\alpha_s}(\overline{m}_s + 1)) \ll$$

$$\ll \frac{1}{N} \sum_{m\in H_K} 1 \ll \frac{1}{\beta_K^{1/\lambda}} \ll 1.$$

Therefore, for some $C_{15} > 0$ the function $t_K(x) = C_{15}h_K(x)$ belongs to class $W_2^{\mathrm{r};\alpha}$. It is clear that

$$\|t_K\|_{L^2} \gg \frac{1}{N^{\lambda}(\ln N)^{\lambda(\alpha_1/r_1+\dots+\alpha_s/r_s)}}\beta_K^{1-1/(2\lambda)}. \tag{37}$$

Let

$$\widetilde{\gamma}_N^{(\tau)} = -\frac{\hat{t}_k(m^{(\tau)})}{\widetilde{\varepsilon}_N\eta_N}, \overline{\gamma}_N^{(\tau)} = -\frac{(-\hat{t}_K)(m^{(\tau)})}{\widetilde{\varepsilon}_N\eta_N}(\tau = 1, \dots, N \equiv N(K)),$$

$$\Phi_N = \{(l^{(N)}, \varphi_N) : l_N^{(1)}(f) = \hat{f}(m^{(1)}), \dots, l_N^{(N)}(f) = \hat{f}(m^{(N)})\}$$

for each integer $K > C_{14}$. Since

$$|\widetilde{\gamma}_N^{(\tau)}| \le 1, |\overline{\gamma}_N^{(\tau)}| \le 1, \hat{t}_K(m^{(\tau)}) + \eta_N\widetilde{\gamma}_N^{(\tau)}\widetilde{\varepsilon}_N = 0, (-\hat{t}_K)(m^{(\tau)}) + \eta_N\overline{\gamma}_N^{(\tau)}\widetilde{\varepsilon}_N = 0$$

for any $\tau \in \{1, \dots, N\}$, then for every computing unit $(l^{(N)}, \varphi_N) \in \Phi_N$ we have

$$\sup_{f\in W_2^{\mathrm{r};\alpha}} \sup_{|\gamma_N^{(1)}|\le 1,\dots,|\gamma_N^{(N)}|\le 1} \Big\| f(\cdot) - \varphi_N(\hat{f}(m^{(1)}) + \gamma_N^{(1)}\eta_N\widetilde{\varepsilon}_N, \dots,$$

$$\hat{f}(m^{(N)}) + \gamma_N^{(N)}\eta_N\widetilde{\varepsilon}_N; \cdot)\Big\|_{L^2} \ge$$

$$\ge \max\{\Big\|t_K(\cdot) - \varphi_N(\hat{t}_K(m^{(1)}) + \widetilde{\gamma}_N^{(1)}\eta_N\widetilde{\varepsilon}_N, \dots, \hat{t}_K(m^{(N)}) + \widetilde{\gamma}_N\eta_N\widetilde{\varepsilon}_N; \cdot)\Big\|_{L^2},$$

$$\Big\|(-t_K)(\cdot) - \varphi_N((-\hat{t}_K)(m^{(1)}) + \overline{\gamma}_N^{(1)}\eta_N\widetilde{\varepsilon}_N, \dots, (-\hat{t}_K)(m^{(N)}) + \overline{\gamma}_N^{(N)}\eta_N\widetilde{\varepsilon}_N; \cdot)\Big\|_{L^2}\} =$$

$$= \max\{\|t_K(\cdot) - \varphi_N(0, \dots, 0; \cdot)\|_{L^2}, \|(-t_K)(\cdot) - \varphi_N(0, \dots, 0; \cdot)\|_{L^2}\} \ge \|t_K\|_{L^2}. \tag{38}$$

Comparing inequalities (37) and (38), for any computing unit we obtain

$$\triangle_N(\eta_N\widetilde{\varepsilon}_N, (l^{(N)}, \varphi_N)) \gg \delta_N(L_N) \cdot \beta_K^{1-1/(2\lambda)}. \tag{39}$$

Since $1 - 1/(2\lambda) = \frac{2\lambda-1}{2\lambda} > 0$, then $\overline{\lim\limits_{K\to\infty}}\beta_K^{1-1/(2\lambda)} = +\infty$. Therefore, due to (39), for each $(l^{(N)}, \varphi_N) \in \Phi_N$ we have

$$\overline{\lim_{K\to\infty}} \frac{\triangle_N(\eta_N\widetilde{\varepsilon}_N, (l^{(N)}, \varphi_N))}{\delta_N(L_N)} = +\infty. \tag{40}$$

Since $(\tilde{l}^{(N)}, \widetilde{\varphi}_N) \in \Phi_N$, then from (40) the equality (4) follows. $\square$

7 Conclusion

When considering the class $W_2^{\mathrm{r};\alpha}$ in problems of optimal recovery of functions and finding limiting errors, in contrast to the multidimensional Sobolev classes with a dominant mixed derivative SW, Korobov E and the isotropic Sobolev class W both the exact order ψ_N and the limiting error $\widetilde{\varepsilon}_N$ do not depend on the number s of the variable functions $f(x) = f(x_1, \ldots, x_s) \in W_2^{\mathrm{r};\alpha}$ (see, for example, [4], [13], [14]). Therefore, the theorems formulated and proven here are important results in approximation theory, numerical analysis and computational mathematics.

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AN ELLIPTIC SELF-ADJOINT OPERATOR OF THE SECOND ORDER ON A GRAPH WITH SMALL EDGES

This work is devoted to the study of a second-order elliptic self-adjoint operator on a metric graph with short edges. The underlying structure is constructed by rescaling a given graph by the factor ε^{-1} and attaching it to another fixed graph, where $\varepsilon > 0$ is a small parameter. No substantial restrictions are imposed on the pair of graphs. On this combined structure, we define a general second-order elliptic self-adjoint operator whose differential expression involves derivatives of arbitrary order with variable coefficients and a non-constant potential. The vertex conditions are taken in a general form as well. All coefficients, both in the differential expression and in the vertex conditions, are allowed to depend analytically on the small parameter ε. It was previously established that the components of the resolvent corresponding to the restrictions of the operator to the fixed-length edges and to the short edges are analytic in ε as operators in the corresponding functional spaces, with the restriction on short edges additionally conjugated by dilation operators. Analyticity here means representability of these operator families by Taylor series. The first principal result of the paper is a recursive procedure, reminiscent of the method of matched asymptotic expansions, for determining all coefficients of such Taylor series. The second main result provides a convergent expansion of the resolvent in the form of a Taylor-type series, together with effective estimates of the remainder terms.

Key words: graph, differential operator, resolvent, boundary conditions, Taylor series.

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Шағын доғасы бар графтар бойында анықталған өзіне-өзі түйіндес екінші ретті эллиптикалық дифференциалдық оператор

Бұл жұмыс қысқа доғалары бар метрикалық графтағы екінші ретті эллиптикалық өзіне-өзі түйіндес операторды зерттеуге арналған. Бастапқы құрылым берілген графты ε^{-1} коэффициентіне дейін масштабтау және оны басқа бекітілген графқа жалғау арқылы құрылады, мұндағы $\varepsilon > 0$ - кіші параметр. Графтардың жұбына айтарлықтай шектеулер қойылмайды. Осы біріктірілген құрылымда екінші ретті эллиптикалық өзіне-өзі түйіндес оператор анықталады, дифференциалдық өрнегі айнымалы коэффициенттері бар туындылар және тұрақты емес потенциал арқылы анықталған. Графтың төбелеріндегі шарттар да жалпы түрде беріледі. Дифференциалдық өрнекте және төбелердегі шарттардағы барлық коэффициенттер кіші ε параметріне аналитикалық тәуелді болуы мүмкін. Алдыңғы зерттеулерде оператордың тұрақты ұзындықтағы доғалардағы және қысқа доғалардағы шектеулеріне сәйкес резольвентаның компоненттері ε параметріне қатысты тиісті функционалдық кеңістіктердегі операторлар ретінде аналитикалық екендігі дәлелденген. Сонымен қатар, қысқа доғалардағы шектеулер қосымша түрде дилатация операторларымен үйлестіріледі. Мұндағы аналитикалық дегеніміз - осы операторлар тобын Тейлор қатары арқылы өрнектеу мүмкіндігі. Жұмыстың бірінші негізгі нәтижесі - Тейлор қатарларының барлық коэффициенттерін табуға арналған, келісілген асимптотикалық жіктемелер әдісіне ұқсас рекурсивті процедура болып табылады. Екінші негізгі нәтиже резольвентаны тейлорлық типтегі қатар түрінде жинақты жіктеу мен қалдық мүшелердің тиімді бағалауларын ұсынады.

Түйін сөздер: граф, дифференциалдық оператор, резольвента, шекаралық шарттар, Тейлор қатары.

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Эллиптический самосопряженный оператор второго порядка на графе с малыми ребрами

Данная работа посвящена исследованию эллиптического самосопряжённого оператора второго порядка на метрическом графе с малыми рёбрами. Исходная структура строится путём масштабирования данного графа с коэффициентом ε^{-1} и присоединения его к другому фиксированному графу, где $\varepsilon > 0$ - малый параметр. Существенных ограничений на пару графов не накладывается. На этой комбинированной структуре определяется общий эллиптический самосопряжённый оператор второго порядка, дифференциальное выражение которого включает производные произвольного порядка с переменными коэффициентами и непостоянным потенциалом. Условия в вершинах также задаются в общей форме. Все коэффициенты - как в дифференциальном выражении, так и в условиях в вершинах - допускается зависимыми от малого параметра ε аналитическим образом. Ранее было установлено, что компоненты резольвенты, соответствующие ограничениям оператора на рёбра фиксированной длины и на короткие рёбра, аналитичны по ε как операторы в соответствующих функциональных пространствах, при этом ограничение на коротких рёбрах дополнительно сопрягается с операторами дилатации. Под аналитичностью здесь понимается представимость этих семейств операторов в виде рядов Тейлора.Первым основным результатом работы является рекурсивная процедура, напоминающая метод согласованных асимптотических разложений, для нахождения всех коэффициентов таких рядов Тейлора. Второй главный результат даёт сходящийся разложение резольвенты в виде ряда тейлоровского типа вместе с эффективными оценками остаточных членов.

Ключевые слова: граф, дифференциальный оператор, резольвента, граничные условия, ряд Тейлора.

1 Introduction

One of the actively developing areas of modern spectral theory of operators over the past two decades is the theory of quantum graphs [1–6]. In this theory, special attention is paid to perturbations caused by geometric features of the graph, in particular, the presence of small edges. Early studies concerning Taylor series for resolvents of operators on graphs with short edges were focused on the approximation of certain boundary conditions at vertices using graphs of a special structure with small edges. Such an approximation was understood as uniform convergence of resolvents of the corresponding operators on the original and approximating graphs.

Questions of this kind were considered for Schrödinger operators on a number of simple model graphs in [7, 8]. In particular, in [7], a star graph of three edges was studied, one of which was considered small, with δ- or δ'-type conditions imposed at the central vertex. In [8], a graph of arbitrary structure was added to a graph including a loop and two fixed edges, obtained by scaling the given finite graph with a coefficient ε^{-1}, where ε is a small parameter. In these studies, it was found that the resolvents and spectra of such operators depend on ε analytically, which was unexpected, given the singular nature of such perturbations. Usually, for singular perturbations, it is possible to construct only asymptotic expansions for eigenvalues, while the questions of convergence of such series and, especially, their analytical dependence on a small parameter remain open [9,10]. Advances in the analysis of model operators became the motivation for studying more general operators on graphs with arbitrary topology and small edges. An elliptic self-adjoint operator of the second order with variable coefficients and general boundary conditions on a graph was considered, to which another graph with small edges whose lengths were proportional to the parameter ε was glued. Moreover, both the coefficients of the differential expression and the boundary conditions could analytically depend on ε. In this paper, we continue the study begun in [7,8] and focus on the analysis of the resolvent of the general operator. The main result is the construction of a recurrent procedure for the coefficients of the Taylor series of the resolvent components. Based on the results of [9,10], a uniformly convergent expansion of the resolvent in a Taylor series is obtained, and effective estimates of the remainders of this expansion in various operator norms are given.

1.1 Research methodology

The text does not specify the specific time, place, and conditions of the study, since the work is of a theoretical and mathematical nature. The study was conducted within the framework of mathematical modeling and analytical analysis, without involving experimental data or a sample of subjects. Abstract graphs constructed by compressing one graph and then gluing it to another, as well as the corresponding second-order elliptic operators, were considered as the "material". The main research tool was the methods of functional analysis, operator theory, and asymptotic expansions.

2 Research results

2.1 Statement of the Problem

The main object of the study of this article is a self-adjoint elliptic operator of the second order on a singular perturbed graph. The essence of a singular perturbation is the presence of small edges. A graph with small edges is obtained by gluing in a certain way a small graph to a given graph with edges of a fixed length. The latter graph is denoted by the symbol Γ and is a finite metric graph. This means that it contains a finite number of edges and vertices, on each edge a direction and a corresponding variable are introduced. As a measure on each edge, the standard Lebesgue measure is chosen. The graph Γ is allowed to have edges of infinite length. At the same time, we assume that this graph does not contain isolated edges and vertices.

By γ we denote another finite metric graph without isolated vertices and edges, and now we assume that the graph γ contains edges of only finite length. We compress the graph γ by a factor of ε^{-1}, i.e., we replace each of its edges e of length $|e|$ with an edge $\varepsilon|e|$ while preserving the rest of the graph structure. We denote the resulting graph by γ_ε.

In what follows, we will often identify the original graphs Γ and γ_ε with the corresponding subgraphs of Γ_ε, for which the same notations will be used. The directions and variables on the edges of Γ and γ_ε are preserved after the described gluing. Therefore, each function defined on the graphs Γ and γ_ε is simultaneously considered to be defined on the corresponding subgraphs of Γ_ε. And vice versa, a function defined on Γ_ε, is considered to be defined on the graphs Γ and γ_ε.

On the graph Γ_ε we consider the elliptic operator $\mathcal{H}_\varepsilon$ with the differential expression

$$\widehat{\mathcal{H}}(\varepsilon) := -\frac{d}{dx}p_\varepsilon\frac{d}{dx} + i\Big(\frac{d}{dx}q_\varepsilon + q_\varepsilon\frac{d}{dx}\Big) + V_\varepsilon, \tag{1}$$

where i is the imaginary unit, and the coefficients are given by the equalities

$$p_\varepsilon := \begin{cases} p_\Gamma(\cdot,\varepsilon) & \text{on } \Gamma, \\ S_\varepsilon p_\gamma(\cdot,\varepsilon) & \text{on } \gamma_\varepsilon, \end{cases} q_\varepsilon := \begin{cases} q_\Gamma(\cdot,\varepsilon) & \text{on } \Gamma, \\ \varepsilon_\varepsilon^{-1} p_\gamma(\cdot,\varepsilon) & \text{on } \gamma_\varepsilon, \end{cases} V_\varepsilon := \begin{cases} V_\Gamma(\cdot,\varepsilon) & \text{on } \Gamma, \\ \varepsilon^{-2} S_\varepsilon V_\gamma(\cdot,\varepsilon) & \text{on } \gamma_\varepsilon, \end{cases}$$

Here $p_\Gamma = p_\Gamma(\cdot,\varepsilon) \in W^1_\infty(\Gamma), q_\Gamma = q_\Gamma(\cdot,\varepsilon) \in W^1_\infty(\Gamma), V_\Gamma = V_\Gamma(\cdot,\varepsilon) \in L_2(\Gamma)$ and $p_\gamma = p_\gamma(\cdot,\varepsilon) \in W^1_\infty(\gamma), q_\gamma = q_\gamma(\cdot,\varepsilon) \in W^1_\infty(\gamma), V_\gamma = V_\gamma(\cdot,\varepsilon) \in L_2(\gamma)$–are some real functions defined respectively on the graphs Γ and γ and analytic in ε in the norm of the indicated spaces $\mathcal{S}_\varepsilon : L_2(\gamma) \to L_2(\gamma_\varepsilon)$-is a linear operator, defined by the formula

$$(\mathcal{S}_\varepsilon u)(x) := u\Big(\frac{x}{\varepsilon}\Big),\ x \in e_\varepsilon \tag{2}$$

on each edge e_ε of the graph γ_ε.

The differential expression $\widehat{\mathcal{H}}(\varepsilon)$ is considered uniformly elliptic: taking into account the analyticity of the functions p_Γ and p_γ with respect to ε, we assume that the inequalities

$$p_\Gamma(x,0) \geq c_\mathcal{H} \text{ on } \Gamma, p_\gamma(\xi,0) \geq c_\mathcal{H} \text{ on } \gamma$$

with some fixed constant $c_\mathcal{H} > 0$.

The boundary conditions for the operator $\mathcal{H}_\varepsilon$ are defined as follows. For an arbitrary vertex $M \in \Gamma_\varepsilon$ with degree $d(M) > 0$, we denote by $e_j(M), i = 1, ..., d(M)$, the edges coming out of M. We introduce a pair of $d(M)$-dimensional vectors

$$U_M(u) := \begin{pmatrix} u_1(M) \\ \vdots \\ u_{d(M)}(M) \end{pmatrix}, \quad U'_M(u) := \begin{pmatrix} \frac{du_1}{dx_1}(M) \\ \vdots \\ \frac{du_d}{dx_d}(M) \end{pmatrix}, \tag{3}$$

where x_i-variable on the edge e_i. The boundary conditions at the vertex $M \in \Gamma_\varepsilon$ are specified in the general form:

$$A_M(\varepsilon)U_M(u) + B_M(\varepsilon)U'_M(u) = 0, \tag{4}$$

where $A_M(\varepsilon)$ and $B_M(\varepsilon)$–analytic matrices of size $d(M) \times d(M)$ in ε.

Strictly,$\mathcal{H}_\varepsilon$ is defined as an unbounded operator in $L_2(\Gamma_\varepsilon)$ whose action is described by the differential expression (1) on the domain of definition composed of functions from the space $W^2_2(\Gamma_\varepsilon)$ satisfying the boundary conditions (4); here we use the notation $W^j_2(\cdot) := \oplus W^j_2(e),\ j = 1, 2$. All other operators that are used further in the paper are strictly defined in a similar way based on their differential expressions and boundary conditions.

We restrict our consideration to self-adjoint operators, which means that it is necessary to impose certain conditions on the coefficients of the differential expression (1) and the matrix in the boundary conditions (4). The criterion for the self-adjointness of the operator $\mathcal{H}_\varepsilon$ is the simultaneous fulfillment of the equality

$$\operatorname{rank}\Big(A_M(0)B_M(0)\Big) = d(M) \tag{5}$$

and the presence of self-adjointness of the matrix

$$A_M(\varepsilon)\Pi_M^{-1}(\varepsilon)B_M^*(\varepsilon) + iB_M(\varepsilon)\Pi_M^{-1}(\varepsilon)\Theta_M\Pi_M^{-1}(\varepsilon)B_M^*(\varepsilon),$$

where

$$\begin{aligned}&\textstyle\prod_M(\varepsilon) := \operatorname{diag}\{\vartheta_i(M)p_\varepsilon\big|_{e_i(M)}(M)\}_{i=1,\dots,d(M)},\\ &\Theta_M(\varepsilon) := \operatorname{diag}\{\vartheta_i(M)q_\varepsilon\big|_{e_i(M)}(M)\}_{i=1,\dots,d(M)},\end{aligned} \tag{6}$$

$e_i(M)$ are the edges emanating from the vertex M, and the numbers $\vartheta_i(M)$ are defined as follows: $e_i(M)$, if the direction on the edge $e_i(M)$ inward from the vertex M coincides with the initially chosen direction on this edge, and $\vartheta_i(M) := -1$, if these directions are opposite.

The boundary condition (4), in essence, does not change when multiplying it from the left by non-singular square matrices of size $d(M) \times d(M)$. Taking into account equality (5), we partially limit such freedom in the choice of matrices $A_M(\varepsilon)$ and $B_M(\varepsilon)$ by the following condition. We denote $r(M) := \operatorname{rank} B_M(0)$ and assume that the first $r(M)$ rows of the matrix $B_M(0)$ are linearly independent, and the last $d(M) - r(M)$ rows vanish. We simultaneously assume that the last $d(M) - r(M)$ rows of the matrix $A_M(0)$ are not equal to zero.

The main goal of this paper is to describe in detail the dependence of the resolvent of the operator $\mathcal{H}_\varepsilon$ on the parameter ε. To formulate the main result, we need to introduce auxiliary notations. This will be done in sections 2.2 and 2.3.

2.2 Auxiliary notations and the main condition

For convenience and to simplify a number of technical calculations, we assume throughout the paper that the directions on the edges $e_i, i = 1, ..., d_0$, of the graph Γ, emanating from the vertex M_0, are chosen inside the edges from the vertex M_0. If there is a loop among the edges e_i, then in order to ensure that such a condition is satisfied, we introduce an additional artificial vertex on the loops, on which we set the standard Kirchhoff condition. Such a vertex does not change either the operator $\mathcal{H}_\varepsilon$, or its resolvent, or its spectrum.

We introduce another auxiliary graph γ_∞, which is obtained by attaching edges of infinite length $e_i^\infty, i \in J_j, j = 1, ..., n$, to the vertices $M_j, j = 1, ..., n$, of the graph γ. The variable on the graph γ is denoted by ξ. An auxiliary operator $\mathcal{H}_\infty$ is defined on the graph $\mathcal{H}_\infty$. This is an unbounded operator in $L_2(\gamma_\infty)$ with the differential expression

$$\begin{gathered}\widehat{\mathcal{H}}_\infty := -\tfrac{d}{d\xi}p_\gamma(\cdot,0)\tfrac{d}{d\xi} + i\Big(\tfrac{d}{d\xi}q_\gamma(\cdot,0) + q_\gamma(\cdot,0)\tfrac{d}{d\xi}\Big) + V_\gamma(\cdot,0) \ \text{ on } \gamma,\\ \widehat{\mathcal{H}}_\infty := -p_i(0)\tfrac{d^2}{d\xi^2} \ \text{ on } \ e_i^\infty,\ i \in J_j,\ j = 1, ..., n, \quad p_i(\varepsilon) := p_\Gamma\big|_{e_i}(M_0,\varepsilon),\end{gathered}$$

and boundary conditions

$$A_M^{(0)}U_M(u) + B_M^{(0)}U_M'(u) = 0 \ \text{ at the vertices } \ M \in \gamma_\infty.$$

Here the vectors $U_M(u)$ and $U'_M(u)$ are introduced similarly to (3) with the replacement of derivatives $\frac{du_i}{dx_i}$ on $\frac{du_i}{d\xi_i}$. The matrices $A_M^{(0)}$ and $B_M^{(0)}$ are defined by the formulas

$$A_M^{(0)} := \begin{pmatrix} 0 \\ A_M^-(0) \end{pmatrix}, B_M^{(0)} := \begin{pmatrix} B_M^+(0) \\ \frac{dB_M^-}{d\varepsilon}(0) \end{pmatrix},$$

where $A_M^-(\cdot)$ and $B_M^-(\cdot)$ are the matrices composed of the last $d(M) - r(M)$ rows of the matrices $A_M(\cdot)$ and $B_M(\cdot)$ and the matrix $B_M^+(\cdot)$ is formed by the first $r(M)$ rows of the matrix B_M. The operator $\mathcal{H}_\infty$ is self-adjoint, and its essential spectrum is the semi-axis $[0, \infty)$.

The fundamental condition imposed on the operator $\mathcal{H}_\varepsilon$ is expressed in terms of the operator $\mathcal{H}_\infty$:

(A) *The operator $\mathcal{H}_\infty$ has no embedded eigenvalues at the edge of its essential spectrum.*

Equivalently, condition (A) can be reformulated as follows. Consider the boundary value problem

$$\widehat{\mathcal{H}}_\infty \psi = 0 \text{ on } \gamma_\infty, \ \ A_M^{(0)} U_M(\psi) + B_M^{(0)} U'_M(\psi) = 0 \text{ at the vertices } M \in \gamma_\infty. \tag{7}$$

By virtue of the definition of the differential expression $\widehat{\mathcal{H}}_\infty$ on semi-infinite edges e_i^∞, the solution of this problem on these edges is given by a linear function. Therefore, the absence of an embedded eigenvalue on the edge of the essential spectrum of the operator $\mathcal{H}_\infty$ is equivalent to the absence of nontrivial solutions $\psi \in W_2^2(\gamma) \oplus W_{2,loc}^2(e_i^\infty)$ of problem (7), which identically vanish on all edges e_i^∞. At the same time, the presence of nontrivial solutions that are constant on these edges and do not simultaneously vanish on all edges e_i^∞, is not excluded. In other words, the presence of a virtual level is allowed on the edge of the essential spectrum of the operator $\mathcal{H}_\infty$.

Let $\psi^{(j)}, j = 1, ..., k$, be linearly independent solutions of problem (7), constant on the edges e_i^∞. By condition (A), these functions do not vanish identically simultaneously on all edges e_i^∞. It is clear that $k \leq d_0$. If there are no such solutions, then we set $k := 0$.

For an arbitrary function u defined on the edges $e_i^\infty, i \in J_j, j = 1, ..., n$, at least in the neighborhood of the vertex M_j and continuous up to this vertex, we introduce the notation

$$U_\gamma(u) := \left(u\Big|_{e_i^\infty}(M_j) \right)_{i \in J_j, j=1,\ldots,n}.$$

We denote $\Psi^{(j)} := U_\gamma(\psi^{(j)}), j = 1, ..., k$ and we choose the functions $\psi^{(j)}$ so that the introduced vectors $\Psi^{(j)}$ are orthonormal in C^{d_0}. If $k < d_0$, then we additionally choose arbitrary vectors $\Psi^{(j)} \in C^{d_0}$, $j = k+1, ..., d_0$, so that the entire set of vectors $\Psi^{(j)} \in C^{d_0}$, $j = 1, ..., d_0$, forms an orthonormal basis in C^{d_0}. This means that the matrix $\Psi := \left(\Psi^{(1)} ... \Psi^{(k)} \Psi^{(k+1)} ... \Psi^{(d_0)} \right)$ is unitary.

