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REGULARIZED TRACE OF A SECOND-ORDER DIFFERENTIAL OPERATOR ON A TREE WITH DIRICHLET CONDITIONS

This paper considers the problem of calculating the regularized trace of a second-order differential operator on a tree with Dirichlet conditions at the boundary vertices and matching conditions at the interior vertices. The original graph is a tree consisting of seven vertices and six oriented arcs. The relevance of the study is determined by the need to further develop methods of the spectral theory of differential operators on graphs and to generalize known results obtained for operators on a finite interval. To solve the problem, the original tree is divided into two subgraphs, each of which is a star graph. This approach makes it possible to apply established methods of spectral analysis to individual parts of the original graph. Particular attention is paid to constructing the characteristic determinants of the Dirichlet and Neumann problems on the considered subgraphs, as well as to studying their asymptotic behavior. The obtained representations are used to establish asymptotic formulas for the eigenvalues of the original problem. By applying the residue theorem, contour integration methods, and methods of complex analysis, a final formula for the regularized trace of the Sturm–Liouville operator on the considered tree is obtained. The obtained result generalizes known results for differential operators on a finite interval to the case of graph-trees. The study extends existing approaches to the analysis of spectral characteristics of operators on graphs and provides a basis for further investigation of regularized traces on more complex graphs and other types of differential operators.
Key words: regularized trace, second-order differential operator, Dirichlet conditions, Neumann conditions, matching conditions, characteristic determinant, eigenvalues.

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Дирихле шарттарымен ағаштағы екінші ретті дифференциалдық оператордың өңделген ізі

Бұл жұмыста шекаралық төбелерде Дирихле шарттары және ішкі төбелерде сәйкестік шарттары қойылған ағашта екінші ретті дифференциалдық оператордың регуляризацияланған ізін есептеу мәселесі қарастырылады. Бастапқы граф жеті төбеден және алты бағытталған доғадан тұратын ағаш болып табылады. Зерттеудің өзектілігі графтардағы дифференциалдық операторлардың спектрлік теориясының әдістерін одан әрі дамыту және кесіндідегі операторлар үшін алынған белгілі нәтижелерді жалпылау қажеттілігімен байланысты. Қойылған мәселені шешу үшін бастапқы ағашты екі ішкі графқа бөлу әдісі қолданылады, олардың әрқайсысы жұлдызша граф болып табылады. Мұндай тәсіл бастапқы графтың жеке бөліктеріне спектрлік талдаудың белгілі әдістерін қолдануға мүмкіндік береді. Зерттеуде қарастырылып отырған ішкі графтардағы Дирихле және Нейман есептерінің сипаттамалық анықтауыштарын құруға, сондай-ақ олардың асимптотикалық өзгерісін зерттеуге ерекше көңіл бөлінеді.

Алынған өрнектер бастапқы есептің меншікті мәндері үшін асимптотикалық формулаларды анықтауда қолданылады. Қалдықтар теоремасын, контур бойынша интегралдау әдісін және комплекс анализ әдістерін қолдану арқылы қарастырылып отырған ағаштағы Штурм–Лиувилл операторының регуляризацияланған ізі үшін қорытынды формула алынды. Алынған нәтиже кесіндідегі дифференциалдық операторлар үшін белгілі нәтижелерді граф-ағаштар жағдайына жалпылайды. Жүргізілген зерттеу графтардағы операторлардың спектрлік сипаттамаларын зерттеудің қолданыстағы тәсілдерін кеңейтеді және күрделірек графтар мен дифференциалдық операторлардың басқа да типтеріндегі регуляризацияланған іздерді одан әрі зерттеуге негіз қалайды.
Түйін сөздер: өңделген із, екінші ретті дифференциалдық оператор, Дирихле шарттары, Нейман шарттары, келісу шарттары, сипаттамалық анықтауыш, меншікті мәндер.

