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## ALGEBRAIC VARIETIES DEFINED BY TWO KELLER POLYNOMIALS IN TWO VARIABLES

We study pairs of Keller polynomials in two variables and the associated polynomial mapping

$$\Phi(x, y) = (f(x, y), g(x, y))$$

under the constant Jacobian condition. We establish that this condition implies strong local geometric constraints on the mapping. In particular, we prove that all fibers of  $\Phi$  are discrete and locally finite, and admit uniform bounds on compact subsets. We further describe the graph of  $\Phi$  as a regular surface in  $\mathbf{R}^4$ , providing a geometric framework for the analysis. In addition, we derive a differential identity along parametrized rays, which yields an orthogonality relation between the value of the mapping and its differential, together with a non-collinearity constraint under natural nondegeneracy assumptions.

**Keywords:** Keller polynomial, polynomial automorphism; Jacobian conjecture, algebraic variety, two variables.

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## Екі айнымалыдағы екі Келлер көпмүшесімен анықталатын алгебралық көптүрліліктер

Бұл жұмыста екі айнымалыдан тұратын Келлер көпмүшелерінің жұптары және оларға сәйкес

$$\Phi(x, y) = (f(x, y), g(x, y))$$

полиномдық бейнелеу Якобианның тұрақты шарты кезінде зерттеледі. Аталған шарт бейнелеудің локал геометриялық құрылымына қатаң шектеулер қоятындығы көрсетіледі. Атап айтқанда,  $\Phi$  бейнелеуінің барлық талшықтары дискретті және локал түрде ақырлы болатыны дәлелденеді, әрі ықшам жиындарда олардың қуатына біркелкі шектеулер бар екені анықталады. Сонымен қатар,  $\Phi$  бейнелеуінің графигі  $\mathbf{R}^4$  кеңістігінде регулярлы бет ретінде сипатталады, бұл геометриялық талдау үшін табиғи негіз береді. Бұдан бөлек, параметрленген сәулелер бойымен дифференциалдық тождество алынып, ол бейнелеудің мәні мен оның дифференциалы арасындағы ортогоналдық қатынасты, ал табиғи неvyрождендік шарттарда — коллинеар еместік қасиетін береді.

**Түйін сөздер:** Келлер көпмүшесі, көпмүшелі автоморфизм, Якобиан болжамы, алгебралық көптүрлілік, екі айнымалы.

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## Алгебраические многообразия, определяемые двумя полиномами Келлера от двух переменных

В данной работе исследуются пары многочленов Келлера от двух переменных и соответствующее полиномиальное отображение

$$\Phi(x, y) = (f(x, y), g(x, y))$$

при условии постоянства якобиана. Показано, что это условие накладывает сильные локальные геометрические ограничения на отображение. В частности, доказано, что все слои отображения  $\Phi$  являются дискретными и локально конечными, а на компактных множествах допускают равномерные оценки мощности. Кроме того, граф отображения  $\Phi$  описывается как регулярная поверхность в  $\mathbf{R}^4$ , что дает геометрическую основу для анализа. Также получено дифференциальное тождество вдоль параметризованных лучей, из которого следует ортогональное соотношение между значением отображения и его дифференциалом, а при естественных условиях невырожденности — условие неколлинеарности.

**Ключевые слова:** полином Келлера, полиномиальный автоморфизм, гипотеза Якобиана, алгебраическое многообразие, две переменные.

### 1 Introduction

The Jacobian conjecture, proposed by Keller in 1939 [1], asks whether a polynomial mapping

$$F : \mathbb{K}^n \rightarrow \mathbb{K}^n$$

with constant nonzero Jacobian determinant must have a polynomial inverse. The problem remains open and has motivated extensive work on polynomial mappings and polynomial automorphisms. Early results on injective polynomial mappings and Keller-type problems were obtained in [2, 3, 6, 7]. Fundamental reduction methods and structural approaches were developed in [8–10, 14]. Polynomial automorphisms and their geometric properties were studied in [11–13, 17]. More recent developments connect the Jacobian conjecture with the Dixmier conjecture, quantization, and automorphism groups; see [15, 18]. Related real and complex geometric aspects are considered in [19–21].