An important role will be played by another auxiliary operator on yet another graph, denoted by γ_{ex} and obtained by attaching unit edges e_i^{ex}, $i \in J_j$, $j = 1, ..., n$, to vertices M_j, $j = 1, \ldots, n$, of the graph γ. Vertices M_j will be considered the beginning of edges e_i^∞, i.e., the direction on these edges is chosen inward from M_j. Vertices that are the ends of edges

e_i^∞ will be denoted by M_i^{ex}, $i \in J_j,\ j=1,...,n$. The mentioned auxiliary operator is denoted by $\mathcal{H}_{ex}(\varepsilon)$ and is determined by the differential expression

$$\begin{aligned} \widehat{\mathcal{H}}_{ex}(\varepsilon) &:= -\tfrac{d}{d\xi}p_\gamma(\cdot,\varepsilon)\tfrac{d}{d\xi} + i\Big(\tfrac{d}{d\xi}q_\gamma(\cdot,\varepsilon) + q_\gamma(\cdot,\varepsilon)\tfrac{d}{d\xi}\Big) + V_\gamma(\cdot,\varepsilon) \text{ on } \gamma \\ \widehat{\mathcal{H}}_{ex}(\varepsilon) &:= -p_i(\varepsilon)\tfrac{d^2}{d\xi^2} + 2i\varepsilon q_i(\varepsilon) \text{ on } e_i^{ex},\ i \in J_j,\ j=1,...n, \end{aligned} \tag{8}$$

with boundary conditions

$$\varepsilon A_M(\varepsilon)U_M(u) + B_M(\varepsilon)U'_M(u) = 0 \tag{9}$$

at the vertices $M \in \gamma_\infty$ and boundary conditions of the third type

$$\Pi_{\Gamma,M_0}(\varepsilon)U'_{ex}(u) - i\varepsilon\Theta_{\Gamma,M_0}(\varepsilon)U_{ex}(u) = 0$$

at the vertices M_i^{ex}, where it is denoted

$$U'_{ex}(u) := \begin{pmatrix} \frac{du|_{e_1^{ex}}}{d\xi_1}(M_1^{ex}) \\ \vdots \\ \frac{du|_{e_{d_0}^{ex}}}{d\xi_{d_0}}(M_{d_0}^{ex}) \end{pmatrix}, U_{ex}(u) := \begin{pmatrix} u(M_1^{ex}) \\ \vdots \\ u(M_{d_0}^{ex}) \end{pmatrix}, q_i(\varepsilon) := \vartheta_i(M_0)q_\Gamma|_{e_i}(M_0,\varepsilon).$$

2.3 Parts of the resolvent and known results

Let $\mathcal{R}_\Gamma : L_2(\Gamma_\varepsilon) \to L_2(\Gamma)$ and $\mathcal{R}_{\gamma_\varepsilon} : L_2(\Gamma_\varepsilon) \to L_2(\gamma_\varepsilon)$–restriction operators to subgraphs of Γ and γ_ε, namely $\mathcal{R}_\Gamma f := f|_\Gamma$, $\mathcal{R}_{\gamma_\varepsilon} f := f|_{\gamma_\varepsilon}$. It is clear that

$$L_2(\Gamma_\varepsilon) = L_2(\Gamma) \oplus L_2(\gamma_\varepsilon), \mathcal{R}_\Gamma \oplus \mathcal{R}_{\gamma_\varepsilon} = \mathcal{J}_{\Gamma_\varepsilon}, \tag{10}$$

where $\mathcal{J}_{\Gamma_\varepsilon}$- is the identity operator in $L_2(\Gamma_\varepsilon)$.

Since the operator $\mathcal{H}_\varepsilon$ is self-adjoint, its resolvent $(\mathcal{H}_\varepsilon - \lambda)^{-1}$ is well defined for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Therefore, we can introduce a pair of operators

$$R_\Gamma(\varepsilon,\lambda) := \mathcal{R}_\Gamma(\mathcal{H}_\varepsilon - \lambda)^{-1}(\mathcal{J}_\Gamma \oplus \mathcal{S}_\varepsilon), R_\gamma(\varepsilon,\lambda) := \mathcal{S}_\varepsilon^{-1}\mathcal{R}_{\gamma_\varepsilon}(\mathcal{H}_\varepsilon - \lambda)^{-1}(\mathcal{J}_\Gamma \oplus \mathcal{S}_\varepsilon),$$

where $\mathcal{J}_\Gamma$– is the identity operator in $L_2(\Gamma)$. These operators are linear and bounded as acting from $L_2(\Gamma) \oplus L_2(\gamma)$ to $W_2^2(\Gamma)$ and $W_2^2(\gamma)$, respectively.

Let us explain the action of the operators R_Γ and R_γ. For an arbitrary pair of functions $(f_\Gamma, f_\gamma) \in L_2(\Gamma) \oplus L_2(\gamma)$, we construct a function $f \in L_2(\Gamma_\varepsilon)$ by the rule $f := f_\Gamma$ on Γ and $f := \mathcal{S}_\varepsilon f_\gamma$ on γ_ε. Next, the resolvent is applied to f and restrictions of the result to the subgraphs of Γ and γ_ε are considered, i.e., the functions $\mathcal{R}_\Gamma(\mathcal{H}_\varepsilon - \lambda)^{-1}f$ and $\mathcal{R}_{\gamma_\varepsilon}(\mathcal{H}_\varepsilon - \lambda)^{-1}f$. The first of these functions is the action of the operator $\mathcal{R}_\Gamma(\varepsilon,\lambda)$ on (f_Γ, f_γ). To the second restriction we additionally apply the operator $\mathcal{S}_\varepsilon^{-1}$: the resulting function $\mathcal{S}_\varepsilon^{-1}\mathcal{R}_{\gamma_\varepsilon}(\mathcal{H}_\varepsilon - \lambda)^{-1}f$- is the action of the operator $\mathcal{R}_\gamma(\varepsilon,\lambda)$ on (f_Γ, f_γ). We also note the formula

$$(\mathcal{H}_\varepsilon - \lambda)^{-1}f = \big(R_\Gamma(\varepsilon,\lambda) \oplus \mathcal{S}_\varepsilon R_\gamma(\varepsilon,\lambda)\big)\big(\mathcal{R} \oplus \mathcal{S}_\varepsilon^{-1}\mathcal{R}_{\gamma_\varepsilon}\big) \tag{11}$$

Next, we need another auxiliary operator on the graph Γ, considered as a separate graph. This operator is denoted by $\mathcal{H}_0$, and it corresponds to the differential expression $\widehat{\mathcal{H}}(0)$ and the boundary conditions

$$A_M^{(0)}U_M(u) + B_M^{(0)}U_M'(u) = 0 \text{ at the vertices } M \in \Gamma. \tag{12}$$

For $M \neq M_0$, the matrices $A_M^{(0)}$ and $B_M^{(0)}$ are introduced simply: $A_M^{(0)} := A_M(0), B_M^{(0)} := B_M(0)$. In the case of $M \doteq M_0$ the description of the matrices $A_{M_0}^{(0)}$ and $B_{M_0}^{(0)}$ is much more cumbersome. Namely, these matrices are of size $d_0 \times d_0$ and they have the form

$$A_{M_0}^{(0)} := \begin{pmatrix} Q & 0 \\ 0 & I_{d_0-k} \end{pmatrix} \Psi^* + i \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Psi^* \Theta_{\Gamma,M_0}(0),\ B_{M_0}^{(0)} := -\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Psi^* \Pi_{\Gamma,M_0}(0) \tag{13}$$

where for an arbitrary d the symbol I_d denotes the identity matrix of size $d \times d$, and the symbol 0 denotes the zero matrices of the corresponding sizes. The matrices Π_{Γ,M_0} and Θ_{Γ,M_0} are described by the formulas

$$\Pi_{\Gamma,M_0}(\varepsilon) := diag\{\vartheta_i(M_0) p_\Gamma|_{e_i}(M_0,\varepsilon)\}_{i=1,\dots,d_0},$$
$$\Theta_{\Gamma,M_0}(\varepsilon) := diag\{\vartheta_i(M_0) q_\Gamma|_{e_i}(M_0,\varepsilon)\}_{i=1,\dots,d_0}$$

where, recall, e_i are the edges of the graph Γ emanating from the vertex M_0, the numbers $\vartheta_i(M_0)$ are defined as and in (6). The matrix Q has the form

$$Q := \begin{pmatrix} Q^{(11)} & \dots & Q^{(k1)} \\ \dots & \dots & \dots \\ Q^{(1k)} & \dots & Q^{(kk)} \end{pmatrix},\ Q^{(ij)} := Q_\gamma^{(ij)} + \sum_{M \in \gamma_\infty} Q_M^{(ij)} \tag{14}$$

$$\begin{aligned} Q^{(ij)} := &\left(\tfrac{dp_\gamma}{d\varepsilon}(\cdot,0)\tfrac{d\psi^{(i)}}{d\xi}, \tfrac{d\psi^{(j)}}{d\xi}\right)_{L_2(\gamma)} + \left(\tfrac{d\psi^{(i)}}{d\xi}, i\tfrac{dq_\gamma}{d\varepsilon}(\cdot,0)\psi^{(j)}\right)_{L_2(\gamma)} + \\ &+ \left(i\tfrac{dq_\gamma}{d\varepsilon}(\cdot,0)\psi^{(j)}, \tfrac{d\psi^{(j)}}{d\xi},\right)_{L_2(\gamma)} + \left(\tfrac{dV_\gamma}{d\varepsilon}(\cdot,0)\psi^{(i)}, \psi^{(j)}\right)_{L_2(\gamma)}, \end{aligned} \tag{15}$$

$$Q_M^{(ij)} := \left(L_M(\psi^{(i)}), U_M(\psi^{(j)})\right)_{C^{d(M)}} - \frac{i}{2}\left(\varepsilon_M(\psi^{(i)}), \vartheta_M(\psi^{(j)}),\right)_{C^{d(M)}}, \tag{16}$$

$$\mathcal{L}_M(\psi^{(i)}) := \frac{d\Pi_{\gamma,M}}{d\varepsilon}(0)U_M'(\psi^{(i)}) - i\frac{d\Theta_{\gamma,M}}{d\varepsilon}(0)U_M(\psi^{(i)}) + \left(Y_M + I_{d(M)}\right)^{-1} P_M^\perp \varepsilon_M(\psi^{(i)}), \tag{17}$$

$$\varepsilon_M(\cdot) := 2iC_M\left(A_M^{(1)}U_M(\cdot) + B_M^{(1)}U_M'(\cdot)\right), \tag{18}$$

$$U_M := -C_M\left(\tilde{A}_M + i\tilde{B}_M\right),\ \vartheta_M(\cdot) := \Pi_{\gamma,M}(0)U_M'(\cdot) - \Theta_{\gamma,M}(0)U_M(\cdot), \tag{19}$$

$$\tilde{A}_M := A_M^{(0)} + iB_M^{(0)}\Pi_{\gamma,M}^{-1}(0)\Theta_{\gamma,M}(0),\ \tilde{B}_M := B_M^{(0)}\Pi_{\gamma,M}^{-1}(0), C_M := \left(\tilde{A}_M - i\tilde{B}_M\right)^{-1},$$

$$A_M^{(1)} := \begin{pmatrix} A_M^+(0) \\ \frac{dA_M^-}{d\varepsilon}(0) \end{pmatrix}, B_M^{(1)} := \begin{pmatrix} \frac{dB_M^+}{d\varepsilon}(0) \\ \frac{1}{2}\frac{d^2B_M^-}{d\varepsilon^2}(0), \end{pmatrix},$$

$$\Pi_{\gamma,M_0}(\varepsilon) := diag\{\vartheta_i(M_0)p_\gamma|_{e_i}(M,\varepsilon)\}_{i=1,...,d_M},$$
$$\Theta_{\gamma,M_0}(\varepsilon) := diag\{\vartheta_i(M_0)q_\gamma|_{e_i}(M,\varepsilon)\}_{i=1,...,d_M}$$

where the matrix $A_M^+(\cdot)$ is formed by the first $r(M)$ rows of the matrix $A_M(\cdot), e_i(M)$ are the edges emanating from the vertex M, the numbers $\vartheta_i(M)$ are defined in (6), and the functions p_γ and q_γ are extended to the edges e_i^∞, $i \in J_j$, $j = 1, ..., n$, by the formulas $p_\gamma(\cdot,\varepsilon) \equiv p_i(\varepsilon)$, $q_\gamma(\cdot,\varepsilon) \equiv \varepsilon q_i(\varepsilon)$.

By P_M we denote the projection in $\mathbb{C}^{d(M)}$ onto the eigenspace of the matrix U_M corresponding to the eigenvalue -1, and we also set $P_M^\perp := I_{d(M)} - P_M$.

Let's define more spaces:

$$\tau(\cdot) := \oplus_{e\in\cdot} C(\overline{e}) \cap L_\infty(e), \|u\|_{\tau(\overline{e})} := \sum_{e\in\cdot} \|u\|_{L_{\infty(e)}}$$

$$\tau^1(\cdot) := \oplus_{e\in\cdot} C^1(\overline{e}) \cap W^1_\infty(e), \|u\|_{\tau^1(\overline{e})} := \sum_{e\in\cdot} \|u\|_{W^1_{\infty(e)}}.$$

Theorem 1 *Let the matrices $A_M(\varepsilon), B_M(\varepsilon)$ satisfy the above conditions. Then the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ are self-adjoint. Let condition (A) also be satisfied. Then the operators $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ are linear and bounded as acting from $L_2(\Gamma) \oplus L_2(\gamma)$ to $W_2^2(\Gamma)$ and $W_2^2(\gamma)$ and to $\tau^1(\Gamma)$ and $\tau^1(\gamma)$. For each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists $\varepsilon_0(\lambda) > 0$, such that for $\varepsilon_0(\lambda) > 0$ the operators $\mathcal{R}_\Gamma(\varepsilon,\lambda)$ and $\mathcal{R}_\gamma(\varepsilon,\lambda)$ are analytic in ε as are the operators from $L_2 \oplus L_2(\gamma)$ in $W_2^2(\Gamma)$ and $W_2^2(\gamma)$ and in $\tau^1(\Gamma)$ and $\tau^1(\gamma)$. In both cases the first terms of the Taylor series of these operators are of the form*

$$\mathcal{R}_\Gamma(\varepsilon,\lambda) = \big(\mathcal{H}_0 - \lambda\big)^{-1}\mathcal{P}_\Gamma + O(\varepsilon), \mathcal{R}_\Gamma(\varepsilon,\lambda) = \mathcal{R}_\gamma^0\mathcal{P}_\Gamma + O(\varepsilon),$$
$$\mathcal{R}_\gamma^0(\lambda)f := \sum_{i=1}^k c_i(f)\psi^{(i)},\ \big(c_1(f),...,c_k(f)\big)^t := \Big(\Psi^{(1)},...,\Psi^{(k)}\Big)^* U_{M_0}\Big(\big(\mathcal{H}_0 - \lambda\big)^{-1}f\Big). \tag{20}$$

2.4 Main result

The main result of this paper describes all the coefficients of the Taylor series for the operators $\mathcal{R}_\Gamma$ and $\mathcal{R}_\gamma$, as well as the analogue of the Taylor series for the resolvent of the operator $\mathcal{H}_\varepsilon$.

Let's define a family of auxiliary functions - solutions to problems

$$\Big(\mathcal{H}(0) - \lambda\Big)\vartheta_{i,\Gamma} = 0,\ \text{on}\,\Gamma,$$
$$A_M^{(0)}U_M(\vartheta_{i,\Gamma}) + B_M^{(0)}U_M'(\vartheta_{i,\Gamma}) = 0,\ \text{in}\ M \neq M_0,\ U_{M_0}(\vartheta_{i,\Gamma}) = \Psi^{(i)}. \tag{21}$$

The application of Lemma 2 from Section 3 guarantees the unique solvability of these problems. Let us consider two systems of boundary value problems. The first is introduced on the graph Γ, considered as an independent graph ($p \geq 0$):

$$\Big(\widehat{\mathcal{H}}(0) - \lambda\Big)u_p^\Gamma = -\sum_{q=1}^{p} \frac{1}{q!}\frac{d^q\widehat{\mathcal{H}}}{d\varepsilon^q}(\cdot,0)u_{p-q}^\Gamma \text{ in } \Gamma, \tag{22}$$

$$\begin{gathered} A_M^{(0)}U_M(\vartheta_{i,\Gamma}) + B_M^{(0)}U_M'(\vartheta_{i,\Gamma}) = \\ -\sum_{q=1}^{p}\tfrac{1}{q!}\left(\tfrac{d^qA_M}{d\varepsilon^q}(0)U_M\big(u_{p-q}^\Gamma\big) + \tfrac{d^qB_M}{d\varepsilon^q}(0)U_M'\big(u_{p-q}^\Gamma\big)\right)\ M\in\Gamma,\ M\neq M_0. \end{gathered} \tag{23}$$

$$\widehat{\mathcal{H}}_{ex}(0)u_p^\gamma = \delta_{2p}\chi_\gamma f_\gamma - \sum_{q=1}^{p}\frac{1}{q!}\frac{d^q\widehat{\mathcal{H}}_{ex}}{d\varepsilon^q}(0)\chi_\gamma u_{p-q}^\gamma + \lambda\chi_\gamma u_{p-2}^\gamma \text{ on } \gamma_{ex}, p\geq 0, \tag{24}$$

$$P_MU_M(u_p^\gamma) = -P_MC_Mg_p^\gamma,\ P_M^\perp\vartheta_M(u_p^\gamma) + K_M^\perp U_M(u_p^\gamma) = 2iC_M\big(U_M + E_{d(M)}\big)^{-1}C_Mg_p^\gamma, \tag{25}$$

$$g_p^\gamma := \sum_{i=1}^{p}\left(A_M^{(i)}U_M(u_{p-i}^\gamma) + B_M^{(i)}U_M'(u_{p-i}^\gamma)\right),$$

$$A_M^i := \begin{pmatrix} \frac{1}{(i-1)!}\frac{d^{i-1}A_M^+}{d\varepsilon^{i-1}}(0) \\ \frac{1}{i!}\frac{d^iA_M^-}{d\varepsilon^i}(0)\end{pmatrix}, B_M^i := \begin{pmatrix} \frac{1}{i!}\frac{d^iB_M^+}{d\varepsilon^i}(0) \\ \frac{1}{(i+1)!}\frac{d^{i+1}B_M^-}{d\varepsilon^{i+1}}(0)\end{pmatrix}, M\in\gamma_\infty$$

where we set $u_p^\gamma := 0$ for $p \leq -1$, χ_γ denotes the characteristic function of the graph γ, δ_{qp} is the Kronecker-Capelli symbol and

$$K_M^\perp := i\Big(U_M + E_{d(M)}\Big)^{-1}P_M^\perp\Big(U_M - E_{d(M)}\Big).$$

The main result about Taylor series for the operators $\mathcal{R}_\Gamma$ and $\mathcal{R}_\gamma$ is as follows.

Theorem 2 *Let all the above conditions on the coefficients of the differential expression $\widehat{\mathcal{H}}(\varepsilon)$ and the matrices $A_M(\varepsilon), B_M(\varepsilon)$ be satisfied, and let condition (A) be satisfied. For each pair $(f_\Gamma, f_\gamma) \in L_2(\Gamma)\oplus L_2(\gamma)$ the Taylor series of the functions $\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)$ and $\mathcal{R}_\gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)$ have the form*

$$\begin{aligned} \mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma) &= \sum_{p=0}^{\infty}\varepsilon^pu_p^\Gamma,\ \ u_0^\Gamma := \big(\mathcal{H}_0 - \lambda\big)^{-1}f_\Gamma \\ \mathcal{R}_\gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma) &= \sum_{p=0}^{\infty}\varepsilon^pu_p^\gamma,\ \ u_0^\gamma := \sum_{i=1}^{k}c_i(f_\Gamma)\psi^{(i)}, \end{aligned} \tag{26}$$

and converge uniformly in ε in the spaces $W_2^2(\Gamma)$ and $W_2^2(\gamma)$, as well as in the norms of the spaces $\tau^2(\Gamma)$ and $\tau^2(\gamma)$.

The coefficients of these series are given by the formulas

$$u_p^\Gamma = u_{p,*}^\Gamma + \sum_{i=1}^{k} c_{i,p}\vartheta_{i,\Gamma},\ u_p^\gamma = u_{p,*}^\Gamma + \sum_{i=1}^{k} c_{i,p}\psi^{(i)},\ p \geq 1, \tag{27}$$

where $u_{p,*}^\Gamma$ is the unique solution of problem (22), (23) with boundary conditions

$$U_{M_0}\big(u_{p,*}^\Gamma\big) = U_\gamma\big(u_{p,*}^\gamma\big), \tag{28}$$

and $u_{p,*}^\gamma$ is a particular solution of problem (24), (25) with the boundary condition

$$U'_{\gamma_{ex}}\big(u_{p,*}^\gamma\big) = U'_{M_0}\big(u_{p-1}^\Gamma\big), \tag{29}$$

determined by the orthogonality conditions

$$\Big(U_\gamma\big(u_{p,*}^\gamma\big), \Psi^{(j)}\Big)_{\mathbb{C}^{d_0}} = 0,\ j = 1, ..., k. \tag{30}$$

The constants $c_{i,p}$, $p \geq 1$, are given by the formulas

$$c_p = \big(Q + L\big)^{-1} h_p, \tag{31}$$

$$c_p := \begin{pmatrix} c_{1,p} \\ \vdots \\ c_{k,p} \end{pmatrix}, h_p := \begin{pmatrix} h_{1,p} \\ \vdots \\ h_{k,p} \end{pmatrix}, L := \begin{pmatrix} \big(\vartheta_0(\vartheta_{1,\Gamma}), \Psi^{(1)}\big)_{\mathbb{C}^{d_0}} & \dots & \big(\vartheta_0(\vartheta_{k,\Gamma}), \Psi^{(1)}\big)_{\mathbb{C}^{d_0}} \\ \dots & \dots & \dots \\ \big(\vartheta_0(\vartheta_{1,\Gamma}), \Psi^{(k)}\big)_{\mathbb{C}^{d_0}} & \dots & \big(\vartheta_0(\vartheta_{k,\Gamma}), \Psi^{(k)}\big)_{\mathbb{C}^{d_0}} \end{pmatrix} \tag{32}$$

$$\begin{gathered} h_{j,p} := -\sum_{q=2}^{p} \tfrac{1}{q!}\Big(\tfrac{d^q\widehat{\mathcal{H}}_{ex}}{d\varepsilon^q}(0)u_{p-q}^\gamma, \psi^{(i)}\Big)_{L_2(\gamma)} - \Big(\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)u_{p-1,*}^\gamma, \psi^{(j)}\Big)_{L_2(\gamma)} + \\ +\delta_{1p}(f_\gamma, \psi^{(j)})_{L_2(\gamma)} + \lambda\big(u_{p-2}^\gamma, \psi^{(j)}\big)_{L_2(\gamma)} - \Big(\Pi_{\Gamma,M_0}(0)U'_{M_0}\big(u_{p-1,*}^\Gamma, \Psi^{(j)}\big)\Big)_{\mathbb{C}^{d_0}} - \\ -\sum_{M\in\gamma}\Bigg(\Big(P_M g_{M,j}, P_M U'_M(\psi^{(j)})\Big)_{\mathbb{C}^{d(M)}} + 2i\Big(\big(U_M(0) + E_{d(M)}\big)^{-1} P_M^\perp g_{M,j}, P_M^\perp U_M(\psi^{(j)})\Big)_{\mathbb{C}^{d(M)}}\Bigg), \end{gathered}$$

$$\begin{gathered} g_{M,j} := P_M C_M\Bigg(\textstyle\sum_{i=2}^p \Big(A_M^{(i)} U_M(u_{p-i}^\gamma) + B_M^{(i)} U'_M(u_{p-1,*}^\gamma) + A_M^{(1)} U_M(u_{p-1,*}^\gamma) + B_M^{(1)} U'_M(u_{p-1,*}^\gamma)\Big)\Bigg), \\ \vartheta_0(\cdot) := \Pi_{\Gamma,M_0}(0)U'_{M_0}(\cdot) - i\Theta\Gamma, M_0(0)U_{M_0}(\cdot) \end{gathered} \tag{33}$$

The second main result describes an analogue of the Taylor series for the resolvent of the operator $\mathcal{H}_\varepsilon$.

Theorem 3 *Let all the above conditions on the coefficients of the differential expression $\widehat{\mathcal{H}}(\varepsilon)$ and the matrices $A_M(\varepsilon), B_M(\varepsilon)$, be satisfied, and let condition (A) be satisfied. For each* *function* $f \in L_2(\Gamma_\varepsilon)$ *the function* $(\mathcal{H}_\varepsilon - \lambda)^{-1} f$ *can be represented by a series converging in* $W_2^2(\Gamma_\varepsilon)$ *and* $\tau^2(\Gamma_\varepsilon)$

$$(\mathcal{H}_\varepsilon - \lambda)^{-1} f = \sum_{p=0}^{\infty} \varepsilon^p u_p^\Gamma \oplus \mathcal{S}_\varepsilon^{-1} u_p^\gamma, \tag{34}$$

where the functions u_p^Γ *and* u_p^γ *are the coefficients of the series* (26) *with* $f_\Gamma := \mathcal{P}_\Gamma f,\ f_\gamma := \mathcal{S}_\varepsilon \mathcal{P}_{\gamma_\varepsilon} f$.
For an arbitrary $N \in \mathbb{Z}_+$ *the following estimates are valid:*

$$\|\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1} f - \sum_{p=0}^{N} \varepsilon^p u_p^\Gamma \|_{W_2^2(\Gamma)} \le C^{N+1} \varepsilon^{N+1/2} \|f\|_{L_2(\Gamma_\varepsilon)} \tag{35}$$

$$\|\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1} f - \sum_{p=0}^{N} \varepsilon^p u_p^\Gamma \|_{\tau^2(\Gamma)} \le C^{N+1} \varepsilon^{N+1/2} \|f\|_{L_2(\Gamma_\varepsilon)} \tag{36}$$

$$\|\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1} f - \sum_{p=0}^{N} \varepsilon^p \mathcal{S}_\varepsilon u_p^\Gamma \|_{L_2(\gamma_\varepsilon)} \le C^{N+1} \varepsilon^{N+1/2} \|f\|_{L_2(\Gamma_\varepsilon)} \tag{37}$$

$$\|\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1} f - \sum_{p=0}^{N} \varepsilon^p \mathcal{S}_\varepsilon u_p^\Gamma \|_{W_2^i(\gamma_\varepsilon)} \le C^{N+1} \varepsilon^{N+1/2} \|f\|_{L_2(\Gamma_\varepsilon)}\ i = 1, 2, \tag{38}$$

$$\|\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1} f - \sum_{p=0}^{N} \varepsilon^p \mathcal{S}_\varepsilon u_p^\Gamma \|_{\tau(\gamma_\varepsilon)} \le C^{N+1} \varepsilon^{N+1/2} \|f\|_{L_2(\Gamma_\varepsilon)} \tag{39}$$

$$\|\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1} f - \sum_{p=0}^{N} \varepsilon^p \mathcal{S}_\varepsilon u_p^\Gamma \|_{\tau^1(\gamma_\varepsilon)} \le C^{N+1} \varepsilon^{N-1/2} \|f\|_{L_2(\Gamma_\varepsilon)} \tag{40}$$

where C *is some fixed constant independent of* ε, N *and* f.

3 Auxiliary lemmas

To prove the main result, we will need a series of auxiliary lemmas and facts, which are presented in this section.

