IRSTI 27.39.21 DOI: https://doi.org/10.26577/JMMCS130220267

D. Eshmamatova[1,2,*] **Sh. Seytov**[3] **N. Boboyarova**[4] **M. Islamova**[5]
[1]Tashkent State Transport University, Tashkent, Uzbekistan
[2]V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan
[3]Tashkent State University of Economics, Tashkent, Uzbekistan
[4]Urgench technological university RANCH, Urgench, Uzbekistan
[5]Urgench State University named after Al-Beruni, Urgench, Uzbekistan
*e-mail: 24dil@mail.ru

AN ANALYTICAL APPROACH TO BIFURCATIONS IN SOME TWO-DIMENSIONAL CUBIC MAPPINGS

This paper investigates the dynamics of two-dimensional mixed-type mappings that combine quadratic and cubic nonlinearities. Extending classical results on purely quadratic or cubic systems, we study the geometric and topological properties of the corresponding Mandelbrot and Julia sets. An analytical framework is developed to examine the stability of fixed points, bifurcations, and the structure of the parameter space. Particular emphasis is placed on the interplay between the algebraic form of the mapping and the shape of the filled Julia set. Our results include criteria for the existence of attractors, the identification of bifurcation curves, and the classification of parameter regions based on dynamical behavior.
Keywords: Julia set, Mandelbrot set, bifurcation, nonlinear mapping, cubic-quadratic dynamics, stability analysis, dynamical systems.

Д. Эшмаматова[1,2,*], Ш. Сейтов[3], Н Бобоярова[4], М. Исламова[5]
[1]Ташкент мемлекеттік көлік университеті, Ташкент, Өзбекстан
[2]В.И. Романовский атындағы математика институты, Өзбекстан Ғылым академиясы, Ташкент, Өзбекстан
[3]Ташкент мемлекеттік экономика университеті, Ташкент, Өзбекстан
[4]Үргеніш технологиялық университеті RANCH, Үргеніш, Өзбекстан
[5]Үргеніш мемлекеттік университеті, Үргеніш, Өзбекстан
*e-mail: 24dil@mail.ru

Екі өлшемді кейбір кубик бейнелеулердегі бифуркацияларға аналитикалық жақындау

Бұл мақалада квадратик және кубик бейсызықтықтарды біріктіретін екіөлшемді аралас типтегі бейнелеулердің динамикасы зерттеледі. Тек қана квадратик немесе тек қана кубик жүйелерге қатысты классикалық нәтижелерді кеңейте отырып, сәйкес келетін Мандельброт және Жюлия жиындарының геометриялық және топологиялық қасиеттері талданады. Тұрақты нүктелердің орнықтылығын, бифуркациялардың пайда болуын және параметрлік кеңістіктің құрылымын қарастыру үшін аналитикалық негіздеме жасалды. Әсіресе бейнелеудің алгебралық формасы мен толтырылған Жюлия жиынының геометриясы арасындағы өзара байланысқа ерекше назар аударылады. Негізгі нәтижелер ретінде аттракторлардың бар болуына арналған критерийлер, бифуркация қисықтарының сипаттамасы және параметрлік кеңістіктің динамикалық мінез-құлқына байланысты аймақтарының классификациясы ұсынылады.
Кілт сөздер: Жюлия жиыны, Мандельброт жиыны, бифуркация, бейсызық бейнелеу, кубик-квадраттық динамика, орнықтылықты талдау, динамикалық жүйелер.

Д. Эшмаматова[1,2,*], Ш. Сейтов[3], Н Бобоярова[4], М. Исламова[5]
[1]Ташкентский государственный транспортный университет, Ташкент, Узбекистан
[2] Институт математики имени В.И. Романовского, Академия наук Республики Узбекистан, Ташкент, Узбекистан
[3]Ташкентский государственный экономический университет, Ташкент, Узбекистан
[4]Ургенчский технологический университет RANCH, Ургенч, Узбекистан
[5]Ургенчский государственный университет, Ургенч, Узбекистан
*e-mail: 24dil@mail.ru

Аналитический подход к бифуркациям в некоторых двумерных кубических отображениях

В данной статье исследуется динамика двумерных отображений смешанного типа, которые сочетают квадратичные и кубические нелинейности. Расширяя классические результаты для чисто квадратичных или кубических систем, мы изучаем геометрические и топологические свойства соответствующих множеств Мандельброта и Жюлиа. Разработан аналитический подход для анализа устойчивости неподвижных точек, бифуркаций и структуры параметрического пространства. Особое внимание уделяется взаимосвязи между алгебраической формой отображения и формой заполненного множества Жюлиа. Наши результаты включают критерии существования аттракторов, определение бифуркационных кривых и классификацию областей параметров на основе динамического поведения.
Ключевые слова: Множество Жюлиа, множество Мандельброта, бифуркация, нелинейное отображение, кубико-квадратичная динамика, анализ устойчивости, динамические системы.

1 Introduction

Mandelbrot and Julia sets are fundamental objects in the theory of dynamical systems. While their properties in one dimension have been studied extensively, extensions to higher dimensions and nonlinear mappings remain a challenging and important direction of research. In particular, a large body of work [1–4] has focused on the construction and analysis of these sets for quadratic functions.

The monograph [5] investigates one-dimensional differential equations of the form $\dot{x} = F(\lambda, x)$, where $F : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ and $F(\lambda, x) \in \mathbb{C}^2$ or $\mathbb{C}^3$, with special attention to equilibria exhibiting quadratic and cubic degeneracies. It shows that, in the cubic case, complex dynamics arise near the zero equilibrium and that a full description of the system requires going beyond linear and quadratic terms in the Taylor expansion.

Another line of research is presented in [6], where logistic-type models of population growth, including delayed versions, are analyzed. This work demonstrates the role of Hopf bifurcation in the emergence of oscillatory solutions and highlights how bifurcation theory can explain transitions in applied systems.

In contrast to these studies [1–6], which largely focus on one-dimensional systems or specific population models, the present paper develops an analytical framework for two-dimensional difference equations with both quadratic and cubic degeneracies. Our main contributions are as follows:

We explicitly construct bifurcation curves and the corresponding bifurcation diagram in the parameter plane, thereby providing an analytical counterpart of the Mandelbrot set [5,6].

We introduce and investigate new properties of Julia sets arising in the phase plane of the system.

We propose an additional equation that allows for effective control of bifurcation dynamics, offering a new mechanism for moderating the complexity typically observed in one-dimensional logistic maps.

Earlier works [7, 8] examined Mandelbrot and Julia sets for certain quadratic and cubic mappings. However, the case of two-dimensional cubic mappings has remained analytically underexplored due to their inherent complexity. The novelty of our approach lies in combining a rigorous analytical construction of Mandelbrot- and Julia-type sets with a systematic study of bifurcation phenomena in a genuinely two-dimensional cubic setting.

Building on and extending previous results, this paper advances the theoretical understanding of multidimensional cubic mappings and provides new insights into the interplay between bifurcation structures and the geometry of Mandelbrot and Julia sets.

2 Preliminaries

Let $F_{c_1c_2}(x,y) : \mathbb{R}^2 \to \mathbb{R}^2$ be an arbitrary two-dimensional mapping, where $(x,y) \in \mathbb{R}^2$ and $(c_1,c_2) \in \mathbb{R}^2$. We recall the following definitions, which will be used throughout this paper [9–14].