In this paper, we study Keller pairs in two variables and the associated polynomial mapping

$$\Phi(x, y) = (f(x, y), g(x, y)), \quad f_x g_y - f_y g_x = \text{const} \neq 0.$$

The nonvanishing Jacobian condition implies local invertibility by the inverse function theorem [5, 16]. However, our aim is not only to record this local fact, but to organize several geometric consequences of the Keller condition in a unified framework. The language of

differential topology and differential geometry provides a natural setting for this viewpoint; see [4, 5, 16].

We prove that the fibers of  $\Phi$  are discrete and locally finite, and that their cardinalities are uniformly bounded on compact subsets. We also describe the graph of  $\Phi$  as a regular two-dimensional surface in  $\mathbf{R}^4$ . Finally, we derive a differential identity along parametrized rays, which yields an orthogonality relation between the value of the mapping and its differential, together with a non-collinearity constraint under natural nondegeneracy assumptions.

Thus, the paper emphasizes the geometric restrictions imposed by the Keller condition on fibers, graphs, and differential behavior of polynomial mappings. Section 2 contains the main results, while Section 3 presents examples, and discussion.

## 2 Main results

In this section we introduce Keller pairs and derive several local geometric consequences of the Jacobian condition. From an algebraic point of view, this condition characterizes Keller mappings; see [17]. We begin by establishing local invertibility of the associated polynomial mapping, Translation invariance, then analyze the structure of its fibers, and finally derive a differential identity along parametrized rays.

### 2.1 Keller pairs and the associated mapping

We first fix the basic class of polynomial mappings considered throughout the paper.

**Definition 1** *Let  $\mathbf{R} = \mathbb{R}$  be the field of real numbers, and let  $f, g \in \mathbf{R}[x, y]$ . The pair  $(f, g)$  is called a Keller pair if*

$$f_x g_y - f_y g_x = \text{const} \neq 0. \quad (1)$$

To such a pair we associate the polynomial mapping

$$\Phi : \mathbf{R}^2 \rightarrow \mathbf{R}^2, \quad \Phi(x, y) = (f(x, y), g(x, y)). \quad (2)$$

We also write

$$J(\Phi)(x, y) = \det D\Phi(x, y) \quad (3)$$

for the Jacobian determinant of  $\Phi$ , where  $D\Phi(x, y) = \begin{pmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{pmatrix}$ .

The Keller condition is precisely the nondegeneracy condition for the differential of the associated mapping.

**Lemma 1** *If Equation (1) holds, then  $D\Phi(x, y)$  is invertible for every  $(x, y) \in \mathbf{R}^2$ .*

**Proof.** By Definition 1,

$$\det D\Phi(x, y) = f_x(x, y)g_y(x, y) - f_y(x, y)g_x(x, y) \neq 0$$

for every  $(x, y) \in \mathbf{R}^2$ . Hence  $D\Phi(x, y)$  is invertible.

The preceding lemma allows us to apply the inverse function theorem at every point.

**Theorem 1** *The mapping  $\Phi$  is a local  $C^\infty$ -diffeomorphism on  $\mathbf{R}^2$ .*

**Proof.** Since  $f$  and  $g$  are polynomials, the mapping  $\Phi$  is of class  $C^\infty$ . Let  $(a, b) \in \mathbf{R}^2$  be arbitrary. By Lemma 1, the differential  $D\Phi(a, b)$  is invertible. Therefore, by the inverse function theorem (see [16], Theorem C.34), there exist open neighborhoods

$$U \subset \mathbf{R}^2 \text{ of } (a, b), \quad V \subset \mathbf{R}^2 \text{ of } \Phi(a, b),$$

such that the restriction

$$\Phi|_U : U \rightarrow V$$

is a  $C^\infty$ -diffeomorphism.

Since  $(a, b)$  was arbitrary, the above conclusion holds at every point of  $\mathbf{R}^2$ . Hence  $\Phi$  is a local  $C^\infty$ -diffeomorphism on  $\mathbf{R}^2$ .