Lemma 1 *Suppose that condition (A) is satisfied. For an arbitrary family of vectors $g_M \in P_M\mathbb{C}^{d(M)}, g_{M,\perp} \in P_{M,\perp}\mathbb{C}^{d(M)}$, $M \in \gamma_\infty$, an arbitrary vector $g_{ex} \in \mathbb{C}^{d_0}$ and an arbitrary function $g \in L_2(\gamma_{ex})$ the boundary value problem*

$$\begin{gathered}\widehat{\mathcal{H}}_{ex}(0)u = g, \text{ in } \gamma_{ex},\ \Pi_{\Gamma,M_0}(0)U'_{ex}(u) = \mathfrak{g}_{ex},\\ P_M U_M(u) = \mathfrak{g}_M,\ P_M^{\perp}\vartheta_M(u) + K_M^{\perp}U_M(u) = \mathfrak{g}_{M,\perp} \text{ in } M \in \gamma_\infty\end{gathered} \tag{41}$$

is solvable in $W_2^2(\gamma_{ex})$, if and only if for all $j = 1, \ldots, k$ the equality holds

$$\begin{gathered}\big(g, \psi^{(j)}\big)_{L_2(\gamma_{ex})} = -\Big(\mathfrak{g}_{ex}, U_\gamma\big(\psi^{(j)}\big)\Big)_{\mathbb{C}^{d_0}} +\\ \sum_{M\in\gamma_\infty}\Big(\mathfrak{g}_{M,\perp}, U_\gamma\big(\psi^{(j)}\big)\Big)_{\mathbb{C}^{d(M)}} - \sum_{M\in\gamma_\infty}\Big(\mathfrak{g}_M, \vartheta_M\big(\psi^{(j)}\big)\Big)_{\mathbb{C}^{d(M)}}.\end{gathered} \tag{42}$$

Lemma 2 *For an arbitrary family of vectors $\mathfrak{g}_M \in \mathbb{C}^{d(M)}, M \in \Gamma$, an arbitrary function $g \in L_2(\Gamma)$ and each $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the boundary value problem*

$$\begin{gathered}\Big(\widehat{\mathcal{H}}(0) - \lambda\Big)u = g \text{ on } \Gamma,\\ U_{M_0}(u) = \mathfrak{g}_{M_0} \text{ in } M \in \Gamma,\ M \neq M_0\end{gathered}$$

is uniquely solvable in $W_2^2(\Gamma)$.

The following auxiliary lemma guarantees the invertibility of the matrix $Q + L$ from the formulation of Theorem 2 (see (31)).

Lemma 3 *The matrix $Q + L$ is non-singular.*

We multiply the equation in (21) by $\vartheta_{j,\Gamma}$ scalarly in $L_2(\Gamma)$ and integrate by parts twice, taking into account the boundary conditions from (21). Then we obtain that the matrix $L - iIm\lambda G_\Gamma$ is self-adjoint, where G_Γ is the positive definite self-adjoint Gram matrix of the functions $\vartheta_{i,\Gamma}$. Since the matrix Q is also self-adjoint, it follows from here that for all $c \in \mathbb{C}^{d_0}$ we have

$$Im\Big(\big(Q + L\big)c, c\Big)_{\mathbb{C}^k} = -Im\lambda\big(G_\Gamma c, c\big)_{\mathbb{C}^k} \neq 0$$

which immediately implies the non-degeneracy of the matrix Q and completes the proof of the lemma.

4 Taylor series for parts of the resolvent

In this section we prove Theorem 2. According to Theorem 1, the operators $\mathcal{R}_\Gamma$ and $\mathcal{R}_\gamma$ are analytic in ε as operators from $L_2(\Gamma)\oplus L_2(\gamma)$ in $W_2^2(\Gamma)$ and $W_2^2(\gamma)$ and in $\tau^1(\Gamma)$ and $\tau^1(\gamma)$. This means that for an arbitrary pair $(f_\Gamma, f_\gamma) \in L_2(\Gamma) \oplus L_2(\gamma)$ the functions $u_\varepsilon^\Gamma := \mathcal{R}_\Gamma(\varepsilon,\gamma)(f_\Gamma, f_\gamma)$ and $u_\gamma^\varepsilon := \mathcal{R}_\gamma(\varepsilon,\gamma)(f_\Gamma, f_\gamma)$ are represented by series (26) converging uniformly in ε in the norms $\tau^1(\Gamma)$ and $\tau^1(\gamma)$. It also follows from formula (11) that in the sense of decompositions (10) the equality

$$\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1} f =: u_\varepsilon = u_\varepsilon^\Gamma \oplus \mathcal{S}_\varepsilon u_\varepsilon^\gamma,\ f := f_\Gamma \oplus \mathcal{S}_\varepsilon f_\gamma \tag{43}$$

From this and from the equation for the resolvent $\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1}$ it follows that the function u_ε is a solution of the differential equation $\big(\widehat{\mathcal{H}}_\varepsilon - \lambda\big)^{-1} u_\varepsilon^\Gamma = f_\Gamma$ on the graph Γ and satisfies the boundary conditions (4) at the vertices $M \in \Gamma$, $M \neq M_0$. We substitute the series for $\mathcal{R}_\Gamma$ from (26) into this equation and the boundary conditions, expand all the coefficients in Taylor series in ε and collect the coefficients at the same powers of ε. Then we obtain a recurrent system of boundary value problems (22), (23) for the functions u_p^Γ.

Our next step is to obtain similar boundary value problems for the functions u_p^γ. To do this, we first extend these functions from the graph γ to the graph γ_{ex} according to the following rule:

$$u_p^\gamma(\xi_i) := \frac{du_{p-1}^\Gamma|_{e_i}}{dx_i}(M_0)\vartheta_i(M_0)\xi_i + u_p^\Gamma|_{e_i}(M_0),\ i \in J_j,\ j = 1, ..., n. \tag{44}$$

Due to such continuation, equality (43) and continuous differentiability of the function u_ε, we immediately conclude that the following equalities must be satisfied:

$$U_{M_0}\big(u_p^\Gamma\big) = U_\gamma\big(u_p^\gamma\big),\ p \geq 1, \tag{45}$$

$$U'_{\gamma_{ex}}\big(u_0^\gamma\big) = 0,\ U'_{\gamma_{ex}}\big(u_p^\gamma\big) = U'_{M_0}\big(u_{p-1}^\Gamma\big)\ \ p \geq 1, \tag{46}$$

These relations are continuity conditions connecting the restrictions of the function u_ε to the subgraphs Γ and γ on the edges e_i; here we should keep in mind the replacement $\xi = x_i\varepsilon^{-1}$, connecting the variables on the edges e_i^{ex} and e_i. Further, we also consider these continuity conditions as boundary conditions for the functions u_p^Γ and u_p^γ.

From formulas (43), the equation for the resolvent $\big(\mathcal{H}_\varepsilon - \lambda\big)^{-1}$, the definition of the operator $\mathcal{H}_{ex}(\varepsilon)$ in Section 3 and the continuation formulas (43) it follows that the function u_ε^γ is a solution of the differential equation $\big(\mathcal{H}_{ex}(\varepsilon) - \varepsilon^2\lambda\big)u_\gamma^\varepsilon = \varepsilon^2 f$ on γ with boundary conditions (9). We substitute into this problem the series for $\mathcal{R}$ from (26), expand all the coefficients into Taylor series in ε and collect the coefficients at the same powers of ε. Then, taking into account the continuation formulas (44) and the definition (8) of the differential expression $\widehat{\mathcal{H}}_{ex}(0)$ on the edges e_i^{ex}, we obtain a recurrent system of boundary equations (24) for the functions u_p^γ with the boundary conditions

$$\begin{gathered} A_M^{(0)} U_M\big(u_p^\gamma\big) + B_M^{(0)} U'_M\big(u_p^\gamma\big) = -g_p^\gamma, \\ g_p^\gamma := \sum_{i=1}^{p} \Big(A_M^{(i)} U_M\big(u_{p-i}^\gamma\big) + B_M^{(0)} U'_M\big(u_{p-i}^\gamma\big)\Big),\ M \in \gamma_\infty. \end{gathered} \tag{47}$$

We now investigate the solvability of problems (22), (23), (45) and (24), (25), (46). The function u_0^Γ is already defined in (26). Since problem (24), (25), (46) for u_0^γ is homogeneous, its solution is a linear combination of functions $\psi^{(i)} : u_0^\gamma = \sum_{i=1}^{k} c_{i,0}\psi^{(i)}$. Due to the boundary conditions for u_0^Γ at the vertex M_0 and the definition of the vectors $\Psi^{(i)}$, $i \leq k+1$ we obviously have $\Big(U_{M_0}\big(u_0^\Gamma\big), \Psi^{(i)}\Big)_{\mathbb{C}^{d_0}} = 0$ for $j \geq k+1$. Consequently,

$$c_{0,i} = c_i(f_\Gamma),\ U_{M_0}\big(u_0^\Gamma\big) = \sum_{i=1}^{k} c_i(f_\Gamma)\Psi^{(i)},$$

where the functionals $c_i(f)$ were defined in (20). This leads to the formula for u_0^γ from (26).

Let us now consider problem (24), (25), (46) for u_1^γ. The right-hand side $U'_{M_0}(u_0^\Gamma)$ in the boundary condition (46) with $p = 1$ is a known quantity, since the function u_0^Γ is already completely defined. This problem is a special case of problem (41) with

$$f = -\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)u_0^\gamma,\ \mathfrak{g}_{M^{ex}} = \Pi_{\Gamma,M_0}(0)U'_{M_0}(u_0^\Gamma)$$
$$\mathfrak{g}_M = \tfrac{i}{2}P_M\varepsilon_M(u_0^\gamma),\ \mathfrak{g}_{M,\perp} = \Big(U_M + E_{d(M)}\Big)^{-1}P_M^\perp\varepsilon_M(u_0^\gamma),$$

where the operator EM is defined in (18). The solvability condition for this problem is given by equality (42), which in this case takes the form

$$0 = -\Big(\Pi_{\Gamma,M_0}(0)U'_{M_0}(u_0^\Gamma), U_\gamma(\psi^{(j)})\Big)_{\mathbb{C}^{d_0}} + \Big(\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)u_0^\gamma, (\psi^{(j)})\Big)_{L_2(\gamma)} -$$
$$- \sum_{M\in\gamma_\infty}\tfrac{i}{2}\Big(P_M\varepsilon_M(u_0^\gamma), \vartheta_M(\psi^{(j)})\Big)_{\mathbb{C}^{d(M)}} + \sum_{M\in\gamma_\infty}\Big(\Big(U_M + E_{d(M)}\Big)^{-1}P_M^\perp\varepsilon_M(u_0^\gamma), U_M(\psi^{(j)})\Big)_{\mathbb{C}^{d(M)}} \tag{48}$$

Using the obvious equalities

$$\frac{d\psi^{(i)}}{d\xi} = 0,\ \text{on } \gamma\setminus\gamma_{ex},\ \frac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)\psi^{(j)} = \begin{cases}\frac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)\chi_\gamma(\psi^{(j)}) & \text{on, } \gamma\\ 0 & \text{on } \gamma\setminus\gamma_{ex}\end{cases}$$

and definition (15) quantities $Q_\gamma^{(ij)}$, by integrating by parts we verify that

$$\begin{gathered}\Big(\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)\chi_\gamma(\psi^{(j)})\Big)_{L_2(\gamma)} = \Big(\tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0)\chi_\gamma(\psi^{(j)})\Big)_{L_2(\gamma)_{ex}} = \\ = Q_\gamma^{(ij)} + \sum_{M\in\gamma_\infty}\Big(\tfrac{d\Pi_{\gamma,M}}{d\varepsilon}(0)U'_M(\psi^{(i)}) - i\tfrac{d\Theta_{\gamma,M}}{d\varepsilon}(0)U_M(\psi^{(i)}), U_M(\psi^{(i)})\Big)_{\mathbb{C}^{d(M)}}.\end{gathered} \tag{49}$$

Taking into account the last relation and the definition of the function u_0^γ in (26), it is easy to see that the solvability condition (48) of the problem for u_1^γ is equivalent to the boundary condition (12) at the vertex M_0 with matrices (13). Since this condition is satisfied, the problem for u_1^γ is solvable and its general solution has the form

$$u_1^\gamma = u_{1,*}^\gamma + \sum_{i=1}^k c_{i,1}\psi^{(i)}, \tag{50}$$

where $u_{1,*}^\gamma$ is a particular solution of problem (24), (25), (46) satisfying the orthogonality condition (30), and $c_{i,1}$ are some constants that will be found later.

Having found the function u_1^γ, we can already determine the function u_1^Γ. It is found as a solution to problem (22), (23), (45). According to Lemma 2, problem (24), (47), (44) for u_1^Γ is uniquely solvable and its solution has the form

$$u_1^\gamma = u_{1,*}^\gamma + \sum_{i=1}^k c_{i,1}\vartheta_{i,\Gamma}, \tag{51}$$

where $u^{\Gamma}_{1,*}$ is the solution of problem (22), (23) with the boundary condition $U_{M_0}\big(u^{\Gamma}_{1,*}\big) = U_{\gamma}\big(u^{\gamma}_{1,*}\big)$.

Let us now study problem (24), (25), (46). We substitute formula (50) into the right-hand sides of equalities (24), (25), and we substitute equality (51) into the right-hand side of (46). The resulting problem is a special case of problem (41), the solvability of which is determined by condition (42). Writing out this condition for this problem and taking into account relations (49) and (14)-(17), we obtain

$$\sum_{i=1}^{k} Q^{(ij)} c_{i,1} + \sum_{i=1}^{k} c_{i,1} \Big(\vartheta_0\big(\vartheta_{i,\Gamma}\big), \Psi^{(j)}\Big)_{\mathbb{C}^{d_0}} = h_{j,1}, \tag{52}$$

where the numbers $h_{j,1}$ are defined by the equalities

$$\begin{aligned} h_{j,1} := \left(f_{\gamma} - \tfrac{1}{2} \tfrac{d^2\widehat{\mathcal{H}}_{ex}}{d\varepsilon^2}(0)\chi_{\gamma} u^{\gamma}_0 + \lambda u^{\gamma}_0 - \tfrac{d\widehat{\mathcal{H}}_{ex}}{d\varepsilon}(0) u^{\gamma}_{1,*}, \psi^{(i)} \right)_{L_2(\gamma)} - \\ \Big(\Pi_{\Gamma,M_0}(0) u'_{M_0}\big(u^{\Gamma}_{1,*}\big), \Psi^{(j)}\Big)_{\mathbb{C}^{d_0}} - \\ -2i \sum_{M\in\gamma_\infty} \bigg(\Big(U_M(0) + E_{d(M)}\Big)^{-1} P^{\perp}_M C_M \Big(A^{(1)}_M U_M\big(u^{\gamma}_{1,*}\big) + B^{(1)}_M U'_M\big(u^{\gamma}_{1,*}\big) + \\ + A^{(2)}_M U_M\big(u^{\gamma}_0\big) + B^{(2)}_M U'_M\big(u^{\gamma}_0\big), P^{\perp}_M U_M\big(\psi^{(j)}\big)\Big)\bigg)_{\mathbb{C}^{d(M)}} - \\ - \textstyle\sum_{M\in\gamma_\infty} \bigg(P_M C_M \Big(A^{(1)}_M U_M\big(u^{\gamma}_{1,*}\big) + B^{(1)}_M U'_M\big(u^{\gamma}_{1,*}\big) + \\ + A^{(2)}_M U_M\big(u^{\gamma}_0\big) + B^{(2)}_M U'_M\big(u^{\gamma}_0\big), P_M \vartheta_M\big(\psi^{(j)}\big)\Big)\bigg)_{\mathbb{C}^{d(M)}}. \end{aligned}$$

In matrix form, equalities (52) are rewritten to the equation $(Q+L)c_1 = h_1$, and Lemma 3 allows us to uniquely solve this equation by finding the coefficients $c_{i,1}$. This leads to formulas (31), (32) with $p=1$.

The remaining functions, u^{Γ}_p and u^{γ}_p, are defined in a similar way. Namely, the function u^{γ}_p is defined up to a linear combination of the functions $\psi^{(j)}$ with some coefficients $c_{i,p}$ by the formula from (27). Then problem (22), (23) for the function u^{Γ}_p is uniquely solvable and its solution has the form (27). Now we can solve problem (24), (46), (47) for u^{γ}_{p+1},since all the right-hand sides in this problem are expressed through the functions already found. The solvability condition for the last problem is given by equality (42) and leads to a system of linear equations $(Q+L)c_p = h_p$, where the vector h_p is from (32) with coefficients from (33). This system is uniquely solved thanks to Lemma 3 and the solution is given by formula (31). The described procedure allows us to determine all the coefficients of the Taylor series (26).

Theorem 2 is completely proved.

5 Analogue of the Taylor series for the resolvent

This section is devoted to the proof of Theorem 3. Equality (34) follows immediately from (26) and (11); it suffices to substitute the series (26) into formula (11).

Since, by Theorem 1, the operators $\mathcal{R}_\Gamma(\varepsilon,\lambda)$ and $\mathcal{R}_\Gamma(\varepsilon,\lambda)$ are analytic in ε and, according to Theorem 2, their Taylor series are given by equalities (26), the following inequalities hold:

$$\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|^2_{W_2^2(\Gamma)}\leq C^{2N+2}\varepsilon^{2N+2}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma)}\Big), \tag{53}$$

$$\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|^2_{W_2^2(\gamma)}\leq C^{2N+2}\varepsilon^{2N+2}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma)}\Big), \tag{54}$$

$$\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|^2_{\tau^2(\Gamma)}\leq C^{N+1}\varepsilon^{N+1}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma)}\Big)^{\frac{1}{2}}, \tag{55}$$

$$\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\|^2_{\tau^2(\gamma)}\leq C^{N+1}\varepsilon^{N+1}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma)}\Big)^{\frac{1}{2}}, \tag{56}$$

where C is a fixed constant independent of ε, N, f_Γ and f_γ. From definition (2) of the operator $\mathcal{S}_\varepsilon$ it follows that

$$\Big\|\frac{d^i\mathcal{S}_\varepsilon u}{dx^i}\Big\|^2_{L_2(\gamma_\varepsilon)}=\varepsilon^{1-2i}\Big\|\frac{d^i u}{dx^i}\Big\|^2_{L_2(\gamma)},\ \Big\|\frac{d^i\mathcal{S}_\varepsilon u}{dx^i}\Big\|^2_{\tau(\gamma_\varepsilon)}=\varepsilon^{-i}\Big\|\frac{d^i u}{dx^i}\Big\|_{\tau(\gamma)}, \tag{57}$$

and, in particular,

$$\Big\|f_\gamma\Big\|^2_{L_2(\gamma)}=\varepsilon^{-1}\Big\|f_\gamma\Big\|^2_{L_2(\gamma_\varepsilon)} \tag{58}$$

We also note that from formula (11) follows the equality

$$\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\oplus\mathcal{S}_\varepsilon u_p^\gamma=\Big(\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\Big)\oplus\mathcal{S}_\varepsilon\Big(\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\gamma\Big)\Big) \tag{59}$$

Then from (53), (56) we have

$$\Big\|\big(\mathcal{H}_\varepsilon-\lambda\big)^{-1}f-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\Big\|^2_{W_2^2(\Gamma)}=\Big\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\Big\|^2_{W_2^2(\Gamma)}\leq$$
$$\leq C^{2N+2}\varepsilon^{2N+2}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\|f_\Gamma\|^2_{L_2(\gamma_\varepsilon)}\Big)$$

and from here it follows (35). Similarly, using (55) instead of (53), it is easy to prove estimate (36).

As above, applying inequality (54), the first formula in (57) and equalities (59), (**??**) we obtain estimates (37), (38):

$$\left\|(\mathcal{H}_\varepsilon-\lambda)^{-1}f-\sum_{p=0}^{N}\varepsilon^p u_p^\Gamma\right\|^2_{W_2^i(\gamma_\varepsilon)}=\left\|\mathcal{S}_\varepsilon\left(\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\gamma\right)\right\|^2_{W_2^i(\gamma_\varepsilon)}\leq$$
$$\leq\varepsilon^{1-2i}\left\|\mathcal{R}_\Gamma(\varepsilon,\lambda)(f_\Gamma,f_\gamma)-\sum_{p=0}^{N}\varepsilon^p u_p^\gamma\right\|^2_{W_2^i(\gamma_\varepsilon)}\leq C^{2N+2}\varepsilon^{2N+2}\Big(\|f_\Gamma\|^2_{L_2(\Gamma)}+\varepsilon^{-1}\|f_\Gamma\|^2_{L_2(\gamma_\varepsilon)}\Big),$$

where $i=0,1,2$, and for $i=0$ for convenience we set $W_2^i:=L_2$. Reasoning as above and using inequality (56) instead of (54) and the second equality in (57) instead of the first, we easily prove inequalities (39), (40). Theorem 3 is completely proven.

6 Discussion

The results obtained show that even in the case of graphs with complex structure and arbitrary boundary conditions, the behavior of the resolvent of an elliptic operator can be accurately described using analytical methods. This is important for further study of the spectral properties of operators on quantum graphs, as well as for applications in physics, where such structures model real systems with small scales. The work develops and refines previously known approaches, offering a more general and flexible analysis scheme without strict restrictions on the graph structure or the type of coefficients. The results are consistent with previous studies in the field of asymptotic analysis, but take a step forward due to the rigorous description of the remainder terms. Prospects include extending the methods to nonlinear operators, systems with variable scales, and numerical implementations for specific applications.

7 Conclusion

In this paper, we study the behavior of an elliptic self-adjoint operator of the second order on a graph with small edges, depending on a small parameter ε. It is established that the resolvent of such an operator is analytic in ε and can be represented as a convergent series with the possibility of exact calculation of all coefficients. An effective method, similar to the matching of asymptotic expansions, is developed for constructing this series and estimates of the remainder terms are obtained. The results confirm the possibility of a rigorous description of the spectral properties of the operator as $\varepsilon\to 0$ and open the way to further studies of operators on graphs of complex structure.

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IMPROVEMENT IN VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS BY ADOMIAN DECOMPOSITION METHOD

The Adomian Decomposition Method (ADM) is widely recognized as a powerful and versatile semi-analytical tool designed to solve a broad range of problems, including linear and nonlinear differential equations, as well as integral equations. This method has been extensively applied across various scientific and engineering disciplines due to its simplicity and efficiency in generating accurate approximate solutions. In this note, we introduce an enhanced and refined scheme based on the ADM framework to obtain approximate solutions for Volterra-Fredholm integro-differential equations (IDEs) with specified initial conditions. Our proposed scheme not only simplifies the computational process but also ensures improved accuracy and convergence. Additionally, we rigorously prove the uniqueness of the solutions to the Volterra-Fredholm IDEs by leveraging the mathematical foundation of Banach's Fixed Point Theorem, providing theoretical validity to our approach. To validate the effectiveness of the enhanced scheme, we apply it to a diverse set of linear and nonlinear Volterra-Fredholm IDEs with initial conditions. The numerical results obtained are systematically compared with those from existing methods reported in the literature. Our findings reveal that the proposed approach demonstrates remarkable accuracy, efficiency, and reliability in solving complex IDEs. Consequently, this method represents a significant advancement in the field of integro-differential equations.

Key words: Adomian decomposition method, Volterra-Fredholm Integrodifferential equation, Approximate solution, Uniqueness solution, Adomian poliynomials, Banach's fixed point theorem.

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Адомианның жіктеу әдісі арқылы Вольтерра–Фредгольм интегро-дифференциалдық теңдеулерін жетілдіру

Адомиандық декомпозиция әдісі (ADM) желілік және сызықтық емес дифференциалдық теңдеулерді, сондай-ақ интегралдық теңдеулерді қоса алғанда, есептердің кең ауқымын шешуге арналған қуатты және әмбебап жартылай аналитикалық құрал ретінде кеңінен танылды. Бұл әдіс нақты жуық шешімдерді шығарудағы қарапайымдылығы мен тиімділігіне байланысты әртүрлі ғылыми және инженерлік пәндерде кеңінен қолданылады. Бұл жазбада біз белгіленген бастапқы шарттармен Вольтерра-Фредгольм интегро-дифференциалдық теңдеулеріне (IDE) жуық шешімдерді алуға арналған ADM негізіне негізделген жетілдірілген және нақтыланған схеманы ұсынамыз. Біздің ұсынылған схема есептеу процесін жеңілдетіп қана қоймайды, сонымен қатар жоғарылатылған дәлдік пен конвергенцияны қамтамасыз етеді.

Сонымен қатар, біз Вольтерра-Фредхольм IDE шешімдерінің бірегейлігін Банах бекітілген нүкте теоремасының математикалық негізін пайдалана отырып, біздің көзқарасымызды теориялық негіздей отырып, қатаң дәлелдейміз. Жақсартылған схеманың тиімділігін тексеру үшін біз оны әр түрлі сызықтық және сызықтық емес Вольтерра-Фредхольм бастапқы мән теңдеулерінің жиынтығына қолданамыз. Алынған сандық нәтижелер әдебиетте сипатталған бар әдістердің нәтижелерімен жүйелі түрде салыстырылады. Нәтижелеріміз ұсынылған тәсіл күрделі ӨЖБ шешуде керемет дәлдік, тиімділік және беріктік көрсететінін көрсетеді. Демек, бұл әдіс интегро-дифференциалдық теңдеулер саласындағы айтарлықтай ілгерілеушілікті білдіреді.
Түйін сөздер: Адомиандық кеңейту әдісі, Вольтерра-Фредгольм интегро-дифференциалдық теңдеуі, жуық шешім, бірегей шешім, Адомиян полиномдары, Банахтың қозғалмайтын нүкте теоремасы

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Улучшение интегро-дифференциальных уравнений Вольтерра–Фредгольма методом разложения Адомиана

Метод разложения Адомиана (ADM) широко признан как мощный и универсальный полу-аналитический инструмент, предназначенный для решения широкого круга задач, включая линейные и нелинейные дифференциальные уравнения, а также интегральные уравнения. Этот метод широко применяется в различных научных и инженерных дисциплинах благодаря своей простоте и эффективности в создании точных приближенных решений. В этой заметке мы представляем усовершенствованную и улучшенную схему, основанную на структуре ADM, для получения приближенных решений для интегро-дифференциальных уравнений (ИДУ) Вольтерры-Фредгольма с указанными начальными условиями. Наша предлагаемая схема не только упрощает вычислительный процесс, но и обеспечивает повышенную точность и сходимость. Кроме того, мы строго доказываем уникальность решений ИДУ Вольтерры-Фредгольма, используя математическую основу теоремы Банаха о неподвижной точке, обеспечивая теоретическую обоснованность нашего подхода. Чтобы подтвердить эффективность усовершенствованной схемы, мы применяем ее к разнообразному набору линейных и нелинейных ИДУ Вольтерры-Фредгольма с начальными условиями. Полученные численные результаты систематически сравниваются с результатами существующих методов, описанных в литературе. Наши результаты показывают, что предложенный подход демонстрирует замечательную точность, эффективность и надежность при решении сложных ИДУ. Следовательно, этот метод представляет собой значительный прогресс в области интегро-дифференциальных уравнений.
Ключевые слова: Метод разложения Адомиана, Интегродифференциальное уравнение Вольтерра-Фредгольма, Приближенное решение, единственное решение, многочлены Адомиана, теорема Банаха о неподвижной точке

1 Introduction

In this paper, we consider a class of Volterra-Fredholm integro-differential equations of the type:

$$\sum_{j=0}^{m} \delta_j (x) u^{(j)} (x) = f (x) + \lambda_1 \int_a^x K_1 (x,t) F_1 (u (t)) dt + \lambda_2 \int_a^b K_2 (x,t) F_2 (u (t)) dt, \quad (1)$$

with the initial conditions

$$u^{(j)}(a) = b_j, \quad j = 0, 1, 2, \ldots (m-1) \quad (2)$$

Where $u^j(x)$ is the j^{th} derivative of the unknown function $u(x)$ that will be determined $K_r(x,t)$, $r = 1, 2$ are the kernels of the equation (1), $f(x)$ and $\delta_j(x)$ are analytic functions, $F_1(u(t))$ and $F_2(u(t))$ are nonlinear continuous functions and $a, b, \lambda_1, \lambda_2, b_j$ are constants.

