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THE RIEMANN-HILBERT PROBLEM FOR A MULTIDIMENSIONAL SYSTEM GENERALIZING THE CAUCHY-RIEMANN SYSTEM

In this article, we construct a multidimensional — specifically, five-dimensional — analogue of the well-known Cauchy–Riemann system for analytic functions, using quaternions. The classical Cauchy–Riemann system plays a fundamental role in complex analysis and the theory of harmonic functions, and its higher-dimensional generalizations have attracted considerable attention in recent decades. Quaternions, with their non-commutative algebraic structure, provide a natural and effective framework for extending these ideas to five independent variables. We provide a representation of the solutions of the resulting system in terms of the derivatives of harmonic functions. This representation is of particular importance because it connects the solutions of the original first-order elliptic system with the well-studied theory of harmonic functions, thereby allowing classical methods to be applied. In addition, we study the corresponding Riemann–Hilbert boundary value problem and reduce it to the classical oblique derivative problem for harmonic functions. This reduction enables us to apply known results on the solvability and regularity of oblique derivative problems, and it clarifies the structure of the original boundary value problem for the five-dimensional system.
Keywords: holomorphic vector, Cauchy-Riemann system, Moisil-Teodorescu system, Riemann-Hilbert boundary value problem, oblique derivative problem, quaternions.

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Коши-Риман жүйесін жалпылайтын, бірінші ретті дифференциалдық теңдеулердің көп өлшемді жүйесі үшін Риман-Гильберт есебі

Бұл жұмыста біз кватерниондарды қолдана отырып, аналитикалық функцияларға арналған белгілі Коши–Риман жүйесінің көп өлшемді — атап айтқанда, бес өлшемді — аналогын құрастырамыз. Классикалық Коши–Риман жүйесі комплекс талдау мен гармоникалық функциялар теориясында іргелі рөл атқарады, ал оның жоғары өлшемді жалпылаулары соңғы онжылдықтарда едәуір назар аудартып келеді. Кватерниондар өзінің коммутативті емес алгебралық құрылымымен бұл идеяларды бес тәуелсіз айнымалыға табиғи және тиімді түрде кеңейтуге мүмкіндік береді. Біз алынған жүйенің шешімдерін гармоникалық функциялардың туындылары арқылы өрнектейміз. Бұл өрнектеу ерекше маңызға ие, себебі ол бастапқы бірінші ретті эллиптикалық жүйенің шешімдерін жақсы зерттелген гармоникалық функциялар теориясымен байланыстырып, классикалық әдістерді қолдануға мүмкіндік береді. Сонымен қатар, біз Риман–Гильберт шекаралық есебін зерттеп, оны гармоникалық функциялар үшін көлбеу туындысы бар классикалық есепке келтіреміз. Бұл келтіру көлбеу туынды есептерінің шешілімділігі мен регулярлылығы туралы белгілі нәтижелерді қолдануға мүмкіндік береді және бес өлшемді жүйе үшін бастапқы шекаралық есептің құрылымын айқындай түседі.
Түйін сөздер: голоморфтық вектор, Коши–Риман жүйесі, Моисил–Теодореску жүйесі, Риман– Гильберт шекаралық есебі, көлбеу туындысы бар есеп, кватерниондар.

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Задача Римана-Гильберта для многомерной системы дифференциальных уравнений первого порядка, обобщающей систему Коши-Римана

В данной статье мы строим многомерный — а именно пятимерный — аналог хорошо известной системы Коши–Римана для аналитических функций, используя кватернионы. Классическая система Коши–Римана играет фундаментальную роль в комплексном анализе и теории гармонических функций, а её многомерные обобщения привлекают значительное внимание в последние десятилетия. Кватернионы, обладающие некоммутативной алгебраической структурой, предоставляют естественную и эффективную основу для распространения этих идей на пять независимых переменных. Мы приводим представление решений полученной системы через производные гармонических функций. Это представление имеет особое значение, поскольку оно связывает решения исходной эллиптической системы первого порядка с хорошо изученной теорией гармонических функций и тем самым позволяет применять классические методы. Кроме того, мы исследуем соответствующую краевую задачу Римана – Гильберта и сводим её к классической задаче с наклонной производной для гармонических функций. Такое сведение даёт возможность использовать известные результаты о разрешимости и регулярности задач с наклонной производной и проясняет структуру исходной краевой задачи для пятимерной системы.
Ключевые слова: голоморфный вектор, система Коши — Римана, система Моисила — Теодореску, краевая задача Римана — Гильберта, задача с наклонной производной, кватернионы.

1 Introduction

The Cauchy–Riemann equations relate two harmonic components by a first-order differential operator. In three dimensions, Moisil and Théodoresco [1] used a system coupling a scalar and a vector field. For this system, Polkovnikov and Tarkhanov [2] showed that a Riemann–Hilbert problem satisfies the Lopatinskii–Shapiro condition exactly when the prescribed vector field is nowhere tangent to the boundary. In four dimensions, the Cauchy–Fueter operator defines quaternionic regularity [3, 4]. Clifford analysis provides an algebraic setting for constructing Dirac operators in more dimensions and for factorizing the Laplacian [5, 6].