Definition 1 *The* filled Julia set *of the mapping $F_{c_1c_2}$ is defined as the set of all points (x,y) whose orbits remain bounded under iteration:*

$$K(F_{c_1c_2}) = \left\{(x,y) \in \mathbb{R}^2 \,:\, \|F^n_{c_1c_2}(x,y)\| < \infty \ \text{as}\ n \to \infty\right\}.$$

See [9] for more details.

Definition 2 *The* Julia set *of $F_{c_1c_2}$ is the boundary of its filled Julia set:*

$$J(F_{c_1c_2}) = \partial K(F_{c_1c_2}).$$

Definition 3 *The* Mandelbrot set $M_{F_{c_1c_2}}$ *of the mapping $F_{c_1c_2}$ is defined as the set of all parameter values $(c_1,c_2) \in \mathbb{R}^2$ for which the orbits of all critical points of $F_{c_1c_2}$ are bounded.*

Definition 4 *Let $F_{c_1c_2}$ be a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. A point $(x_0,y_0) \in \mathbb{R}^2$ is called an* attractive fixed point *(or* attractor*) of $F_{c_1c_2}$ if it satisfies $F_{c_1c_2}(x_0,y_0) = (x_0,y_0)$, and there exists a neighborhood U of (x_0,y_0) such that for every $(x,y) \in U$, the iterates*

$$F^n_{c_1c_2}(x,y) \to (x_0,y_0) \quad \text{as}\ n \to \infty.$$

See [9] for details.

Definition 5 *The* basin of attraction *of an attractive fixed point (x_0,y_0) is the set of all initial points $(x,y) \in \mathbb{R}^2$ whose orbits under $F_{c_1c_2}$ converge to (x_0,y_0):*

$$B(x_0,y_0) = \left\{(x,y) \in \mathbb{R}^2 \,:\, F^n_{c_1c_2}(x,y) \to (x_0,y_0) \ \text{as}\ n \to \infty\right\}.$$

See [9].

Remark 1 *The basin of attraction $B(x_0, y_0)$ can be highly nontrivial in structure. In many nonlinear systems, especially those exhibiting chaotic dynamics, the basin may be disconnected, fractal, or possess a complicated boundary. In particular, when multiple attractors exist, their basins may be intertwined in a highly intricate way, making long-term predictions sensitive to initial conditions.*

Definition 6 *A point $(x_0, y_0) \in \mathbb{R}^2$ is called a* periodic point *of period $p \in \mathbb{N}$ for the mapping $F_{c_1c_2}$ if*

$$F^p_{c_1c_2}(x_0, y_0) = (x_0, y_0) \quad \text{and} \quad F^k_{c_1c_2}(x_0, y_0) \neq (x_0, y_0) \quad \text{for all } 0 < k < p.$$

The orbit $\{(x_0, y_0), F_{c_1c_2}(x_0, y_0), \dots, F^{p-1}_{c_1c_2}(x_0, y_0)\}$ is called a periodic orbit.

Definition 7 *A periodic orbit $\{(x_0, y_0), \dots, F^{p-1}_{c_1c_2}(x_0, y_0)\}$ of period p is called* attracting *(or* stable*) if there exists a neighborhood U of the orbit such that for all $(x, y) \in U$, the iterates $F^n_{c_1c_2}(x, y)$ converge to the periodic orbit as $n \to \infty$.*

If, in contrast, orbits nearby diverge from the periodic orbit, it is called repelling *(or* unstable*).*

Definition 8 *A* bifurcation *occurs in a dynamical system when a small variation in the parameters (c_1, c_2) of the mapping $F_{c_1c_2}$ causes a qualitative change in the behavior of the system, such as the number or stability of fixed points or periodic orbits.*

A common example is the transition from a stable fixed point to a periodic orbit, or the onset of chaos.

Definition 9 *[13] Let $P : X \to X$ be a mapping. A point $x_0 \in X$ is called a* periodic point of prime period *m (or* minimal period *m) if*

$$P^m(x_0) = x_0 \quad \text{and} \quad P^k(x_0) \neq x_0 \text{ for all } 1 \leq k < m.$$

The set of all periodic points of period m is denoted by

$$\mathrm{Per}^m(P) = \left\{ x \in X : P^m(x) = x,\ P^k(x) \neq x \text{ for } 1 \leq k < m \right\}.$$

Definition 10 *[13] Let $P : X \to X$ be a mapping and let $p \in X$ be a fixed point, i.e., $P(p) = p$. The point p is called an* attracting fixed point *if there exists $\varepsilon > 0$ such that for all $x \in N_\varepsilon(p)$, the iterates $P^k(x)$ converge to p as $k \to \infty$:*

$$\lim_{k \to \infty} P^k(x) = p.$$

In this case, p is also called an attractor.

Definition 11 *[9] Let $P : X \to X$ be a mapping and $p \in X$ a fixed point such that $P(p) = p$. The point p is called a* repelling fixed point *if there exists $\varepsilon > 0$ such that every point $x \in N_\varepsilon(p)$ with $x \neq p$ eventually maps outside the neighborhood $N_\varepsilon(p)$ under iteration:*

$$\exists\, k_0 \in \mathbb{N} \text{ such that } P^{k_0}(x) \notin N_\varepsilon(p).$$

In this case, p is called a repeller.

Remark 2 *Fixed points that are neither attracting nor repelling are often referred to as* saddle points *or* indifferent fixed points*, depending on their local behavior.*

3 Quadratic-Cubic Mixed Mappings

Consider a two-dimensional mapping $F_{c_1c_2} : \mathbb{R}^2 \to \mathbb{R}^2$ of mixed quadratic-cubic type, defined by:

$$F_{c_1c_2} : \begin{cases} x' = y^2 + c_1, \\ y' = x^3 + c_2, \end{cases} \tag{1}$$

where $(x, y) \in \mathbb{R}^2$ and $(c_1, c_2) \in \mathbb{R}^2$ are parameters.

According to [7], in order to find the fixed points of the mapping (1), we must solve the system

$$x = y^2 + c_1, \qquad y = x^3 + c_2.$$

Substituting the second equation into the first yields a single algebraic equation:

$$x = (x^3 + c_2)^2 + c_1 = x^6 + 2c_2x^3 + c_2^2 + c_1.$$

Rewriting this, we define the polynomial

$$f(x) = x^6 + 2c_2x^3 - x + c_2^2 + c_1,$$

whose real roots determine the x-coordinates of fixed points of $F_{c_1c_2}$.

Now consider the mapping in a general form:

$$\begin{cases} x' = f(y, c_1), \\ y' = g(x, c_2), \end{cases} \tag{2}$$

where the first component depends only on y and the parameter c_1, while the second component depends only on x and the parameter c_2.

Lemma 1 *Let (x_0, y_0) and (x_1, y_1) be fixed points of the mapping (2). Then there exist two periodic points, namely (x_0, y_1) and (x_1, y_0), such that under iteration of the mapping, the point (x_0, y_1) maps to (x_1, y_0) after one step, and vice versa. Thus, both points form a periodic orbit of period two.*

Proof. This result follows directly from the definition of a periodic point of period $n = 2$ (see Definition 7). Indeed, by applying the mapping (2) to (x_0, y_1), we obtain:

$$F(x_0, y_1) = (f(y_1, c_1), g(x_0, c_2)) = (x_1, y_0).$$

Similarly, applying the mapping to (x_1, y_0) gives:

$$F(x_1, y_0) = (f(y_0, c_1), g(x_1, c_2)) = (x_0, y_1).$$

Thus, both points form a 2-cycle under F. The proof is complete.