Linear and nonlinear IDEs are the class of mathematical problems appearing in many engineering and science areas. These types of equations are challenging to solve analytically. Therefore, numerical methods are required to obtain approximate solutions [1–12]. The Adomian Decomposition Method (ADM) is one of the powerful tools to obtain semi-analytical solutions of operator equations and has been widely used in recent years due to its simplicity and accuracy in solving linear and nonlinear differential equations, integral equations and IDEs with initial and boundary conditions [13–28].

Besides that we have achieved very good results in this direction in our previous articles [29–31].

Our main aim is to obtain numerical solution of Eq. (1)-(2) by standard ADM and modified ADM and analysis the behaviour of error terms for large number of iterations. The proposed scheme is tested on various linear and nonlinear IDEs and compared with other existing methods, to demonstrate its high accuracy and efficiency. This paper provides a valuable tool for researchers in engineering and science, enabling them to obtain high accurate approximation solutions of Volterra-Fredholm IDEs with initial conditions using the standard and modified ADM.

2 Methodology

2.1 Application of adomian decomposition method to nonlinear volterra-fredholm integro-differential equations

Now, we can rewrite Eq. (1) in the form

$$u^{(m)}(x) = \frac{f(x)}{\delta_m(x)} + \lambda_1 \int_0^x \frac{K_1(x,t)}{\delta_m(x)} F_1(u(t))dt + \lambda_2 \int_a^b \frac{K_2(x,t)}{\delta_m(x)} F_2(u(t))dt - \sum_{j=0}^{m-1} \frac{\delta_j(x)}{\delta_m(x)} u^{(j)}(x), \quad (3)$$

Since $Lu = \frac{d^m u}{dx^m}$ is the differential operator of order m., to reduce IDEs (1) into integral equations (IEs), we integrate both sides of equation (3) m-times in the interval $[a, x]$ with respect to x to obtain

$$u(x)=\sum_{r=0}^{m-1}\tfrac{1}{r!}(x-a)^r b_r-\sum_{j=0}^{m-1}L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)}u^{(j)}(x)\right)+L^{-1}\left(\tfrac{f(x)}{\delta_m(x)}\right)$$
$$+\lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)}F_1(u(t))dt\right)+\lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)}F_2(u(t))dt\right) \tag{4}$$

where L^{-1} is the inverse operator of L and can be computed by Leibniz rule

$$L^{-1}(F)=\int_a^x\int_a^{x_1}\dots\int_a^{x_{m-1}}(F)dx_m dx_{m-1}\dots dx_1=\frac{1}{(m-1)!}\int_0^x(x-t)^{m-1}(F)dt. \tag{5}$$

For nonlinear terms in (4), we apply Adomian polynomial series

$$M_1(u(x))=\sum_{n=0}^{\infty}A_n, \quad M_2(u(x))=\sum_{n=0}^{\infty}B_n. \tag{6}$$

where A_n, B_n; $n\geq 0$ are the Adomian polynomials determined formally as follows:

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\gamma^n}F_1\left(\sum_{i=0}^{\infty}\gamma^i u_i\right)\right]\Bigg|_{\gamma=0}, \qquad B_n=\frac{1}{n!}\left[\frac{d^n}{d\gamma^n}F_2\left(\sum_{i=0}^{\infty}\gamma^i u_i\right)\right]\Bigg|_{\gamma=0} \tag{7}$$

The Adomian polynomials were introduced in [1-3] as:

$$\begin{gathered}
A_0=F_1(u_0),\\
A_1=u_1F_1{}'(u_0),\\
A_2=u_2F_1{}'(u_0)+\tfrac{1}{2!}u_1{}^2F_1{}''(u_0),\\
A_3=u_3F_1{}'(u_0)+u_1u_2F_1{}''(u_0)+\tfrac{1}{3!}u_1{}^2F_1{}'''(u_0),\\
A_4=u_4F_1{}'(u_0)+u_1u_3F_1{}''(u_0)+\tfrac{1}{2!}u_2{}^2F_1{}''(u_0)+\tfrac{1}{2!}u_1{}^2u_2F_1{}'''(u_0)+\tfrac{1}{4!}u_1{}^{(IV)}F_1{}^{(IV)}(u_0),\\
\dots\dots\dots\dots
\end{gathered} \tag{8}$$

and

$$\begin{gathered}
B_0=F_2(u_0),\\
B_1=u_1F_2{}'(u_0),\\
B_2=u_2F_2{}'(u_0)+\tfrac{1}{2!}u_1{}^2F_2{}''(u_0),\\
B_3=u_3F_2{}'(u_0)+u_1u_2F_2{}''(u_0)+\tfrac{1}{3!}u_1{}^2F_2{}'''(u_0),\\
B_4=u_4F_2{}'(u_0)+u_1u_3F_2{}''(u_0)+\tfrac{1}{2!}u_2{}^2F_2{}''(u_0)+\tfrac{1}{2!}u_1{}^2u_2F_2{}'''(u_0)+\tfrac{1}{4!}u_1{}^{(IV)}F_2{}^{(IV)}(u_0),\\
\dots\dots\dots\dots
\end{gathered} \tag{9}$$

The standard decomposition technique represents the solution of u as the following series:

$$u=\sum_{i=0}^{\infty}u_i \tag{10}$$

By substituting (6) and (10) into (4), we obtain

$$\begin{aligned}\sum_{i=0}^{\infty} u_i(x) = \sum_{k=0}^{m-1} \tfrac{1}{r!}(x-a)^r b_r - \sum_{i=0}^{\infty}\sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_i}^{(j)}(x)\right) + L^{-1}\left(\tfrac{f(x)}{\delta_m(x)}\right)\\ +\lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_i(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_i(t)dt\right)\end{aligned} \tag{11}$$

The components $u_0,\ u_1,\ u_2,\ldots$ are usually determined recursively by

$$\begin{aligned} u_0(x) &= L^{-1}\left(\tfrac{f(x)}{\delta_m(x)}\right) + \sum_{k=0}^{m-1} \tfrac{1}{r!}(x-a)^r b_r,\\ u_1(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_0(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_0(t)dt\right)\\ &\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_0}^{(j)}(x)\right),\\ u_2(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_1(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_1(t)dt\right)\\ &\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_1}^{(j)}(x)\right),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,\\ u_n(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_{n-1}(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_{n-1}(t)dt\right)\\ &\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_{n-1}}^{(j)}(x)\right). \end{aligned} \tag{12}$$

For the modified ADM, we decompose $f(x) = f_1(x) + f_2(x)$ in Eq. (11) and do search solution as (10) and performing operations similarly as (12) we arrive at

$$\begin{aligned} u_0(x) &= L^{-1}\left(\tfrac{f_1(x)}{\delta_m(x)}\right) + \sum_{k=0}^{m-1} \tfrac{1}{r!}(x-a)^r b_r,\\ u_1(x) &= L^{-1}\left(\tfrac{f_2(x)}{\delta_m(x)}\right) + \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_0(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_0(t)dt\right)\\ &\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_0}^{(j)}(x)\right),\\ u_2(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_1(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_1(t)dt\right)\\ &\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_1}^{(j)}(x)\right),\\ &\cdots\cdots\cdots\cdots\cdots\cdots ,\\ u_n(x) &= \lambda_1 L^{-1}\left(\int_0^x \tfrac{K_1(x,t)}{\delta_m(x)} A_{n-1}(t)dt\right) + \lambda_2 L^{-1}\left(\int_a^b \tfrac{K_2(x,t)}{\delta_m(x)} B_{n-1}(t)dt\right)\\ &\quad - \sum_{j=0}^{m-1} L^{-1}\left(\tfrac{\delta_j(x)}{\delta_m(x)} {u_{n-1}}^{(j)}(x)\right),\quad n \geq 1. \end{aligned} \tag{13}$$

Then, $u(x) = \sum_{i=0}^{n} u_i(x)$ will be taken as the approximate solution. Eqs (12) and (13) are called standard and modified ADM respectively.

2.2 Uniqueness solution of ides

In this section, we show that under which conditions the solution of the Eq. (1)-(2) is unique. In Eshkuvatov [30] proved the following lemma and theorem regarding to the problem (1)-(2). Before starting and proving the main results, we introduce the following hypotheses:

H_1: There exist two constants $L_1 > 0,\ L_2 > 0$ such that for any $u_1,\, u_2 \in C(J,R),\quad J = [a,b]$

$$\begin{gathered} \left|F_1\left(u_1\left(t\right)\right) - F_1\left(u_2\left(t\right)\right)\right| \le L_1\left|u_1 - u_2\right|, \\ \left|F_2\left(u_1\left(t\right)\right) - F_2\left(u_2\left(t\right)\right)\right| \le L_2\left|u_1 - u_2\right|, \\ \left|D^j\left(u_1\left(t\right)\right) - D^j\left(u_2\left(t\right)\right)\right| \le \gamma_j\left|u_1 - u_2\right|,\ \ j = \{0,1,\ldots,m-1\}, \end{gathered}$$

where D^j is a differential operator and γ_j are Lipschitz constants.

H_2: There exist two functions K_1^*, K_2^* for the kernel functions $K_1, K_2 \in C\left(D,R\right)$ the set of all positive functions continuous on $D = \{(t,x) \in R : a \le x \le t \le b\}$ such that

$$K_1^* = \sup_{t\in[a,b]} \int_a^t |K_1(t,x)dx| < \infty,\ \ K_2^* = \sup_{t\in[a,b]} \int_a^b |K_2(t,x)dx| < \infty.$$

H_3: The functions $\delta_j\left(t\right),\ j = \{1,2,\ldots,m-1\}$ and $f(t)$ mapping $J \to R$ are continuous functions.

Theorem 1 ***(Banach's Fixed Point Theorem)****. Let (X,d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x \in X$ such that $T(x) = x$. The following lemma is proven in Eshkuvatov [30].*

Lemma 1 *Let $\varphi(t) \in C(J,R^+)$ then $u(t) \in C(J,R^+)$ is a solution of the problem (1)-(2) if u is satisfying.*

$$\begin{cases} u(t) = \varphi(t) - \sum\limits_{j=1}^{m-1} \frac{1}{(m-1)!} \int\limits_a^t (t-x)^{m-1} \left(\delta_j(x) D^j\left(u(x)\right)\right) dx \\ \qquad + \frac{1}{(m-1)!} \int\limits_a^t (t-x)^{m-1} \lambda_1 \left[\int\limits_0^x K_1(x,r) F_1(u(r)) dr\right] dx \\ \qquad + \frac{1}{(m-1)!} \int\limits_a^b (t-x)^{m-1} \lambda_2 \left[\int\limits_0^x K_2(x,r) F_2(u(r)) dr\right] dx, \end{cases} \tag{14}$$

for $t \in J = [a,b]$ and

$$\varphi(t) = \sum_{j=1}^{m-1} \frac{\delta_k}{k!}(t-a)^k + \frac{1}{(m-1)!} \int_a^t (t-x)^{m-1} f\left(t\right) dx \tag{15}$$

Let us prove the following theorems based on the Lemma 1.

Theorem 2 *Assume that the hypothesizes H_1, H_2 and H_3 hold. If*

$$\varepsilon^* = \left[\frac{\gamma^*\delta^*}{m!} + \frac{\lambda_1 K_1^*(L_1)}{m!} + \frac{\lambda_2 K_2^*(L_2)}{m!}\right](b-a)^m < 1, \tag{16}$$

where $\gamma^* = \max_{1\le j\le m-1}\gamma_j$ and $\delta^* = \max_{1\le j\le m-1}|\delta_j(t)|$ then there exists a unique solution $u(x) \in C(J)$ for Eq. (1)-(2).

Proof of Theorem 2. Let the operator $T : C(J,R) \to C(J,R)$ be defined by

$$\begin{cases} (Tu)(t) = \varphi(t) - \sum\limits_{j=1}^{m-1} \frac{1}{(m-1)!}\int\limits_a^t (t-x)^{m-1}(\delta_j(x)D^j(u(x)))dx \\ \qquad +\frac{1}{(m-1)!}\int\limits_a^t (t-x)^{m-1}\left[\lambda_1\int\limits_0^x K_1(x,r)F_1(u(r))\,dr\right]dx \\ \qquad +\frac{1}{(m-1)!}\int\limits_a^b (t-x)^{m-1}\left[\lambda_2\int\limits_0^x K_2(x,r)F_2(u(r))\,dr\right]dx, \end{cases}$$

where $\varphi(t)$ is defined by (15).

It is known by Lemma 1, that a function u is a solution to (1)-(2) if u satisfies Eq.(14). Now we prove that T has a fixed point u in $C(J,R)$ under condition (16). To do this end, let $u_1, u_2 \in C(J,R)$ then for any $t \in [a,b]$.

$$\begin{aligned} \|(Tu_1)(t) - (Tu_2)(t)\| &\le \sum_{j=1}^{m-1}\frac{1}{(m-1)!}\int\limits_a^t (t-x)^{m-1}\|\delta_j(x)\|\,\|D^ju_1(x) - D^ju_2(x)\|dx \\ &+\frac{1}{(m-1)!}\int\limits_a^t (t-x)^{m-1}\left[\lambda_1\int\limits_0^x \|K_1(x,r)\|\,\|F_1(u_1(r)) - F_1(u_2(r))\|\,dr\right]dx \\ &+\frac{1}{(m-1)!!}\int\limits_a^b (t-x)^{m-1}\left[\lambda_2\int\limits_0^x \|K_2(x,r)\|\,\|F_2(u_1(r)) - F_2(u_2(r))\|\,dr\right]dx, \\ &\le \frac{\delta^*}{(m-1)!}\sum_{j=1}^{m-1}\gamma_j\|u_1-u_2\|\int\limits_a^t (t-x)^{m-1}dx + \frac{\lambda_1}{(m-1)!}\left[K_1^*L_1\|u_1-u_2\|\right]\int\limits_a^t (t-x)^{m-1}dx \\ &+\frac{\lambda_2}{(m-1)!}\left[K_2^*L_2\|u_1-u_2\|\right]\int\limits_a^b (t-x)^{m-1}dx \\ &\le \left[\frac{\gamma^*\delta^*(b-a)^m}{m!} + \frac{\lambda_1K_1^*L_1(b-a)^m}{m!} + \frac{\lambda_2K_2^*L_2(b-a)^m}{m!}\right]\|u_1-u_2\| \\ &= \varepsilon\|u_1-u_2\|,\ \varepsilon < 1. \end{aligned}$$

Thus, operator T is the contraction map. By the Banach contraction principle we can conclude that T has a unique fixed point u in $C(J,R)$.

3 Results

3.1 Illustrative examples

Example 1 (Hamoud et al. [23]). Consider the following Fredholm IDEs of order one with the initial condition.

$$u^{'}(x) = e^x(1+x) - x + \int_0^1 xu(t)dt, \quad u(0) = 0. \tag{17}$$

The exact solution of (17) is $u(x) = xe^x$

Solution: To apply ADM, we need to convert Eq. (17) into integral equations which yields $u(x) = xe^x - \frac{x^2}{2} + \frac{x^2}{2}\int_0^1 u(\tau)d\tau$

Using the standard ADM we get $U_2 = xe^x - \frac{x}{72}$; $U_5 = xe^x - \frac{x}{15552}$; $U_{10} = xe^x - \frac{x}{120932352}$; $U_n = xe^x - \frac{x}{2\cdot 6^n}$.

The numerical results of Example 1 are given in Table 1.

Table 1: Numerical results of Example 1

X	Exact solution	Error for ADM [9] (n=3)	Error for ADM (n=10)	Error for ADM (n=30)
0.1	0.11051709	1.3×10^{-4}	8.27×10^{-11}	2.26×10^{-26}
0.2	0.24428055	5.56×10^{-4}	3.31×10^{-10}	9.05×10^{-26}
0.3	0.40495764	1.25×10^{-3}	7.44×10^{-10}	2.04×10^{-25}
0.4	0.59672987	2.22×10^{-3}	1.32×10^{-9}	3.62×10^{-25}
0.5	0.82436063	3.47×10^{-3}	2.07×10^{-9}	5.65×10^{-25}
0.6	1.09327128	5.00×10^{-3}	2.98×10^{-9}	8.14×10^{-25}
0.7	1.40962689	6.80×10^{-3}	4.05×10^{-9}	1.11×10^{-24}
0.8	1.78043274	8.89×10^{-3}	5.29×10^{-9}	1.45×10^{-24}
0.9	2.21364280	1.12×10^{-2}	6.70×10^{-9}	1.83×10^{-24}
1.0	2.71828182	–	8.27×10^{-9}	2.26×10^{-24}

Remark 1. From the Table 1, we can conclude that ADM is very high accurate semi-analytical method to solve linear IDEs of order one. Hamoud et al. [23] found error of ADM for two iteration only. We are able to run the iteration upto 30 and improve accuracy upto 10^{-24}.

Example 2 (Alao et al. [20]). Solve the following Fredholm integrodifferential equation

$$u^{'}(x) = 1 - \frac{x}{3} + \int_0^1 xtu(t)dt, \quad u(0) = 0. \tag{18}$$

The exact solution is $u(x) = x$.

Solution: Convert IDEs (18) into integral equations $u(x) = x - \frac{x^2}{6} + \frac{x^2}{2}\int_0^1 \tau u(\tau)d\tau$

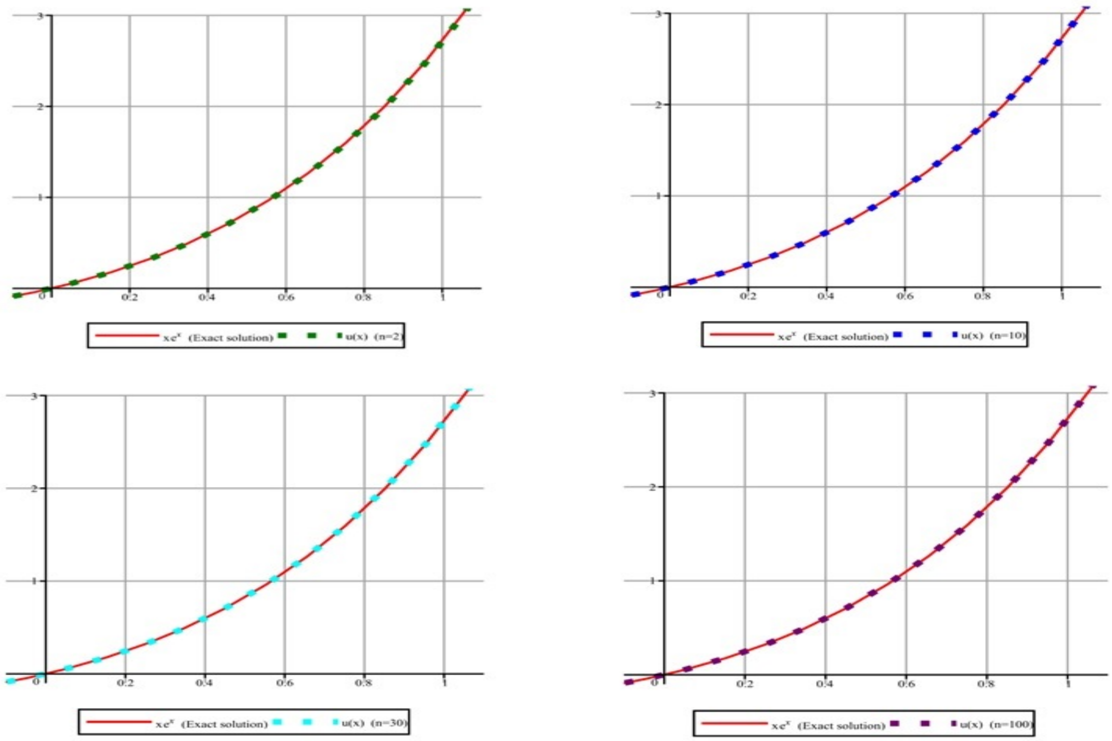

Figure 1: Graph for Example 1.

Using the standard ADM we get

$$\begin{aligned} U_2 &= x - \tfrac{x^2}{384}, \\ U_5 &= x - \tfrac{x^2}{196608}, \\ U_{10} &= x - \tfrac{x^2}{6442450944}; \\ U_n &= x - \tfrac{x^2}{6 \cdot 8^n}. \end{aligned}$$

The numerical results of Example 2 are given in Table 2.

Remark 2. Table 2 shows that ADM is again very high accurate semi-analytical method to solve linear IDEs of order one. Alao et al. [8] found error of ADM for seven iteration only. We are able to run the iteration upto 30 and improve accuracy is 10^{-28}.

Example 3. (Hamoud [24]). Consider the following non-linear Volterra-Fredholm integro-differential equation.

$$\begin{gathered} u'(x) + xu(x) = 2x + x^3 - \tfrac{x^5}{5} - \tfrac{0.9^7}{7}x + \int_0^x u^2(t)dt + \int_0^{0.9} xu^3(t)dt, \\ u(0) = 0. \end{gathered} \tag{19}$$

with the exact solution is $u(x) = x^2$.

Solution: To apply ADM, we convert it to integral equations

$$u(x) = x^2 + \frac{x^4}{4} - \frac{x^6}{30} - \frac{0.9^7}{14}x^2 + \int_0^t u^2(\tau)d\tau + \frac{x^2}{2}\int_0^{0.9} u^3(\tau)d\tau - \int_0^x tu(t)dt$$

Table 2: Numerical results of Example 2

X	Exact solution	Error for ADM [17] (n=7)	Error for ADM (n=10)	Error for ADM (n=30)
0.1	0.10	7.95×10^{-10}	1.55×10^{-12}	1.35×10^{-30}
0.2	0.20	3.18×10^{-9}	1.62×10^{-12}	5.39×10^{-30}
0.3	0.30	7.15×10^{-9}	1.40×10^{-11}	1.21×10^{-29}
0.4	0.40	1.27×10^{-8}	2.48×10^{-11}	2.15×10^{-29}
0.5	0.50	1.99×10^{-8}	3.88×10^{-11}	3.37×10^{-29}
0.6	0.60	2.86×10^{-8}	5.59×10^{-11}	4.85×10^{-29}
0.7	0.70	3.89×10^{-8}	7.61×10^{-11}	6.60×10^{-29}
0.8	0.80	5.09×10^{-8}	9.93×10^{-11}	8.62×10^{-29}
0.9	0.90	6.44×10^{-8}	1.26×10^{-10}	1.09×10^{-28}
1.0	1.00	7.95×10^{-8}	1.55×10^{-10}	1.35×10^{-28}

The summary of numerical results of Example 3 are given in Table 3.

Table 3: Numerical results of Example 3

X	Exact solution	Error for ADM [17] (n=2)	Error for ADM (n=3)	Error for ADM (n=10)
0.1	0.0100	2.40×10^{-5}	2.55×10^{-6}	4.7×10^{-11}
0.2	0.0400	3.94×10^{-4}	9.87×10^{-6}	1.9×10^{-10}
0.3	0.0900	1.96×10^{-3}	2.07×10^{-5}	4.4×10^{-10}
0.4	0.1600	8.72×10^{-3}	3.27×10^{-5}	9.0×10^{-10}
0.5	0.2500	6.24×10^{-3}	4.16×10^{-5}	1.2×10^{-9}
0.6	0.3600	9.12×10^{-3}	4.17×10^{-5}	1.7×10^{-9}
0.7	0.4900	6.31×10^{-3}	2.78×10^{-5}	2.4×10^{-9}
0.8	0.6400	9.74×10^{-3}	9.61×10^{-5}	3.1×10^{-9}
0.9	0.8100	8.53×10^{-3}	3.63×10^{-5}	4.0×10^{-9}

Remark 3. In Hamoud [24] consider nonlinear IDEs with initial conditions and demonstrated the error terms for two iteration only. In Table 3, we are able to run Maple coding until 10 iterations and got favourable decreasing the error terms. ADM is again demonstrated the suitable and high accurate semi-analytical method to solve nonlinear IDEs of order one.

Example 4. (Olayiwola et al. [28]). Consider the following non-linear Volterra IDEs.

$$\begin{aligned} u'''(x) &= 1 + x + \tfrac{x^3}{6} + \int_0^x (x-t)\, u(t)dt, \\ u(0) &= 1,\ u'(0) = 0,\ u''(0) = 1. \end{aligned} \tag{20}$$

The exact solution of Eq. (17) is $u(x) = e^x - x$.

Solution. Convert (20) into integral equations

$$u(x) = 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^6}{720} + \frac{1}{2}\int_0^x (x-t)^2 \left(\int_0^t (t-\tau)\, u(\tau) d\tau \right) dt.$$

The numerical results of Example 4 are given in Table 4.

Table 4: Numerical results of Example 4

X	Exact solution	Error for ADM [17] (n=5)	Error for ADM (n=5)	Error for ADM (n=30)
0.0	1.0051709	1.0×10^{-10}	0.0	0.0
0.1	1.0051709	3.2×10^{-5}	3.7×10^{-63}	1.2×10^{-149}
0.2	1.0214027	2.3×10^{-4}	4.0×10^{-54}	2.0×10^{-149}
0.3	1.0498588	6.8×10^{-4}	7.8×10^{-49}	1.0×10^{-149}
0.4	1.0918246	1.4×10^{-3}	4.3×10^{-45}	3.0×10^{-149}
0.5	1.1487212	2.4×10^{-3}	3.5×10^{-42}	5.7×10^{-199}
0.6	1.2221188	3.7×10^{-3}	8.3×10^{-40}	1.6×10^{-199}
0.7	1.3137527	5.3×10^{-3}	8.5×10^{-38}	1.9×10^{-199}
0.8	1.4255409	7.3×10^{-3}	4.6×10^{-36}	3.2×10^{-199}
0.9	1.5596031	1.2×10^{-2}	1.5×10^{-34}	4.7×10^{-199}
1.0	1.7182818	1.2×10^{-2}	3.7×10^{-33}	2.2×10^{-274}

Remark 4. In Example 4, M.O.Olayiwola et al. [28] consider linear Volterra IDEs with initial conditions and solved in the Collocation method. In Table 4, we are able to get results until 5 iterations and got the gradually decreasing error terms. Numerical results revealed that ADM is very accurate semi-analytical method to solve linear IDEs of order one.

