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REGULAR SELF-ADJOINT PROBLEMS FOR THE LAPLACE EQUATION

The construction of solutions of regular boundary value problems for the Laplace equation has great theoretical and applied significance. Therefore, it is an urgent problem, and numerous studies have been devoted to this problem. Unlike other researches, in the work of T.Sh. Kalmenov, when performing a priori estimates for solving correctly solvable problems, using Riesz's theorem for a Hilbert space with a scalar product with a parameter, the potential of a simple layer was transformed into an integral operator over the domain, depending on the right-hand side. Using this, integral representations of solutions of coercively solvable problems were constructed, including criteria for their boundary value. In this paper, Green's functions of self-adjoint problems are explicitly obtained through the fundamental solution of the Laplace equation.
Keywords: Green function, boundary value problems, Laplace equation, integral representations, self-adjoint problems.

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Лаплас теңдеуі үшін регулярлы өзіне-өзі түйіндес есептер

Лаплас теңдеуі үшін регулярлы шеттік есептердің шешімдерін құру үлкен теориялық және қолданбалы маңызға ие. Сондықтан бұл өзекті мәселе болып табылады және осы проблемаға көптеген зерттеулер арналған. Басқа зерттеулерден айырмашылығы, Т. Ш. Кальменовтың жұмысында корректілі шешілімді есептердің шешімдері үшін априорлық бағалауларды орындау кезінде, параметрі бар скаляр көбейтіндімен берілген Гильберт кеңістігі үшін Рисс теоремасын пайдалана отырып, жай қабат потенциалы оң жақ бөлікке тәуелді облыс бойынша интегралдық операторға түрлендірілді. Осыны пайдалана отырып, коэрцитивті шешілімді есептер шешімдерінің интегралдық көріністері құрылды; соның ішінде олардың шекаралық болу критерийлері алынды. Осы жұмыста Лаплас теңдеуінің фундаменталды шешімі арқылы өзіне өзі түйіндес есептер үшін Грин функциялары айқын түрде алынды.
Түйін сөздер: Грин функциясы, шеттік есептер, Лаплас теңдеуі, интегралдық көріністер, өзіне-өзі түйіндес есептер.

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Регулярные самосопряженные задачи для уравнения Лапласа

Построение решений регулярных краевых задач для уравнения Лапласа имеет большую теоретическую и прикладную значимость. Поэтому она представляет актуальную задачу, и этой проблеме посвящены многочисленные исследования. В отличие от других исследований, в работе Кальменова Т. Ш. при выполнении априорных оценок для решении корректно разрешимых задач, пользуясь теоремой Рисса для гильбертова пространства со скалярным произведением с параметром, потенциал простого слоя был преобразован в интегральный оператор по области, зависящий от правой части. Используя это, были построены интегральные представления решений коэрцитивно разрешимых задач; в том числе получены критерии их граничности. В настоящей работе в явном виде получено функции Грина для самосопряжённых задач через фундаментальное решение уравнения Лапласа.
Ключевые слова: Функция Грина, краевые задачи, уравнение Лапласа, интегральные представления, самосопряжённые задачи.

1 Introduction

The theory of boundary value problems for elliptic equations occupies a central place in modern analysis and mathematical physics. In particular, the Laplace equation is one of the fundamental models in the study of potential theory, spectral theory, and the qualitative analysis of partial differential equations. The construction of solutions to regular boundary value problems for the Laplace equation is therefore of both theoretical and applied importance.

An important role in the analysis of boundary value problems for the Laplace equation is played by potential theory and Green functions. Green functions provide integral representations of solutions and allow one to describe the influence of boundary conditions explicitly.

In particular, coercively solvable problems for the Laplace equation were studied in [1], where integral representations of solutions were obtained by using a priori estimates and the Riesz representation theorem in a Hilbert space with a parameter-dependent scalar product. A method for constructing the Green function of the Dirichlet problem for the Laplace equation was proposed in [2]. In [3], a method for transforming the double layer potential into a volume potential was developed and applied to the description of coercive problems for the Laplace equation.