## 2.2 Translation invariance

Translation invariance shows that the Keller condition is stable under changes of coordinates induced by translations. This property allows one to shift the analysis to arbitrary base points while preserving the geometric structure of the mapping.

**Theorem 2** *Let  $(f, g)$  be a Keller pair and let  $\Phi$  be defined by Equation (2). For  $(a, b) \in \mathbf{R}^2$ , define*

$$\Psi(x, y) = \Phi(x + a, y + b) - \Phi(a, b).$$

*Then  $\Psi$  is a polynomial mapping satisfying*

$$J(\Psi)(x, y) = J(\Phi)(x + a, y + b).$$

*In particular,  $\Psi$  is again a Keller mapping.*

**Proof.** By definition,

$$\Psi(x, y) = (f(x + a, y + b) - f(a, b), g(x + a, y + b) - g(a, b)).$$

Differentiating, we obtain

$$D\Psi(x, y) = D\Phi(x + a, y + b).$$

Hence

$$J(\Psi)(x, y) = J(\Phi)(x + a, y + b).$$

Since  $(f, g)$  is a Keller pair,  $J(\Phi)$  is a nonzero constant. Therefore  $J(\Psi)$  is also a nonzero constant, and thus  $\Psi$  is again a Keller mapping.

### 2.3 Structure of the fibers

The local invertibility of  $\Phi$ , which follows from the Jacobian condition, has direct consequences for the structure of its fibers. Such local properties play a central role in reduction approaches to the Jacobian conjecture; see, for instance, [14].

The following result makes this consequence explicit: the Keller condition prevents fibers from having local accumulation points, and it also gives a uniform finiteness statement on compact sets.

**Theorem 3 (Local structure and finiteness of fibers)** *Let  $(f, g)$  be a Keller pair and let  $\Phi$  be defined by Equation (2). Then, for every  $q \in \mathbf{R}^2$ , the fiber  $\Phi^{-1}(q)$  is a closed discrete subset of  $\mathbf{R}^2$ . Consequently, the following statements hold:*

- (i) *For every bounded set  $\Omega \subset \mathbf{R}^2$ , the set  $\Phi^{-1}(q) \cap \Omega$  is finite.*
- (ii) *For every compact set  $K \subset \mathbf{R}^2$ , there exists a positive integer  $N$  such that*

$$\#(\Phi^{-1}(q) \cap K) \leq N \quad \text{for all } q \in \mathbf{R}^2.$$

**Proof.** Fix  $q \in \mathbf{R}^2$ .

We first show that  $\Phi^{-1}(q)$  is closed. Since  $\Phi$  is continuous and  $\{q\}$  is closed in  $\mathbf{R}^2$ , the preimage  $\Phi^{-1}(q)$  is closed in  $\mathbf{R}^2$ .

We next show that  $\Phi^{-1}(q)$  is discrete. Let  $x_0 \in \Phi^{-1}(q)$ . By Theorem 1, there exists an open neighborhood  $U$  of  $x_0$  such that the restriction  $\Phi|_U : U \rightarrow \Phi(U)$  is injective. We claim that

$$U \cap \Phi^{-1}(q) = \{x_0\}.$$

Indeed, if  $x \in U \cap \Phi^{-1}(q)$ , then

$$\Phi(x) = q = \Phi(x_0).$$

Since both  $x$  and  $x_0$  belong to  $U$  and  $\Phi|_U$  is injective, it follows that  $x = x_0$ . Thus  $x_0$  is isolated in  $\Phi^{-1}(q)$ . Since  $x_0$  was arbitrary, the fiber  $\Phi^{-1}(q)$  is discrete.

This proves the first assertion of the theorem.

*Proof of (i).* Let  $\Omega \subset \mathbf{R}^2$  be bounded, and let

$$K = \overline{\Omega}.$$

Then  $K$  is compact. Since  $\Phi^{-1}(q)$  is closed in  $\mathbf{R}^2$ , the set

$$\Phi^{-1}(q) \cap K$$

is a closed subset of the compact set  $K$ , and hence it is compact.