Example 5. Consider the following non-linear Volterra integro-differential equation.

$$\begin{aligned} u''(x) - u'(x) &= \tfrac{3}{2} - x + \tfrac{e^{-2x}}{2} + \int_0^x u^2(t)dt \\ u(0) = 0, &\quad u'(0) = -1. \end{aligned} \tag{21}$$

with the exact solution is $u(x) = e^{-x} - 1$.

Solution: To apply ADM, we convert it to integral equations

$$u(x) = -\frac{1}{8} - \frac{3}{4}x + \frac{3}{4}x^2 - \frac{x^3}{6} + \frac{e^{-2x}}{8} + \int_0^x (x-t) \left(\int_0^t u^2(\tau)d\tau \right) dt + \int_0^x u(t)dt$$

This example was solved by SADM and MADM.

The solution with SADM is as follows

$$u_0(x) = -\frac{1}{8} - \frac{3}{4}x + \frac{3}{4}x^2 - \frac{x^3}{6} + \frac{e^{-2x}}{8};$$
$$u_1(x) = \int_0^x (x-t)\left(\int_0^t A_0 d\tau\right) dt + \int_0^x u_0(t)dt$$
$$\dots$$
$$u_n(x) = \int_0^x (x-t)\left(\int_0^t A_{n-1} d\tau\right) dt + \int_0^x u_{n-1}(t)dt$$

So that n-term approximate solution
$U_n(x) = u_0(x) + u_1(x) + u_2(x) + \ldots + u_n(x)$
Also, the solution with MADM is as follows

$$u_0(x) = -\frac{1}{8} - \frac{3}{4}x + \frac{e^{-2x}}{8};$$
$$u_1(x) = \frac{3}{4}x^2 - \frac{x^3}{6} + \int_0^x (x-t)\left(\int_0^t A_0 d\tau\right) dt + \int_0^x u_0(t)dt;$$
$$u_2(x) = \int_0^x (x-t)\left(\int_0^t A_1 d\tau\right) dt + \int_0^x u_1(t)dt$$
$$\dots$$
$$u_n(x) = \int_0^x (x-t)\left(\int_0^t A_{n-1} d\tau\right) dt + \int_0^x u_{n-1}(t)dt$$

The n-term approximate solution for this also looks the same

$$U_n(x) = u_0(x) + u_1(x) + u_2(x) + \ldots + u_n(x)$$

The summary of numerical results of Example 5 are given in Table 5.

Table 5: Numerical results of Example 5

X	Exact solution	Error for ADM (n=5)	Error for MADM (n=5)	Error for ADM (n=10)	Error for MADM (n=10)
0.1	-0.09516258	-1.95×10^{-11}	-9.91×10^{-12}	-2.06×10^{-21}	1.04×10^{-21}
0.2	-0.18126924	-2.47×10^{-9}	-9.92×10^{-9}	-8.23×10^{-18}	-9.91×10^{-17}
0.3	-0.25918177	-4.14×10^{-8}	-9.41×10^{-8}	-1.0×10^{-15}	-4.28×10^{-17}
0.4	-0.32967995	-3.03×10^{-7}	-4.69×10^{-7}	-2.82×10^{-14}	-9.10×10^{-14}
0.5	-0.39346934	-1.4×10^{-6}	-1.57×10^{-6}	-3.36×10^{-13}	-9.54×10^{-13}
0.6	-0.45118836	-4.87×10^{-6}	-4.04×10^{-6}	-2.08×10^{-12}	-6.46×10^{-12}
0.7	-0.50341469	-1.30×10^{-5}	-8.60×10^{-6}	-5.95×10^{-12}	-3.27×10^{-11}
0.8	-0.55067103	-3.31×10^{-5}	-1.56×10^{-5}	-1.39×10^{-11}	-1.35×10^{-10}
0.9	-0.59343034	-7.10×10^{-5}	-2.47×10^{-5}	-2.60×10^{-10}	-4.78×10^{-10}

Remark 5. This example was established by the authors and solved by ADM and MADM. It can be seen that the error in both methods worked well and not much difference each other. When $n = 10$ iterations, error of both methods reached to 10^{-10}. By increasing number of iterations we get more decreasing errors.

4 Conclusion

In this study, we have developed an enhanced scheme based on the Adomian Decomposition Method (ADM) for solving both linear and nonlinear integro-differential equations (IDEs) of arbitrary order nnn. Through a series of numerical experiments, we observed that the approximation error associated with ADM decreases progressively as the number of iterations increases, indicating improved convergence behavior (see Tables 1–4). Notably, our results demonstrate that ADM performs particularly well for linear IDEs when a larger number of iterations is employed (see Tables 1–2), with a significant improvement in the accuracy of the solution. To evaluate the effectiveness of the proposed method, we compared our results with those obtained using existing approaches, including the collocation method and the variational iteration method. The comparison reveals that our improved ADM-based scheme achieves superior accuracy across various test problems. Looking ahead, we intend to extend this work by applying the improved ADM-based scheme in combination with the Series Solution Method (SSM) to tackle systems of complex Volterra-Fredholm integro-differential equations. This future direction aims to further enhance the applicability and efficiency of analytical and semi-analytical methods for solving advanced IDE systems arising in applied mathematics and engineering contexts.

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2-бөлім **Раздел 2** **Section 2**

Қолданбалы математика **Прикладная математика** **Applied mathematics**

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NUMERICAL SIMULATION OF DECAYING TURBULENCE IN THE PRESENCE OF FREELY MOVING BUOYANT SOLID PARTICLES OF FINITE-SIZE

We present interface-resolved direct numerical simulations of decaying homogeneous isotropic turbulence laden with finite-size spherical particles. The fluid phase is solved using the lattice Boltzmann method (LBM) coupled with the interpolated bounce-back (IBB) scheme to impose no-slip boundary conditions at moving particle surfaces. The accuracy of the method is verified against benchmark cases, including the settling sphere experiment of ten Cate et al. and pseudo-spectral simulations of single-phase turbulence. Simulations are performed at an initial Taylor-scale Reynolds number in the range of $Re_\lambda = 20 - 45$ for particle volume fraction of 2.5%. The results show that finite-size particles enhance small-scale flow structures and accelerate the decay of turbulent kinetic energy compared to the single-phase case. Energy spectra analysis reveals a redistribution of energy from large to small scales. These findings provide new insights into turbulence modulation mechanisms in particle-laden flows and demonstrate the applicability of LBM for fully resolved particle–turbulence interaction studies.
Keywords: particle-laden, turbulence, multiphase, LBM, DNS.

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Шекті өлшемдегі еркін қозғалатын қалқымалы қатты бөлшектердің қатысуымен ыдырайтын турбуленттікті сандық модельдеу

Осы жұмыста біз шекті өлшемді сферикалық бөлшектермен жүктелген ыдырайтын біртекті изотропты турбуленттіліктің тікелей сандық модельдеу нәтижелерін ұсынамыз. Сұйық фазаның динамикасы торлы Больцман әдісімен (LBM) шешіледі, ол қозғалыстағы бөлшектер бетінде жабысып тұру шекаралық шарттарын қою үшін интерполяцияланған шағылысу (IBB) сұлбасы қолданылған. Әдістің дәлдігі сынақтық есептерде тексерілді, соның ішінде сфералық бөлшектін шөгу тәжірибесімен және бірфазалы турбуленттілікке арналған псевдоспектралды әдіс нәтижелерімен салыстыру орындалды. Есептеулер Тейлор масштабы бойынша бастапқы Рейнольдс сандарының мәні $Re_\lambda = 20-45$ диапазонында болғанда және бөлшектердің көлемдік үлесі 2.5% кезінде жүргізілді. Нәтижелер бөлшектердің шекті өлшемдері ағынның ұсақмасштабты құрылымдарын күшейтіп, турбуленттік кинетикалық энергияның бірфазалы жағдаймен салыстырғанда тезірек өшуін жеделдететінін көрсетті. Энергетикалық спектрлерді талдау энергияның үлкен масштабтардан кіші масштабтарға ауысуын айқындады. Бұл нәтижелер бөлшектер бар ағындардағы турбуленттілікті модуляциялау механизмдері туралы жаңа мәліметтер береді және «бөлшек–турбуленттілік» өзара әрекеттесуін толық шешілетін зерттеулерге LBM әдісінің қолданбалығын көрсетеді.
Түйін сөздер: батырылған бөлшектер, турбуленттілік, көп фазалық ағындар, LBM, DNS.

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Численное моделирование затухающей турбулентности в присутствии свободно движущихся плавучих твердых частиц конечного размера

В данной работе нами представлены результаты прямого численного моделирования затухающей изотропной однородной турбулентности нагруженной сферическими частицами конечного размера. Динамика жидкой фазы разрешается методом решётчатого Больцмана (LBM), совмещённой с интерполированной схемой отражения (IBB) для задания граничных условий прилипания на поверхностях движущихся частиц. Точность метода подтверждается на тестовых задачах, включая эксперимент по оседанию сферы и при сравнении с результатами псевдоспектрального метода для однофазной турбулентности. Расчёты выполнены при начальных числах Рейнольдса по масштабу Тейлора в диапазоне $Re_\lambda = 20{-}45$ при объёмной доле частиц 2.5%. Результаты показывают, что частицы конечного размера усиливают мелкомасштабные структуры течения и ускоряют затухание турбулентной кинетической энергии по сравнению с однофазным случаем. Анализ энергетических спектров выявляет перераспределение энергии от больших к малым масштабам. Эти результаты предоставляют новые сведения о механизмах модуляции турбулентности в потоках с частицами и демонстрируют применимость LBM для полностью разрешённых исследований взаимодействия частицы–турбулентность.
Ключевые слова: погруженные частицы, турбулентность, многофазные течения, LBM, DNS.

1 Introduction

Turbulence laden with solid particles is a phenomenon of fundamental and practical importance in a wide range of natural and industrial processes, including sediment transport, atmospheric dust dynamics, spray combustion, and chemical mixing. Understanding the dynamics of such flows is particularly challenging due to the complex interplay between turbulent eddies and dispersed particles, especially when the particles are of finite size and can modulate the turbulence field [2].

Recent advances in numerical analysis, and computing hardware allowed researchers to gain more knowledge about two-way interactions of the flow and dispersed solid phase through interface-resolved direct numerical simulations (DNS), where all the flow details around moving solid particles are explicitly resolved, without a need to rely to empirical models or assumptions. The configuration of the turbulent flows in such studies is often described by the decaying homogeneous isotropic turbulence (DHIT) case, which offers a simple environment without of mean shear, wall effects, and external forcing. It enables the isolation and detailed examination of turbulence modulation mechanisms induced by particles, including attenuation or enhancement of turbulent kinetic energy (TKE), spectral energy redistribution, and modifications of small-scale intermittency.

[15] presented particle-resolved direct numerical simulations of homogeneous isotropic turbulence containing small, fixed spheres to investigate their impact on turbulence structure and decay. By resolving the flow around each sphere, detailed near-field and wake dynamics were captured which were absent in point-particle models[4]. The results show that the presence of fixed particles enhances small-scale dissipation and alters the energy spectrum, leading to increased energy at smaller scales and a faster decay of turbulent kinetic energy. The study further demonstrates that wake-induced fluctuations are a dominant mechanism for turbulence modification and provides quantitative measures of the additional dissipation generated by particle wakes.

One of the recent works by [12] employed direct numerical simulations of finite-size spherical particles in decaying homogeneous isotropic turbulence, focusing on particle-induced modulation of turbulence and the underlying mechanisms. The results show that particles generally enhance small-scale dissipation and alter the decay rate of turbulent kinetic energy, with the effect depending on particle size relative to the Kolmogorov length scale and the solid volume fraction. Energy spectra reveal a transfer of energy from large to small scales in the presence of particles. The study also quantifies additional dissipation from particle wakes and demonstrates that particle Reynolds number and wake dynamics play a central role in modifying turbulence decay.

Oka and Goto [9] investigated the attenuation of turbulence in a periodic cube by finite-size spherical solid particles through direct numerical simulations using an immersed boundary method to resolve flow around each particle, with fixed volume fraction $\phi = 8.2 \times 10^{-3}$ and systematically varying particle diameters d and Stokes numbers St. They found that turbulent kinetic energy attenuation is governed by the additional energy dissipation rate ϵ_p in particle wakes. Based on their simulation results authors proposed conditions for turbulence attenuation by finite-size particles.

While previous works incorporated discretized form of the Navier-Stokes equations to solve for the motion of the fluids [7], [5] used the Lattice Boltzmann Method (LBM) as fluid solver, combined with the interpolated bounce-back (IBB) boundary scheme to account for dynamic fluid-solid interface. The authors demonstrated that such numerical method can archive second order spatial accuracy and good computational efficiency, proposing its usage for turbulent flows laden with finite-size particles.

From the above mentioned examples, it is evident that the study of detailed mechanisms of turbulence decay in the presence of finite-size particles remain an active research area. Presence of the finite-size particles can significantly affect the temporal decay of turbulence. These effects depend strongly on several dimensionless parameters, such as the particle Stokes number, volume fraction, density ratio, and Reynolds number. In particular, when the particles are large compared to the Kolmogorov length scale, the interaction becomes two-way or even four-way coupled, involving significant feedback from particles to the fluid and particle-particle interactions [2].

In this study, we perform interface-resolved DNS of decaying particle-laden turbulence using the LBM method with fully resolved flow around spherical solid particles. Our aim is to investigate the effect of finite-size particles on the decay of turbulence at different flow Reynolds numbers, along this way we also demonstrate that kinetic methods, such as LBM are suitable for such complex cases as particle-laden turbulent flows. In the next section we provide the details of the combined LBM-IBB method we use, followed by Section 3, where validation cases and simulation results for particle-laden DHIT problem cases are presented. Finally, in the end we present conclusions to this work.

2 Numerical Methods

In this section the numerical methods used to simulate fluid and solid (dispersed) phases are described, including the details of the particle-particle collision treatment and scheme to enforce a no-slip boundary condition on the surface of the freely moving finite-size particles.

2.1 Lattice Boltzmann Method for Fluid Flow

The fluid motion is simulated using the Lattice Boltzmann Method, a kinetic approach that solves the discrete Boltzmann equation on a cubic mesh. We employ the three-dimensional, 27-velocity lattice model (D3Q27, see Fig. 2.1) , where the evolution of the particle distribution function $f_i(\mathbf{x}, t)$ is governed by the lattice Boltzmann equation with the Bhatnagar-Gross-Krook (BGK) approximation:

$$f_i(\mathbf{x} + \mathbf{c}_i \delta t, t + \delta t) = f_i(\mathbf{x}, t) - \frac{1}{\tau} \left(f_i(\mathbf{x}, t) - f_i^{\text{eq}}(\mathbf{x}, t) \right), \tag{1}$$

where $\mathbf{c}_i$ is the discrete velocity in the i-th direction, τ is the relaxation time, δt is the timestep ($\delta t = 1$ in present work) and f_i^{eq} is the equilibrium distribution function given by:

$$f_i^{\text{eq}} = w_i \rho \left(1 + \frac{\mathbf{c}_i \cdot \mathbf{u}}{c_s^2} + \frac{(\mathbf{c}_i \cdot \mathbf{u})^2}{2c_s^4} - \frac{\mathbf{u} \cdot \mathbf{u}}{2c_s^2} \right), \tag{2}$$

where w_i are the lattice weights, ρ is the fluid density, $\mathbf{u}$ is the macroscopic velocity, and c_s is the lattice speed of sound. The D3Q27 discrete velocity model is utilized because it is found to be more accurate and stable for turbulent flows in complex configurations [13].

The macroscopic flow variables are recovered via the moments of the discrete disctribution function:

$$\rho = \sum_{i=0}^{26} f_i, \qquad \rho \mathbf{u} = \sum_{i=0}^{26} f_i \mathbf{c}_i. \tag{3}$$

The kinematic viscosity ν is related to the relaxation time τ through:

$$\nu = c_s^2 \left(\tau - \frac{1}{2} \right) \delta t. \tag{4}$$

During numerical simulations, it is common to advance Eq. (1) in two step procedure consisting of *collision* and *streaming* steps.

Collision:

$$f_i^*(\mathbf{x}, t) = f_i(\mathbf{x}, t) - \frac{\delta t}{\tau} \left[f_i(\mathbf{x}, t) - f_i^{\text{eq}}(\rho, \mathbf{u}) \right], \tag{5}$$

Streaming:

$$f_i(\mathbf{x} + \mathbf{c}_i \delta t, t + \delta t) = f_i^*(\mathbf{x}, t), \tag{6}$$

where f_i^* is the post-collision distribution function, and δt is the time step size.

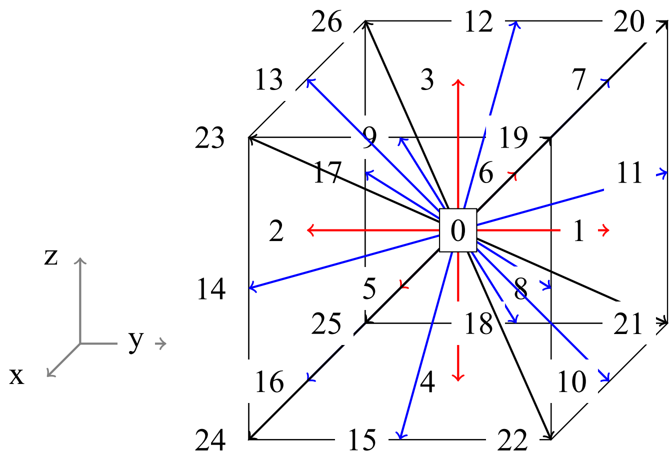

Figure 1: The D3Q27 discrete velocity model is used in the current work. The fluid particles are allowed to move only in the shown discrete directions. The cube centered at the origin $(0, 0, 0)$ has side length of 1.

2.2 Interpolated Bounce-Back Scheme for Moving Boundaries

The standard bounce-back scheme is a popular choice to enforce no-slip boundary condition in kinetic methods, such as LBM. However, it assumes that solid boundary is always located exactly half-way between fluid nodes. An improvement made by Bouzidi et al. [1] allowed to use bounce-back scheme on curvilinear boundaries with arbitrary location of solid boundary relative to fluid nodes. The key point in the improvement of Bouzidi et al. [1] is to use linear or quadratic interpolation to recover unknown fluid particles coming from the boundary. The interpolated bounce-back scheme is found to posses second-order of accuracy and is a common choice in laminar and turbulent flow simulations.

The idea behind IBB is demonstrated using one-dimensional example in Fig. 2. When a lattice link between a fluid node (x_f) and a solid node (x_b) intersects the particle boundary, the IBB scheme determines the exact intersection point and computes parameter q. The value of q defines which of two ways is used to recover unknown distributoin function values, heading off the solid boundary (that is, moving in the right direction at node x_f). In case $q \leq 0.5$, the value of f_i is equal to the pre streaming value of f_i at the node x_i, while in the case $q > 0.5$, f_i at the near boundary node x_f is recovered using interpolation through the values at nodes x_{fff}, x_{ff} and x_i. The required values at the node x_i also found using the interpolation.

The hydrodynamic force and torque acting on a particle are computed from the momentum exchange (Peng et al. [10]) at each fluid-solid interface:

$$\mathbf{F}_{\text{hydro}} = \sum_{x_f} \sum_{i} \Delta \mathbf{p}, \qquad \mathbf{T}_{\text{hydro}} = \sum (\mathbf{r} - \mathbf{r}_c) \times \Delta \mathbf{p}, \tag{7}$$

where particle nature of LBM is exploited, and $\Delta \mathbf{p}$ is the momentum exchanged across a lattice link, $\Delta \mathbf{p} = \left(\tilde{f}_i^t + f_{\bar{i}}^{t+\Delta t} \right) \cdot \mathbf{c}_i$, $\mathbf{r}_c$ is the center of the particle, and the summation extends over all links intersecting the particle surface.

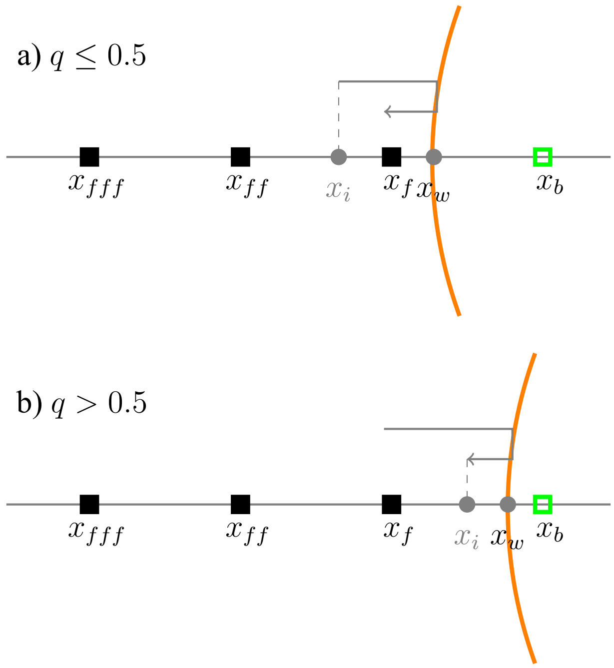

Figure 2: An example of the interpolated bounce-back scheme for two possible cases, depending on the parameter q which defines relative solid boundary intersection with velocity links, defined as $q = (x_w - x_f)/(x_b - x_f)$: a) $q \leq 0.5$, the unknown fluid particles are coming from the node x_i, b) $q > 0.5$, the unknown fluid particles ar node x_f are reconstructed from the data from the nodes x_{fff}, x_{ff} and x_i.

2.3 Particle Motion and Coupling

Particles are modeled as rigid spheres, and their motion is governed by the Newton-Euler equations:

$$m_p \frac{d\mathbf{u}_p}{dt} = \mathbf{F}_{\text{hydro}} + \mathbf{F}_{\text{coll}}, \tag{8}$$

$$\mathbf{I}_p \frac{d\boldsymbol{\omega}_p}{dt} = \mathbf{T}_{\text{hydro}} + \mathbf{T}_{\text{coll}}, \tag{9}$$

where m_p and $\mathbf{I}_p$ are the particle mass and moment of inertia, $\mathbf{u}_p$ and $\boldsymbol{\omega}_p$ are the translational and angular velocities of the particle, and $\mathbf{F}$, $\mathbf{T}$ represent forces and torques acting on a particle due to interaction with the surrounding fluid (subscript hydro) and collision with other particles (subscript coll).

The fluid-particle interaction is resolved at each time step. First, the LBM updates the fluid flow field (Eqs. (5, 6)), after which hydrodynamic forces and torques are computed via the IBB scheme. The particle motion is then integrated based on the computed forces, and updated positions and velocities are used in the next iteration of the IBB scheme. Eqs. (8, 9) are discretized using second-order scheme.

2.4 Collision and Lubrication Modeling

To account for short-range hydrodynamic interactions (Fig. 3) that are under-resolved on the lattice, particularly during close particle-particle approaches, a lubrication correction model is employed. Specifically, a soft-sphere collision model with linear spring-dashpot mechanics combined with a lubrication force correction is employed in this work. This approach provides physically correct contact mechanics and helps to avoid nonphysical particle overlapping.

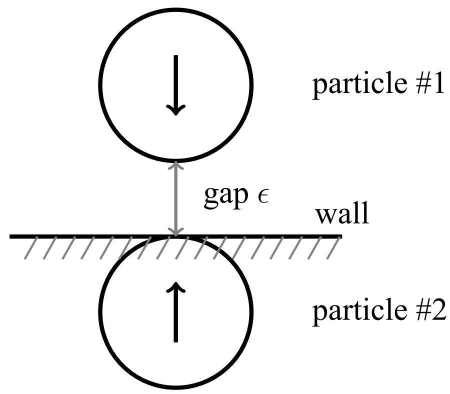

Figure 3: An example of particle-particle or particle-wall interaction. The gap between particles or between particle and wall is denoted as ϵ.

The particle interaction force is split into two constituents $\boldsymbol{F}_{coll} = \boldsymbol{F}_l + \boldsymbol{F}_s$, $\boldsymbol{F}_l$ accounts for lubrication correction and $\boldsymbol{F}_s$ for soft-sphere collision. The magnitude of $\boldsymbol{F}_l$ is then:

$$F_l(\epsilon, u_n) = -6\pi\mu_f R u_n[\lambda(\epsilon) - \lambda(\epsilon_{sw})], \tag{10}$$

where u_n - relative velocity, ϵ - gap width, and ϵ_{sw} is the distance at which correction is activated. The particular forms for λ depend on the type of interaction, analytical forms of which were given by Brenner [3], and for particle-particle interactions $\lambda = \lambda_{pp}$:

$$\lambda_{pp} = \frac{1}{2\epsilon} - \frac{9}{20}ln(\epsilon) - \frac{3}{56}\epsilon ln(\epsilon) + 1.346 + \mathcal{O}(\epsilon). \tag{11}$$

The soft-sphere contact collision force takes the form of a spring-dashpot system with the magnitude of $\boldsymbol{F}_s$ given as:

$$F_s = -k_n\delta - \beta_n u_n \tag{12}$$

where k_n is the spring stiffness parameter, β_n is the dashpot resistance parameter and $t = N_c\Delta t$ - collision time:

$$k_n = -\frac{m_e(\pi^2 + [\ln e_d]^2)}{[N_c\Delta t]^2}, \beta_n = -\frac{2m_e[\ln e_d]}{[N_c\Delta t]}. \tag{13}$$

The coefficients k_n and β_n are the functions of the collision time $t = N_c\Delta t$, effective mass m_e and of the coefficient of dry restitution. They form spring-dashpot system and guarantee that particle re-bounces during the collision with duration of $t = N_c\Delta t$ with the velocity $u_c^* = e_d u_c$, where u^* is the velocity at the end of the contact interaction.

3 Simulation results

As a validation of the code first the single-phase Decaying Homogeneous Isotropic Turbulence (DHIT) problem was compared to spectral simulation results. For this case, Rogallo's procedure [11] was used to generate an initial divergence-free velocity field with the prescribed energy spectrum. The initial energy spectrum is given by [16]

$$E_0(k) = Ak^4 e^{-0.14k^2},$$

with energy containing wavenumbers $k = [k_a, k_b] = [3, 8]$ and coefficient $A = 1.1474 \times 10^{-2}$. The flow is prescribed for the DHIT problem using the Taylor microscale-based Reynolds number Re_λ defined as

$$Re_\lambda = \frac{u_{rms}\lambda}{\nu},$$

where u_{rms} is *Root-Mean-Square* (RMS) velocity, λ is Taylor micro length scale, and ν is fluid kinematic viscosity.