The Moisil–Teodorescu system, as well as the systems satisfied by the components of a holomorphic quaternion, are first-order systems generalizing the Cauchy–Riemann system. The representation of all solutions of the three-dimensional and four-dimensional generalizations of the Cauchy–Riemann system in terms of derivatives of harmonic functions allows one to reduce the Riemann–Hilbert problem for these systems to a problem with an oblique derivative for harmonic functions.

In the present paper, the Riemann–Hilbert problem is considered for a more general extension of the Cauchy–Riemann system. A five-dimensional generalization of the Cauchy–Riemann system is constructed and a representation of its general solution in terms of derivatives of harmonic functions is derived. With the help of this representation, the Riemann–Hilbert problem is reduced to a problem with an oblique derivative for four harmonic functions of five independent variables.

2 Main result

It is known that one of the ways to construct multidimensional analogues of holomorphic functions of a complex variable consists in generalizing the Cauchy–Riemann conditions [7], [8], [9], [10], [11], [12]. For example, consider a system of differential equations in four-dimensional Euclidean space, which is a four-dimensional generalization of the Cauchy–Riemann system:

$$\sum_{j=1}^{4} B_j \frac{\partial U}{\partial x_j} = 0, \tag{1}$$

where B_j, $j = 1, 2, 3, 4$, are 2×2 matrices with constant complex entries, and $U = (u, v)$ is the unknown complex column vector composed of complex-valued functions $u = u(x)$, $v = v(x)$, $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. System (1) is called a four-dimensional analogue of the Cauchy–Riemann system if its components u and v are harmonic functions. Denote by $U = (u_1, u_2, u_3, u_4)$ the column vector composed of the real and imaginary parts of the components of the solution $U = (u, v)$ of system (1):

$$u_1 = \operatorname{Re} u, \quad u_2 = \operatorname{Im} u, \quad u_3 = \operatorname{Re} v, \quad u_4 = \operatorname{Im} v.$$

Having performed a homotopy classification, one can write the real variant of system (1) in canonical form, as in [13]:

$$\begin{cases} u_{1x_1} + u_{2x_2} + \rho b_1 u_{3x_3} - \rho b_2 u_{4x_3} - \rho b_2 u_{3x_4} - \rho b_1 u_{4x_4} = 0, \\ u_{2x_1} - u_{1x_2} + \rho b_2 u_{3x_3} + \rho b_1 u_{4x_3} - \rho b_1 u_{3x_4} - \rho b_2 u_{4x_4} = 0, \\ u_{3x_1} - u_{4x_2} - \rho b_1 u_{1x_3} - \rho b_2 u_{2x_3} - \rho b_2 u_{1x_4} - \rho b_1 u_{2x_4} = 0, \\ u_{4x_1} + u_{3x_2} + \rho b_2 u_{1x_3} - \rho b_1 u_{2x_3} + \rho b_1 u_{1x_4} + \rho b_2 u_{2x_4} = 0, \end{cases} \tag{2}$$

where $b = b_1 + ib_2$ is an arbitrary complex number, $\rho = (b_1^2 + b_2^2)^{-1}$. Setting $b_1 = 1$, $b_2 = 0$ in this system and assuming that the functions u_j, $j = 1, 2, 3, 4$, do not depend on x_4, we obtain the Moisil–Teodorescu system [14], [15]:

$$\begin{cases} u_{1x_1} + u_{2x_2} + u_{3x_3} = 0, \\ u_{2x_1} - u_{1x_2} + u_{4x_3} = 0, \\ u_{3x_1} - u_{4x_2} - u_{1x_3} = 0, \\ u_{4x_1} + u_{3x_2} - u_{2x_3} = 0. \end{cases} \tag{3}$$

If in system (2) we set $b_1 = 1$, $b_2 = 0$ and assume that all functions u_j, $j = 1, 2, 3, 4$, depend on all four arguments x_1, x_2, x_3, x_4, then we obtain the system:

$$\begin{cases} u_{1x_1} + u_{2x_2} + u_{3x_3} - u_{4x_4} = 0, \\ u_{2x_1} - u_{1x_2} + u_{4x_3} - u_{3x_4} = 0, \\ u_{3x_1} - u_{4x_2} - u_{1x_3} - u_{2x_4} = 0, \\ u_{4x_1} + u_{3x_2} - u_{2x_3} + u_{1x_4} = 0. \end{cases} \tag{4}$$

System (3) is a three-dimensional generalized analogue of the Cauchy–Riemann system, and its solutions are called holomorphic vectors. System (4) represents a four-dimensional generalization of the Cauchy–Riemann system and occurs in the theory of the holomorphic quaternion.