Lemma 2 *The polynomial $f(x) = x^6 + 2c_2x^3 - x + c_2^2 + c_1$ has no triple complex root.*

Proof. Assume, for contradiction, that $z_0 \in \mathbb{C}$ is a triple root of $f(x)$. Then its complex conjugate $\overline{z_0}$ is also a triple root, since $f(x)$ has real coefficients. Thus, the six roots of $f(x)$ are:

$$z_0, z_0, z_0, \overline{z_0}, \overline{z_0}, \overline{z_0}.$$

By Viete's formula, the sum of the roots satisfies:

$$\sum_{i=1}^{6} z_i = 3z_0 + 3\overline{z_0} = 3(z_0 + \overline{z_0}) = 6\operatorname{Re}(z_0) = 0,$$

which implies that $\operatorname{Re}(z_0) = 0$, i.e., $z_0 = ai$ for some $a \in \mathbb{R}$.

Then $f(x)$ must factor as:

$$f(x) = (x - ai)^3(x + ai)^3 = [(x - ai)(x + ai)]^3 = (x^2 + a^2)^3.$$

Thus,

$$f(x) = x^6 + 3a^2x^4 + 3a^4x^2 + a^6.$$

But by the original form of $f(x)$,

$$f(x) = x^6 + 2c_2x^3 - x + c_2^2 + c_1.$$

Comparing the two expressions, we see that $f(x)$ cannot simultaneously be equal to both forms for any real values of c_1 and c_2, due to mismatch in degrees and missing terms (e.g., x^3 and x terms are absent in $(x^2 + a^2)^3$). Therefore, such a triple complex root cannot exist. The proof is complete.

We recall from [12, 15] that the following determinant

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & \dots & a_n & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_n & \dots & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & \dots & 0 & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_m & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_m & \dots & 0 \\ \vdots & & \ddots & & & \ddots & \vdots \\ 0 & \dots & 0 & b_0 & b_1 & \dots & b_m \end{vmatrix}$$

is called the *resultant* of the polynomials

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n, \qquad g(x) = b_0x^m + b_1x^{m-1} + \dots + b_m.$$

According to [6], the *discriminant* of a polynomial $f(x)$ of degree d can be written (up to sign) as the resultant of $f(x)$ and its derivative $f'(x)$:

$$D(f) = \prod_{i>j}(z_i - z_j)^2 = \pm R(f, f'),$$

where z_i are the roots of $f(x)$.

For the polynomial

$$f(x) = x^6 + 2c_2x^3 - x + c_2^2 + c_1,$$

we compute the discriminant explicitly as:

$$D(f) = R(f, f') = -46656c_1^5 - 93312c_1^4c_2^2 - 54000c_1^2c_2 - 97200c_1c_2^3 - 46656c_2^5 + 3125.$$

According to Lemma 2, the polynomial $f(x)$ has no triple complex roots. Therefore, the equation $D(f) = 0$ corresponds to the condition for $f(x)$ to have a multiple **real** root. Geometrically, this is equivalent to the curves

$$x = y^2 + c_1, \qquad y = x^3 + c_2$$

having a common point with a shared tangent — i.e., a tangency point corresponding to a multiple real root.

Hence, the equation

$$-46656c_1^5 - 93312c_1^4c_2^2 - 54000c_1^2c_2 - 97200c_1c_2^3 - 46656c_2^5 + 3125 = 0 \tag{3}$$

defines the bifurcation curve in the parameter plane (c_1, c_2) where multiplicities of fixed points change. This curve is implicitly defined. The natural question is:

How many distinct ordinary (i.e., single-valued) functions $c_2 = \phi(c_1)$ can be locally defined from the implicit equation (3)?

4 Parameter Plane and Bifurcations

Let us begin the investigation of the implicit equation (3). We compute the discriminant of this equation, considered as a polynomial in the variable c_2, with coefficients depending on the parameter c_1.

According to [16]– [19], the discriminant D of the polynomial in c_2 from equation (3) is given by:

$$D = 4738381338321616896 \cdot \left(3125 + 2916c_1^5\right)^2 \cdot \left(3125 + 9216c_1^5\right)^3.$$

For a general polynomial of degree 5 with real coefficients, classical results (see [8, 9]) imply the following:

- If $D > 0$, then the polynomial has either five distinct real roots, or one real root and two distinct pairs of complex conjugate roots.
- If $D < 0$, then the polynomial has exactly three distinct real roots and one pair of complex conjugate roots.
- If $D = 0$, then the polynomial has multiple (i.e., repeated) roots; these may be real or complex.

The sign and vanishing of the discriminant therefore determine the structure of the real root set and indicate the occurrence of bifurcations in the parameter plane (c_1, c_2). In particular, the zero set of the discriminant defines the bifurcation locus — the set of parameter values where multiple roots appear and qualitative changes occur in the dynamical behavior of the system. Based on the analysis above, we establish the following result.

Theorem 1 *Let equation* (3) *be considered as an implicit relation defining* c_2 *as a function of* c_1*. Then the following statements hold:*

(i) If $c_1 > -\dfrac{5}{4\sqrt[5]{9}}$*, then equation* (3) *defines a single real-valued function* $c_2 = \phi(c_1)$*.*

(ii) If $-\dfrac{5}{3\sqrt[5]{12}} < c_1 < -\dfrac{5}{4\sqrt[5]{9}}$*, then the equation defines three real branches of* $c_2 = \phi_i(c_1)$*,* $i = 1, 2, 3$*.*

(iii) If $c_1 < -\dfrac{5}{3\sqrt[5]{12}}$*, then equation* (3) *also defines three real branches, two of which are close to each other in value (see Figure 4).*

The values $c_1 = -\dfrac{5}{3\sqrt[5]{12}}$ *and* $c_1 = -\dfrac{5}{4\sqrt[5]{9}}$ *are bifurcation points at which the number of real solutions (branches) changes.*

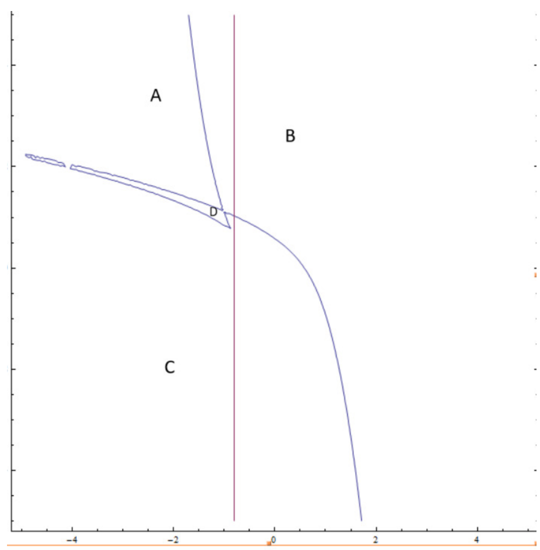

Figure 1. Bifurcation curve for mapping (1).

To clarify the general results, let us consider a concrete example.