We now show that

$$\Phi^{-1}(q) \cap K$$

is finite. Since  $\Phi^{-1}(q)$  is discrete, every point

$$p \in \Phi^{-1}(q) \cap K$$

is isolated in  $\Phi^{-1}(q)$ . Therefore, for each such  $p$ , there exists an open neighborhood  $U_p \subset \mathbf{R}^2$  such that

$$U_p \cap \Phi^{-1}(q) = \{p\}.$$

Set

$$O_p := U_p \cap (\Phi^{-1}(q) \cap K).$$

Then  $O_p = \{p\}$ , and  $O_p$  is open in the relative topology of the compact space

$$\Phi^{-1}(q) \cap K.$$

Hence the family

$$\{O_p : p \in \Phi^{-1}(q) \cap K\}$$

is an open cover of

$$\Phi^{-1}(q) \cap K.$$

By compactness, there exists a finite subcover

$$O_{p_1}, \dots, O_{p_m}.$$

Since each set  $O_{p_i}$  is the singleton  $\{p_i\}$ , it follows that

$$\Phi^{-1}(q) \cap K = \{p_1, \dots, p_m\},$$

and therefore

$$\Phi^{-1}(q) \cap K$$

is finite. Hence its subset

$$\Phi^{-1}(q) \cap \Omega$$

is also finite.

The second assertion strengthens the preceding finiteness statement by choosing finitely many injectivity neighborhoods uniformly over the compact set  $K$ .

*Proof of (ii).* Let  $K \subset \mathbf{R}^2$  be compact. For each  $x \in K$ , Theorem 1 provides an open neighborhood  $U_x$  of  $x$  such that the restriction  $\Phi|_{U_x}$  is injective. The family  $\{U_x\}_{x \in K}$  is an open cover of  $K$ . By compactness, there exist finitely many points

$$x_1, \dots, x_m \in K$$

such that

$$K \subset U_{x_1} \cup \dots \cup U_{x_m}.$$

We claim that, for every  $q \in \mathbf{R}^2$  and every  $i \in \{1, \dots, m\}$ , the set

$$\Phi^{-1}(q) \cap U_{x_i}$$

contains at most one point. Indeed, if

$$z_1, z_2 \in \Phi^{-1}(q) \cap U_{x_i},$$

then

$$\Phi(z_1) = q = \Phi(z_2),$$

and the injectivity of  $\Phi|_{U_{x_i}}$  implies

$$z_1 = z_2.$$

Now fix  $q \in \mathbf{R}^2$ . Every point of

$$\Phi^{-1}(q) \cap K$$

lies in at least one of the sets

$$U_{x_1}, \dots, U_{x_m}.$$

Since each  $U_{x_i}$  contributes at most one point of

$$\Phi^{-1}(q) \cap K,$$

we obtain

$$\#(\Phi^{-1}(q) \cap K) \leq m.$$

Thus the conclusion holds with  $N := m$ .

## 2.4 Geometric realization of the graph

We now turn to the graph of the mapping  $\Phi$  and describe it as a regular surface in  $\mathbf{R}^4$ .

**Definition 2** *Define*

$$P = \{(x, y, u, v) \in \mathbf{R}^4 : u = f(x, y), v = g(x, y)\}.$$

Thus  $P$  is the graph of the mapping  $\Phi$ .

The following theorem shows that the graph associated with a Keller mapping naturally carries the structure of a smooth regular surface.

**Theorem 4** *The set  $P$  is a regular two-dimensional surface in  $\mathbf{R}^4$ .*

**Proof.** Define

$$r : \mathbf{R}^2 \rightarrow \mathbf{R}^4, \quad r(x, y) = (x, y, f(x, y), g(x, y)).$$

Since  $f$  and  $g$  are polynomials, the map  $r$  is of class  $C^\infty$ . By Definition 2, its image is exactly  $P$ .