First, we express the general solution of system (3) in terms of derivatives of two harmonic functions. To this end, we introduce the complex variable

$$\zeta = x_1 + ix_2$$

and the complex unknown functions

$$p = u_4 + iu_3, \quad q = u_2 + iu_1.$$

Then system (3) can be written in the form:

$$\begin{pmatrix} \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2} \\ -\dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \end{pmatrix} \begin{pmatrix} u_4 + iu_3 \\ u_2 + iu_1 \end{pmatrix} = 0,$$

or, equivalently, system (3) can be rewritten as follows:

$$\frac{\partial p}{\partial x_3} + 2\frac{\partial q}{\partial \bar{\zeta}} = 0, \qquad -2\frac{\partial p}{\partial \zeta} + \frac{\partial q}{\partial x_3} = 0, \tag{5}$$

where

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

We seek the solution of the system in the form:

$$p = \frac{\partial \varphi}{\partial x_3} - 2\frac{\partial \varphi}{\partial \bar{\zeta}}, \qquad q = \frac{\partial \varphi}{\partial x_3} + 2\frac{\partial \varphi}{\partial \zeta}, \tag{6}$$

where $\varphi(x_1, x_2, x_3)$ is a harmonic function of three variables. Substituting (6) into (5), we obtain for the function $\varphi(x_1, x_2, x_3)$ the Laplace equation:

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial x_3^2} = 0.$$

Thus, in order for formulas (6) to represent the general solution of system (3), the function $\varphi(x_1, x_2, x_3)$ must have the following complex form:

$$\varphi(x_1, x_2, x_3) = \sigma(x_1, x_2, x_3) + i\omega(x_1, x_2, x_3),$$

where $\sigma(x_1, x_2, x_3)$ and $\omega(x_1, x_2, x_3)$ are arbitrary real harmonic functions. Separating the real and imaginary parts in formulas (6), we find:

$$\begin{cases} u_1 = \omega_{x_1} - \sigma_{x_2} + \omega_{x_3}, \\ u_2 = \sigma_{x_1} + \omega_{x_2} + \sigma_{x_3}, \\ u_3 = -\omega_{x_1} - \sigma_{x_2} + \omega_{x_3}, \\ u_4 = -\sigma_{x_1} + \omega_{x_2} + \sigma_{x_3}. \end{cases}$$

Now, to represent the solution of system (4) in terms of derivatives of a harmonic function, following the method presented above, we introduce two independent complex variables

$$\zeta = x_1 + ix_4, \qquad \eta = x_2 + ix_3,$$

and two complex unknown functions

$$p = u_4 + iu_1, \qquad q = u_3 + iu_2.$$

We rewrite system (4) as follows:

$$\begin{pmatrix} \dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_4} & \dfrac{\partial}{\partial x_2} + i\dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_2} - i\dfrac{\partial}{\partial x_3} & -\dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_4} \end{pmatrix} \begin{pmatrix} u_4 + iu_1 \\ u_3 + iu_2 \end{pmatrix} = 0,$$

or, using the new notation, we write it in the form:

$$\frac{\partial p}{\partial \zeta} + \frac{\partial q}{\partial \bar{\eta}} = 0, \qquad \frac{\partial p}{\partial \eta} - \frac{\partial q}{\partial \bar{\zeta}} = 0, \tag{7}$$

where

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_4}\right), \quad \frac{\partial}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_4}\right),$$
$$\frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial x_3}\right), \quad \frac{\partial}{\partial \bar{\eta}} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3}\right).$$

The solution of system (7) can be represented by the formulas:

$$p = \frac{\partial \varphi}{\partial \bar{\zeta}} + \frac{\partial \varphi}{\partial \bar{\eta}}, \qquad q = \frac{\partial \varphi}{\partial \eta} - \frac{\partial \varphi}{\partial \zeta}, \tag{8}$$

where the function $\varphi(x_1, x_2, x_3, x_4)$ is assumed to be analytic in all four variables. Substituting formulas (8) into the system of equations (7), we find that the complex function $\varphi(x_1, x_2, x_3, x_4)$ satisfies the Laplace equation:

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial x_3^2} + \frac{\partial^2 \varphi}{\partial x_4^2} = 0.$$

Consequently, in order for formulas (8) to represent a solution of system (7), the function $\varphi(x_1, x_2, x_3, x_4)$ must have the form:

$$\varphi(x_1, x_2, x_3, x_4) = \sigma(x_1, x_2, x_3, x_4) + i\omega(x_1, x_2, x_3, x_4).$$

Both the real and imaginary parts of the function $\varphi(x_1, x_2, x_3, x_4)$ are harmonic functions. Separating the real and imaginary parts in formulas (8), we obtain a representation of the solution of system (4) in terms of derivatives of two harmonic functions σ and ω:

$$\begin{cases} u_1 = \sigma_{x_2} - \omega_{x_1} - \sigma_{x_3} + \omega_{x_4}, \\ u_2 = \omega_{x_2} + \sigma_{x_1} - \omega_{x_3} - \sigma_{x_4}, \\ u_3 = \omega_{x_3} - \sigma_{x_4} + \omega_{x_2} - \sigma_{x_1}, \\ u_4 = \sigma_{x_3} + \omega_{x_4} + \sigma_{x_2} + \omega_{x_1}. \end{cases}$$