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Регуляризованный след дифференциального оператора второго порядка на дереве с условиями Дирихле

В данной работе рассматривается задача вычисления регуляризованного следа дифференциального оператора второго порядка на дереве с условиями Дирихле в граничных вершинах и условиями согласования во внутренних вершинах. Исходный граф представляет собой дерево с семью вершинами и шестью ориентированными дугами. Актуальность исследования обусловлена необходимостью дальнейшего развития методов спектральной теории дифференциальных операторов на графах и обобщения известных результатов, полученных для операторов на конечном отрезке. Для решения поставленной задачи используется разбиение исходного дерева на два подграфа, каждый из которых представляет собой звёздный граф. Такой подход позволяет применять известные методы спектрального анализа к отдельным частям исходного графа. Особое внимание уделяется построению характеристических определителей задач Дирихле и Неймана на рассматриваемых подграфах, а также исследованию их асимптотического поведения. Полученные представления используются для установления асимптотических формул для собственных значений исходной задачи. С помощью теоремы о вычетах, метода интегрирования по контуру и методов комплексного анализа получена итоговая формула для регуляризованного следа оператора Штурма–Лиувилля на рассматриваемом дереве. Полученный результат обобщает известные результаты для дифференциальных операторов на конечном отрезке на случай графов-деревьев. Проведённое исследование расширяет существующие подходы к изучению спектральных характеристик операторов на графах и создаёт основу для дальнейшего исследования регуляризованных следов на более сложных графах и других типах дифференциальных операторов.
Ключевые слова: регуляризованный след, дифференциальный оператор второго порядка, условия Дирихле, условия Неймана, условия согласования, характеристический определитель, собственные значения.

1 Introduction

The study was conducted to calculate the regularized trace of the Sturm-Liouville operator on a tree consisting of seven vertices and six oriented edges. This question is relevant in the context of the theory of differential equations and the application of Sturm-Liouville operators. Second-order differential equations defined on the edges of a tree and their eigenvalues under various boundary conditions were investigated. The main goal was to calculate the limit of the regularized trace of the Sturm-Liouville operator on a tree using various mathematical methods such as characteristic determinants and deduction theorems.

2 Research methodology

Consider a tree Γ consisting of seven vertices and six oriented edges (Fig. 1). Let us denote

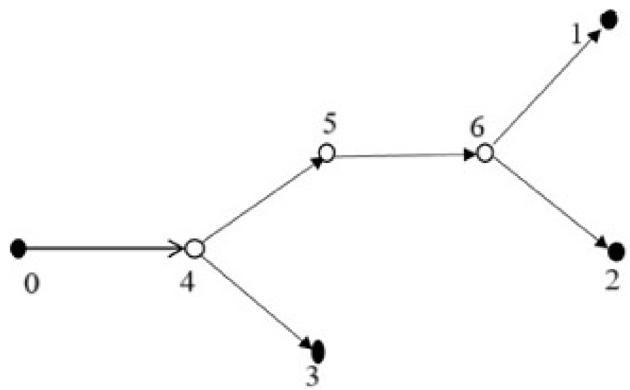

Figure 1: The tree.

the edges of the tree by e_1, e_2, e_3, e_4, e_5, e_6. Each edge has a starting and ending vertex. For example, an edge e_3 starts from vertex number 4 and enters the vertex number 3. In general, the edge e_j ends at vertex number j. The following homogeneous second-order differential equation is defined on each edge:

$$-y_j''(x_j) = \lambda y_j(x_j), \qquad x_j \in e_j, \qquad j = 1,2,3,4,5,6. \tag{1}$$

The length of the edge e_j is equal to $b_j > 0$. The numbers b_1, b_2, b_3, b_4, b_5, b_6 satisfy the following requirement:

Requirement 1. For all k, j from the set $\{1,2,3,4,5,6\}$, the ratio $\frac{b_k}{b_j}$ is irrational.

To the system of equations (1), we add the Dirichlet boundary conditions at the boundary vertices:

$$y_j(b_j) = 0 \qquad \text{by} \quad j = 1,2,3 \qquad \text{and} \qquad y_4(0) = 0. \tag{2}$$

We also add the matching conditions at the interior vertices of the tree to the system of equations (1):

$$\begin{aligned}
&y_4(b_4) = y_5(0) = y_3(0), \\
&y_5(b_5) = y_6(0), \\
&y_6(b_6) = y_1(0) = y_2(0), \\
&y_4'(b_4) = y_5'(0) + y_3'(0) + k_4 y_3(0), \\
&y_5'(b_5) = y_6'(0) + y_6(0), \\
&y_6'(b_6) = y_1'(0) + y_2'(0) + k_6 y_2(0).
\end{aligned} \tag{3}$$

The numbers k_4, k_6 are real, the eigenvalues of problem (1)–(3) are real [1]. Since the lengths b_1, b_2, b_3, b_4, b_5, b_6 satisfy Requirement 1, the eigenvalues of problem (1)–(3) are simple. These conclusions can be found in [2].