Example. Consider the mapping (1) with parameter values $c_1 = 0$ and $c_2 = 0$, that is:

$$x' = y^2, \qquad y' = x^3.$$

To find the fixed points of this two-dimensional mapping, we solve the system:

$$x = y^2, \qquad y = x^3.$$

Substituting $x = y^2$ into the second equation yields: $y = (y^2)^3 = y^6$. Hence,

$$y^6 - y = y(y^5 - 1) = 0,$$

which has the real solutions $y = 0$, $y = 1$, and $y = -1$. The corresponding x-coordinates are:

$$x = y^2 \quad \Rightarrow \quad x = 0,\ 1,\ 1.$$

Thus, the fixed points of the mapping are:

$$(0,0), \quad (1,1), \quad (1,-1).$$

To study their stability, we compute the Jacobian matrix J of the mapping. According to [20, 21], the Jacobian is given by:

$$J = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix}.$$

Since $x' = y^2$ and $y' = x^3$, the partial derivatives are:

$$\frac{\partial x'}{\partial x} = 0, \qquad \frac{\partial x'}{\partial y} = 2y, \qquad \frac{\partial y'}{\partial x} = 3x^2, \qquad \frac{\partial y'}{\partial y} = 0.$$

Thus, the Jacobian matrix becomes:

$$J = \begin{bmatrix} 0 & 2y \\ 3x^2 & 0 \end{bmatrix}.$$

Evaluation at Each Fixed Point

We now evaluate the Jacobian matrix and determine the stability of the fixed points.

1. **At** $(0,0)$:

   The Jacobian matrix is:

   $$J = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{so} \quad \lambda_1 = 0,\ \lambda_2 = 0.$$

   Both eigenvalues are zero; hence, the fixed point $(0,0)$ is **non-hyperbolic (neutral)**. Its stability cannot be determined via linearization, and further nonlinear analysis is required.

2. **At** $(1,1)$:

   The Jacobian matrix becomes:

   $$J = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}.$$

   The characteristic equation is:

   $$\det(J - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 - 6 = 0,$$

   which gives eigenvalues $\lambda = \pm\sqrt{6}$. Since the eigenvalues are real and of opposite sign, the fixed point is a **saddle** and therefore **unstable**.

3. **At** $(1, -1)$:

   Similarly, the Jacobian matrix is:

$$J = \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}.$$

   The characteristic equation is:

$$\lambda^2 + 6 = 0 \quad \Rightarrow \quad \lambda = \pm i\sqrt{6}.$$

   This gives purely imaginary eigenvalues, implying that the fixed point is a **center**. However, since the system is nonlinear, further investigation is needed to determine whether it is a stable or unstable focus or a true center.

Summary of Fixed Point Classification:

- $(0, 0)$: neutral (non-hyperbolic); linearization inconclusive.
- $(1, 1)$: saddle point (unstable).
- $(1, -1)$: center (non-hyperbolic); requires nonlinear analysis for full classification.

Stability Analysis and Phase Portraits Near Fixed Points

Let us now analyze the local behavior near the fixed points of the mapping (1) using both linearization and nonlinear qualitative methods.

1. Behavior near the origin $(0, 0)$**.**

At this point, the Jacobian matrix is zero:

$$J = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \lambda_1 = \lambda_2 = 0.$$

Since both eigenvalues are zero, the linearization is inconclusive. We proceed with a nonlinear analysis. Substituting $x = y^2$ into the second equation yields:

$$y' = x^3 = (y^2)^3 = y^6.$$

Thus, $y' > 0$ for all $y \neq 0$, and similarly $x' = y^2 > 0$ for all $y \neq 0$.

This indicates that both x and y increase along the trajectories (unless they are at rest on the axes). Hence, the origin behaves as a **nonlinear repelling point** — trajectories move away from it in all directions, even though the Jacobian vanishes.

2. Behavior near the point $(1, -1)$**.**

The Jacobian matrix is:

$$J = \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}, \quad \text{with eigenvalues } \lambda = \pm i\sqrt{6}.$$

This implies a purely imaginary spectrum and suggests the point is a **center** in the linear approximation.

To justify this classification, we compute the divergence of the vector field $F = (y^2, x^3)$:

$$\operatorname{div}(F) = \frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} = 0 + 0 = 0.$$

Since the divergence is zero, the system is **conservative**, and the point $(1, -1)$ is indeed a **nonlinear center**. Trajectories in a neighborhood of this point form closed orbits, and the motion is locally periodic.

3. Refined Summary of Fixed Points

- $(0, 0)$: *non-hyperbolic*, unstable (nonlinear repeller).
- $(1, 1)$: *saddle point*, unstable.
- $(1, -1)$: *center*, stable in the sense of Lyapunov (non-asymptotically).

These results will be supported by phase portraits in the next section.

Stability and Local Bifurcations

Let us analyze how the stability of the fixed points changes as the parameters (c_1, c_2) vary. For the map

$$F_{c_1,c_2} : \begin{cases} x' = y^2 + c_1, \\ y' = x^3 + c_2, \end{cases}$$

the Jacobian matrix at a fixed point (x^*, y^*) is

$$J = \begin{pmatrix} 0 & 2y^* \\ 3(x^*)^2 & 0 \end{pmatrix}, \qquad \lambda_{1,2} = \pm\sqrt{6x^{*2}y^*}.$$

Hence, the type of equilibrium depends on the sign of y^*:

- If $y^* > 0$, the eigenvalues are real and of opposite signs — the point is a *saddle*.
- If $y^* < 0$, the eigenvalues are purely imaginary — the point behaves as a *center*.
- If $y^* = 0$, the equilibrium is *degenerate*, and higher-order terms determine its behavior.

Varying (c_1, c_2) causes transitions between these regimes. When y^* changes sign, a *saddle–center bifurcation* occurs. The following table summarizes representative cases.

Case	c_1	c_2	Fixed Point(s)	Stability Type
A	0	0	$(0, 0)$, $(1, -1)$	neutral / saddle
B	-0.1	0.1	$(0.83, -0.9)$	center
C	0.15	-0.1	$(1.12, 0.9)$	saddle

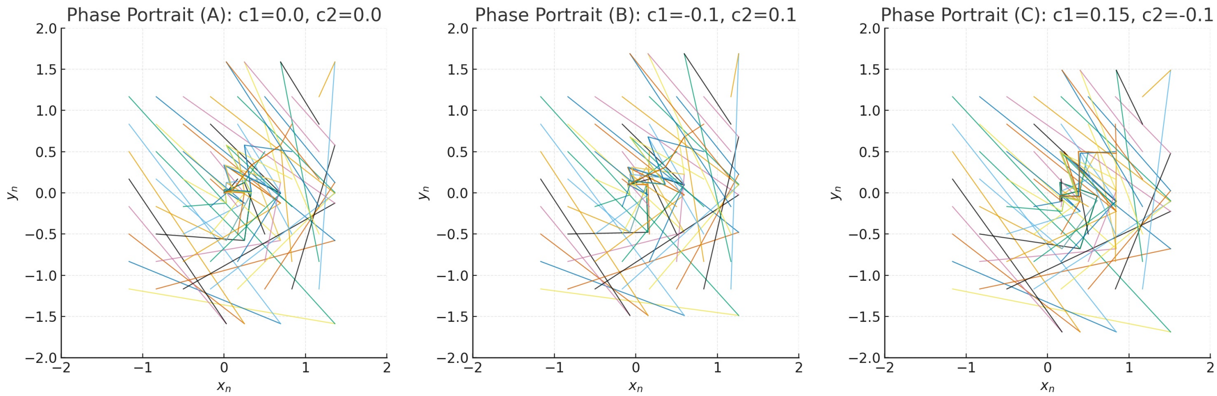

Figure 2. Bifurcation curve for mapping (1). Phase portraits of map F_{c_1,c_2} for three representative cases: (A) $(c_1, c_2) = (0, 0)$, (B) $(c_1, c_2) = (-0.1, 0.1)$, (C) $(c_1, c_2) = (0.15, -0.1)$