We first show that  $r$  is injective. If  $r(x_1, y_1) = r(x_2, y_2)$ , then comparison of the first two coordinates gives  $x_1 = x_2$  and  $y_1 = y_2$ . Hence  $r$  is one-to-one.

Next, we compute the partial derivatives:

$$r_x = (1, 0, f_x, g_x), \quad r_y = (0, 1, f_y, g_y).$$

Suppose that  $\alpha r_x + \beta r_y = 0$  for some  $\alpha, \beta \in \mathbf{R}$ . Comparing the first two coordinates, we obtain  $\alpha = 0$  and  $\beta = 0$ . Thus  $r_x$  and  $r_y$  are linearly independent at every point, and therefore  $r$  is an immersion.

It remains to show that  $r$  is a homeomorphism onto its image. Define

$$\pi : P \rightarrow \mathbf{R}^2, \quad \pi(x, y, u, v) = (x, y).$$

This map is continuous as the restriction of the standard projection  $\mathbf{R}^4 \rightarrow \mathbf{R}^2$ . Moreover,

$$\pi \circ r = \text{id}_{\mathbf{R}^2}.$$

Conversely, if  $(x, y, u, v) \in P$ , then by Definition 2 we have  $u = f(x, y)$  and  $v = g(x, y)$ . Hence

$$r \circ \pi(x, y, u, v) = r(x, y) = (x, y, u, v).$$

Therefore

$$r \circ \pi = \text{id}_P.$$

Thus  $\pi$  is the inverse of  $r$  on  $P$ , so  $r$  is a homeomorphism of  $\mathbf{R}^2$  onto  $P$ .

Therefore  $r$  is a global regular parametrization of  $P$ , and  $P$  is a regular two-dimensional surface in  $\mathbf{R}^4$ .

**Corollary 1** *The graph  $P \subset \mathbf{R}^4$  of a Keller mapping is connected.*

**Proof.** By Theorem 4, the graph  $P$  is the image of the continuous map

$$r : \mathbf{R}^2 \rightarrow \mathbf{R}^4.$$

Since  $\mathbf{R}^2$  is connected and continuous images of connected spaces are connected, it follows that  $P$  is connected.

## 2.5 A differential identity and a geometric constraint

We now derive a differential identity along parametrized rays and then deduce a geometric constraint.

Fix  $(a, b) \in \mathbf{R}^2$  and define

$$S(t) = (a^2 + b^2) \|\Phi(at, bt)\|^2, \quad t \in [0, 1]. \quad (4)$$

where  $\|\cdot\|$  denotes the standard Euclidean norm on  $\mathbf{R}^2$ .

The following lemma provides a differential identity associated with the behavior of the mapping along parametrized rays.

**Lemma 2** *For every  $t \in [0, 1]$ ,*

$$S'(t) = 2(a^2 + b^2) \Phi(at, bt) \cdot D\Phi(at, bt) \begin{pmatrix} a \\ b \end{pmatrix}.$$

**Proof.** By Equation (4),

$$S(t) = (a^2 + b^2) \Phi(at, bt) \cdot \Phi(at, bt),$$

the product rule and the chain rule yield

$$\frac{d}{dt} \Phi(at, bt) = D\Phi(at, bt) \begin{pmatrix} a \\ b \end{pmatrix}.$$

Therefore

$$S'(t) = 2(a^2 + b^2) \Phi(at, bt) \cdot D\Phi(at, bt) \begin{pmatrix} a \\ b \end{pmatrix}.$$

We now use this identity to derive an orthogonality relation and a geometric restriction on the differential of the mapping.