The general theory of elliptic boundary value problems has its roots in classical works devoted to generalized boundary conditions and operator methods. Some simple generalizations of linear elliptic boundary value problems were considered in [4]. General boundary value problems for elliptic differential equations were studied in [5]. Questions concerning extensions and restrictions of operators were investigated in [6]. A criterion for the boundaryness of integral operators was obtained in [7], and spectral questions related to the volume potential were studied in [8]. Eigenfunctions of the Laplacian and their applications to data analysis and representation on general domains were considered in [9].

Boundary value problems for partial differential equations are closely related not only to elliptic equations, but also to hyperbolic, parabolic, and nonlocal problems. Initial-boundary value problems for the wave equation were investigated in [10]. Nonlocal boundary value problems for the multidimensional heat equation in noncylindrical domains were studied in [11]. The Sommerfeld problem and inverse problems for the Helmholtz equation were considered in [12]. Solvability questions for the mixed Cauchy problem for the multidimensional Gellerstedt equation were analyzed in [13].

Potential theory and integral operators also play an essential role in the study of qualitative properties of solutions. Isoperimetric inequalities for the heat potential operator were obtained in [14]. A method for solving ill-posed nonlocal problems for elliptic equations with data on the whole boundary was proposed in [15]. The overdetermined Cauchy problem for the hyperbolic Gellerstedt equation was studied in [16]. A representation of the solution of the Dirichlet problem for the Laplace equation in the form of a generalized convolution was obtained in [17].

Inverse and abstract Cauchy problems form another closely related direction. Multidimensional inverse Cauchy problems for evolution equations were investigated in [18], while inverse abstract Cauchy problems were studied in [19]. Explicit formulas for Green functions are also important in concrete boundary value problems. In particular, an explicit form of the Green function of the Robin problem for the Laplace operator in a circle was obtained in [20]. Green's formula for integrodifferential operators was considered in [21].

Further developments include fractional elliptic problems and boundary value problems with integral conditions. Lyapunov, Hartman–Wintner and De La Vallée Poussin-type inequalities for fractional elliptic boundary value problems were studied in [22]. Lyapunov-type inequalities for nonlinear fractional boundary value problems were obtained in [23]. The Gellerstedt equation with integral perturbation in the Cauchy data was considered in [24]. Well-posed problems for the fractional Laplace equation with integral boundary conditions were investigated in [25].

Operator-theoretic approaches to boundary value problems are also important for understanding self-adjointness, correct extensions, and restrictions. The correctness of the definition of the Laplace operator with delta-like potentials was studied in [26]. Hyponormal and dissipative correct extensions and restrictions were investigated in [27]. Classical foundations of equations of mathematical physics are presented in [28]. More recent works include the study of the Klein–Gordon potential in characteristic coordinates [29] and modified Cauchy problems with nonlocal boundary conditions for degenerate hyperbolic equations [30]. Overdetermined problems for elliptic equations were considered in [31].

The present paper is devoted to regular self-adjoint problems for the Laplace equation. The main purpose is to construct Green functions for these problems explicitly in terms of the fundamental solution of the Laplace equation. This approach makes it possible to obtain integral representations of solutions and to describe regular self-adjoint boundary value problems through their corresponding Green kernels.

2 Main results

Let $\Omega \subset R^n$ is a simply connected finite domain with a smooth boundary, $u(x) = L_N^{-1} f$ is the Newtonian (volumetric) potential

$$u(x) = \int\limits_{\Omega} \varepsilon(x, y) f(y) dy, \tag{1}$$

where $\varepsilon(x, y)$ is the main fundamental solution of the Laplace equation

$$-\Delta_x \varepsilon(x, y) = \delta(x, y), \tag{2}$$

represented as

$$\varepsilon(x, y) = \begin{cases} -\frac{1}{2\pi} \ln|x - y|, & n = 2 \\ \frac{1}{\omega_n (n-2)|x-y|^{n-2}}, & n > 2 \end{cases}, \tag{3}$$