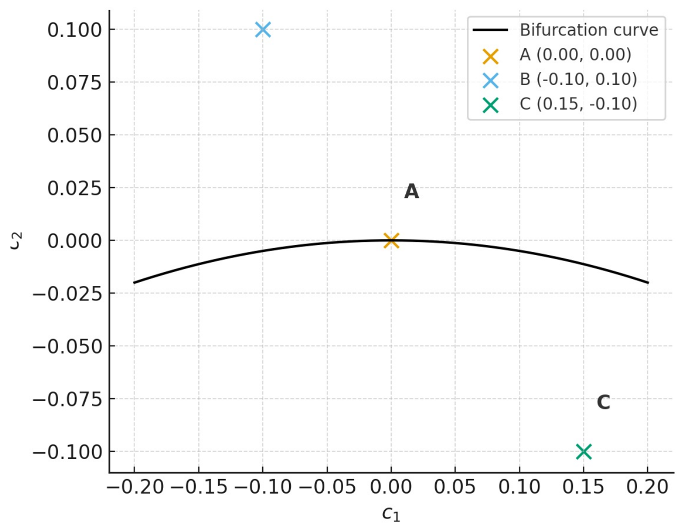

Figure 3. Bifurcation curve in the (c_1, c_2)-plane with marked points A, B, and C corresponding to different stability regimes

The Figure 2 illustrates the corresponding phase portraits for these parameter sets. In the vicinity of the bifurcation curve defined by equation (3), the trajectories exhibit a slow drift along neutral directions, indicating the onset of quasi-periodic or chaotic dynamics.

The qualitative difference between Figures 2. (A–C) and Figure 3. lies in the level of representation of the system's dynamics. Figures 2 (A–C) illustrate the local phase portraits in the (x, y)–plane for three fixed parameter pairs (c_1, c_2), demonstrating how trajectories behave near equilibrium points—either as centers or saddles. In contrast, Figure 3. depicts the global bifurcation structure in the (c_1, c_2)–parameter space. The bifurcation curve separates regions corresponding to different qualitative behaviors: on one side, the fixed point acts as a center (stable oscillations), while on the other, it becomes a saddle (unstable divergence). The marked points A, B, and C on the bifurcation diagram correspond exactly to the parameter values used in Figures 2 (A–C), thus linking local dynamics with global parametric organization of the mapping.

5 Possible Applications and Interpretation

Although the study is primarily theoretical, the results are applicable to various nonlinear models where polynomial feedback occurs.

- **Population dynamics:** x and y may represent normalized populations of two interacting species, with cubic nonlinearity describing saturation or competition.
- **Nonlinear electronic circuits:** In feedback systems with cubic elements, similar maps describe discrete voltage evolution; parameters c_1 and c_2 serve as control gains.
- **Control and bifurcation prediction:** The analytical bifurcation curve (Eq. (3)) provides a compact criterion for predicting qualitative changes in system dynamics.

Thus, the analytical expression for the bifurcation boundary can serve as a predictive tool for identifying stability loss or the onset of complex dynamics in real nonlinear systems.

6 Structure of Julia Sets

Based on the results obtained, we now examine the structure of the Julia set for the mapping (1) (see [9]).

Theorem 2 (Structure of the filled Julia set for cross–dependent maps) *Let*

$$F_{c_1,c_2} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad F_{c_1,c_2}(x,y) = (f(y),\, g(x)),$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are polynomial maps and (c_1, c_2) denote parameters entering f and g (for example $f(y) = y^2 + c_1$, $g(x) = x^3 + c_2$). Define the second–iterate one-dimensional maps

$$h(x) := f(g(x)), \qquad k(y) := g(f(y)).$$

Let K_h and K_k be the filled Julia sets of h and k, respectively:

$$K_h = \{x_0 \in \mathbb{R} : \{h^n(x_0)\}_{n\geq 0} \text{ is bounded}\}, \quad K_k = \{y_0 \in \mathbb{R} : \{k^n(y_0)\}_{n\geq 0} \text{ is bounded}\}.$$

Then the filled Julia set of the two–dimensional map F_{c_1,c_2} satisfies

$$K_F = \{(x,y) \in \mathbb{R}^2 : \{F^n(x,y)\}_{n\geq 0} \text{ is bounded}\} = K_h \times K_k.$$

In particular, for the map from equation (1),

$$h(x) = (x^3 + c_2)^2 + c_1, \qquad k(y) = \left(y^2 + c_1\right)^3 + c_2,$$

and therefore the 2D filled Julia set coincides with the Cartesian product of the filled sets of the degree-6 polynomials h and k. If, for some parameter region, K_h and K_k are intervals $[a,b]$ and $[c,d]$, respectively, then $K_F = [a,b] \times [c,d]$ (a closed rectangle).

Proof 1 *Recall the definitions:*

$$F(x,y) = (f(y), g(x)), \qquad h(x) = f(g(x)), \quad k(y) = g(f(y)).$$

Define

$$K_F = \{(x,y) \in \mathbb{R}^2 : \{F^n(x,y)\}_{n\geq 0} \text{ is bounded}\},$$

$$K_h = \{x \in \mathbb{R} : \{h^n(x)\}_{n\geq 0} \text{ is bounded}\}, \qquad K_k = \{y \in \mathbb{R} : \{k^n(y)\}_{n\geq 0} \text{ is bounded}\}.$$

1) Inclusion $K_F \subseteq K_h \times K_k$.

Let $(x,y) \in K_F$. *Then the full orbit* $\{F^n(x,y)\}_{n\geq 0}$ *is bounded, and in particular its subsequence of even iterates is bounded:*

$$F^{2n}(x,y) = (h^n(x),\, k^n(y)), \qquad n = 0,1,2,\ldots.$$

Hence both sequences $\{h^n(x)\}_{n\geq 0}$ *and* $\{k^n(y)\}_{n\geq 0}$ *are bounded, that is,* $x \in K_h$ *and* $y \in K_k$. *Therefore* $(x,y) \in K_h \times K_k$, *and consequently* $K_F \subseteq K_h \times K_k$.

2) Inclusion $K_h \times K_k \subseteq K_F$.