The numerical simulation results for the evolution of turbulent kinetic energy and energy dissipation at $Re_\lambda = 26$, obtained from the LBM and the benchmark pseudo-spectral simulation [6], are presented in Fig. 4. The excellent agreement between the kinetic method and the pseudo-spectral simulation at a mesh resolution of 256^3 strongly supports the applicability of the LBM to turbulent flow simulations.

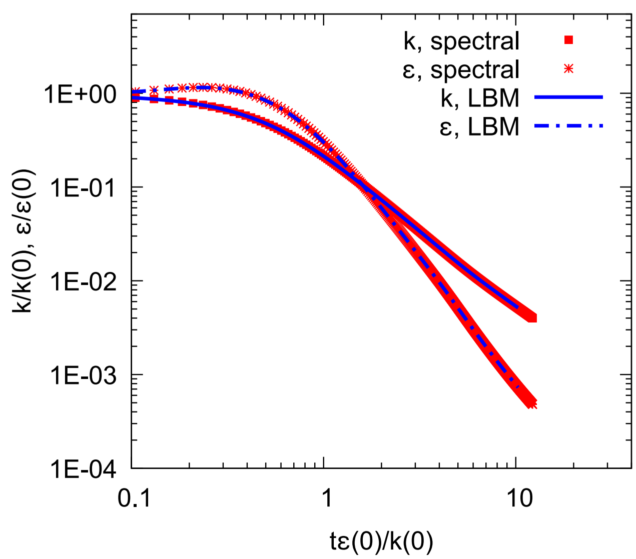

Figure 4: Evolution of the turbulent kinetic energy (k) and it's dissipation rate (ϵ) for DHIT problem.

To validate the accuracy of the numerical method for two-phase cases, we compare our simulation results with the experimental data reported by ten Cate et al. [8] for the solid particle settling problem. In their study, the motion of a single solid sphere settling under gravity in a quiescent viscous fluid was investigated using particle image velocimetry (PIV). The experiments were conducted in a square tank with dimensions $100 \times 100 \times 160\,\text{mm}^3$, and the diameter of the sphere was $d = 15\,\mathrm{mm}$. The sphere was released at the height of 120 mm above bottom of the tank and was allowed to settle freely. The fluid used was a water–glycerol mixture with varying viscosities, and the density ratio between the particle and fluid was in the range of $\rho_p/\rho_f = 1.15 - 1.16$. The Reynolds number based on terminal velocity ranged from $Re \approx 1.5$ to 32, covering both creeping and moderately inertial regimes. In Fig. 5 the evolution of the particle settling velocity is shown. Here, again very good agreement between numerical results from LBM and experimental data indicate sufficient validation and performance of the newly implemented kinetic approach.

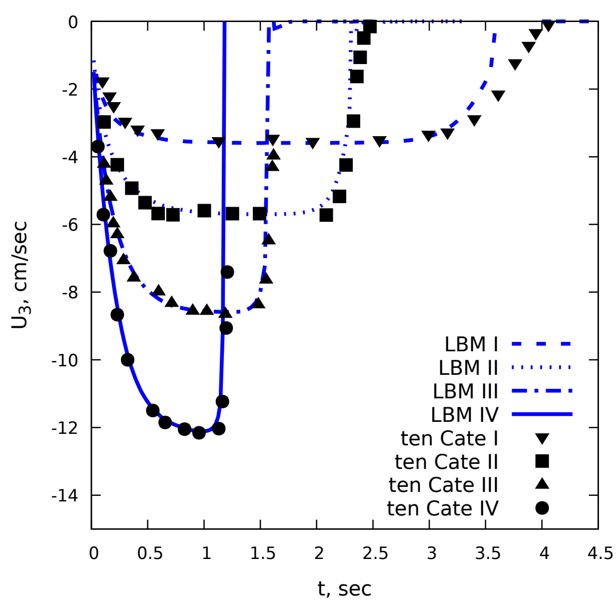

Figure 5: Evolution of the particle settling velocity compared with experimental data of cases I-IV from Ten Cate et al. [14].

After above mentioned validation cases we now present DHIT results for particle-laden and particle-free cases. In total six simulations were performed, three of which are single-phase cases and other three are particle-laden counterparts of fluid only cases. The carrier flow properties are shown in Table 1, whereas suspended solid phase properties are shown in Table 2.

#	L	N	k_0	u_{rms}	ν	Re_λ
I	256	256	$3.52 \cdot 10^{-4}$	$1.53209 \cdot 10^{-2}$	0.0121	20
II	256	256	$3.52 \cdot 10^{-4}$	$1.53209 \cdot 10^{-2}$	0.0101	30
III	256	256	$3.52 \cdot 10^{-4}$	$1.53209 \cdot 10^{-2}$	0.0068	45

Table 1: Fluid phase properties for single-phase and particle-laden cases.

The DHIT problem was defined in the cubic domain with periodic boundary conditions in all three directions with the length of the cube side as L and the mesh resolution of N. The initial kinetic energy of the flow is k_0, while RMS velocity of the initial flow is u_{rms}. Turbulence was characterized using the Taylor microscale Reynolds number Re_λ. In particle-laden cases the volume fraction of

#	N_p	ρ_p/ρ_f	ϕ	$d/\Delta x$	d/λ	d/η
I	50	1.0	2.5%	25.6	12.93	12.7
II	50	1.0	2.5%	25.6	12.93	13.91
III	50	1.0	2.5%	25.6	12.93	14.91

Table 2: Solid-phase properties for particle-laden counterparts of the single-phase flows shown in Table 1.

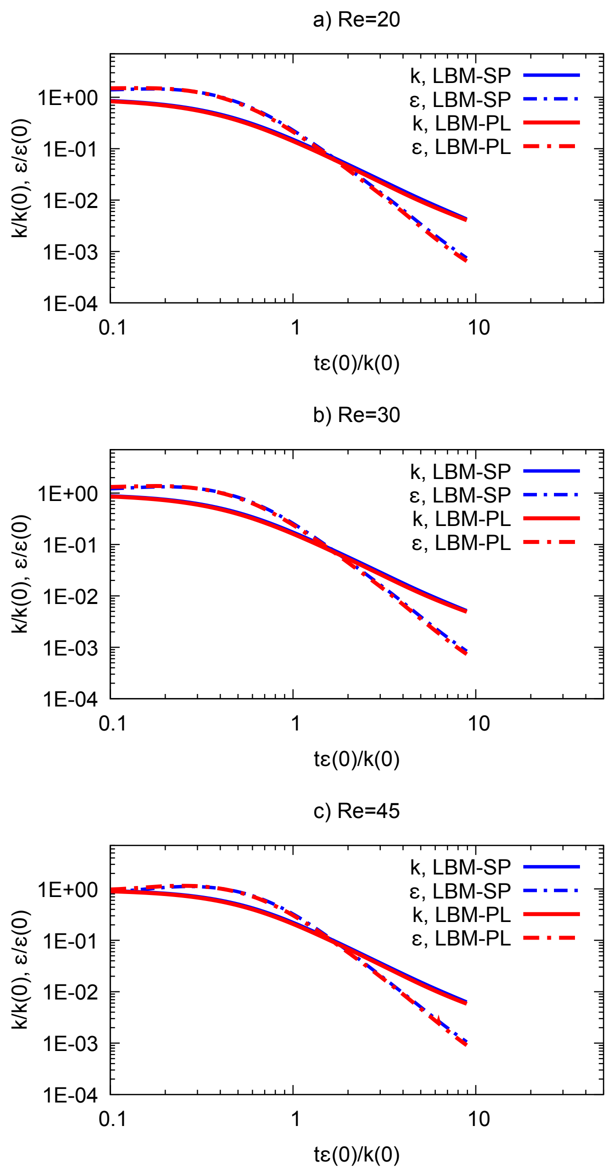

Figure 6: Evolution of the flow turbulent kinetic energy and dissipation rate for single-phase (SP) and particle-laden (PL) cases at different Reynolds numbers.

the solid-phase is set to 2.5% with the number of buoyant particles equal to 50. Particle diameter (d) and mesh resolution was large enough to allow properly resolve all flow scales, including Taylor (λ)

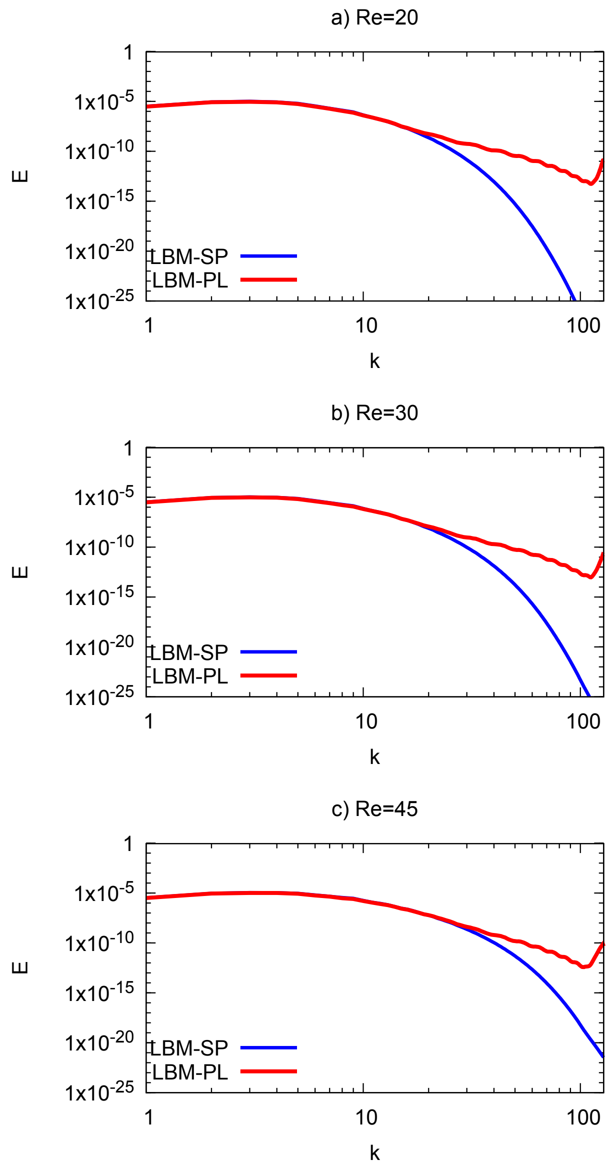

Figure 7: Comparison of turbulence energy spectra at $t^* = 1$ between single-phase (SP) and particle-laden cases (PL).

and Kolmogorov (η) length scales. Particle-laden cases were identical to single-phase cases, except that buoyant particles were introduced to the flow after one step of LBM algorithm.

Evolution of the flow turbulent kinetic energy and dissipation rate for single-phase and particle-laden cases at different Reynolds numbers are shown in Fig. 6. For all three considered cases which differ only in the initial flow Reynolds numbers, increase of dissipation rate due to the introduction of the particles is visible at the beginning of the simulation. However, at later time the energy dissipation rate becomes lower for PL cases. For all PL cases slight reduction of the TKE is observable. This decrease of TKE is usually related to enhanced energy dissipation at the surface of freely moving particles.

Overall, evolution of TKE and energy dissipation is quite close to the ones in the single-phase cases due to the relatively low volume fraction of suspended phase. At the same time, no significant Reynolds number dependence in the behavior of the energy evolution is visible.

Comparison of the turbulence energy spectra at non-dimensional time $t^* = 1$ between single-phase (SP) and particle-laden cases (PL) is shown in Fig. 7. Immediately two observations can be underlined. First, we see an increase in energy of the small structures in PL cases relative to SP cases, related to a high wavenumber part of the spectra for all three Reynolds number cases. Second, energy distribution at high wavenumber part of the spectra has noticeable wiggling, which is related to velocity discontinuity on the solid-phase boundaries. Such osccillations can be considered as numerical artifact and will be ignored here. Level of energy increase of small structures is larger for lower Reynolds number cases. This can be explained due to enhanced introduction of the smaller structures to the flow by fixed-size particles. At larger Reynolds numbers, the flow already contains such small strucutres, and their enhancement due to the freely moving particles is minor.

4 Conclusions

From the performed work we can conclude that kinetic method of LBM is an efficient tool to study turbulence in multiphase flows, particularly, for flows with freely moving suspended finite-size particles. The second-order accuracy and stability of LBM is well suited to perform direct numerical simulations of turbulent flows. The decaying homogeneous isotropic turbulence problem studied here in single-phase and particle-laden cases revealed that modulation of turbulence is present, visible from energy enhancement at higher end of spectra and decreased overall TKE. For all three Reynolds numbers considered, such behaviour lasts, indicating that, at least under considered Reynolds number range ($Re_\lambda = 20-45$) no significant Reynolds number dependence is visible. For future works we suggest to hold similar studies at wider range of flow and solid-phase parameters such as ones shown in Tables 1 and 2.

Acknowledgments

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3-бөлім **Раздел 3** **Section 3**

Механика **Механика** **Mechanics**

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NUMERICAL ANALYSIS OF FLUIDIZED BED HYDRODYNAMICS WITH OPENFOAM

Gas–solid fluidized beds play a vital role in energy production, chemical processing, and thermal management due to their excellent mixing and transport properties. Despite their importance, predicting fluidized bed hydrodynamics remains a major challenge because of the highly coupled and nonlinear interactions between gas and particle phases. Computational fluid dynamics (CFD) has become an indispensable tool for analyzing such systems, but the reliability of predictions depends strongly on solver formulation, closure models, and postprocessing strategies. This study revisits the benchmark experiment of Taghipour et al. [1], which provides high-quality measurements of pressure drop and bed expansion BER, and applies it to the most recent release of OpenFOAM (v12). An Euler–Euler two-fluid approach is employed, incorporating kinetic theory of granular flow for solid-phase stresses and the Gidaspow drag correlation for interphase momentum exchange. Simulations are performed on a two-dimensional rectangular bed fluidized with air and Geldart B particles. Pressure drop and bed expansion ratio (BER) are selected as the main indicators for validation. Beyond conventional postprocessing methods, a new mass-conservation-based approach for estimating BER is introduced, which takes into account data from the entire computational domain. The work aims to evaluate the predictive capacity of OpenFOAM v12 in reproducing well-established benchmarks and to advance postprocessing techniques for more reliable characterization of fluidized bed hydrodynamics.
Key words: Fluidized bed, OpenFOAM, postprocessing methods, pressure drop, bed expansion ratio.

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OpenFOAM көмегімен сұйытылған қабат гидродинамикасының сандық талдауы

Газ–қатты фазалы сұйытылған қабаттар энергия өндіру, химиялық технологиялар және жылуалмасу процестерінде тиімді араластыру және массаны/жылуды тасымалдау қабілеттеріне байланысты кеңінен қолданылады. Алайда мұндай жүйелердегі гидродинамикалық құбылыстарды дәл болжау газ және қатты бөлшектер фазалары арасындағы күрделі әрі сызықты емес байланысқандықтан әлі күнге дейін күрделі мәселе. Қазіргі таңда CFD сұйытылған қабаттарды зерттеудің негізгі құралдарының бірі, бірақ модельдеу нәтижелерінің сенімділігі шешуші теңдеулердің нұсқасына, қолданылған жабу модельдеріне және деректерді өңдеу әдістеріне тікелей тәуелді. Осы зерттеуде Taghipour et al. [1] қысым түсуі мен қабаттың ұлғаюы жөніндегі жоғары дәлдіктегі эксперименттік деректері қарастырылып, OpenFOAM бағдарламасының соңғы нұсқасында (v12) сандық талдау жүргізіледі. Екі фазалы Эйлер–Эйлер моделі қолданылып, қатты фаза кернеулері гранулалық ағынның кинетикалық теориясымен, ал фазалар арасындағы импульс алмасу Gidaspow тарту корреляциясымен сипатталады. Сандық модельдеу ауа ағынымен сұйытылған Geldart B санатына жататын бөлшектері бар екі өлшемді тікбұрышты қабат үшін орындалды. Негізгі валидациялық көрсеткіштер ре-

тінде қысым түсуі және қабаттың ұлғаю коэффициенті (BER) алынған. Бұған қоса, есептеу ауданындағы деректерді толық ескере алатын, BER-ді масса сақталуы принципіне негізделген жаңа есептеу тәсілі ұсынылады. Зерттеу нәтижелері OpenFOAM v12 нұсқасының жақсы белгілі эталондық гидродинамикалық сипаттамаларды қайта өндіру қабілетін бағалауға және сұйытылған қабаттарды талдауға арналған деректерді өңдеу әдістерін жетілдіруге бағытталған.
Түйін сөздер: Сұйытылған қабат, OpenFOAM, екі-фазалы модель, деректерді өңдеу, қысымның түсуі, қабаттың ұлғаю коэффициенті.

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Численный анализ гидродинамики псевдоожиженного слоя с использованием OpenFOAM

Газо–твердые псевдоожиженные слои широко применяются в энергетике, химической промышленности и теплотехнике благодаря высоким характеристикам смешения, тепло- и массообмена. Тем не менее, точное прогнозирование гидродинамики таких систем остаётся сложной задачей из-за интенсивных, нелинейных и тесно связанных взаимодействий газовой и твёрдой фаз. Вычислительная гидродинамика (CFD) сегодня является ключевым инструментом для анализа псевдоожиженных систем, однако достоверность численного моделирования в значительной степени зависит от выбора численного решателя, моделей замыкания и применяемых методов постпроцессинга. В настоящей работе повторно рассмотрены высокоточные экспериментальные данные по падению давления и расширению слоя, представленные Taghipour и соавторами [1], и проведено их численное воспроизведение в последней версии OpenFOAM (v12). Используется двухфазный Эйлер–Эйлеровский подход, где напряжения твёрдой фазы описываются кинетической теорией гранулярного потока, а межфазный обмен импульсом — корреляцией сопротивления Gidaspow. Моделирование выполнено для двумерного прямоугольного псевдоожиженного слоя, аэрируемого воздухом и содержащего частицы типа Geldart B. В качестве основных критериев валидации приняты падение давления и коэффициент расширения слоя (BER). Кроме того, предложен новый метод оценки BER, основанный на законе сохранения массы, который позволяет учитывать данные всей вычислительной области. Результаты исследования направлены на оценку способности OpenFOAM v12 воспроизводить общепринятые эталонные характеристики гидродинамики псевдоожиженного слоя и на совершенствование методов постпроцессинга для более надёжного анализа таких систем.
Ключевые слова: Псевдоожиженный слой, OpenFOAM, двухфазная модель, постпроцессинг, падение давления, коэффициент расширения слоя.

1 Introduction

Fluidized beds are widely applied in energy conversion, chemical processing, and waste heat recovery due to their excellent mixing and transport properties [2]. However, predicting their hydrodynamic behavior remains challenging because of the complex interactions between gas and solid phases. Computational fluid dynamics (CFD) has become an essential tool for analyzing these systems, but numerical predictions often diverge from experiments, making reliable benchmarks crucial for model validation. In a related context, recent CFD studies of thermal energy storage systems have demonstrated that two-dimensional numerical models, validated against experimental data, can accurately capture complex flow structures and performance indicators such as temperature stratification, efficiency metrics, and mixing behavior, highlighting the broader applicability of CFD methodologies beyond fluidized beds [3]. One of the most widely used benchmarks is the experiment of Taghipour et al. [1], which provides high-quality measurements of global flow quantities such as pressure drop and bed expansion ratio (hereafter BER) in a two-dimensional gas–solid fluidized bed. These data have been used extensively to test Euler–Euler two-fluid models (TFM) combined with drag correlations such as Wen–Yu [4], Syamlal–O'Brien [5], and Gidaspow [6]. While many studies have reproduced this benchmark, results remain sensitive to choices of drag law, boundary conditions. Previous evaluations of OpenFOAM have shown mixed conclusions: some reported insufficient accuracy, while others demonstrated close agreement with experiments. Importantly, no study has yet assessed the most recent release, OpenFOAM v12, which incorporates updates in solver robustness and numerical schemes. Addressing this gap, the present work investigates whether the multiPhaseEuler solver in OpenFOAM v12 can reliably reproduce Taghipour's [1] benchmark. Pressure drop and BER are selected as the main hydrodynamic indicators. In addition, a new mass-conservation-based method for calculating BER is proposed and compared with conventional approaches. The objective of this study is to validate the predictive capability of OpenFOAM v12 for gas–solid fluidized bed hydrodynamics and to provide improved postprocessing strategies that can support industrial-scale applications.

2 Materials and Methods

This study reproduces the benchmark experiment of Taghipour et al. [1] using the Euler–Euler two-fluid model in OpenFOAM v12. The objective is to assess the solver's ability to predict two key hydrodynamic indicators: pressure drop and bed expansion ratio (BER). The simulated system is a two-dimensional rectangular bed with dimensions of 1.0 m $\times$ 0.28 m $\times$ 0.025 m. The bed is initially filled to 40% of its height with Geldart B particles (mean diameter 275 μm, density 2500 kg/m^3). Air serves as the fluidizing medium, with density $1.225\,\mathrm{kg\,m^{-3}}$ and kinematic viscosity $1.485 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$. The superficial gas velocity was varied between 0.025 and 0.51 m/s, covering the transition from fixed to bubbling and turbulent regimes. The simulation utilized a multifluid Eulerian approach that solves conservation equations for mass and momentum across gas and fluid phases. For modeling solid-phase stresses, the kinetic theory of granular flow was implemented, providing closure through conservation of solid fluctuation energy [7]. The governing equations can be summarized as follows:

Mass conservation equations of gas (g) and solid (s) phases:

$$\frac{\partial}{\partial t}(\alpha_g\rho_g) + \nabla \cdot (\alpha_g\rho_g\vec{v}_g) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\alpha_s\rho_s) + \nabla \cdot (\alpha_s\rho_s\vec{v}_s) = 0. \tag{2}$$

Momentum conservation equations of gas (g) and solid (s) phases:

$$\frac{\partial}{\partial t}\big(\alpha_g\rho_g\vec{u}_g\big) + \nabla \cdot \big(\alpha_g\rho_g\vec{u}_g\vec{u}_g\big) = -\alpha_g\nabla p + \nabla \cdot \overline{\tau}_g + \alpha_g\rho_g\vec{g} + K_{gs}(\vec{u}_g - \vec{u}_s) \tag{3}$$

$$\frac{\partial}{\partial t}(\alpha_s \rho_s \vec{u}_s) + \nabla \cdot (\alpha_s \rho_s \vec{u}_s \vec{u}_s) = -\alpha_s \nabla p - \nabla p_s + \nabla \cdot \overline{\tau}_s + \alpha_s \rho_s \vec{g} + K_{gs}(\vec{u}_g - \vec{u}_s) \tag{4}$$

where K_{gs} represents the interphase momentum transfer was described by the Gidaspow drag law (Eq. (5-7)), while particle–wall interactions were represented by the Johnson–Jackson boundary condition (Eq. (8-9)) with a restitution coefficient of 0.9 and specularity coefficient of 0.2.

Gidaspow drag model [6]:

For $\varepsilon_g > 0.8$:

$$K_{gs} = \frac{3}{4} C_D \frac{\varepsilon_s \varepsilon_g \rho_g \left| \vec{u}_s - \vec{u}_g \right|}{d_p} \varepsilon_g^{-2.65} \tag{5}$$

For $\varepsilon_g \leq 0.8$:

$$K_{gs} = 150 \frac{\varepsilon_s^2 \mu_g}{\varepsilon_g d_s^2} + 1.75 \frac{\varepsilon_s \rho_g \left| \vec{u}_s - \vec{u}_g \right|}{d_s} \tag{6}$$

$$C_D = \frac{24}{\varepsilon_g Re_s} \left(1 + 0.15(\varepsilon_g Re_s)^{0.687}\right) \tag{7}$$

Johnson and Jackson partial slip model [8]:

$$\mu_s \frac{\partial U_s}{\partial x} = \frac{\pi \phi_s \rho_s \alpha_s g_0 \sqrt{\theta_s}}{2\sqrt{3} \alpha_s^{\max}} U_s, \tag{8}$$

$$\kappa_s \frac{\partial \theta_s}{\partial x} = -\frac{\pi \phi_s U_s^2 \rho_s \alpha_s g_0 \sqrt{\theta_s}}{2\sqrt{3} \alpha_s^{\max}} - \frac{\pi \sqrt{3}\, \phi_s \rho_s \alpha_s g_0 \left(1 - e_w^2\right) \sqrt{\theta_s}}{4 \alpha_s^{\max}} \theta_s. \tag{9}$$

where μ_s and κ_s are the viscosity and conductivity of the solid phase, and ϕ_s and e_w are the specularity coefficient and the particle–wall coefficient of restitution, respectively.

Numerical simulations were performed on a structured mesh of 56 × 200 cells (11,200 total), corresponding to a uniform grid spacing of 5 mm. Time integration employed an implicit first-order scheme with a base time step of $\delta t = 10^{-3}$s. Postprocessing was conducted in ParaView and with OpenFOAM utilities. Pressure drop and BER were computed and compared against both experimental data and results from recent CFD studies. To improve reliability, a new mass-conservation-based approach for BER calculation was tested alongside conventional midline methods.

3 Literature Review

The benchmark of Taghipour et al. [1] remains a cornerstone for validating CFD models of gas–solid fluidized beds, providing reliable experimental data on pressure drop and BER. Their work demonstrated the applicability of Eulerian–Eulerian two-fluid models (TFM) with drag laws such as Wen–Yu [4], Syamlal–O'Brien [5], and Gidaspow [6]. Because of its reproducibility, this case has been widely adopted in later validation studies. Several authors have extended the benchmark using different CFD solvers. Herzog et al. [9] compared Fluent, MFIX, and OpenFOAM, finding that global parameters like pressure drop were captured reasonably well. Londono [10] reported larger deviations, while Kusriantoko et al. [11] emphasized sensitivity to mesh resolution, boundary conditions, and particle restitution. Fatti [7] focused on OpenFOAM and showed that pressure drop was stable across numerical setups, but BER was strongly influenced by wall models. Liu [12] highlighted the role of adaptive time-stepping and averaging windows, which improved stability of results. A major source of disagreement across studies is the calculation of the BER. Herzog [9] and Liu [12] used pressure-drop methods, whereas Kusriantoko [11] applied a midline solid fraction approach, which tended to overpredict expansion. This inconsistency underlines the need for more reliable postprocessing techniques. More recent works expanded validation to different solvers and scales. Shi et al. [13] compared 2D and 3D models in Fluent, recommending 3D for accuracy but retaining 2D for sensitivity analysis. Reyes-Urrutia et al. [14] compared OpenFOAM and MFIX for fluidized beds with heat transfer, finding both reliable but MFIX slightly more accurate. Patil [15] and Armstrong [16] validated similar cases using CFX and Fluent, further confirming the robustness of Taghipour's benchmark [1] across platforms. In summary, the literature demonstrates that predictive accuracy depends strongly on drag models, boundary conditions, and numerical settings. Earlier OpenFOAM studies (Herzog [9], Londono [10]) reported significant deviations, whereas more recent works (Fatti [7], Kusriantoko [11]) suggest that careful parameter selection can yield accurate results. However, no study has yet applied the latest OpenFOAM v12 to this benchmark. The present work addresses this gap by assessing solver performance and proposing a new mass-conservation-based BER calculation method as an alternative to existing approaches.