Theorem 1 *[1]. Let for any $f \in L_2(\Omega)$ there is a solution $u(x) \in W_2^2(\Omega)$ of the equation $-\Delta u = f$, satisfying the inequality*

$$\|u\|_{W_2^2(\Omega)} \leq c\|f\|_0, \tag{4}$$

then there are functions $\tilde{\mu}(\xi, y) \in W_2^{1/2}(\partial\Omega) \cap L_2(\Omega)$ *u* $\mu_0(y) \in L_2(\Omega)$ *such that this solution* $u(x)$ *is given as an integral operator* L^{-1}

$$u(x) = L^{-1}f = \int\limits_{\Omega} \varepsilon(x, y)f(y)dy + \int\limits_{\partial\Omega}\int\limits_{\Omega} \varepsilon(x, \xi)\tilde{\mu}(\xi, y)f(y)dS_\xi dy + \int\limits_{\Omega} \mu_0(y)f(y)dy. \quad (5)$$

Inversely, for any functions $\tilde{\mu}(\xi, y) \in W_2^{1/2}(\partial\Omega) \cap L_2(\Omega)$ *and* $\mu_0(y) \in L_2(\Omega)$ *there is a solution* $u \in W_2^2(\Omega)$ *of the equation* $-\Delta u = f$*, given by formula (5) and satisfying the inequality*

$$\|u\|_{W_2^2(\Omega)} \leq C\|f\|_0 \left(1 + \|\tilde{\mu}_0\|_0 + \|\tilde{\mu}(\xi, y)\|_{W_2^{1/2}(\partial\Omega)\times L_2(\Omega)}\right). \quad (6)$$

Here is a brief proof.

Proof. The general solution $u(x) \in W_2^2(\Omega)$ of the equation $-\Delta u = f$ has the form

$$u = L_N^{-1}f + u_\mu(x) + u_0, \quad \text{т.е.} \quad u_\mu(x) = u - L_N^{-1}f - u_0, \quad (7)$$

where u_0 is constant and

$$u_\mu(x) = \int\limits_{\partial\Omega}\int\limits_{\Omega} \varepsilon(x, \xi)\tilde{\mu}(\xi, y)f(y)dS_\xi dy,$$

$$L_N^{-1}f = \int\limits_{\Omega} \varepsilon(x, y)f(y)dy \in W_2^2(\Omega),$$

$$\|L_N^{-1}f\|_0 \leq c_1\|f\|_0. \quad (8)$$

Functions u_μ и u_0 are linearly independent, and since $u_0 = \text{const} \in W_2^2(\Omega)$, it follows that $u_\mu \in W_2^2(\Omega)$. Moreover, both u_μ and u_0 depend linearly and continuously on f, that is, there exists $\mu_0 \in L_2(\Omega)$ such that

$$u_0 = (\mu_0, f)_0, \quad \|u_0\|_{W_2^2(\Omega)} \leq \|\mu_0\|_0\|f\|_0. \quad (9)$$

Taking into account (4) and (7), and from the inequality

$$\|u_\mu\|_{W_2^2(\Omega)} \leq C\|f\|_0 \quad (10)$$

we conclude that $u_\mu \in W_2^2(\Omega)$, by virtue of (10) it follows that

$$c^{-1}\|\mu\|_{W_2^{1/2}(\partial\Omega)} \leq \|u_\mu\|_{W_2^2(\Omega)} \leq c\|f\|_0, \quad (11)$$

which shows the linear and continuous dependence of μ on f.

Using inequality (11), based on Theorem 4.1 from [1], there exists a function $V(\xi)\tilde{\mu}(\xi, y) \in W_2^{1/2}(\partial\Omega) \cap L_2(\Omega)$ such that

$$\mu(\xi) = \int\limits_{\Omega} V(\xi)\tilde{\mu}(\xi, y)f(y)dy$$

and

$$u_\mu(x) = \int\limits_{\partial\Omega} \varepsilon(x,\xi) \int\limits_{\Omega} V(\xi)\tilde{\mu}(\xi,y)f(y)dydS_\xi, \tag{12}$$

$$\|\mu\|_{W_2^{1/2}(\partial\Omega)} \le \|V(\xi)\tilde{\mu}(\xi,y)\|_{W_2^{1/2}(\partial\Omega)\cap L_2(\Omega)}\|f\|_0,$$

$$\|u_\mu\|_{W_2^2(\Omega)} \le c_4\|V(\xi)\tilde{\mu}(\xi,y)\|_{W_2^{1/2}(\partial\Omega)\cap L_2(\Omega)}\|f\|_0. \tag{13}$$