Let $x \in K_h$ *and* $y \in K_k$. *Then the sequences* $\{h^n(x)\}$ *and* $\{k^n(y)\}$ *are bounded; that is, there exist positive constants* M_1, M_2 *such that*

$$|h^n(x)| \leq M_1, \qquad |k^n(y)| \leq M_2, \qquad \text{for all } n \geq 0.$$

Thus the even iterates satisfy

$$F^{2n}(x,y) = (h^n(x), k^n(y)),$$

and therefore the set of all even iterates lies within the rectangle

$$[-M_1, M_1] \times [-M_2, M_2].$$

We now verify that all iterates (including the odd ones) are bounded. For the odd iterates we have

$$F^{2n+1}(x,y) = F(F^{2n}(x,y)) = (f(k^n(y)),\, g(h^n(x))).$$

Since f *and* g *are polynomial (and hence continuous) maps, the image of a bounded set under* f *or* g *is also bounded. Therefore, there exist positive constants* M_3, M_4 *such that for all* n

$$|f(k^n(y))| \leq M_3, \qquad |g(h^n(x))| \leq M_4.$$

It follows that the sequence of odd iterates is bounded as well. Hence the entire orbit $\{F^n(x,y)\}_{n\geq 0}$ *is bounded, implying* $(x,y) \in K_F$.

Therefore, $K_h \times K_k \subseteq K_F$.

Combining both inclusions, we obtain the required equality

$$K_F = K_h \times K_k.$$

Remark. Two simple observations are essential in the above proof:

- the explicit formula for the second iterate, $F^2(x,y) = (h(x), k(y))$, which separates the coordinates on even steps;

- the continuity of the polynomial maps: the image of a bounded set under a polynomial map is bounded, so the boundedness of even subsequences implies boundedness of odd ones via one application of f and g.

In the particular case of your map $f(y) = y^2 + c_1,\ g(x) = x^3 + c_2$, we have

$$h(x) = (x^3 + c_2)^2 + c_1, \qquad k(y) = (y^2 + c_1)^3 + c_2,$$

and consequently $K_F = K_h \times K_k$, i.e., the filled set of the two-dimensional map is the Cartesian product of the filled sets of the corresponding power polynomials. If, for some parameter region, the sets K_h and K_k are intervals, then K_F is a rectangle. This completes the proof.

Corollary 1 *Under the assumptions of Theorem 2, the boundary of the two-dimensional filled Julia set satisfies the standard product-boundary identity:*

$$\partial K_F \;=\; (\partial K_h \times K_k) \;\cup\; (K_h \times \partial K_k).$$

Proof For arbitrary subsets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$, the topological boundary of their Cartesian product is given by

$$\partial(A \times B) = (\partial A \times B) \cup (A \times \partial B).$$

Applying this general identity with $A = K_h \subset \mathbb{R}$ and $B = K_k \subset \mathbb{R}$, and using Theorem 2 which states $K_F = K_h \times K_k$, we obtain the desired result. This completes the proof.

Remark 3 *Several comments on the geometric nature of ∂K_F are in order.*

1. *If, for a certain parameter region, the one-dimensional filled Julia sets are closed intervals, $K_h = [a, b]$ and $K_k = [c, d]$, then*

   $$K_F = [a, b] \times [c, d], \qquad \partial K_F = (\{a, b\} \times [c, d]) \cup ([a, b] \times \{c, d\}),$$

   that is, the boundary of K_F consists of four line segments and four corner points.

2. *In generic parameter regimes of polynomial maps, the sets K_h and K_k possess fractal (Cantor-type) boundaries. Consequently, ∂K_F also has a fractal structure and is precisely the union described in Corollary 1. Therefore, in Theorem 2, the term "rectangle" should be interpreted in the sense of a* Cartesian product structure *$K_F = K_h \times K_k$, rather than implying a geometrically smooth rectangular boundary.*

3. ***On boundary smoothness.*** *A smooth (C^1 or analytic) boundary ∂K_F occurs only in exceptional situations when both ∂K_h and ∂K_k are smooth one-dimensional manifolds (e.g., boundaries of intervals). In typical dynamical settings, these boundaries are highly irregular or fractal, reflecting complex or chaotic behavior in the corresponding one-dimensional maps.*

The geometric interpretation of the discriminant D of a degree-five polynomial is as follows:
• $D > 0$: the polynomial has either two pairs of complex conjugate roots and one real root, or five distinct real roots;
• $D < 0$: the polynomial has one pair of complex conjugate roots and three distinct real roots;
• $D = 0$: at least two roots coincide, either real or complex (in which case their conjugates also coincide).

Based on the behavior of the discriminant $D(f)$ described above, we arrive at the following conclusion:

Theorem 3 *The algebraic curve defined by equation* (3) *partitions the parameter plane* (c_1, c_2) *into four distinct regions, each corresponding to a different qualitative root structure of the polynomial* $f(x)$.

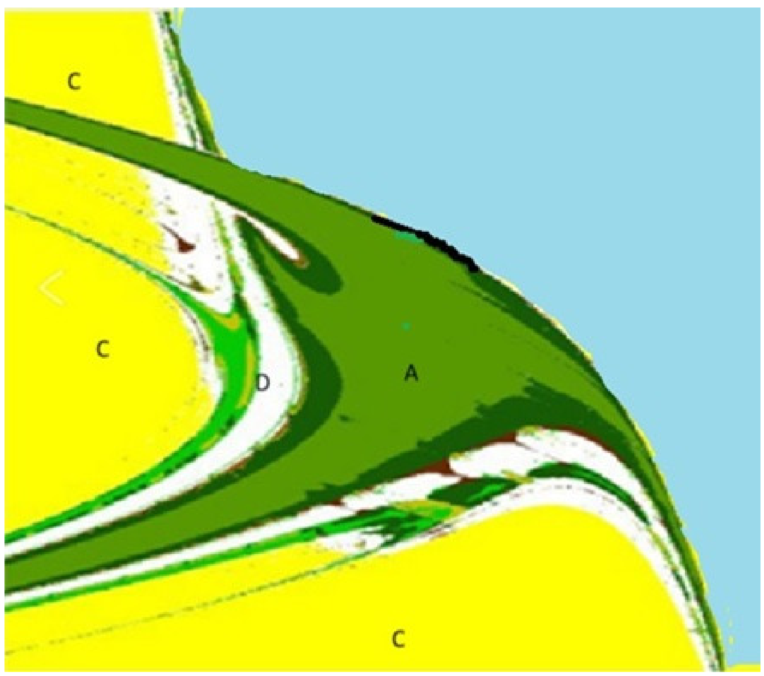

Figure 4. Bifurcation diagram of mapping (1)

We can clearly see in Figure 1 that the implicit function (3) splits the parameter plane (c_1, c_2) into four open regions, denoted by A, B, C, and D. These curves correspond to the first bifurcation of the system.

Figure 4 illustrates how the bifurcation curves divide the region in which fixed points of the mapping are located. Figure 4 presents the bifurcation diagram, which corresponds to the Mandelbrot set for the mapping (1) (see [7]).

Let us now describe the color coding used in Figure 4:

- **Blue region:** The mapping (1) has no fixed points.
- **Green region:** The filled Julia set is connected.
- **Yellow region:** The Julia set is disconnected.
- **White region:** The system exhibits chaotic behavior.

7 Numerical Simulations

In this section, we numerically investigate the filled Julia sets of the mapping and explore the behavior of selected orbits.