4 Results and Discussion

The inherently transient processes of bubble coalescence and breakup generate significant pressure-drop oscillations within the fluidized bed [9]. To avoid the influence of these initial fluctuations, the pressure drop used for comparison was calculated as a time-averaged quantity only after the flow had reached a statistically steady state. Consistent with the procedure reported by Taghipour [1], the averaging of global parameters commenced after 3 s of simulated time.

The pressure drop results obtained by various researchers using OpenFOAM for Taghipour's setup are summarized in Tab. 1. All simulations considered here correspond to an inflow gas velocity of 0.38 m/s. Earlier numerical studies using OpenFOAM have reported pressure-drop values between 5027 Pa and 8064 Pa, compared with Taghipour's experimental measurement of 5423.398 Pa. The simulations by Herzog [9] and Londono [10] show the greatest departure from the experiment, predicting 7072 Pa and 8064 Pa, respectively. Conversely, results on the lower end of the reported range tend to more closely match the experimental value. Although Herzog (2012), Londono (2012), Fatti (2021), and Kusriantoko (2024) all used OpenFOAM, their reported values differ due to changes in solver versions and the availability of specific models. A noticeable trend appears: as the simulations become more recent, their deviation from the experimental benchmark decreases. The most up-to-date results, produced by Fatti and Kusriantoko, are in close agreement; however, because Fatti provides more extensive methodological details, their findings are adopted as the primary

Table 1: Pressure drop results obtained by various authors

Sources	Pressure [Pa]
Experiment	5423.398
Herzog	7072.423
Londono	8064.067
Fatti J&J	5067.370
Kusriantoko	5072.617

Table 2: Comparison

Time discretization	This work's dP [Pa]	Deviation from results of
10^{-5}s euler	5145.96	Fatti et al. is 1.5%
10^{-3}s euler	5132.88	Kusriantoko et al. is 1.2%

point of comparison in this work.

To assess the reliability of the present simulations, the parameter set used by Fatti was replicated. This resulted in a predicted pressure drop of 5145.96 Pa, differing from Fatti's value by only 1.5%. Likewise, reproducing the conditions reported by Kusriantoko produced a pressure drop of 5132.88 Pa, corresponding to a 1.2% deviation (Tab. 2). These results demonstrate that the solver version employed in this study (v12) yields predictions that closely align with the recent OpenFOAM investigations of Fatti and Kusriantoko et al., and that the discrepancies are considerably smaller than those observed in earlier works, such as those by Herzog and Londono.

According to the Ergun equation, which is applied in the Gidaspow drag model for $\varepsilon \leq 0.8$ (Eq. (6)), the pressure drop across a bed increases with increasing superficial gas velocity [17]. When the gas velocity reaches a value at which the drag force on the particles balances their weight ($m \times g$), the bed becomes fluidized. This velocity is referred to as the minimum fluidization velocity u_{mf}. Figure 1 presents the dependence of bed pressure drop on inflow air velocity reported by various authors. The results of the current simulations, as well as those of Kusriantoko and Herzog, are consistent with the prediction according to the Ergun equation, showing an increase in pressure drop with increasing velocity until the inflow velocity reaches u_{mf}. In contrast, the plots reported by Fatti and Londono exhibit deviations from this trend at low velocities. However, it is noticeable that Kusriantoko's pressure drop begins to level off later than both the current simulation results and Fatti's results. This discrepancy may be due to a lack of velocity points: there are insufficient data near u_{mf}, making it difficult to determine whether the pressure drop increases exactly up to u_{mf}. For velocities above $u_{mf} = 0.62$ m/s, the results of the current work closely match those of Fatti et al. and Kusriantoko et al.

Tab. 3 presents BER reported by different authors alongside the experimental value of 1.491 obtained by Taghipour. The results show significant variation across studies, with reported values ranging from 1.343 (Taghipour) to 1.721 (Kusriantoko's OpenFOAM).

It is clear that pressure drop is the difference of time-averaged and spatial-averaged (along boundary) pressure between inlet and outlet boundaries. However BER can be determined in three ways: from the pressure drop along a vertical midline of the bed (*midline* ΔP method);

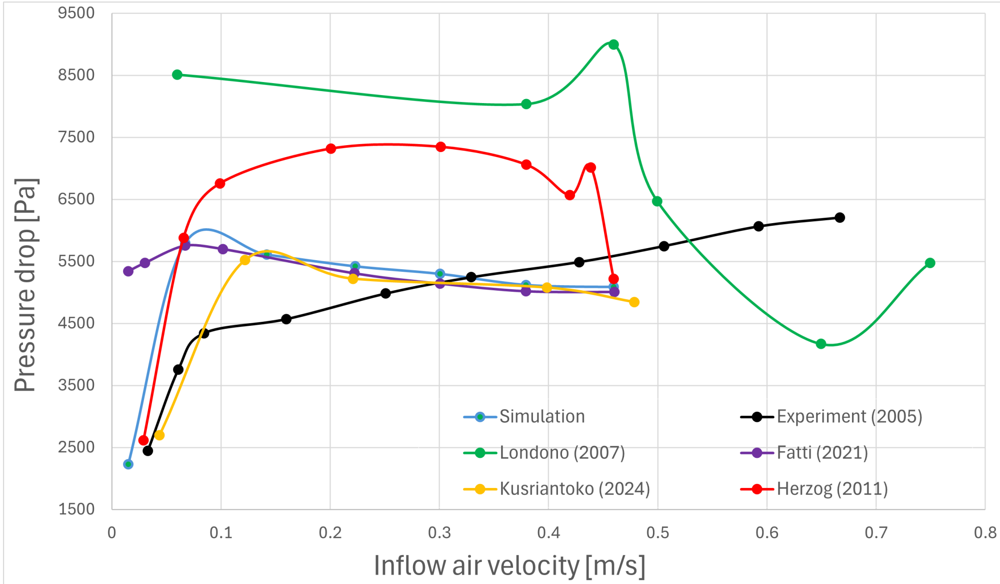

Figure 1: Bed pressure drop – inflow velocity relation comparison with the results of other researchers

Table 3: Bed expansion ratio BER results obtained by various authors

Sources	BER
Experiment	1.491
Herzog	1.538
Londono	1.380
Fatti J&J	1.554
Kusriantoko	1.721

from time-averaged gas fraction along midline (*midline gas fr.* method); from calculating sum of particles time-averaged mass (*domain mass* method).

Herzog [9] and Liu [12] applied the *midline* ΔP but did not specify the criterion referred to as the *threshold*, calculated by Eq. (10). The bed ratio is defined as the ratio between the bed height at which the threshold reaches 1 and the initial bed height:

$$\text{threshold} = \frac{P_{\text{inlet}} - P(h)}{P_{\text{inlet}} - P_{\text{outlet}}}. \tag{10}$$

Regarding the *midline gas fr.* method, Kusriantoko [11] proposed a criterion based on the gas fraction $\varepsilon_g(h)$ along the midline. In this method, the height at which the *threshold* that is gas fraction equals 0.99 defines the extent of BER, and the BER is computed as the ratio between this height and the initial bed height.

The *domain mass* method was developed in the present work. It is based on integrating the solid mass (or, equivalently for incompressible flow, the solid volume fraction) over all cells of the computational domain. The bed ratio is defined as the height below which the total solid mass in the bed is nearly equal to the total solid mass in the entire domain. The corresponding *threshold* is computed using Eq. (11):

$$\text{threshold} = \frac{\sum_{j=1}^{j(h)} \sum_{i=1}^{56} \varepsilon_s}{\varepsilon_{s,\text{init}}\, H_0}, \tag{11}$$

where i and j denote the horizontal and vertical cell indices, respectively.

Kusriantoko used the $\delta t = 10^{-3}$s euler scheme for time discretization. He determined BER at height where the air fraction reached 0.99 and got $H/H_0 = 1.721$. The value of the current work by the scheme and method with a threshold such as Kusriantoko is equal to 1.658. That's deviation from Kusriantoko's is 3.6%. However, the values taken by Kusriantoko overcome the values of the results of all other authors. Moreover, the current work value by Kusriantoko's method overcomes the current work values by other methods too according to Tab. 4. For this reason, the Kusriantoko's method is discarded for the following studies.

Table 4: Bed expansion ratio BER calculated by time discretization of $\delta t = 10^{-3}$s, scheme euler

Threshold/Method	midline ΔP	midline gas fr.	domain mass
0.950	1.382	1.621	1.438
0.980	1.482	1.646	1.525
0.990	1.533	1.658	1.562

Fatti used $\delta t = 10^{-5}$s euler and got $H/H_0 = 1.554$. Among the results by different methods for H/H_0 shown in Tab. 5, results taken by *midline* ΔP method with 0.99 threshold and by *domain mass* method with 0.98 threshold are closest to Fatti's result. It approves the use of *midline* ΔP method with 0.99 threshold or domains mass method with 0.98 threshold.

Table 5: Bed expansion ratio BER calculated by time discretization of $\delta t = 10^{-5}$s, scheme euler

Threshold/Method	midline ΔP	domain mass
0.950	1.382	1.462
0.980	1.482	1.537
0.990	1.533	1.587

Fig. 2 shows BER dependence on inflow air velocity of different authors. Across all cases, the BER increases with inflow velocity, and the growth patterns similar. Kusriantoko's OpenFOAM results show the highest expansion values overall, due to the use of method *midline gas fr.*. Plot of Fatti and this works simulation almost overlaps each other, meaning that they validate each other. This simulations plot was done by method *domain mass* (plots of by method *midline* ΔP and by method *domain mass* gave the same plot). Both of these plots are close to experimental result. Herzog reported not as smooth as Kusriantoko, but closely matching experimental and simulation trends. In contrast, Londono [10] observed comparable behavior at low velocities but significant fluctuations above 0.4 m/s, with expansion ratios oscillating between 1.45 and 1.8. These deviations suggest flow instabilities not present in Herzog's data.

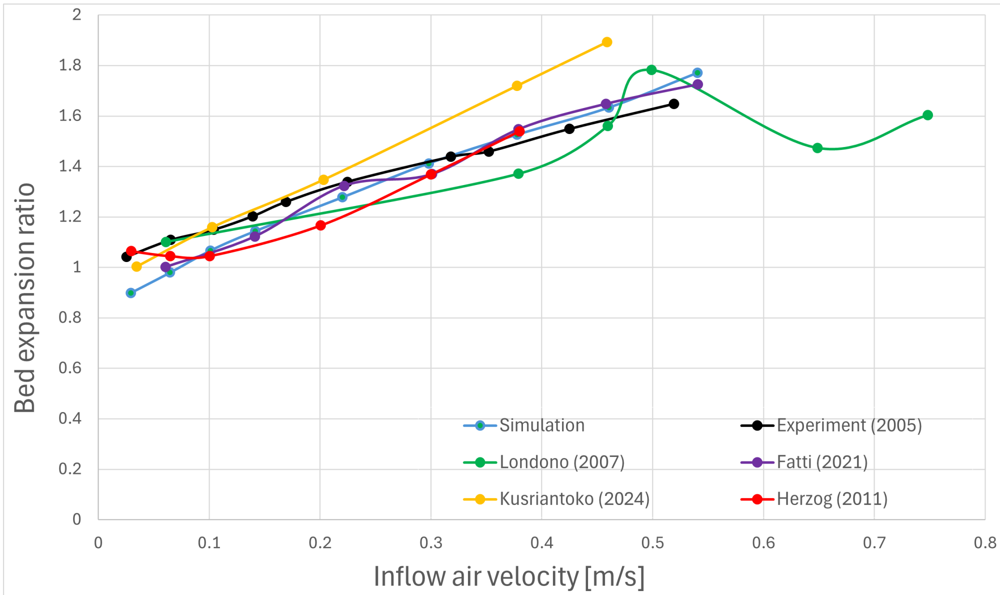

Figure 2: Bed expansion ratio BER – inflow velocity relation comparison with the results of other researchers

5 Conclusion

This work assessed the capability of OpenFOAM v12 to simulate gas–solid fluidized bed hydrodynamics against the benchmark of Taghipour et al. Results confirmed that the solver provides accurate pressure drop predictions, with deviations under 6% relative to experiments and under 2% compared with recent CFD studies. The newly proposed mass-conservation-based method for calculating BER was shown to deliver more consistent and experimentally aligned results than conventional midline solid fraction approaches, which tend to overpredict expansion. Overall, the study demonstrates that OpenFOAM v12 can reliably reproduce key hydrodynamic indicators of fluidized beds while offering improved postprocessing strategies. These advances support the application of CFD in industrial design, scale-up, and optimization of fluidized bed reactors.

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EXACT SOLUTIONS OF EQUATIONS OF THE TWO-PRIMARY-BODY PROBLEM IN THE RESTRICTED THREE-BODY PROBLEM WITH VARIABLE MASSES

This study investigates the translational-rotational motion of a non-stationary, axisymmetric body of variable mass in the Newtonian gravitational field of two primary spherical bodies with variable masses, formulated within the framework of the restricted three-body problem with variable masses in a barycentric coordinate system. The masses of the bodies vary isotropically. The small axisymmetric body may change its size and shape while remaining axially symmetric throughout the process. The restricted formulation implies that the small body does not influence the motion of the two primary spherical bodies with variable masses. The study focuses on the secular perturbations of translational-rotational motion in the considered three-body system. Since the exact solutions for the translational-rotational motion of the two primary spherical bodies with variable masses in the barycentric coordinate system are unknown, the differential equations of the two-body problem and those of the non-stationary small body are investigated jointly. Due to the complexity of the problem, the translational-rotational motion of the three-body system is studied using perturbation theory in analogues of Delaunay-Andoyer variables. Exact analytical solutions of the differential equations for the secular perturbations of the translational-rotational motion in the problem of two primary spherical bodies in terms of Delaunay-Andoyer variable analogues are obtained. These exact solutions open the possibility of further investigating the secular perturbations of the translational-rotational motion of a non-stationary, axisymmetric body within the restricted three-body problem with variable masses.
Key words: Restricted three-body problem, variable mass, translational-rotational motion, analogues of the Delaunay-Andoyer variables, perturbation theory.

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Шектелген үш дене мәселесі аясындағы массалары айнымалы екі негізгі дене мәселесі теңдеулерінің нақты шешімдері

Бұл жұмыста шектелген үш дене есебінің аясында массалары айнымалы екі негізгі сфералық дененің ньютондық тартылыс өрісіндегі массасы айнымалы бейстационар өстік симметриялы кіші дененің барицентрлік координаттар жүйесіндегі ілгерілемелі-айналмалы қозғалысы қарастырылады. Денелердің массалары уақыт бойынша изотропты өзгереді, сондықтан реактивті күштер мен реактивті моменттер пайда болмайды. Кіші өстік симметриялы дененің өлшемі мен пішіні уақыт бойынша өзгереді, бірақ әрқашан өстік симметриялы күйін сақтайды. Шектелген есептің қойылымы - кіші массалы дене екі негізгі сфералық дененің қозғалысына әсер етпейдігін сипаттайды. Бұл жұмыста үш дененің де ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқулары зерттеледі. Массалары айнымалы екі негізгі сфералық дененің барицентрлік координаттар жүйедегі ілгерілемелі-айналмалы қозғалысының дәл шешімдері белгісіз болғандықтан, екі негізгі дененің және бейстационар кіші дененің дифференциалдық теңдеулері бірлесіп қарастырылады. Мәселенің күрделілігіне байланысты үш денелі жүйенің ілгерілемелі-айналмалы қозғалысы Делоне-Андуайе айнымалыларының аналогтарындағы ұйытқу теориясы әдістерімен зерттелді. Нәтижесінде екі негізгі сфералық дененің ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқу теңдеулерінің дәл аналитикалық шешімдері алынды.

Алынған шешімдер шектелген үш дене есебінің аясында массасы айнымалы бейстационар өстік симметриялы дененің ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқуын әрі қарай зерттеуге мүмкіндік береді.
Түйін сөздер: шектелген үш дене мәселесі, айнымалы масса, айналмалы-ілгерілемелі қозғалысы, Делоне-Андуайе айнымалыларының аналогтары, ұйытқу теориясы.

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Точные решения уравнений задачи двух основных тел в ограниченной задаче трех тел с переменными массами

В работе рассмотрено поступательно-вращательное движение нестационарного малого осесимметричного тела переменной массы в ньютоновском поле притяжения двух основных сферических тел с переменными массами в рамках ограниченной задачи трех тел с переменными массами в барицентрической системе координат. Массы тел меняются со временем изотропно, поэтому не появляются реактивные силы и реактивные моменты. Малое осесимметричное тело может менять размеры и формы при этом все время остается осесимметричным. Ограниченная постановка задачи характеризуется тем, что малое тело не влияет на движение двух основных сферических тел с переменными массами. Исследуется вековые возмущения поступательно-вращательного движения в рассматриваемой проблеме всех трех тел. Решения поступательно-вращательного движения двух основных сферических тел с переменными массами в барицентрической системе координат неизвестна, поэтому дифференциальные уравнения задачи двух основных тел и дифференциальные уравнения нестационарного малого тела исследуется совместно. Проблема сложная, поэтому поступательно-вращательное движение системы трех тел исследуется методами теорий возмущений в аналогах переменных Делоне-Андуайе. Были получены точные аналитические решения дифференциальных уравнений вековых возмущений поступательно-вращательного движения задачи двух основных сферических тел в аналогах переменных Делоне-Андуайе. Эти решения открывают возможность дальнейшего исследования вековых возмущений поступательно-вращательного движения нестационарного малого осесимметричного тела в рамках ограниченной задачи трёх тел с переменными массами.

Ключевые слова: ограниченная задача трех тел, переменная масса, поступательно-вращательное движение, аналоги переменных Делоне-Андуайе, теория возмущения.

1 Introduction

The investigation of the impact of the variability of celestial bodies' masses on the dynamic evolution of gravitational systems is a relevant problem in modern astronomy and astrophysics. In this paper, we investigate the celestial-mechanical formulation of the problem of the translational-rotational motion of a non-stationary axisymmetric small body in the Newtonian gravitational field of two primary spherically symmetric bodies with variable masses and radius in a restricted formulation.

The masses of the bodies are variable, with the laws of mass variation being arbitrarily prescribed functions of time - $m_1 = m_1(t), m_2 = m_2(t), m_3 = m_3(t)$. In the general case, the mass changes occur isotropically but at different rates

$$\frac{\dot{m}_1(t)}{m_1(t)} \neq \frac{\dot{m}_2(t)}{m_2(t)} \neq \frac{\dot{m}_3(t)}{m_3(t)}$$

no reactive forces or reactive moments appear. A small axisymmetric body can change its size and shape, has three mutually perpendicular planes of symmetry, and remains axisymmetric at all times. In this work, the restricted form of the problem is characterised by the fact that the small body does not affect the motion of the two primary spherical bodies.

A brief overview of studies related to the present work is provided. The formation and dynamical evolution of planetary systems is a central theme of modern astronomy. The influence of single and binary stars on planet formation is of great importance, since nearly half of all stars are found in binary or multiple stellar systems. One of the most effective ways to assess this influence is to obtain an accurate picture of the population of binary stars. In [1], an extensive database was created as a result of a comprehensive literature survey, in order to carry out a complete census of all known binary stars hosting planets to date. The database includes the characteristics (orbit or separation, stellar masses, dynamical stability, etc.) of 759 systems (including 31 circumbinary systems), which is nine times larger than the previous complete census of binary stars with planets. Among the 728 S-type systems, 651 are binaries, 73 are triples, and 4 are quadruples.

Binary stars are considered key natural laboratories for the study of stellar physics, which explains their inclusion in photometric space observations starting from the very first orbital telescope launched in 1968. The review [2] follows the history of binary star observations and the scientific insights gained, beginning with the early ultraviolet missions, moving through the phase of mission diversification with various satellite projects, and reaching the present stage of large-scale surveys focused on planetary transits. Over this time, detached, semi-detached, and contact binaries have been studied, comprising stars at different evolutionary stages—dwarfs, subgiants, giants, supergiants—as well as compact objects such as white dwarfs and neutron stars, often accompanied by planets or accretion disks. Modern surveys have uncovered a wide range of phenomena, including pulsating stars in eclipsing binaries and systems that host transiting planets. Particular emphasis is placed on eclipsing binaries due to their high scientific value, and on the most recent missions, which, owing to their extensive sky coverage, provide unique opportunities for comprehensive astrophysical research.

A group of researchers from NASA's Eclipsing Binary Patrol citizen science project [3] has published a catalogue containing 10,001 eclipsing binary star systems. This discovery significantly expands our knowledge of stellar physics and formation processes, and opens up new opportunities for the search for exoplanets.

In [4], an interesting object was discovered 70 light years from Earth: the amazing world of the ν Octantis system, an exoplanetary system consisting of binary stars and one planet. The main star there is slightly more massive than the Sun, and its companion is a white dwarf. The planet, squeezed into a narrow space between the binary stars, not only exists in a complex gravitational environment, but also has a retrograde orbit. The main star is a subgiant, 1.6 times more massive than the Sun, and the second is an object with a mass of about half that of the Sun. They orbit each other with a period of 1,050 days.

In [5], a unique planet was discovered in the 2M1510 system. The planet's orbit is almost perpendicular to the plane of the binary stars' orbits. The shape of the planet itself is normal, but it has an unusual orbit. There are three brown dwarfs in the system — too large to be planets, too small to be full-fledged stars. Two dwarfs revolve around each other, while the third, located further away, revolves around both of its companions. Planet 2M1510 b, which has attracted the interest of scientists, is in a polar orbit around the two central brown dwarfs.

In [6], the classical dynamics of binary stars undergoing mass exchange between them is studied. Assuming that one of the stars is more massive than the other, the dynamics of the lighter star is analysed as a function of its mass change over time. Within the framework of approximations and mass transfer models, a general result is obtained which establishes that if the lighter star loses mass, its period increases. If the lighter star gains mass, its period decreases. Such non-stationarity in the dynamics of binary stars can significantly affect the dynamic evolution of planetary systems around binary stars. We also note a two-volume fundamental monograph devoted to close binary stars [7]. In the book [8], the evolution of the rotational motion of a rigid body about its center of mass is examined under the assumption that the body's mass and dimensions remain constant. The rotational motion of a triaxial satellite about its center of mass with moments of inertia close to one another is analyzed, and several interesting results are obtained.

This review shows that the development of celestial-mechanical models of non-stationary binary stars and planets is a relevant topic.

This work is structured as follows. Section 2 presents the formulation of the problem of translational-rotational motion of a non-stationary axisymmetric small body in the Newtonian gravitational field of two massive spherically symmetric bodies with variable masses and radius in a restricted formulation and the equations of motion. Section 3 provides exact analytical solutions for the rotational motion of two primary bodies in variable Eulers. Section 4 provides exact analytical solutions for the equation of secular perturbations of the translational motion of the centre of mass of two primary spherical bodies with variable masses in a barycentric coordinate system. In Section 5, analytical expressions for the coordinates and velocities of two primary spherical bodies with variable masses are obtained based on exact solutions of the differential equations of secular perturbations. The conclusion highlights the main result of the work and further prospects for research.

2 Formulation of the problem

In this work considered translational-rotational motion of three non-stationary celestial bodies with variable masses. There, P_1, P_2 are the primary spherically symmetric bodies with variable masses and variable dimensions, whose motion is determined by the problem of these two bodies with variable masses.

The third body P_3 is axisymmetric small body, and does not affect the motion of the first two bodies. The body P_3 has three mutually perpendicular planes of symmetry. The principal axes of inertia of the own coordinate system are directed along the line of intersection of the three mutually perpendicular planes, and this position is preserved during evolution.

Assumptions and differential equations of the problem in the barycentric coordinate system were obtained in [9]. The equations of translational motion of two spherically symmetric bodies in the barycentric coordinate system are as follows

$$\ddot{\vec{r}}_i = -f\tilde{m}_i\frac{\vec{r}_i}{r_i^3} + \tilde{A}_i\dot{\vec{r}}_i + \tilde{B}_i\vec{r}_i, \tag{1}$$

where, f - gravitational constant, $\tilde{m}_i = \tilde{m}_i(t) = m\nu_j^3 = m_j^3(t)/m^2, \nu_j = m_j/m, m = m_1 + m_2, \tilde{A}_i = 2\dot{\nu}_j/\nu_j, \tilde{B}_i = \ddot{\nu}_j/\nu_j - 2\dot{\nu}_j^2/\nu_j^2, i \neq j, i, j = 1, 2$.

The translational- rotational motion of two non-stationary bodies can be conveniently studied in analogues of Delaunay-Andoyer variables [10], [11].