Based on 9 and 12, equality 7 is written as

$$u(x) = \int\limits_{\Omega} \varepsilon(x,y)f(y)dy + \int\limits_{\partial\Omega} \varepsilon(x,\xi) \int\limits_{\Omega} V(\xi)\tilde{\mu}(\xi,y)f(y)dydS_\xi + \int\limits_{\Omega} \mu_0(y)f(y)dy. \tag{14}$$

Considering the inequalities (4), (9), (10) from (14) we obtain the validity of the inequality

$$\|u\|_{W_2^2(\Omega)} \le \|f\|_0 \left(1 + \|V(\xi)\tilde{\mu}(\xi,y)\|_{W_2^{1/2}(\partial\Omega)\cap L_2(\Omega)} + \|\mu_0\|_0\right). \tag{15}$$

Since the core of the integral operator L^{-1}, according to (14), is given by the formula

$$K(x,y) = \varepsilon(x,y) + \int\limits_{\partial\Omega} \varepsilon(x,\xi)V(\xi)\tilde{\mu}(\xi,y)d\xi + \mu_0(y) = \varepsilon(x,,y) + \int\limits_{\partial\Omega} V(\xi)\varepsilon(x,\xi)\tilde{\mu}(\xi,y)d\xi + \mu_0(y), \tag{16}$$

the kernel of the operator conjugate to L^{-1} has the following form

$$K^*(x,y) = K(y,x) = \varepsilon(y,x) + \int\limits_{\partial\Omega} \varepsilon(y,\xi)V(\xi)\tilde{\mu}(\xi,x)d\xi + \mu_0(x). \tag{17}$$

Here $\tilde{\mu}(\xi,x) = V(\xi)\varepsilon(\xi,x)$.

Thus proved

Theorem 2 *Let L^{-1} is a coercively self-adjoint operator. Then the Green's function of this operator is given by the formula:*

$$G(x,y) = \varepsilon(x,y) + \int\limits_{\partial\Omega} V(\xi)\varepsilon(x,\xi)\varepsilon(\xi,y)dS_\xi. \tag{18}$$

Here $V(\xi)$ is a linear operator in the spaces of the corresponding generalized functions.

Proof. The Green's function $K(x,y)$ of an arbitrary coercive solvable problem and its adjoint problem are given by formulas (16)-(17), respectively, therefore, by virtue of self-adjointness, we have $K(x,y) = K^*(x,y)$, hence we have $\mu(\xi,x) = \varepsilon(\xi,x)$ and $\mu_0(x) \equiv \mu_0(y) \equiv 0$. Taking these equalities into account, we will make sure that

$$G(x,y) = K(x,y) = \varepsilon(x,y) + \int\limits_{\partial\Omega} V(\xi)\varepsilon(x,\xi)\varepsilon(\xi,y)dS_\xi.$$

Theorem 2 is proved.

Let us find the boundary condition of the self-adjoint operator

$$u(x) = \int_{\Omega} G(x,y) f(y)\, dy = \int_{\Omega} G(x,y)(-\Delta u)\, dy =$$

$$= \int_{\partial\Omega} \left(u(y) \frac{\partial G}{\partial n_y}(x,y) - G(x,y) \frac{\partial u}{\partial n_y} \right) dS_y + \int_{\Omega} \Delta_y G(x,y) u(y)\, dy. \tag{19}$$

Taking into account

$$\langle \Delta_y G(x,y), u(x) \rangle_{L_2(\Omega)} = \langle \delta(x-y), u(y) \rangle = u(x),$$

from (19) we find the boundary conditions of the general self-adjoint problem:

$$\int_{\partial\Omega} \left(u(y) \frac{\partial G}{\partial n_y}(x,y) - G(x,y) \frac{\partial u}{\partial n_y} \right) dS_y = 0. \tag{20}$$

Here

$$G(x,y) = \varepsilon(x,y) + \int_{\partial\Omega} V(\xi) \varepsilon(x,\xi) \varepsilon(\xi,y)\, d\bar{\xi}. \tag{21}$$

The boundary condition (20) is an analog of the boundary condition of a one-dimensional problem:

$$iy^{(1)}(x) = f(x), \qquad y(0) = \mu y(1),$$

where μ is a complex number, $|\mu| = 1$, $\mu^* \mu = |\mu| = 1$ gives all self-adjoint boundary conditions for the equation $iy^{(1)} = i\frac{d}{dx}, \quad x \in (0,1)$.