In Figure 5, the yellow region represents the filled Julia set of the mapping (1) for the parameter values $c_1 = 0.2$ and $c_2 = -0.696$. Points shown in black escape to infinity under iteration. All points inside the filled Julia set tend toward the approximate fixed point (Figure 5.):

$$(x^*, y^*) \approx (0.504,\ -0.544).$$

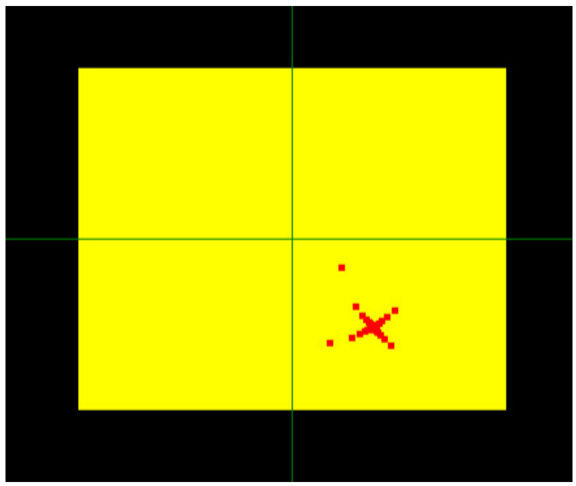

Figure 5. Attractor fixed point in the filled Julia set

Let us now consider the case $c_1 = -0.936$, $c_2 = -0.32$. As shown in Figure 6, the previously attracting fixed point $(x^*, y^*) \approx (0.504,\ -0.544)$ becomes a repeller. Instead, four new stable periodic points of period 4 appear at the vertices of a rectangle. The orbit of these points is as follows (Figure 6.):

$$(-0.832,\ -0.328) \to (-0.832,\ -0.896) \to (-0.136,\ -0.896) \to$$

$$\to (-0.136,\ -0.328) \to (-0.832,\ -0.328).$$

Next, consider the parameter values $c_1 = -1.096$, $c_2 = -0.176$. In Figure 6, the previously attracting periodic points of period 4 have become repelling. Two new stable periodic orbits of period 8 emerge: one located within the green region, and the other within the yellow region. All points in the yellow region tend toward the corresponding stable orbit in that region, and similarly for points in the green region (Figure 7.).

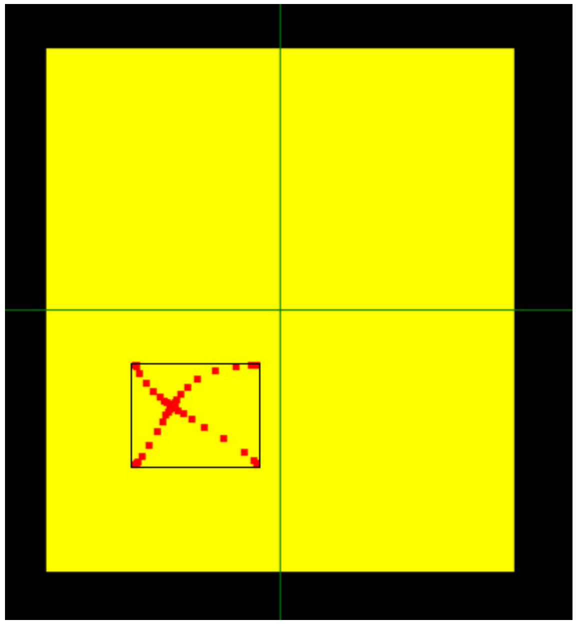

Figure 6. Attractor periodic points with period 4 in the filled Julia set

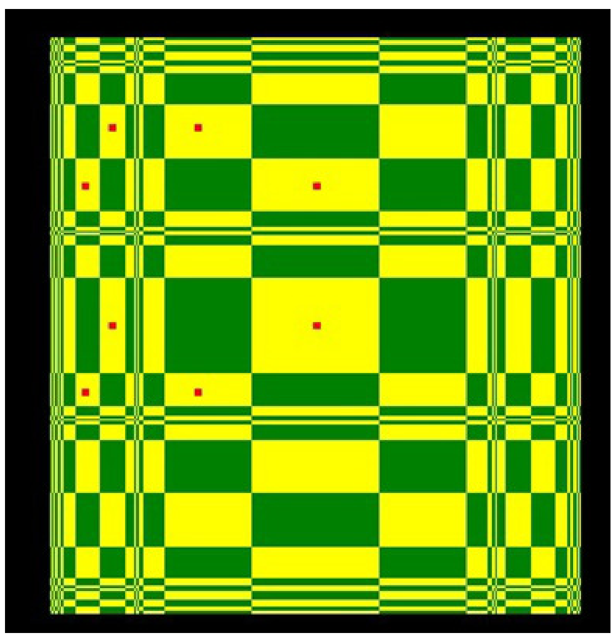

Figure 7. Attractor periodic points with period 8 in the filled Julia set

8 Conclusion

In this paper, we analyzed a two-dimensional system of difference equations incorporating both quadratic and cubic nonlinearities. We constructed bifurcation curves that divide the parameter plane (c_1, c_2) into four distinct regions, including one in which no fixed points exist. These curves define the boundary of the Mandelbrot set for the system under consideration. Additionally, using computational methods, we generated a complete bifurcation diagram.

For the given system, we proved that the filled Julia set has the shape of a rectangle with sides parallel to the coordinate axes. Through numerical simulations, we demonstrated that the filled Julia set may contain attracting fixed points and periodic orbits, depending on the parameter values. Furthermore, we analyzed and classified the basins of attraction associated with these attractors.

In the purely quadratic case, the system exhibits highly chaotic dynamics for certain parameter values. Since one of the goals of this study was to mitigate or control chaotic behavior, we introduced a cubic term into the system, acting as a stabilizing mechanism.

In contrast to previous studies [22]– [27], we introduced an operator that combines both cubic and quadratic polynomial terms. In particular, works such as [23]– [26] consider the quadratic operator as a discrete analogue of the classical Kermack–McKendrick compartmental model. To slow down the onset of chaotic behavior and enhance model flexibility, we incorporated a cubic term into the dynamics.

This class of mappings holds potential for applications in mathematical modeling of real-world systems, particularly in ecology and population genetics, where interactions of different nonlinear orders often arise. In future work, we aim to further develop this framework for use in ecological modeling.

References

[1] Mandelbrot, B. B. *The Fractal Geometry of Nature*. San Francisco: W. H. Freeman, 1982. P. 443.

[2] Douady, A. Julia Sets and the Mandelbrot Set. In *The Beauty of Fractals*; Peitgen, H.-O., Richter, P.H., Eds.; Springer: Berlin/Heidelberg, Germany, 1986; pp. 161–174.

[3] Falconer, K.J. *Fractal Geometry: Mathematical Foundations and Applications*, 3rd ed.; Wiley: Chichester, UK, 2013; 368 p.

[4] Devaney, R.L. *The Mandelbrot and Julia Sets: A Toolkit of Dynamics Activities*; Key Curriculum Press: Emeryville, CA, USA, 2000; 121 p.

[5] Hale, J.K.; Kocak, H. *Dynamics and Bifurcations*; Springer: New York, NY, USA, 1991; 568 p.

[6] Martcheva, M. *An Introduction to Mathematical Epidemiology*. Springer: New York, NY, USA, 2015; 453 p.

[7] Ganikhodzhaev, R. N., and Sh. J. Seytov. "Coexistence Chaotic Behavior on the Evolution of Populations of the Biological Systems Modeling by Three-Dimensional Quadratic Mappings."*Global and Stochastic Analysis* **8**, no. 3 (2021): 41–45.

[8] Seytov, Sh. J., N. B. Narziyev, A. I. Eshniyozov, and S. N. Nishonov. "The Algorithms for Developing Computer Programs for the Sets of Julia and Mandelbrot."*AIP Conference Proceedings* **2789**, no. 1 (2023): 050021.