**Theorem 5 (Orthogonality and directional constraint)** *Let  $t_0 \in (0, 1)$  satisfy  $S'(t_0) = 0$ . Then*

$$\Phi(at_0, bt_0) \cdot D\Phi(at_0, bt_0) \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

*If, in addition,*

$$(a, b) \neq (0, 0) \quad \text{and} \quad \Phi(at_0, bt_0) \neq (0, 0),$$

*then there does not exist  $\lambda \in \mathbf{R}$  such that*

$$D\Phi(at_0, bt_0) \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \Phi(at_0, bt_0).$$

*In particular, the vectors*

$$\Phi(at_0, bt_0) \quad \text{and} \quad D\Phi(at_0, bt_0) \begin{pmatrix} a \\ b \end{pmatrix}$$

*are not collinear.*

**Proof.** By Lemma 2,

$$0 = 2(a^2 + b^2) \Phi(at_0, bt_0) \cdot D\Phi(at_0, bt_0) \begin{pmatrix} a \\ b \end{pmatrix}.$$

If  $(a, b) \neq (0, 0)$ , then  $a^2 + b^2 > 0$ , and hence

$$\Phi(at_0, bt_0) \cdot D\Phi(at_0, bt_0) \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

The case  $(a, b) = (0, 0)$  is trivial. This proves the first assertion.

Assume now that

$$(a, b) \neq (0, 0) \quad \text{and} \quad \Phi(at_0, bt_0) \neq (0, 0).$$

Suppose that

$$D\Phi(at_0, bt_0) \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \Phi(at_0, bt_0)$$

for some  $\lambda \in \mathbf{R}$ . Taking the inner product with  $\Phi(at_0, bt_0)$  and using the first assertion, we obtain

$$0 = \lambda \|\Phi(at_0, bt_0)\|^2.$$

Since  $\Phi(at_0, bt_0) \neq (0, 0)$ , it follows that  $\lambda = 0$ . Hence

$$D\Phi(at_0, bt_0) \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

By Lemma 1, the matrix  $D\Phi(at_0, bt_0)$  is invertible, which implies  $(a, b) = (0, 0)$ , a contradiction. Therefore the two vectors are not collinear.

### 3 Examples and discussion

In this section we present several standard examples of Keller pairs in two variables. These examples illustrate how the results of Section 2 apply in concrete situations. We also record a basic invariance property of the Jacobian condition and summarize the geometric implications of the obtained results.

#### 3.1 Examples

We conclude this section with several standard examples of Keller mappings. These examples illustrate how the geometric properties established above appear naturally in classical polynomial automorphisms and provide concrete realizations of the fiber and graph structures studied in this paper.

The first example corresponds to an affine polynomial automorphism with constant Jacobian determinant.

**Example 1** Let  $f(x, y) = x + y$  and  $g(x, y) = x - y$ . Then  $f_x = 1$ ,  $f_y = 1$ ,  $g_x = 1$ , and  $g_y = -1$ , so

$$J(\Phi) = f_x g_y - f_y g_x = -2 \neq 0.$$

Thus  $(f, g)$  is a Keller pair.

The associated mapping  $\Phi(x, y) = (x + y, x - y)$  is invertible with inverse

$$\Phi^{-1}(u, v) = \left( \frac{u + v}{2}, \frac{u - v}{2} \right).$$

The next example illustrates a triangular polynomial automorphism, which plays an important role in the structure theory of polynomial automorphisms.

**Example 2** Let  $f(x, y) = x + y^2$  and  $g(x, y) = y$ . Then  $f_x = 1$ ,  $f_y = 2y$ ,  $g_x = 0$ , and  $g_y = 1$ , and therefore

$$J(\Phi) = 1 \neq 0.$$

Thus  $(f, g)$  is a Keller pair.

The associated mapping  $\Phi(x, y) = (x + y^2, y)$  is invertible with inverse

$$\Phi^{-1}(u, v) = (u - v^2, v).$$

The following example is a Hénon-type polynomial automorphism, representing a nontrivial nonlinear Keller mapping.

**Example 3** Let  $f(x, y) = y$  and  $g(x, y) = x + y^2$ . Then  $f_x = 0$ ,  $f_y = 1$ ,  $g_x = 1$ , and  $g_y = 2y$ , so

$$J(\Phi) = -1 \neq 0.$$

Thus  $(f, g)$  is a Keller pair.