In a similar way, we obtain representations of the solutions of system (2). To this end, as above, we introduce independent variables and unknown complex functions in the following form:

$$\zeta = x_1 + ix_2, \qquad \eta = x_3 + ix_4, \qquad U = u_1 + iu_2, \qquad V = u_3 + iu_4.$$

Then system (2) can be rewritten as follows:

$$\frac{\partial U}{\partial \zeta} + \rho b \frac{\partial V}{\partial \bar{\eta}} = 0, \qquad -\rho \bar{b} \frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \bar{\zeta}} = 0, \tag{9}$$

where

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$
$$\frac{\partial}{\partial \eta} = \frac{1}{2}\left(\frac{\partial}{\partial x_3} - i\frac{\partial}{\partial x_4}\right), \quad \frac{\partial}{\partial \bar{\eta}} = \frac{1}{2}\left(\frac{\partial}{\partial x_3} + i\frac{\partial}{\partial x_4}\right).$$

If $\varphi = \varphi(\zeta, \bar{\zeta}, \eta, \bar{\eta})$ is an analytic function of all four arguments, then the solution of the system can be represented in the form:

$$U = \rho b \varphi_{\bar{\zeta}} + \varphi_{\bar{\eta}}, \qquad V = -\rho \bar{b} \varphi_{\zeta} + \varphi_{\eta}. \tag{10}$$

Indeed, substituting (10) into (9), for the complex function $\varphi(x_1, x_2, x_3, x_4)$ we obtain the Laplace equation:

$$\frac{1}{4}\Delta\varphi \equiv \frac{\partial^2 \varphi}{\partial \zeta \partial \bar{\zeta}} + \frac{\partial^2 \varphi}{\partial \eta \partial \bar{\eta}} = 0.$$

Thus, in order for formulas (10) to represent a solution of system (9), the function $\varphi(\zeta, \bar{\zeta}, \eta, \bar{\eta})$ must have the form:

$$\varphi(x_1, x_2, x_3, x_4) = \sigma(x_1, x_2, x_3, x_4) + i\omega(x_1, x_2, x_3, x_4),$$

where σ and ω are harmonic functions. Separating the real and imaginary parts in formula (10), we find the solution of system (2):

$$\begin{cases} u_1 = \rho b_1(\sigma_{x_1} - \omega_{x_2}) - \rho b_2(\omega_{x_1} + \sigma_{x_2}) + \sigma_{x_3} - \omega_{x_4}, \\ u_2 = \rho b_2(\sigma_{x_1} - \omega_{x_2}) + \rho b_1(\omega_{x_1} + \sigma_{x_2}) + \sigma_{x_4} + \omega_{x_3}, \\ u_3 = -\rho b_1(\sigma_{x_1} + \omega_{x_2}) - \rho b_2(\omega_{x_1} - \sigma_{x_2}) + \sigma_{x_3} + \omega_{x_4}, \\ u_4 = -\rho b_1(\omega_{x_1} - \sigma_{x_2}) + \rho b_2(\sigma_{x_1} + \omega_{x_2}) + \omega_{x_3} - \sigma_{x_4}. \end{cases} \tag{11}$$

In particular, it is easy to verify that for $b_1 = 1$, $b_2 = 0$ this solution yields the solution of system (4). If $b_1 = 1$, $b_2 = 0$ and the functions u_1, u_2, u_3, u_4 do not depend on the variable x_4, then solution (11) also yields the solution of system (3).

An analogous construction of the solution can be carried out for an elliptic first-order system with five independent variables, generalizing the Cauchy–Riemann system, the three-dimensional and four-dimensional Moisil–Teodorescu systems, as well as system (2):