Let the eigenvalues of problem (1)–(3) for admissible k_4, k_6 be denoted by λ_n, $n \geq 1$. If $k_4 = k_6 = 0$ then the corresponding eigenvalues of problem (1)–(3) will be denoted by λ_n^0, $n \geq 1$. It follows from the results of N. P. Bondarenko [3] that the following asymptotic relations hold:

$$\lim_{n\to\infty} \frac{\lambda_n}{n^2} = \lim_{n\to\infty} \frac{\lambda_n^0}{n^2} = \frac{(b_1 + b_2 + b_3 + b_4 + b_5 + b_6)^2}{\pi^2}. \tag{4}$$

The aim of the present work is to calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} \left(\lambda_k - \lambda_k^0 - t_k\right),$$

where the sequences $\{m_n\}$ and $\{a_n\}$ are chosen in a special way. The resulting formula, in the terminology of V. A. Sadovnichii [4], is called the regularized trace of the Sturm–Liouville operator on a tree.

Let us choose the interior vertex numbered 5 and divide the original graph Γ into two subgraphs Γ_1 and Γ_2. In fact, the subgraphs Γ_1 and Γ_2 are star graphs. For these subgraphs, vertex 5 is a boundary vertex. On the subgraph Γ_1, we consider two different problems. The Dirichlet problem on the subgraph Γ_1 means that all boundary conditions at vertices 0 and 3 preserved, while at vertex 5, the Dirichlet condition $y_5(b_5) = 0$ is imposed. We also introduce the Neumann problem on the subgraph Γ_1: all boundary conditions at vertices 0 and 3 remain unchanged, and the matching conditions at vertex 4 are also satisfied, while the Neumann condition $y_5'(b_5) = 0$ is imposed at vertex 5. It follows from the results of [2] that the characteristic determinants of the corresponding Neumann and Dirichlet problems are given by the following formulas:

$$\begin{aligned}
\Delta_1^{(N)}\left(z^2\right) &= \frac{1}{z}\sin zb_5 \sin zb_3 \sin zb_4 - \frac{1}{z}\sin zb_3 \cos zb_4 \cos zb_5 \\
&\quad - \frac{1}{z}\sin zb_4 \cos zb_3 \cos zb_5 + k_4 \frac{\sin zb_3}{z}\frac{\sin zb_4}{z}\cos zb_5, \\
\Delta_1^{(D)}\left(z^2\right) &= -\frac{\sin zb_3}{z}\frac{\sin zb_4}{z}\cos zb_5 - \frac{\sin zb_3}{z}\frac{\sin zb_5}{z}\cos zb_4 \\
&\quad - \frac{\sin zb_4}{z}\frac{\sin zb_5}{z}\cos zb_3 + k_4 \frac{\sin zb_3}{z}\frac{\sin zb_4}{z}\frac{\sin zb_5}{z}.
\end{aligned}$$

Similarly, the Dirichlet and Neumann problems are introduced on the subgraph Γ_2. Their characteristic determinants are calculated using the following formulas:

$$\begin{aligned}
\Delta_2^{(N)}\left(z^2\right) &= -\frac{1}{z}\sin zb_2 \sin zb_1 \sin zb_6 + \frac{1}{z}\sin zb_2 \cos zb_1 \cos zb_6 \\
&\quad + \frac{1}{z}\sin zb_1 \cos zb_2 \cos zb_6 - k_6 \frac{\sin zb_2}{z}\frac{\sin zb_1}{z}\cos zb_6, \\
\Delta_2^{(D)}\left(z^2\right) &= -\frac{\sin zb_2}{z}\frac{\sin zb_1}{z}\cos zb_6 - \frac{\sin zb_2}{z}\frac{\sin zb_6}{z}\cos zb_1 \\
&\quad - \frac{\sin zb_1}{z}\frac{\sin zb_6}{z}\cos zb_2 + k_6 \frac{\sin zb_2}{z}\frac{\sin zb_1}{z}\frac{\sin zb_6}{z}.
\end{aligned}$$