Define the operator $V(\xi)$ as follows:

$$V(\xi)\, \varepsilon(x,\xi) \big|_{x \in \partial\Omega,\ \xi \in \partial\Omega} = -\delta(x-\xi). \tag{22}$$

Then the function

$$G(x,y) = \varepsilon(x,y) + \int_{\partial\Omega} V(\xi)\, \varepsilon(x,\xi) \varepsilon(\xi,y)\, dS_\xi \tag{23}$$

is the Green's function of the Dirichlet problem.

Indeed, to do this, we will check the boundary condition for the function $G(x,y)$ using the method from [28]:

$$G(x,y) \big|_{x \in \partial\Omega} = \varepsilon(x,y) \big|_{x \in \partial\Omega} + \int_{\partial\Omega} V(\xi)\, \varepsilon(x,\xi) \big|_{x \in \partial\Omega} \varepsilon(\xi,y)\, dS_\xi =$$

$$= \varepsilon(x,y)\big|_{x\in\partial\Omega} + \int\limits_{\partial\Omega} \Big(V(\xi),\, \varepsilon(x,\xi)\big|_{x\in\partial\Omega}\Big)\, \varepsilon(\xi,y)\, dS_\xi =$$

$$= \varepsilon(x,y)\big|_{x\in\partial\Omega} - \int\limits_{\partial\Omega} \Big(\delta(x-\xi),\, \varepsilon(\xi,y)\Big)\, dS_\xi =$$

$$= \varepsilon(x,y)\big|_{x\in\partial\Omega} - \varepsilon(x,y)\big|_{x\in\partial\Omega} = 0.$$

Note that in [1] the operator $V(\xi)$ is defined through the spectral decomposition of the function $\varepsilon(x,\xi)\big|_{x\in\partial\Omega,\ \xi\in\partial\Omega}$.

Using the methods of adjoint problems, the properties of the Newtonian potential and the potential of the double layer, it is possible to obtain a representation of the Green function of the Dirichlet problem, that is, the following is proved.

Theorem 3 *The Green function of the Dirichlet problem in the domain of Ω is given by the formula*

$$G(x,y) = \chi_- G^-(x,y) - \chi_+ G^+(x,y), \tag{24}$$

where

$$G^-(x,y) = \varepsilon(x,y) + \int_{\partial\Omega} \frac{\partial}{\partial n_\xi^-}\varepsilon(x,\xi)\varepsilon(\xi,y), \quad x\in\Omega^- = \Omega, \tag{25}$$

$$G^+(x,y) = \int_{\partial\Omega} \frac{\partial}{\partial n_\xi^+}\varepsilon(x,\xi)\varepsilon(\xi,y), \quad x\in\Omega^+ = R^n\backslash\Omega, \tag{26}$$

here $\frac{\partial}{\partial n_\xi^-}$ is the derivative on the external normal to $\partial\Omega^-$, and $\frac{\partial}{\partial n_\xi^+}$ is the derivative on the normal to $\partial\Omega^+$,

$$\chi_- = \begin{cases} 1 & x\in\overline{\Omega^-} \\ 0 & x\in\Omega^+ \end{cases} \tag{27}$$

$$\chi_+ = \begin{cases} 1 & x\in\overline{\Omega^+} \\ 0 & x\in\Omega^- \end{cases} \tag{28}$$

Proof. By direct verification, we can verify that the function (24) is the Green function of the Dirichlet problem in the domain of Ω, i.e.

$$-\Delta_x G(x,y) = \delta(x-y), \;\; x\in\Omega, \tag{29}$$

$$G|_{x\in\partial\Omega} = 0. \tag{30}$$

Remark 1 *Using the property of the function $\delta(x-\xi)$, where $x\in\Omega,\ \xi\in\partial\Omega$, the Green's function $G(x,y)$ of the Dirichlet problem can be represented in the form*

$$G(x,y) = \varepsilon(x,y) - \langle\delta(x-\xi), \varepsilon(\xi,y)\rangle_{L_2(\partial\Omega)}. \tag{31}$$

3 Conclusion

In this paper, regular self-adjoint boundary value problems for the Laplace equation were considered. Using the fundamental solution of the Laplace equation and the representation of coercively solvable problems, an explicit form of the Green function for self-adjoint problems was obtained.