$$\begin{cases} u_{1x_5}+\rho b_1(u_{5x_1}-u_{6x_2}-u_{7x_3}-u_{8x_4})-\rho b_2(u_{6x_1}+u_{5x_2}+u_{8x_3}-u_{7x_4}) \\ \quad -\rho b_3(u_{7x_1}-u_{8x_2}+u_{5x_3}+u_{6x_4})-\rho b_4(u_{8x_1}+u_{7x_2}-u_{6x_3}+u_{5x_4})=0, \\ u_{2x_5}+\rho b_2(u_{5x_1}-u_{6x_2}-u_{7x_3}-u_{8x_4})+\rho b_1(u_{6x_1}+u_{5x_2}+u_{8x_3}-u_{7x_4}) \\ \quad +\rho b_3(u_{8x_1}+u_{7x_2}-u_{6x_3}+u_{5x_4})-\rho b_4(u_{7x_1}-u_{8x_2}+u_{5x_3}+u_{6x_4})=0, \\ u_{3x_5}+\rho b_3(u_{5x_1}-u_{6x_2}-u_{7x_3}-u_{8x_4})+\rho b_4(u_{6x_1}+u_{5x_2}+u_{8x_3}-u_{7x_4}) \\ \quad +\rho b_1(u_{7x_1}-u_{8x_2}+u_{5x_3}+u_{6x_4})-\rho b_2(u_{8x_1}+u_{7x_2}-u_{6x_3}+u_{5x_4})=0, \\ u_{4x_5}+\rho b_4(u_{5x_1}-u_{6x_2}-u_{7x_3}-u_{8x_4})+\rho b_3(u_{6x_1}+u_{5x_2}+u_{8x_3}-u_{7x_4}) \\ \quad +\rho b_2(u_{7x_1}-u_{8x_2}+u_{5x_3}+u_{6x_4})+\rho b_1(u_{8x_1}+u_{7x_2}-u_{6x_3}+u_{5x_4})=0, \\ u_{5x_5}-\rho b_1(u_{1x_1}+u_{2x_2}+u_{3x_3}+u_{4x_4})-\rho b_2(u_{2x_1}-u_{1x_2}-u_{4x_3}+u_{3x_4}) \\ \quad -\rho b_3(u_{3x_1}+u_{4x_2}-u_{1x_3}-u_{2x_4})-\rho b_4(u_{4x_1}-u_{3x_2}+u_{2x_3}-u_{1x_4})=0, \\ u_{6x_5}+\rho b_2(u_{1x_1}+u_{2x_2}+u_{3x_3}+u_{4x_4})-\rho b_1(u_{2x_1}-u_{1x_2}-u_{4x_3}+u_{3x_4}) \\ \quad -\rho b_4(u_{3x_1}+u_{4x_2}-u_{1x_3}-u_{2x_4})+\rho b_3(u_{4x_1}-u_{3x_2}+u_{2x_3}-u_{1x_4})=0, \\ u_{7x_5}+\rho b_3(u_{1x_1}+u_{2x_2}+u_{3x_3}+u_{4x_4})+\rho b_4(u_{2x_1}-u_{1x_2}-u_{4x_3}+u_{3x_4}) \\ \quad -\rho b_1(u_{3x_1}+u_{4x_2}-u_{1x_3}-u_{2x_4})-\rho b_2(u_{4x_1}-u_{3x_2}+u_{2x_3}-u_{1x_4})=0, \\ u_{8x_5}+\rho b_4(u_{1x_1}+u_{2x_2}+u_{3x_3}+u_{4x_4})-\rho b_3(u_{2x_1}-u_{1x_2}-u_{4x_3}+u_{3x_4}) \\ \quad +\rho b_2(u_{3x_1}+u_{4x_2}-u_{1x_3}-u_{2x_4})-\rho b_1(u_{4x_1}-u_{3x_2}+u_{2x_3}-u_{1x_4})=0, \end{cases} \tag{12}$$

where $b=b_1+ib_2+jb_3+kb_4$ is a quaternionic constant; i,j,k are quaternionic units obeying the rules $ij=k$, $i^2=j^2=k^2=-1$; $u_1,\dots,u_8$ are the unknown functions; $x_1,\dots,x_5$ are independent variables.

For $b_1=1$, $b_2=b_3=b_4=0$ we obtain a system of first-order differential equations with five independent variables, which is a generalization of the Moisil–Teodorescu system:

$$\begin{cases} u_{1x_5}+u_{5x_1}-u_{6x_2}-u_{7x_3}-u_{8x_4}=0, \\ u_{2x_5}+u_{6x_1}+u_{5x_2}+u_{8x_3}-u_{7x_4}=0, \\ u_{3x_5}+u_{7x_1}-u_{8x_2}+u_{5x_3}+u_{6x_4}=0, \\ u_{4x_5}+u_{8x_1}+u_{7x_2}-u_{6x_3}+u_{5x_4}=0, \\ u_{5x_5}-u_{1x_1}-u_{2x_2}-u_{3x_3}-u_{4x_4}=0, \\ u_{6x_5}-u_{2x_1}+u_{1x_2}+u_{4x_3}-u_{3x_4}=0, \\ u_{7x_5}-u_{3x_1}-u_{4x_2}+u_{1x_3}+u_{2x_4}=0, \\ u_{8x_5}-u_{4x_1}+u_{3x_2}-u_{2x_3}+u_{1x_4}=0. \end{cases} \tag{13}$$

In this system, each of the functions $u_l(x_1,x_2,x_3,x_4,x_5)$, $l=1,2,\dots,8$, is harmonic. For example, to prove the harmonicity of the function u_7, we differentiate the first equation of the system with respect to x_3, the second with respect to x_4, the third with respect to x_1, the fourth with respect to x_2, and the seventh with respect to x_5; then we change the signs in the first two differentiated equations to the opposite. Adding the obtained equations, we get $\Delta u_7=0$. The harmonicity of the remaining functions is established analogously.