[9] Eshmamatova, D. B., Sh. J. Seytov, and N. B. Narziev. "Basins of Fixed Points for Composition of the Lotka–Volterra Mappings and Their Classification."*Lobachevskii Journal of Mathematics* **44**, no. 2 (2023): 558–569.

[10] Seytov, Sh. J., and D. B. Eshmamatova. "Discrete Dynamical Systems of Lotka–Volterra and Their Applications on the Modeling of the Biogen Cycle in Ecosystem."*Lobachevskii Journal of Mathematics* **44**, no. 4 (2023): 1471–1485.

[11] Ganikhodzhaev, R. N., N. B. Narziyev, and Sh. J. Seytov. "Multi-dimensional Case of the Problem of Von Neumann–Ulam."*Uzbek Mathematical Journal* **3**, no. 1 (2015): 11–23.

[12] Devaney, R. L. *An Introduction to Chaotic Dynamical Systems*. 3rd ed.; Taylor & Francis: Boca Raton, FL, USA, 2022; 416 p.

[13] Ewens, W. J. *Mathematical Population Genetics*. 2nd ed.; Springer: New York, NY, USA, 2004; 417 p.

[14] Elaydi, S. N. *Discrete Chaos with Applications in Science and Engineering*. 2nd ed.; Taylor & Francis: Boca Raton, FL, USA, 2007; 440 p.

[15] Brauer, F., and C. Castillo-Chavez. *Mathematical Models in Population Biology and Epidemiology*. 2nd ed.; Springer: New York, NY, USA, 2012; 534 p.

[16] Kuznetsov, Y.A. *Elements of Applied Bifurcation Theory*, 4th ed.; Springer: Cham, Switzerland, 2023. 691 p.

[17] Strogatz, S. H. *Nonlinear Dynamics and Chaos*. 2nd ed.; Taylor & Francis: Boca Raton, FL, USA, 2019; 513 p.

[18] Gamelin, T. *Complex Analysis*. Springer: New York, NY, USA, 2001; 478 p.

[19] Müller, J., and C. Kuttler. *Methods and Models in Mathematical Biology: Deterministic and Stochastic Approaches*. Springer: Berlin/Heidelberg, Germany, 2015; 711 p.

[20] Milnor, J. *Dynamics in One Complex Variable*. 3rd ed.; Princeton University Press: Princeton, NJ, USA, 2006; 320 p.

[21] Sharkovsky, A. N. *Attractors, Orbits, and Their Basins*. Kiev: Naukova Dumka, 2013. 320 p.

[22] Eshmamatova, D. B., and F. A. Yusupov. "Dynamics of Compositions of Some Lotka–Volterra Mappings Operating in a Two-Dimensional Simplex."*Turkish Journal of Mathematics* **48**, no. 3 (2024): 391–406.

[23] Ganikhodzhaev, R. N., D. B. Eshmamatova, M. A. Tadzhieva, and B. S. Zakirov. "Some Degenerate Cases of Discrete Lotka–Volterra Dynamical Systems and Their Applications in Epidemiology."*Journal of Mathematical Sciences* **89**, no. 5: (2024): 755-769.

[24] Eshmamatova, D. B. "Discrete Analog of the SIR Model."*AIP Conference Proceedings* **2781**, no. 1 (2023). https://doi.org/10.1063/5.0144884

[25] Eshmamatova, D. B. "Dynamics of a Discrete SIRD Model Based on Lotka–Volterra Mappings."*AIP Conference Proceedings* **3004**, no. 1 (2024). https://doi.org/10.1063/5.0199929

[26] Eshmamatova, D. B. "Compositions of Lotka–Volterra Mappings as a Model for the Study of Viral Diseases."*AIP Conference Proceedings* **3085**, no. 1 (2024). https://doi.org/10.1063/5.0194902

[27] Eshmamatova, D. B., R. N. Ganikhodzhaev, and M. A. Tadzieva. "Dynamics of Lotka–Volterra Quadratic Mappings with Degenerate Skew-Symmetric Matrix."*Uzbek Mathematical Journal* **66**, no. 1 (2022): 85–97.

Information about authors:

Dilfuza Bakhromovna Eshmamatova – Doctor of Physical and Mathematical Sciences, Professor of Tashkent State Transport University, and Leading Researcher at the V.I. Romanovskiy Institute of Mathematics, Academy of Sciences of the Republic of Uzbekistan (Tashkent, Uzbekistan, e-mail: 24dil@mail.ru; eshmamatova.dil@gmail.com).

Seytov Shavkat Jumabayevich – PhD in Physical and Mathematical Sciences, Associate Professor at Tashkent State University of Economics (Tashkent, Uzbekistan, e-mail: sh-seytov@mail.ru).

Boboyarova Nargiza Ashurovna – Doctor of Philosophy in Physical and Mathematical Sciences, Associate Professor of Urgench technological university RANCH (Urgench, Uzbekistan, e-mail: boboyarova@gmail.com).

Islamova Mavluda Ikrambayevna – PhD student of Urgench State University named after Abu Raykhan Biruni (Urgench, Uzbekistan, e-mail: islamovamavluda191@gmail.com).

Авторлар туралы мәлімет:

Ешмаматова Дилфуза Бахромовна – физика-математика ғылымдарының докторы, профессор, Ташкент мемлекеттік көлік университеті; Өзбекстан Республикасы Ғылым академиясының В.И. Романовский атындағы Математика институтының жетекші маманы (Ташкент, Өзбекстан, электрондық пошта: 24dil@mail.ru; eshmamatova.dil@gmail.com).

Сейтов Шавкат Жумабаевич – PhD, Ташкент мемлекеттік экономика университетінің доценті (Ташкент, Өзбекстан, электрондық пошта: sh-seytov@mail.ru).

Species Бобоярова Наргиза Ашуровна – Физика-математика ғылымдары бойынша философия докторы (PhD), Үргеніш технологиялық университеті RANCH (Үргеніш, Өзбекстан, электрондық пошта: boboyarova@gmail.com).

Исламова Мавлуда Икрамбаевна – Әбу Райхан Бируни атындағы Ургенч мемлекеттік университетінің докторанты (PhD) (Ургенч, Өзбекстан, электрондық пошта: islamovamavluda191@gmail.com).

Информация об авторах:

Эшмаматова Дилфуза Бахромовна – доктор физико-математических наук, профессор Ташкентского государственного транспортного университета, ведущий специалист института математики имени В.И.Романовского Академии Наук Республики Узбекистан (Ташкент, Узбекистан, электронная почта: 24dil@mail.ru; eshmamatova.dil@gmail.com).

Сейтов Шавкат Жумабаевич – доктор философии (PhD) по физико-математическим наукам, доцент Ташкентского государственного экономического университета (Ташкент, Узбекистан, электронная почта: sh-seytov@mail.ru).

Бобоярова Наргиза Ашуровна – Доктор философии по физико-математическим наукам, доцент Ургенчский технологический университет RANCH (Ургенч, Узбекистан, электронная почта: boboyarova@gmail.com).

Исламова Мавлуда Икрамбаевна – докторант (PhD) Ургенчского государственного университета имени Абу Райхана Беруни (Ургенч, Узбекистан, электронная почта: islamovamavluda191@gmail.com).

Received: December 10, 2025
Accepted: June 10, 2026