The obtained representation makes it possible to describe regular self-adjoint boundary value problems in terms of their Green functions and the corresponding boundary operators. In particular, the boundary condition of the general self-adjoint problem was derived in integral form. This condition reflects the symmetry of the Green function and gives a natural characterization of the self-adjointness of the corresponding operator.

As an important special case, the Dirichlet problem for the Laplace equation was considered. By choosing the operator $V(\xi)$ appropriately, the Green function of the Dirichlet problem was recovered and its boundary condition was verified directly.

References

[1] Kalmenov, T. Sh. "Integral Representations of Solutions of Coercive Solvable Problems for the Laplace Equation." *Boundary Value Problems* 2025, no. 12 (2025).

[2] Kalmenov, T. S. "On a Method for Constructing the Green Function of the Dirichlet Problem for the Laplace Equation." *Bulletin of the Karaganda University. Mathematics Series* 114, no. 2 (2024): 105–113.

[3] Kalmenov, T., D. Suragan, and A. Kadirbek. "On a Method for Transforming the Double Layer Potential to a Volume Potential and Its Application to the Description of Coercive Problems for the Laplace Equation." *Computational and Applied Mathematics* 25, no. 1 (2026): 170–179.

[4] Bitsadze, A. V., and A. A. Samarskii. "On Some Simplest Generalizations of Linear Elliptic Boundary Value Problems." *Doklady Akademii Nauk SSSR* 185, no. 4 (1969): 739–740.

[5] Vishik, M. I. "On General Boundary Value Problems for Elliptic Differential Equations." *Trudy Moskovskogo Matematicheskogo Obshchestva* 1 (1952): 187–246.

[6] Kokebaev, B. K., M. Otelbaev, and A. N. Shynybekov. "Some Questions of Extensions and Restrictions of Operators." *Reports of the Academy of Sciences of the USSR* 271, no. 6 (1983): 1307–1310.

[7] Kalmenov, T. Sh., and M. Otelbaev. "A Criterion for the Boundedness of Integral Operators." *Reports of the National Academy of Sciences of Kazakhstan* 466, no. 4 (2016): 395–395.

[8] Kalmenov, T. Sh., and D. Suragan. "On the Spectral Questions of the Volume Potential." *Reports of the National Academy of Sciences of Kazakhstan* 428, no. 1 (2009): 16–19.

[9] Saito, N. "Data Analysis and Representation on a General Domain Using Eigenfunctions of the Laplacian." *Applied and Computational Harmonic Analysis* 25, no. 1 (2008): 68–97.

[10] Kalmenov, T. Sh., and D. Suragan. "Initial-Boundary Value Problems for the Wave Equation." *Electronic Journal of Differential Equations* 2014, no. 48 (2014): 1–6.

[11] Kalmenov, T. Sh., and N. E. Tokmagambetov. "On a Nonlocal Boundary Value Problem for the Multidimensional Heat Equation in a Noncylindrical Domain." *Siberian Mathematical Journal* 54, no. 6 (2013): 1023–1028.

[12] Kalmenov, T. Sh., S. I. Kabanikhin, and A. Les. "The Sommerfeld Problem and Inverse Problem for the Helmholtz Equation." *Journal of Inverse and Ill-posed Problems* 29, no. 1 (2021): 49–64.

[13] Kalmenov, T. Sh., A. V. Rogovoy, and S. I. Kabanikhin. "Hadamard's Example and Solvability of the Mixed Cauchy Problem for the Multidimensional Gellerstedt Equation." *Journal of Inverse and Ill-posed Problems* 30, no. 6 (2022): 891–904.