Rotational motion of two spherical bodies with variable masses will first be considered in analogues of Euler's variables. Euler's dynamic equations are greatly simplified due to the spherical symmetry of the bodies and take the form [12]

$$\frac{d}{dt}(A_i p_i) = 0, \qquad \frac{d}{dt}(B_i q_i) = 0, \qquad \frac{d}{dt}(C_i r_i) = 0, \tag{2}$$

Accordingly, Euler's kinematic equations can be written as

$$p_i = \dot{\psi}_i \sin\theta_i \sin\varphi_i + \dot{\theta}_i \cos\varphi_i, \;\; q_i = \dot{\psi}_i \sin\theta_i \cos\varphi_i - \dot{\theta}_i \sin\varphi_i, \;\; r_i = \dot{\psi}_i \cos\theta_i + \dot{\varphi}_i, \tag{3}$$

The equations of translational-rotational motion of a small non-stationary axisymmetric body P_3 have the form

$$\ddot{\vec{r}}_3 = grad_{\vec{r}_3}\tilde{U} + \tilde{A}_{23}\dot{\vec{r}}_2 + \tilde{B}_{23}\vec{r}_2, \tag{4}$$

$$\begin{aligned}
&\frac{d}{dt}(A_3 p_3) - (A_3 - C_3)\, q_3 r_3 = \left[\frac{\partial \tilde{U}}{\partial \psi_3} - \cos\theta_3 \frac{\partial \tilde{U}}{\partial \varphi_3}\right] \frac{\sin\varphi_3}{\sin\theta_3} + \cos\varphi_3 \frac{\partial \tilde{U}}{\partial \theta_3},\\
&\frac{d}{dt}(A_3 q_3) - (C_3 - A_3)\, r_3 p_3 = \left[\frac{\partial \tilde{U}}{\partial \psi_3} - \cos\theta_3 \frac{\partial \tilde{U}}{\partial \varphi_3}\right] \frac{\cos\varphi_3}{\sin\theta_3} - \sin\varphi_3 \frac{\partial \tilde{U}}{\partial \theta_3},\\
&\frac{d}{dt}(C_3 r_3) = 0,
\end{aligned} \tag{5}$$

$$\begin{gathered}
p_3 = \dot{\psi}_3 \sin\theta_3 \sin\varphi_3 + \dot{\theta}_3 \cos\varphi_3, \qquad q_3 = \dot{\psi}_3 \sin\theta_3 \cos\varphi_3 - \dot{\theta}_3 \sin\varphi_3,\\
r_3 = \dot{\psi}_3 \cos\theta_3 + \dot{\varphi}_3,
\end{gathered} \tag{6}$$

$$\tilde{U} = f\left(\frac{m_1}{r_{31}} + \frac{m_2}{r_{32}}\right) + f\,(C_3 - A_3)\,\frac{1}{2}\left(\frac{1 - 3\gamma_{31}^2}{r_{31}^2} + \frac{1 - 3\gamma_{32}^2}{r_{32}^2}\right) \tag{7}$$

$$\tilde{A}_{23} = -\frac{2\tilde{A}}{\nu_1}, \tilde{B}_{23} = -\frac{\tilde{B}}{\nu_1} + 4\tilde{A}\frac{\dot{\nu}_1}{\nu_1^2},$$

$$\begin{aligned}
\gamma_{31} &= a_{13}\frac{x_1 - x_3}{r_{31}} + a_{23}\frac{y_1 - y_3}{r_{31}} + a_{33}\frac{z_1 - z_3}{r_{31}},\\
\gamma_{32} &= a_{13}\frac{x_2 - x_3}{r_{32}} + a_{23}\frac{y_2 - y_3}{r_{32}} + a_{33}\frac{z_2 - z_3}{r_{32}},
\end{aligned} \tag{8}$$

In equations (1) - (3), (4) - (6), the notation used in [9] is retained.

Solutions for the translational-rotational motion of two primary spherical bodies with variable masses in a barycentric coordinate system are unknown [13], [14], therefore the differential equations of the two primary bodies and the differential equations of the non-stationary small body are investigated jointly. The problem is complex, so the translational-rotational motion of a three-body system is investigated using perturbation theory methods [15], [16], [17] in analogues of Delaunay-Andoyer variables.

3 Exact analytical solutions for the rotational motion of two primary spherical bodies in analogues of Euler variables

Note that the rotational motion of spherical bodies P_1, P_2 has a simple solution, since throughout its evolution, a spherically symmetric body retains its spherical density distribution and spherical external shape.

From equations (2) we obtain

$$A_i p_i = const = A_{i0} p_{i0}, \qquad B_i q_i = const = B_{i0} q_{i0}, \qquad C_i r_i = const = C_{i0} r_{i0}, \tag{9}$$

From this follows the module of the kinetic momentum vector $\vec{K}_{i0}$, of the body P_i a constant value

$$A_i^2 p_i^2 + B_i^2 q_i^2 + C_i^2 r_i^2 = A_{i0}^2 \left(p_{i0}^2 + q_{i0}^2 + r_{i0}^2\right) = const = K_{i0}^2 \tag{10}$$

Let the vector $\vec{K}_{i0}$ be directed along the OZ axis, then the following formulas can be written [18]

$$\begin{array}{l} A_i p_i = A_i p_i = K_{i0} \sin\theta_i \sin\varphi_i = K_{iOx}, \\ B_i q_i = A_i q_i = K_{i0} \sin\theta_i \cos\varphi_i = K_{iOy}, \\ C_i r_i = A_i r_i = K_{i0} \cos\theta_i = K_{iOz}, \end{array} \tag{11}$$

Since $C_i r_i = A_i r_i = A_{i0} r_{i0} = K_{i0} \cos\theta_i = K_{iOz}$,

$$\cos\theta_i = \frac{C_i r_i}{K_{i0}} = \frac{K_{iOz}}{K_{i0}} = \frac{A_{i0} r_{i0}}{A_{i0}\sqrt{p_{i0}^2 + q_{i0}^2 + r_{i0}^2}} = \frac{r_{i0}}{\sqrt{p_{i0}^2 + q_{i0}^2 + r_{i0}^2}} = const \tag{12}$$

Therefore,

$$\dot{\theta}_i = 0 \tag{13}$$

Substituting (13) into equations (3), we obtain

$$p_i = \dot{\psi}_i \sin\theta_i \sin\varphi_i, \qquad q_i = \dot{\psi}_i \sin\theta_i \cos\varphi_i, \qquad r_i = \dot{\psi}_i \cos\theta_i + \dot{\varphi}_i, \tag{14}$$

From the first equation (14) and the first equation (11), the following solution of $\dot{\psi}_i$ is obtained

$$p_i = \dot{\psi}_i \sin\theta_i \sin\varphi_i, \qquad A_i p_i = K_{i0} \sin\theta_i \sin\varphi_i, \tag{15}$$

$$\dot{\psi}_i = \frac{K_{i0}}{A_i}, \tag{16}$$

From the last equation (14), we find the solution $\dot{\varphi}_i$. Let us substitute the solution $\dot{\psi}_i$ and obtain the following

$$\dot{\varphi}_i = r_i - \dot{\psi}_i \cos\theta_i = r_i - \frac{K_{i0}}{A_i}\cos\theta_{i0} = \frac{A_i r_i \cos\theta_{i0}}{A_i} = \frac{A_{i0} r_{i0} \cos\theta_{i0}}{A_i}, \tag{17}$$

As a result, we obtain the following results

$$\cos\theta_i = \cos\theta_{i0} = \frac{K_{i0z}}{K_{i0}} = const, \qquad \theta_i = \theta_{i0} = const, \qquad \dot{\theta}_i = 0, \tag{18}$$

$$\dot{\psi}_i = \frac{K_{i0}}{A_i} \neq const, \tag{19}$$

$$\dot{\varphi}_i = \frac{A_{i0} r_{i0} - K_{i0}\cos\theta_{i0}}{A_i} = \frac{A_{i0} r_{i0} - C_{i0} r_{i0}}{A_i} = \frac{A_{i0} r_{i0} - A_{i0} r_{i0}}{A_i} = 0,$$

$$\dot{\varphi}_i = 0, \qquad \varphi_i = \varphi_i\left(t_0\right) = \varphi_{i0} = const \tag{20}$$

Substituting solutions (18), (19) and (20) into equations (14), we obtain the following equations

$$p_i = \frac{K_{i0}}{A_i}\sin\theta_{i0}\sin\varphi_{i0}, \qquad q_i = \frac{K_{i0}}{A_i}\sin\theta_{i0}\cos\varphi_{i0}, \qquad q_i = \frac{K_{i0}}{A_i}\cos\theta_{i0}, \tag{21}$$

Thus, in a non-rotating coordinate system, we get

$$\begin{aligned} \omega_{iX} &= \dot{\theta}_i \cos\psi_i + \dot{\varphi}_i \sin\theta_i \sin\psi_{i0}, \\ \omega_{iY} &= \dot{\theta}_i \sin\psi_i - \dot{\varphi}_i \sin\theta_i \cos\psi_{i0}, \\ \omega_{iZ} &= \dot{\psi}_i + \dot{\varphi}_i \cos\theta_i, \end{aligned} \tag{22}$$

$$\omega_{iX} = 0, \qquad \omega_{iY} = 0, \qquad \omega_{iZ} = \dot{\psi}_i, \tag{23}$$

$$\omega_i = \sqrt{\omega_{iX}^2 + \omega_{iY}^2 + \omega_{iZ}^2} = \dot{\psi}_i \neq const \tag{24}$$

The solutions found for the differential equations of rotational motion retain their form even in secular perturbations. They will be used in calculations of the total kinetic moment of translational-rotational motion of a gravitational system when analysing dynamic evolution within the framework of a restricted three-body problem in analogues of the Delaunay-Andoyer variables.

4 Exact analytical solutions of the equation of secular perturbations of the translational motion of the centers of mass of two primary spherical bodies with variable masses in the baricentric coordinate system.

4.1 Derivation of the perturbation function

Due to its complexity, so the translational-rotational motion of a restricted three-body system is studied via perturbation theory methods in Delaunay-Andoyer type variables and exact analytical solutions are obtained for secular perturbation equations of the translational-rotational motion of two primary spherical bodies in analogues of Delaunay-Andoyer variables.

Using perturbation theory based on aperiodic motion along a quasi-conical section [12], we derive the perturbing functions from equations (1). We rewrite equations (1) as

$$\ddot{\vec{r}}_1 = -f\tilde{m}_1\frac{\vec{r}_1}{r_1^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\dot{\vec{r}}_1 + \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\frac{\dot{\gamma}_1}{\gamma_1}\right]\vec{r}_1 + \vec{F}_{1pert}, \tag{25}$$

$$\ddot{\vec{r}}_1 + f\tilde{m}_1\frac{\vec{r}_1}{r_1^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\dot{\vec{r}}_1 - \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\frac{\dot{\gamma}_1}{\gamma_1}\right]\vec{r}_1 = \vec{F}_{1pert}, \tag{26}$$

$$\vec{F}_{1pert} = \tilde{B}_1\vec{r}_1 - \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\frac{\dot{\gamma}_1}{\gamma_1}\right]\vec{r}_1, \tag{27}$$

The unknown arbitrary function $\gamma_1 = \gamma_1(t)$ is defined by the following conditions

$$\frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right) = \tilde{A}_1 = 2\frac{\dot{\nu}_2}{\nu_2}, \tag{28}$$

Then we get

$$\gamma_1 = \gamma_1(t) = \frac{\tilde{m}_1(t_0)}{\tilde{m}_1(t)}e^{2\int_{t_0}^{t}\tilde{A}_1 dt} = \frac{m_2}{m_{20}}\frac{m_0^2}{m^2}, \tag{29}$$

Taking into account equations (28) and (29) from (27), we obtain

$$\vec{F}_{1pert} = \tilde{B}_1^*(t)\,\vec{r}_1, \tag{30}$$

Accordingly, using γ_1 we can formulate an explicit form of the function $\dot{B}_1^*$

$$\tilde{B}_1^*(t) = \tilde{B}_1 - \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \tilde{A}_1\frac{\dot{\gamma}_1}{\gamma_1}\right], \tag{31}$$

In the result, equation (26) takes the form

$$\ddot{\vec{r}}_1 + f\tilde{m}_1\frac{\vec{r}_1}{r_1^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\dot{\vec{r}}_1 - \left[\frac{\ddot{\gamma}_1}{\gamma_1} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_1}{\tilde{m}_1} + \frac{\dot{\gamma}_1}{\gamma_1}\right)\frac{\dot{\gamma}_1}{\gamma_1}\right]\vec{r}_1 = grad_{\vec{r}_1}U_1, \tag{32}$$

where, $\vec{F}_{1pert} = \tilde{B}_1^*\vec{r}_1 = grad_{\vec{r}_1}U_1$ - perturbing force, U_1 - perturbing function

$$U_1 = \frac{1}{2}\tilde{B}_1^*(t)r_1^2, \tag{33}$$

Similarly, we write the differential equations (1) in a form convenient for using perturbation theory for the body P_2.

$$\ddot{\vec{r}}_2 + f\tilde{m}_2\frac{\vec{r}_2}{r_2^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_2}{\tilde{m}_2} + \frac{\dot{\gamma}_2}{\gamma_2}\right)\dot{\vec{r}}_2 - \left[\frac{\ddot{\gamma}_2}{\gamma_2} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_2}{\tilde{m}_2} + \frac{\dot{\gamma}_2}{\gamma_2}\right)\frac{\dot{\gamma}_2}{\gamma_2}\right]\vec{r}_2 = grad_{\vec{r}_2}U_2, \tag{34}$$

where, $\vec{F}_{2pert} = \tilde{B}_2^*(t)\vec{r}_2 = grad_{\vec{r}_2}U_2$ - perturbing force, U_2 - perturbing function

$$U_2 = \frac{1}{2}\tilde{B}_2^*(t)r_2^2, \tag{35}$$

$$\tilde{B}_2^*(t) = \tilde{B}_2 - \left[\frac{\ddot{\gamma}_2}{\gamma_2} - \tilde{A}_2\frac{\dot{\gamma}_2}{\gamma_2}\right], \qquad \gamma_2 = \gamma_2(t) = \frac{m_1}{m_{10}}\frac{m_0^2}{m^2}, \tag{36}$$

4.2 Unperturbed motion

In the case where $\vec{F}_{1pert} = 0, \vec{F}_{2pert} = 0$ from equations (32), (34) follows unperturbed motion [12]

$$\ddot{\vec{r}}_i + f\tilde{m}_j\frac{\vec{r}_j}{r_j^3} + \frac{1}{2}\left(\frac{\dot{\tilde{m}}_j}{\tilde{m}_j} + \frac{\dot{\gamma}_j}{\gamma_j}\right)\dot{\vec{r}}_j - \left[\frac{\ddot{\gamma}_j}{\gamma_j} - \frac{1}{2}\left(\frac{\dot{\tilde{m}}_j}{\tilde{m}_j} + \frac{\dot{\gamma}_j}{\gamma_j}\right)\frac{\dot{\gamma}_j}{\gamma_j}\right]\vec{r}_j = 0, \tag{37}$$

The solution of the unperturbed motion (37) is well investigated and has the following form

$$\begin{aligned} x_j &= \gamma_j\rho_j\left[\cos u_j\cos\Omega_j - \sin u_j\sin\Omega_j\cos i_j\right], \\ y_j &= \gamma_j\rho_j\left[\cos u_j\sin\Omega_j + \sin u_j\cos\Omega_j\cos i_j\right], \\ z_j &= \gamma_j\rho_j\left[\sin u_j\sin i_j\right], r_j^2 = x_j^2 + y_j^2 + z_j^2 = \gamma_j^2\rho_j^2 \end{aligned} \tag{38}$$

$$\begin{aligned} \dot{x}_j &= \left(\frac{\dot{\gamma}_j}{\gamma_j} + \frac{\dot{\rho}_j}{\rho_j}\right)x_j + \gamma_j\rho_j\dot{u}_j\left[-\sin u_j\cos\Omega_j - \cos u_j\sin\Omega_j\cos i_j\right], \\ \dot{y}_j &= \left(\frac{\dot{\gamma}_j}{\gamma_j} + \frac{\dot{\rho}_j}{\rho_j}\right)y_j + \gamma_j\rho_j\dot{u}_j\left[-\sin u_j\sin\Omega_j - \cos u_j\cos\Omega_j\cos i_j\right], \\ \dot{z}_j &= \left(\frac{\dot{\gamma}_j}{\gamma_j} + \frac{\dot{\rho}_j}{\rho_j}\right)z_j + \gamma_j\rho_j\dot{u}_j\left[\cos u_j\sin i_j\right], \end{aligned} \tag{39}$$

$$\rho_j = \frac{a_j(1-e_j^2)}{1+e_j\cos\theta_j}, j = 1, 2 \tag{40}$$

where θ_j is the true anomaly, and the parameters $a_j, e_j, \omega_j, \Omega_j, i_j$ are analogous to the semi-major axis, eccentricity, inclination, longitude of the perihelion and longitude of the ascending node of the P_1 and P_2 bodies.

$$\dot{\rho}_j = \frac{1}{\gamma_j^2}\sqrt{\frac{\mu_{j0}}{p_j}}e_j\sin\theta_j, \qquad \mu_{j0} = f\tilde{m}_{j0} = const \tag{41}$$

$$\dot{u}_j = \frac{1}{\gamma_j^2}\frac{\sqrt{\mu_{j0}p_j}}{\rho_j^2}, u_j = \theta_j + \omega_j \tag{42}$$

$$\int_0^v \frac{dv}{(1+e\cos\theta)^2} = \frac{\sqrt{\mu_{j0}}}{p^{3/2}}\left[\phi(t) - \phi(\tau)\right], \tag{43}$$

where $\phi(t)$ is the antiderivative of the $(\tilde{m}_i/\tilde{m}_{i0}\gamma_i^3)^{1/2}$ function, and τ_j is the time of passage through pericenter.

4.3 Exact analytical solutions to the equations of secular perturbations of translational motion in osculating analogues of Delaunay variables

The translational motion of the body P_1 in Delaunay variables can be written in the following form:

$$\begin{array}{l} \dot{L}_1 = \dfrac{\partial U_1^*}{\partial l_1},\ \dot{G}_1 = \dfrac{\partial U_1^*}{\partial g_1},\ \dot{H}_1 = \dfrac{\partial U_1^*}{\partial h_1}, \\ \dot{l}_1 = -\dfrac{\partial U_1^*}{\partial L_1},\ \dot{g}_1 = -\dfrac{\partial U_1^*}{\partial G_1},\ \dot{h}_1 = -\dfrac{\partial U_1^*}{\partial H_1}, \end{array} \tag{44}$$

Accordingly, the Hamiltonian U_1^* expression takes the following form:

$$U_1^* = \left(\frac{\tilde{m}_1}{\tilde{m}_{10}\gamma_1^3(t)}\right)^{1/2}\frac{\mu_{10}^2}{2L_1^2} + \left(\frac{\tilde{m}_{10}}{\tilde{m}_1\gamma_1^3(t)}\right)^{1/2}U(...) \tag{45}$$

where $\mu_{10} = f\tilde{m}_{10}$

The perturbing part of the Hamiltonian:

$$U = \frac{1}{2}\tilde{B}_1^*(t)\vec{r}_1^{\,2} \tag{46}$$

The quantity $\vec{r}_1^2$ is expanded into a series in terms of the small parameter e the eccentricity

$$r_1^2 = \gamma_1^2 \rho_1^2 = \gamma_1^2 a_1^2 \left(\frac{\rho_1}{a_1}\right)^2 = \gamma_1^2 a_1^2 \left[1 - 2e_1 \cos M + \frac{e_1^2}{2}(3 - \cos 2M) + ...\right] \tag{47}$$

The relation between the analogues of the Keplerian elements and the Delaunay variables is as follows:

$$a = \frac{L^2}{\mu_0}, \qquad e^2 = 1 - \frac{G^2}{L^2}, \qquad M \equiv l, \qquad \omega = g, \qquad \Omega = h, \qquad \cos i = \frac{H}{G} \tag{48}$$

If the above quantities are substituted, the perturbing function takes the following form:

$$U = \frac{1}{2}\tilde{B}_1^*(t)\vec{r}_1^2 = \frac{1}{2}\tilde{B}_1^*(t)\gamma_1^2 a_1^2 \left[1 - 2e_1 \cos M + \frac{e_1^2}{2}(3 - \cos 2M) + ...\right] \tag{49}$$

By averaging over the mean anomaly $M \equiv l$, we obtained the secular part of the perturbation function in the analogues of Delaunay variables

$$U_{sec} = \frac{1}{2}\tilde{B}_1^*(t)\vec{r}_1^2 = \frac{1}{2}\tilde{B}_1^*(t)\gamma_1^2 \frac{L_1^2}{\mu_{10}^2}\left[1 + \frac{3}{2}\left(1 - \frac{G_1^2}{L_1^2}\right)\right] \tag{50}$$

Substituting the secular part of the perturbation function (50) into equations (44)-(45), we get

$$\begin{aligned} &\dot{L}_1 = 0, && \dot{G}_1 = 0, && \dot{H}_1 = 0, \\ &\dot{l}_1 = -\frac{\partial U_1^*}{\partial L_1} = -\frac{5}{2}\frac{\tilde{B}_1^*(t)\gamma_1^2}{\mu_{10}^2}L_1, && \dot{g}_1 = -\frac{\partial U_1^*}{\partial G_1} = -\frac{3}{2}\frac{\tilde{B}_1^*(t)\gamma_1^2}{\mu_{10}^2}G_1, && \dot{h}_1 = 0, \end{aligned} \tag{51}$$

Hence it follows that

$$\begin{aligned} &L_1 = L_1(t_0) = L_{10} = const, \qquad G_1 = G_1(t_0) = G_{10} = const, \\ &H_1 = H_1(t_0) = H_{10} = const, \qquad h_1 = h_1(t_0) = h_{10} = const, \\ &l_1 = l_1(t) = l_1(t_0) - \frac{5}{2}\frac{L_{10}}{\mu_{10}^2}\int_{t_0}^t \tilde{B}_1^*(t)\gamma_1^2(t)dt, \ \dot{g}_1 = g_1(t_0) - \frac{3}{2}\frac{G_{10}}{\mu_{10}^2}\int_{t_0}^t \tilde{B}_1^*(t)\gamma_1^2(t)dt, \end{aligned} \tag{52}$$

where according to (29) -(31) $\gamma_1 = \gamma_1(t) = m_2 m_0^2 / m_{20} m^2$, $\tilde{B}_1^* = \tilde{B}_1 - \left[\ddot{\gamma}_1/\gamma_1 - \tilde{A}_1\dot{\gamma}_1/\gamma_1\right]$.

Thus, formulas (52), (48), (38), (39) completely determine the coordinates and velocities of body P_1 in a secular perturbation.

Similarly, we obtain the coordinates and velocities of the body P_2 using exact analytical solutions of the equation of secular perturbations for the body P_2.

$$\begin{aligned} &L_2 = L_2(t_0) = L_{20} = const, \qquad G_2 = G_2(t_0) = G_{20} = const, \\ &H_2 = H_2(t_0) = H_{20} = const, \qquad h_2 = h_2(t_0) = h_{20} = const, \\ &l_2 = l_2(t) = l_2(t_0) - \frac{5}{2}\frac{L_{20}}{\mu_{20}^2}\int_{t_0}^t \tilde{B}_2^*(t)\gamma_2^2(t)dt, \ \dot{g}_2 = g_2(t_0) - \frac{3}{2}\frac{G_{20}}{\mu_{20}^2}\int_{t_0}^t \tilde{B}_2^*(t)\gamma_2^2(t)dt, \end{aligned} \tag{53}$$

Where, according to (36) $\tilde{B}_2^*(t) = \tilde{B}_2 - \left[\ddot{\gamma}_2/\gamma_2 - \tilde{A}_2\dot{\gamma}_2/\gamma_2\right], \gamma_2 = \gamma_2(t) = m_1 m_0^2/m_{10}m^2$.

Formulas (53), (48), (38), (39) completely determine the coordinates and velocities of body P_2 in a secular perturbation.

5 Analytical expressions for the coordinates and velocities of two primary spherical bodies with variable masses based on exact solutions of differential equations of secular perturbations

Consequently, taking into account the formulas of unperturbed motion, which retain their form in perturbed motion, coordinates and velocities in the equations of secular perturbations, in analogues of Kepler's variables, appear as follows:

$$\begin{aligned} x_{jsec} &= \gamma_j a_j \left[\left[\left(-\frac{3}{2}e\right)\cos\omega_j\right]\cos\Omega_j - \left[\left(-\frac{3}{2}e\right)\sin\omega_j\right]\sin\Omega_j\cos i_j\right], \\ y_{jsec} &= \gamma_j a_j \left[\left[\left(-\frac{3}{2}e\right)\cos\omega_j\right]\sin\Omega_j + \left[\left(-\frac{3}{2}e\right)\sin\omega_j\right]\cos\Omega_j\cos i_j\right], \\ z_{jsec} &= \gamma_j a_j \left[\left[\left(-\frac{3}{2}e\right)\sin\omega_j\right]\sin i_j\right] \end{aligned} \tag{54}$$

$$\begin{aligned} \dot{x}_{jsec} &= \frac{e_j}{2a_j\gamma_j}\left(\sqrt{\mu_{j0}p_j}\left(\sin\omega_j\cos\Omega_j + \cos\omega_j\sin\Omega_j\cos i_j\right) - 3a_j\dot{\gamma}_j\left(\sin\omega_j\sin\Omega_j + a_j\gamma_j\cos\omega_j\cos\Omega_j\right)\right), \\ \dot{y}_{jsec} &= \frac{e_j}{2a_j\gamma_j}\left(\sqrt{\mu_{j0}p_j}\left(\sin\omega_j\sin\Omega_j - \cos\omega_j\cos\Omega_j\cos i_j\right) + 3a_j\dot{\gamma}_j\left(\sin\omega_j\cos\Omega_j - a_j\gamma_j\cos\omega_j\sin\Omega_j\right)\right), \\ \dot{z}_{jsec} &= \frac{e_j}{2a_j}\left(-\frac{\sqrt{\mu_{j0}p_j}}{\gamma_j}\cos\omega_j + 3a_j^2\dot{\gamma}_j\sin\omega_j\right)\sin i_j, \end{aligned} \tag{55}$$

Further, formulas (54) – (55) will be rewritten in analogues of Delaunay variables using known transformation formulas (48), which will be used in the study of translational-rotational motion of a non-stationary small axisymmetric body.

In equations (4) – (6) of translational-rotational motion of a small non-stationary axisymmetric body P_3, the values $\vec{r}_j\left(x_j, y_j, z_j\right), \dot{\vec{r}}_j\left(\dot{x}_j, \dot{y}_j, \dot{z}_j\right), j = 1, 2$, according to the formulas (54) - (55) found above, are already known functions of time.

Thus, the problem of investigating secular perturbations of the translational -rotational motion of a non-stationary small axisymmetric body of variable mass in a Newtonian gravitational field of two primary spherical bodies with variable masses within the framework of a restricted three-body problem with variable masses in a baricentric coordinate system is reduced only to finding the coordinates x_3, y_3, z_3 and velocity $\dot{x}_3, \dot{y}_3, \dot{z}_3$ of a small non-stationary body.

6 Conclusion

In this paper, we investigated the problem of translational-rotational motion of a non-stationary axisymmetric small body in the Newtonian gravitational field of two primary spherically symmetric bodies with variable masses in a restricted formulation. We studied secular perturbations in analogues of the Delaunay-Andoyer variables.

In general, the solution to the problem of translational-rotational motion of two primary spherical bodies with isotropically varying masses is unknown, so the problem is investigated by jointly considering the differential equations of the problem of two primary bodies with variable masses and the differential equations of motion of a small non-stationary axisymmetric body. The problem is complex, so the problem investigated using perturbation theory methods.

As the main new result of this work, we have found exact analytical solutions to the equations of secular perturbations of the translational-rotational motion of two primary spherical bodies with variable masses.

Due to the results obtained in this work, the problem of investigating secular perturbations of the translational-rotational motion of a non-stationary small axisymmetric body of variable mass in the Newtonian gravitational field of two primary spherical bodies with variable masses within the framework of a restricted three-body problem with variable masses in a barycentric coordinate system is summarised as finding only the coordinates $\vec{r}_3\,(x_3, y_3, z_3)$ and velocity $\dot{\vec{r}}_3\,(\dot{x}_3, \dot{y}_3, \dot{z}_3)$ of a small non-stationary body.

Thus, the investigation of secular perturbations of the problem considered is greatly simplified, as in the classical restricted problem, in the following only the motion of a small non-stationary axisymmetric body will be investigated.

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