According to the results of [5,16], the characteristic determinant $\Delta\left(z^2\right)$ of problem (1)–(3) can be written in the form

$$\Delta\left(z^2\right)=\left|\lambda=z^2\right|=\Delta_1^{(N)}\left(z^2\right)\Delta_2^{(D)}\left(z^2\right)-\Delta_2^{(N)}\left(z^2\right)\Delta_1^{(D)}\left(z^2\right).$$

In the case $k_4=k_6=0$, it is convenient to denote the characteristic determinant $\Delta\left(z^2\right)$ by $\Delta_0\left(z^2\right)$. In this case, we have the representation

$$\Delta\left(z^2\right)=\Delta_0\left(z^2\right)+k_4\Delta_1\left(z^2\right)+k_6\Delta_2\left(z^2\right)+k_4k_6\Delta_3\left(z^2\right). \tag{5}$$

To introduce the function $\Delta_0\left(z^2\right)$, $\Delta_1\left(z^2\right)$, $\Delta_2\left(z^2\right)$, $\Delta_3\left(z^2\right)$, it is convenient to use the following notation. Let I_5 denote the five-dimensional cube representing the set of vertices $\varepsilon=(\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6)$, where each ε_i can take either the value zero or one. Then the function $\Delta_0\left(z^2\right)$ has the following representation:

$$\Delta_0\left(z^2\right)=\frac{1}{16z^3}\sum_{\varepsilon\in I_5}a_\varepsilon^0\sin z\left(b_1+\sum_{i=2}^{6}(-1)^{\varepsilon_i}b_i\right)$$

if

$$\begin{aligned}
&a_\varepsilon^0=-1, && \varepsilon\in\{(1,0,1,0,1),(1,1,0,0,1),(1,1,1,0,1),(0,0,1,0,1),\\
& && \quad (0,1,0,0,1),(0,1,1,0,1)\},\\
&a_\varepsilon^0=3, && \varepsilon\in\{(1,0,0,0,1),(0,0,0,0,1)\},\\
&a_\varepsilon^0=-3, && \varepsilon\in\{(0,1,0,1,0),(0,0,1,1,0),(0,0,0,1,0),(1,1,1,1,0)\},\\
&a_\varepsilon^0=9, && \varepsilon\in\{(0,1,1,1,0)\},\\
&a_\varepsilon^0=1, && \varepsilon\in\{(1,1,0,1,0),(1,0,1,1,0),(1,0,0,1,0)\},\\
&a_\varepsilon^0=0 && \text{for all other}\,\varepsilon\in I_5.
\end{aligned}$$

Similarly, $\Delta_1\left(z^2\right)$ has the following representation:

$$\Delta_1\left(z^2\right)=\frac{1}{16z^4}\sum_{\varepsilon\in I_5}a_\varepsilon^1\cos z\left(b_1+\sum_{i=2}^{6}(-1)^{\varepsilon_i}b_i\right)$$

if

$$\begin{aligned}
&a_\varepsilon^1=1, && \varepsilon\in\{(0,1,0,1,0),(0,0,1,0,0),(0,1,0,0,0),(0,0,1,1,0),(1,1,1,0,0),\\
& && \quad (1,0,0,1,0),(1,1,1,0,1),(1,0,0,1,1),(0,0,1,1,1)\},\\
&a_\varepsilon^1=-1, && \varepsilon\in\{(1,1,1,1,1),(1,0,0,0,1),(1,1,1,1,0),(1,0,0,0,0),\\
& && \quad (0,1,0,1,1),(0,0,1,0,1),(0,1,0,0,1)\},\\
&a_\varepsilon^1=2, && \varepsilon\in\{(0,1,1,1,1),(0,0,0,0,1)\},\\
&a_\varepsilon^1=-2, && \varepsilon\in\{(0,1,1,0,1),(0,0,0,1,0)\},\\
&a_\varepsilon^1=0 && \text{for all other}\,\varepsilon\in I_5.
\end{aligned}$$