If in system (13) we set $u_r = u_r(x_1, x_2, x_3, x_4)$, $r = 1, 2, 3, 4$, and $u_5 = u_6 = u_7 = u_8 = 0$, then it turns into the Moisil–Teodorescu system (4).

Now we construct the general solution of system (12). To this end, we introduce the quaternionic constant

$$b = b_1 + ib_2 + jb_3 + kb_4, \qquad \bar{b} = b_1 - ib_2 - jb_3 - kb_4,$$

the quaternionic variable

$$\zeta = x_1 + ix_2 + jx_3 + kx_4$$

and the real variable x_5, as well as the unknown functions in quaternionic form:

$$U = u_1 + iu_2 + ju_3 + ku_4, \qquad V = u_5 + iu_6 + ju_7 + ku_8.$$

Then system (12) is written in the form:

$$\begin{pmatrix} \dfrac{\partial}{\partial x_5} & \rho b\delta \\ -\rho\bar{b}\bar{\delta} & \dfrac{\partial}{\partial x_5} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = 0, \tag{14}$$

where $\rho = (b_1^2 + b_2^2 + b_3^2 + b_4^2)^{-1}$, and

$$\delta = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + j\frac{\partial}{\partial x_3} + k\frac{\partial}{\partial x_4},$$
$$\bar{\delta} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} - j\frac{\partial}{\partial x_3} - k\frac{\partial}{\partial x_4}$$

are quaternionic differential operators. Indeed, if we write system (13) in the form:

$$\begin{cases} \dfrac{\partial u_1}{\partial x_5} + i\dfrac{\partial u_2}{\partial x_5} + j\dfrac{\partial u_3}{\partial x_5} + k\dfrac{\partial u_4}{\partial x_5} \\ \quad +\rho(b_1 + ib_2 + jb_3 + kb_4)\left(\dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2} + j\dfrac{\partial}{\partial x_3} + k\dfrac{\partial}{\partial x_4}\right)(u_5 + iu_6 + ju_7 + ku_8) = 0, \\ -\rho(b_1 - ib_2 - jb_3 - kb_4)\left(\dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2} - j\dfrac{\partial}{\partial x_3} - k\dfrac{\partial}{\partial x_4}\right)(u_1 + iu_2 + ju_3 + ku_4) \\ \quad + \left(\dfrac{\partial u_5}{\partial x_5} + i\dfrac{\partial u_6}{\partial x_5} + j\dfrac{\partial u_7}{\partial x_5} + k\dfrac{\partial u_8}{\partial x_5}\right) = 0, \end{cases}$$

then, comparing the coefficients of i, j, k and the free terms, we obtain system (12). It is easy to verify that the solutions of system (14), which can be written in the form

$$\frac{\partial U}{\partial x_5} + \rho b\delta V = 0, \qquad -\rho\bar{b}\bar{\delta}U + \frac{\partial V}{\partial x_5} = 0, \tag{15}$$

are given by the formulas:

$$U = \rho b\left(\delta\varphi + \frac{\partial\varphi}{\partial x_5}\right), \qquad V = \bar{\delta}\varphi - \frac{\partial\varphi}{\partial x_5}. \tag{16}$$

Here the quaternionic analytic function φ has the form:

$$\varphi = \varphi_1 + i\varphi_2 + j\varphi_3 + k\varphi_4.$$

Substituting expressions (16) into the first equation of system (15), for the quaternionic function φ we obtain the Laplace equation:

$$\Delta\varphi \equiv \delta\bar{\delta}\varphi + \varphi_{x_5x_5} = 0, \qquad \left(\Delta = \sum_{l=1}^{5} \frac{\partial^2}{\partial x_l^2}\right).$$

Consequently, in order for formulas (16) to represent a solution of system (13), the quaternionic analytic function φ must have the form:

$$\varphi(x) = \varphi_1(x) + i\varphi_2(x) + j\varphi_3(x) + k\varphi_4(x),$$

where $\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)$ are arbitrary real harmonic functions of the variables $x = (x_1, x_2, x_3, x_4, x_5)$. In real form, formulas (16) can be rewritten as follows:

$$\begin{cases}
u_1 = \rho\{b_1(\varphi_{1x_1} - \varphi_{2x_2} - \varphi_{3x_3} - \varphi_{4x_4}) - b_2(\varphi_{2x_1} + \varphi_{1x_2} + \varphi_{4x_3} - \varphi_{3x_4}) \\
\quad -b_3(\varphi_{3x_1} - \varphi_{4x_2} + \varphi_{1x_3} + \varphi_{2x_4}) - b_4(\varphi_{4x_1} + \varphi_{3x_2} - \varphi_{2x_3} + \varphi_{1x_4}) \\
\quad +(b_1\varphi_{1x_5} - b_2\varphi_{2x_5} - b_3\varphi_{3x_5} - b_4\varphi_{4x_5})\}, \\
u_2 = \rho\{b_2(\varphi_{1x_1} - \varphi_{2x_2} - \varphi_{3x_3} - \varphi_{4x_4}) + b_1(\varphi_{2x_1} + \varphi_{1x_2} + \varphi_{4x_3} - \varphi_{3x_4}) \\
\quad -b_4(\varphi_{3x_1} - \varphi_{4x_2} + \varphi_{1x_3} + \varphi_{2x_4}) + b_3(\varphi_{4x_1} + \varphi_{3x_2} - \varphi_{2x_3} + \varphi_{1x_4}) \\
\quad +(b_1\varphi_{2x_5} + b_2\varphi_{1x_5} + b_3\varphi_{4x_5} - b_4\varphi_{3x_5})\}, \\
u_3 = \rho\{b_3(\varphi_{1x_1} - \varphi_{2x_2} - \varphi_{3x_3} - \varphi_{4x_4}) + b_4(\varphi_{2x_1} + \varphi_{1x_2} + \varphi_{4x_3} - \varphi_{3x_4}) \\
\quad +b_1(\varphi_{3x_1} - \varphi_{4x_2} + \varphi_{1x_3} + \varphi_{2x_4}) - b_2(\varphi_{4x_1} + \varphi_{3x_2} - \varphi_{2x_3} + \varphi_{1x_4}) \\
\quad +(b_1\varphi_{3x_5} + b_2\varphi_{4x_5} + b_3\varphi_{1x_5} + b_4\varphi_{2x_5})\}, \\
u_4 = \rho\{b_4(\varphi_{1x_1} - \varphi_{2x_2} - \varphi_{3x_3} - \varphi_{4x_4}) - b_3(\varphi_{2x_1} + \varphi_{1x_2} + \varphi_{4x_3} - \varphi_{3x_4}) \\
\quad +b_2(\varphi_{3x_1} - \varphi_{4x_2} + \varphi_{1x_3} + \varphi_{2x_4}) + b_1(\varphi_{4x_1} + \varphi_{3x_2} - \varphi_{2x_3} + \varphi_{1x_4}) \\
\quad +(b_1\varphi_{4x_5} + b_2\varphi_{3x_5} - b_3\varphi_{2x_5} + b_4\varphi_{1x_5})\}, \\
u_5 = \varphi_{1x_1} + \varphi_{2x_2} + \varphi_{3x_3} + \varphi_{4x_4} - \varphi_{1x_5}, \\
u_6 = \varphi_{2x_1} - \varphi_{1x_2} - \varphi_{4x_3} + \varphi_{3x_4} - \varphi_{2x_5}, \\
u_7 = \varphi_{3x_1} + \varphi_{4x_2} - \varphi_{1x_3} - \varphi_{2x_4} - \varphi_{3x_5}, \\
u_8 = \varphi_{4x_1} - \varphi_{3x_2} + \varphi_{2x_3} - \varphi_{1x_4} - \varphi_{4x_5}.
\end{cases} \tag{17}$$

From this, for $b_1 = 1$, $b_2 = b_3 = b_4 = 0$, we obtain the solution of system (13), which is a five-dimensional generalization of the Moisil–Teodorescu system:

$$\begin{cases}
u_1 = \varphi_{1x_1} - \varphi_{2x_2} - \varphi_{3x_3} - \varphi_{4x_4} + \varphi_{1x_5}, \\
u_2 = \varphi_{2x_1} + \varphi_{1x_2} + \varphi_{4x_3} - \varphi_{3x_4} + \varphi_{2x_5}, \\
u_3 = \varphi_{3x_1} - \varphi_{4x_2} + \varphi_{1x_3} + \varphi_{2x_4} + \varphi_{3x_5}, \\
u_4 = \varphi_{4x_1} + \varphi_{3x_2} - \varphi_{2x_3} + \varphi_{1x_4} + \varphi_{4x_5}, \\
u_5 = \varphi_{1x_1} + \varphi_{2x_2} + \varphi_{3x_3} + \varphi_{4x_4} - \varphi_{1x_5}, \\
u_6 = \varphi_{2x_1} - \varphi_{1x_2} - \varphi_{4x_3} + \varphi_{3x_4} - \varphi_{2x_5}, \\
u_7 = \varphi_{3x_1} + \varphi_{4x_2} - \varphi_{1x_3} - \varphi_{2x_4} - \varphi_{3x_5}, \\
u_8 = \varphi_{4x_1} - \varphi_{3x_2} + \varphi_{2x_3} - \varphi_{1x_4} - \varphi_{4x_5}.
\end{cases} \tag{18}$$

Thus, the following assertion has been proved:

The solution of the elliptic first-order system (11) with five unknown variables, which is a five-dimensional generalization of the Cauchy–Riemann system, is expressed in terms of derivatives of four harmonic functions by formula (15). In particular, for $b_1 = 1$ and $b_2 = b_3 = b_4 = 0$, the solution of system (13), which is a five-dimensional generalization of the Moisil–Teodorescu system, is represented by formula (16).