[14] Kalmenov, T. S., A. Kassymov, and D. Suragan. "Isoperimetric Inequalities for the Heat Potential Operator." *Filomat* 32, no. 3 (2018): 903–910.

[15] Kalmenov, T. S., and B. T. Torebek. "A Method for Solving an Ill-posed Nonlocal Problem for the Elliptic Equation with Data on the Whole Boundary." *Journal of Pseudo-Differential Operators and Applications* 10, no. 1 (2019): 177–185.

[16] Rogovoy, A. V., T. Sh. Kalmenov, and S. I. Kabanikhin. "The Overdetermined Cauchy Problem for the Hyperbolic Gellerstedt Equation." *Journal of Inverse and Ill-posed Problems* 32, no. 5 (2024): 1051–1062.

[17] Kalmenov, T. Sh., and G. D. Arepova. "Representation of the Solution of the Dirichlet Problem for the Laplace Equation in the Form of a Generalized Convolution." *Complex Variables and Elliptic Equations* 64, no. 5 (2019): 816–824.

[18] Karazym, M., T. Ozawa, and D. Suragan. "Multidimensional Inverse Cauchy Problems for Evolution Equations." *Inverse Problems in Science and Engineering* 28, no. 11 (2020): 1582–1590.

[19] Ozawa, T., J. E. Restrepo, and D. Suragan. "Inverse Abstract Cauchy Problems." *Applicable Analysis* 101, no. 14 (2021): 4965–4969.

[20] Sadybekov, M. A., B. K. Turmetov, and B. T. Torebek. "On an Explicit Form of the Green Function of the Robin Problem for the Laplace Operator in a Circle." *Advances in Pure and Applied Mathematics* 6, no. 3 (2015): 163–172.

[21] Tokmagambetov, N., and B. T. Torebek. "Green's Formula for Integro-differential Operators." *Journal of Mathematical Analysis and Applications* 468, no. 1 (2018): 473–479.

[22] Kassymov, A., M. Kirane, and B. T. Torebek. "Lyapunov, Hartman–Wintner and de la Vallée Poussin-Type Inequalities for Fractional Elliptic Boundary Value Problems." *Complex Variables and Elliptic Equations* 67, no. 1 (2020): 246–258.

[23] Kassymov, A., and B. T. Torebek. "Lyapunov-Type Inequalities for a Nonlinear Fractional Boundary Value Problem." *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A: Matemáticas* 115, no. 15 (2021).

[24] Tokmagambetov, N. E. "The Gellerstedt Equation with Integral Perturbation in the Cauchy Data." *Journal of Mathematical Sciences* 213 (2016): 910–916.

[25] Tokmagambetov, N. E., and B. T. Torebek. "Well-posed Problems for the Fractional Laplace Equation with Integral Boundary Conditions." *Electronic Journal of Differential Equations* 2018, no. 90 (2018): 1–10.

[26] Kanguzhin, B. E., and K. S. Tulenov. "Correctness of the Definition of the Laplace Operator with Delta-like Potentials." *Complex Variables and Elliptic Equations* 67, no. 4 (2020): 898–920.

[27] Kakharman, N., K. Tulenov, and L. Zhumanova. "On Hyponormal and Dissipative Correct Extensions and Restrictions." *Mathematical Methods in the Applied Sciences* 45, no. 16 (2022): 9049–9060.

[28] Vladimirov, V. S. *Equations of Mathematical Physics*. Moscow: Nauka, 1981.

[29] Kalmenov, T., and D. Suragan. "Klein–Gordon Potential in Characteristic Coordinates." *Demonstratio Mathematica* 57, no. 1 (2024): Article 20240015.

[30] Kakharman, N., and T. Sh. Kalmenov. "A Modified Cauchy Problem with Nonlocal Boundary Condition for a Degenerate Hyperbolic Equation." *Mathematical Methods in the Applied Sciences* 48, no. 8 (2025): 9313–9319.

[31] Kalmenov, T., and N. Kakharman. "An Overdetermined Problem for Elliptic Equations." *AIMS Mathematics* 9, no. 8 (2024): 20627–20640.

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