The function $\Delta_2(z^2)$ also has an analogous representation:

$$\Delta_2\left(z^2\right) = \frac{1}{16z^4}\sum_{\varepsilon\in I_5} a_\varepsilon^2 \cos z \left(b_1 + \sum_{i=2}^{6}(-1)^{\varepsilon_i} b_i\right)$$

if

$$\begin{aligned}
&a_\varepsilon^2 = -2, && \varepsilon \in \{(0,1,1,1,1),(0,0,0,1,0)\},\\
&a_\varepsilon^2 = 2, && \varepsilon \in \{(0,0,0,0,1),(0,1,1,0,1)\},\\
&a_\varepsilon^2 = 1, && \varepsilon \in \{(0,1,0,1,0),(0,0,1,1,0),(1,1,1,1,1),(1,0,0,0,0),\\
& && \quad (0,1,0,1,1),(1,0,0,1,0),(1,1,1,0,1),(0,0,1,1,1)\},\\
&a_\varepsilon^2 = -1, && \varepsilon \in \{(0,0,1,0,0),(0,1,0,0,0),(1,0,0,0,1),(1,1,1,1,0),\\
& && \quad (0,0,1,0,1),(1,1,1,0,0),(1,0,0,1,1),(0,1,0,0,1)\},\\
&a_\varepsilon^2 = 0 && \text{for all other}\, \varepsilon \in I_5.
\end{aligned}$$

The function $\Delta_3(z^2)$ also has an analogous representation:

$$\Delta_3\left(z^2\right) = \frac{1}{16z^5}\sum_{\varepsilon\in I_5} a_\varepsilon^3 \sin z \left(b_1 + \sum_{i=2}^{6}(-1)^{\varepsilon_i} b_i\right)$$

if

$$\begin{aligned}
&a_\varepsilon^3 = 1, && \varepsilon \in \{(1,0,0,0,1),(1,1,1,0,1),(0,1,1,1,0),(0,0,0,1,0),\\
& && \quad (0,0,1,0,1),(0,1,0,0,1),(1,1,0,1,0),(1,0,1,1,0)\},\\
&a_\varepsilon^3 = -1, && \varepsilon \in \{(1,0,1,0,1),(1,1,0,0,1),(0,1,0,1,0),(0,0,1,1,0),\\
& && \quad (0,0,0,0,1),(0,1,1,0,1),(1,1,1,1,0),(1,0,0,1,0)\},\\
&a_\varepsilon^3 = 0 && \text{for all other}\, \varepsilon \in I_5.
\end{aligned}$$

To calculate the regularized trace of problem (1)–(3), we rewrite the characteristic determinant in the following form:

$$\Delta(\lambda) = \Delta_0(\lambda)\left(1 + k_4\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)} + k_6\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)} + k_4k_6\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}\right). \tag{6}$$

Then,

$$\ln\Delta(\lambda) = \ln\Delta_0(\lambda) + \ln\left(1 + k_4\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)} + k_6\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)} + k_4k_6\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}\right). \tag{7}$$

In the complex z-plane, choose a sequence of circles $\{\gamma_n\}$ centered at the origin, and denote their radii by R_n. First, the circles R_n contain no zeros of the entire functions $\Delta(\lambda)$, $\Delta_0(\lambda)$. Second, $\lim\limits_{n\to\infty} R_n = \infty$. It follows from representation (4) and Rouché's theorem that, for sufficiently large n, the numbers of zeros of the entire functions $\Delta(\lambda)$, $\Delta_0(\lambda)$ inside the circle γ_n are the same. Denote by m_n the number of zeros of the function $\Delta(\lambda)$ inside the circle γ_n. We also note that the functions $\Delta(\lambda)$, $\Delta_0(\lambda)$ are even functions of z. Recall from [1] that the functions $\Delta(\lambda)$, $\Delta_0(\lambda)$ have simple real zeros, which are denoted by λ_s, λ_s^0, $s>0$. Let z_s^0 denote the nonnegative number whose square is equal to λ_s^0. For sufficiently large n, the following estimates hold on the circles γ_n:

$$\left|\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)}\right|+\left|\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)}\right|\leq\left|\frac{C}{z}\right|,\qquad\left|\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}\right|\leq\left|\frac{C}{z^2}\right|,\tag{8}$$

where C is a constant. By virtue of the above estimates, the following asymptotic representations hold on the corresponding circles γ_n:

$$2z\ln\left(1+k_4\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)}+k_6\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)}+k_4k_6\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}\right)=2z\left(k_4\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)}+k_6\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)}\right.$$

$$\left.+k_4k_6\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}\right)-z\left(k_4\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)}+k_6\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)}+k_4k_6\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}\right)^2+o\left(\frac{1}{z}\right)$$

$$=2k_4z\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)}+2k_6z\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)}+2k_4k_6z\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}-zk_4^2\frac{\Delta_1^2(\lambda)}{\Delta_0^2(\lambda)}$$

$$-zk_6^2\frac{\Delta_2^2(\lambda)}{\Delta_0^2(\lambda)}+2k_4k_6z\frac{\Delta_1(\lambda)\Delta_2(\lambda)}{\Delta_0^2(\lambda)}+o\left(\frac{1}{z}\right).\tag{9}$$