The associated mapping  $\Phi(x, y) = (y, x + y^2)$  is invertible with inverse

$$\Phi^{-1}(u, v) = (v - u^2, u).$$

The preceding examples show that Keller mappings include affine, triangular, and Hénon-type polynomial automorphisms. In each case, the mapping is globally invertible, and hence every fiber consists of a single point, in agreement with Theorem 3. Moreover, the graph of each mapping is a regular two-dimensional surface in  $\mathbf{R}^4$ , as described in Theorem 4. Thus these examples illustrate how the fiber structure and graph geometry obtained above appear naturally in standard classes of polynomial automorphisms. In particular, triangular automorphisms constitute a fundamental class in the structure theory of polynomial automorphisms; see [12, 13].

### 3.2 Discussion

The following remarks summarize the geometric consequences obtained in the previous section and explain their relation to the geometric behavior of Keller mappings.

**Remark 1** *The fibers of a Keller mapping are discrete. In particular, each point in a fiber is isolated. Thus the Keller condition excludes local accumulation phenomena inside fibers.*

**Remark 2** *On compact subsets of  $\mathbf{R}^2$ , the cardinality of the fibers is uniformly bounded. In particular, fibers are finite in bounded regions. This gives a semi-global restriction on the distribution of preimages.*

**Remark 3** *The graph of the Keller mapping  $(f, g)$  defines a regular two-dimensional surface in  $\mathbf{R}^4$ , providing a geometric realization of the mapping. This allows one to study Keller mappings through the geometry of their associated surfaces.*

**Remark 4** *The differential identity along rays yields an orthogonality relation between the value of the mapping and its differential, imposing a constraint on their relative position. In particular, under natural nondegeneracy assumptions, certain collinearity configurations are excluded.*

Taken together, these results show that the Keller condition imposes geometric restrictions on several different levels: the local differential structure, the structure of fibers, the geometry of the graph, and the behavior of the mapping along parametrized rays.

This viewpoint is connected with the Jacobian conjecture, where the main difficulty is to understand how the local condition

$$J(\Phi) = \text{const} \neq 0$$

influences the global behavior of a polynomial mapping. The results obtained here do not solve the global injectivity problem, but they describe several local and semi-global consequences of the Keller condition that are compatible with global invertibility.

In this sense, the paper provides a geometric framework for studying Keller mappings in two variables. The discreteness of fibers, the uniform finiteness on compact sets, and the regularity of the associated graph show how the nonvanishing Jacobian condition restricts the possible behavior of polynomial mappings before global invertibility is known. Such restrictions are closely related to rigidity phenomena appearing in the modern theory of polynomial mappings and polynomial automorphisms; see [20, 21].

## 4 Conclusion

In this paper, we studied pairs of Keller polynomials in two variables and the associated mapping

$$\Phi(x, y) = (f(x, y), g(x, y)),$$

under the condition of constant nonzero Jacobian determinant.

We proved that  $\Phi$  is a local diffeomorphism on  $\mathbf{R}^2$ . We also showed that the fibers of  $\Phi$  are closed discrete subsets of  $\mathbf{R}^2$ , and consequently finite on bounded subsets. In addition, we described the graph of  $\Phi$  as a regular two-dimensional surface in  $\mathbf{R}^4$ . Furthermore, we derived a differential identity along parametrized rays, leading to an orthogonality relation between the value of the mapping and its differential, together with a non-collinearity constraint under natural nondegeneracy assumptions.

The main contribution of the paper is the development of a unified geometric viewpoint for Keller mappings in dimension two. In particular, the obtained results connect the local differential structure, the geometry of fibers, and the geometry of the associated graph within a common framework. Although the results obtained here are primarily local and geometric in nature, they are closely related to the Jacobian conjecture, where the central problem is to understand how the nonvanishing Jacobian condition influences the global behavior of polynomial mappings. The discreteness of fibers, the regularity of the graph, and the differential constraints established in this paper illustrate several geometric restrictions imposed by the Keller condition. These observations may be useful for further investigations related to polynomial automorphisms and the Jacobian conjecture.

## 5 Conflict of Interest

The authors declare that they have no conflict of interest.

## 6 Contribution

All authors contributed equally to this research. The order of authors follows the alphabetical order of their surnames.

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