3 Investigation of the Riemann–Hilbert boundary value problem

Using the obtained representation of the solutions of system (13) in terms of derivatives of harmonic functions, one can investigate the corresponding Riemann–Hilbert boundary value problem.

Let the domain D be the half-space $H^\circ = \{x_5 > 0\}$ of the five-dimensional space $\mathbb{R}^5$. The analogue of the Riemann–Hilbert problem for system (14) is formulated as follows: in the domain D, find a regular solution of system (13), u_l, $l = 1, \ldots, 8$, satisfying on the boundary $\partial H^\circ = C = \{x_5 = 0\}$ of the half-space $H^\circ = \{x_5 > 0\}$ the following conditions:

$$\sum_{l=1}^{8} A_{lm} u_l = F_m, \qquad m = 1, 2, 3, 4, \tag{19}$$

where A_{lm}, F_m $(l = 1, \ldots, 8)$ are functions given on C. Substituting the representation of the solution (18) in terms of harmonic functions φ_r, $r = 1, 2, 3, 4$, into the boundary condition (19), we reduce problem (13), (19) to a boundary value problem with an oblique derivative for harmonic functions:

$$\sum_{q=1}^{4} (B_{qm}, \nabla\varphi_q) = F_m, \qquad m = 1, 2, 3, 4, \qquad \left(\nabla \equiv \sum_{p=1}^{5} \frac{\partial}{\partial x_p}\right), \tag{20}$$

where B_{qm} $(q = 1, 2, 3, 4;\ m = 1, 2, 3, 4)$ are well-defined vectors related to the numbers A_{lm} from (19):

$$\begin{cases} B_{1,m} = (A_{1m} + A_{5m},\ A_{2m} - A_{6m},\ A_{3m} - A_{7m},\ A_{4m} - A_{8m},\ A_{1m} - A_{5m}), \\ B_{2,m} = (A_{2m} + A_{6m},\ -A_{1m} + A_{5m},\ A_{4m} + A_{8m},\ -A_{3m} + A_{4m},\ A_{2m} - A_{6m}), \\ B_{3,m} = (A_{3m} - A_{7m},\ A_{4m} - A_{8m},\ -A_{1m} + A_{5m},\ -A_{2m} + A_{6m},\ A_{3m} - A_{7m}), \\ B_{4,m} = (A_{4m} + A_{8m},\ -A_{3m} + A_{7m},\ A_{2m} - A_{6m},\ -A_{1m} + A_{5m},\ A_{4m} - A_{8m}), \end{cases} \tag{21}$$

wheere $(m = 1, 2, 3, 4)$.

Thus, with the help of representation (18), the boundary value problem (13), (19) is reduced to a problem with an oblique derivative for a system of four harmonic functions for $n \geq 3$. It is also known that the Riemann–Hilbert problem for systems generalizing the Moisil–Teodorescu system for $n \geq 3$ does not satisfy the Lopatinskii condition in any domain D; consequently, the Noetherian property of this problem must fail [16], [17]. However, this problem can be considered in the class of regular harmonic functions with constant coefficients in (20), if the vectors B_{lm} $(l = 1, 2, 3, 4;\ m = 1, 2, 3, 4)$ are chosen so that their components

have a special form, namely such that (20) takes the form described in [18]:

$$\frac{\partial\varphi_1}{\partial x_5} = F_1, \qquad \frac{\partial\varphi_2}{\partial x_5} = F_2, \qquad \frac{\partial\varphi_3}{\partial x_5} = F_3, \qquad \frac{\partial\varphi_4}{\partial x_5} = F_4 \tag{22}$$

on C. Then the problem splits into four problems with an oblique derivative. Considering it as a problem in the half-space $x_5 > 0$, we construct a solution tending to zero at infinity, under the assumption that the functions F_m, $m = 1, 2, 3, 4$, also tend to zero at infinity [13]:

$$\varphi_m(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{F_m(z)\,dz_1\,dz_2\,dz_3\,dz_4}{\sqrt{\sum_{i=1}^{4}(x_i - z_i)^2 + x_5^2}}, \qquad m = 1, 2, 3, 4, \tag{23}$$

where $x = (x_1, x_2, x_3, x_4)$, $z = (z_1, z_2, z_3, z_4)$. Substituting (23) into (18), we find the solution of problem (13), (22).

4 Conclusion

A representation of the solution of system (13) in terms of derivatives of harmonic functions has been obtained. Using this representation, we reduce the Riemann–Hilbert problem (13), (19) to the investigation of a problem with an oblique derivative [16], [17] for a system of four harmonic functions. This problem was considered for regular harmonic functions with constant coefficients, and a solution was presented in a particular case.

Authors contribution

Zh.T.: conceptualization, methodology, supervision, writing – original draft preparation. **G.A.**: formal analysis, investigation, writing – original draft preparation. **U.K.**: validation, writing – review and editing. **A.A.**: methodology, writing – review and editing. All authors have read and agreed to the published version of the manuscript.

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