It follows from the residue theorem that

$$\frac{1}{2\pi i}\oint_{\gamma_n}z^2d\ln\Delta(\lambda)=\sum_{s=1}^{m_n}\lambda_s,\qquad\frac{1}{2\pi i}\oint_{\gamma_n}z^2d\ln\Delta_0(\lambda)=\sum_{s=1}^{m_n}\lambda_s^0.$$

Then, the difference

$$\sum_{s=1}^{m_n}\lambda_s-\sum_{s=1}^{m_n}\lambda_s^0=\frac{1}{2\pi i}\oint_{\gamma_n}\ln\left(1+k_4\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)}+k_6\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)}+k_4k_6\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}\right).$$

Here, relation (7) has been taken into account. Using the integration-by-parts formula, we rewrite the last equality in the following form:

$$\sum_{s=1}^{m_n}\lambda_s-\sum_{s=1}^{m_n}\lambda_s^0=-\frac{1}{2\pi i}\oint_{\gamma_n}\ln\left(1+k_4\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)}+k_6\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)}+k_4k_6\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}\right)d\lambda.\tag{10}$$

Taking into account representation (9) and the residue theorem, equality (10) takes the form

$$\sum_{s=1}^{m_n}\left(\lambda_s-\lambda_s^0-k_4\operatorname*{res}_{\lambda_s^0}\left(z\frac{\Delta_1(\lambda)}{\Delta_0(\lambda)}\right)-k_6\operatorname*{res}_{\lambda_s^0}\left(z\frac{\Delta_2(\lambda)}{\Delta_0(\lambda)}\right)\right)$$

$$=k_4k_6\left(\sum_{s=1}^{m_n}\operatorname*{res}_{\lambda_s^0}z\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}+\sum_{s=1}^{m_n}\operatorname*{res}_{\lambda_s^0}z\frac{\Delta_1(\lambda)\Delta_2(\lambda)}{\Delta_0^2(\lambda)}\right)$$

$$-\frac{k_4^2}{2}\sum_{s=1}^{m_n}\operatorname*{res}_{\lambda_s^0}\left(z\frac{\Delta_1^2(\lambda)}{\Delta_0^2(\lambda)}\right)-\frac{k_6^2}{2}\sum_{s=1}^{m_n}\operatorname*{res}_{\lambda_s^0}\left(z\frac{\Delta_2^2(\lambda)}{\Delta_0^2(\lambda)}\right). \tag{11}$$

It follows from the asymptotic formulas (4) and estimates (8) that there exists a finite limit

$$A_1=\lim_{n\to\infty}\sum_{s=1}^{m_n}\operatorname*{res}_{\lambda_s^0}\left(z\frac{\Delta_3(\lambda)}{\Delta_0(\lambda)}+z\frac{\Delta_1(\lambda)\,\Delta_2(\lambda)}{\Delta_0^2(\lambda)}\right).$$

Now, let us calculate the residue

$$\operatorname*{res}_{\lambda_s^0}\left(z\frac{\Delta_1^2(\lambda)}{\Delta_0^2(\lambda)}\right)=\lim_{\lambda\to\lambda_s^0}\frac{d}{dz}\frac{\left(z-z_s^0\right)^2 z\Delta_1^2(\lambda)}{\Delta_0^2(\lambda)}=\lim_{\lambda\to\lambda_s^0}\frac{d}{dz}\frac{\left(z-z_s^0\right)^2 z\Delta_1^2(\lambda)}{(\lambda-\lambda_s^0)^2\left(\Delta_0'(\lambda)\right)^2\left(1+\frac{1}{2}\left(\lambda-\lambda_s^0\right)\frac{\Delta_0''(\lambda)}{\Delta_0'(\lambda)}\right)}$$

$$=\frac{\Delta_1(\lambda_s^0)\,\Delta_1'(\lambda)}{\left(\Delta_0'(\lambda_s^0)\right)^2}+\frac{\left(z_s^0-1\right)\Delta_1^2(\lambda_s^0)}{4\lambda_s^0\left(\Delta_0'(\lambda_s^0)\right)^2}-\frac{\left(z_s^0\right)^2\Delta_1^2(\lambda_s^0)\,\Delta_0''(\lambda_s^0)}{2\lambda_s^0\left(\Delta_0'(\lambda_s^0)\right)^3}.$$

It follows from the asymptotic formulas (4) and estimates (8) that there exists a finite limit

$$A_2=\lim_{n\to\infty}\sum_{s=1}^{m_n}\frac{\left(z_s^0-1\right)\Delta_1^2(\lambda_s^0)}{4\lambda_s^0\left(\Delta_0'(\lambda_s^0)\right)^2},$$

$$A_3=\lim_{n\to\infty}\sum_{s=1}^{m_n}\frac{\left(z_s^0-1\right)\Delta_2^2(\lambda_s^0)}{4\lambda_s^0\left(\Delta_0'(\lambda_s^0)\right)^2}.$$

3 Research results

With the notation introduced above, we obtain the desired formula for the regularized trace

$$\lim_{n\to\infty}\sum_{s=1}^{m_n}\left(\lambda_s-\lambda_s^0-t_s\right)=k_4k_6A_1-\frac{k_4^2}{2}A_2-\frac{k_6^2}{2}A_3, \tag{12}$$

where

$$t_s=k_4\frac{z_s^0\Delta_1(\lambda_s^0)}{\Delta_0'(\lambda_s^0)}+k_6\frac{z_s^0\Delta_2(\lambda_s^0)}{\Delta_0'(\lambda_s^0)}-\frac{k_4^2}{2}\left(\frac{\Delta_1(\lambda_s^0)\,\Delta_1'(\lambda_s^0)}{\left(\Delta_0'(\lambda_s^0)\right)^2}-\frac{1}{2}\frac{\Delta_1^2(\lambda_s^0)\,\Delta_0''(\lambda_s^0)}{\left(\Delta_0'(\lambda_s^0)\right)^3}\right)$$
$$-\frac{k_6^2}{2}\left(\frac{\Delta_2(\lambda_s^0)\,\Delta_2'(\lambda_s^0)}{\left(\Delta_0'(\lambda_s^0)\right)^2}-\frac{1}{2}\frac{\Delta_2^2(\lambda_s^0)\,\Delta_0''(\lambda_s^0)}{\left(\Delta_0'(\lambda_s^0)\right)^3}\right).$$

4 Research results

The results of the study show that the regularized trace of the Sturm–Liouville operator on a tree can be expressed in terms of the limit of characteristic determinants and asymptotic representations, which is of significant importance for the theoretical analysis of operators on graphs. This result confirms the hypothesis concerning the existence of a finite limit of the regularized trace and reveals its dependence on the characteristic functions, thereby expanding our understanding of differential equations and operator theory.

5 Conclusion

This conclusion fits into the broader context of the work of other researchers, such as A. M. Savchuk and E. D. Galkovskii, who studied the computation of regularized traces for operators on intervals. However, unlike these studies, the analysis on graphs represents a more complex case, requiring the application of new methods, such as decomposing the graph into subgraphs and using characteristic determinants for each of them. Thus, the results of the study not only confirm previously known theoretical conclusions but also constitute a step forward in understanding differential equations and operators on more complex structures such as graphs.

Future research may include the study of analogous problems for more complex graphs, as well as the extension of the obtained results to other classes of differential equations, including the case of multidimensional operators. In addition, the possibility of applying the obtained results in other areas, such as mathematical physics and graph theory, opens up new directions for further theoretical and practical research.

In [17], the regularized trace of the double differentiation operator with non-local matching conditions on a star-shaped graph consisting of edges of the same length was studied. In [18, 19], the finding of eigenvalues and a system of eigenfunctions on one stratified set and on a star graph was studied. Formula (12) generalizes the results obtained by A. M. Savchuk [14] and E. D. Galkovskii [10]. In these works, regularized traces were calculated for differential operators on intervals. Formula (12) represents the regularized trace formula for the Sturm–Liouville operator on a specific tree.

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