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REGULAR BOUNDARY VALUE PROBLEMS FOR A SINGULAR SECOND-ORDER EQUATION

This paper is devoted to the study of regular boundary value problems for a singular second-order differential equation with a degeneracy at one endpoint of the interval. Such equations arise naturally in various problems of mathematical physics and require a special analysis, since the singular behavior of the coefficient near the degenerate point prevents the direct application of standard methods from the classical theory of boundary value problems.

We introduce the maximal and minimal operators associated with the given singular equation and identify the natural weighted boundary traces that arise in this context. We then study the corresponding singular Cauchy problem, establish its unique solvability in an appropriate weighted Sobolev space, and obtain an explicit representation of the inverse operator. This operator plays the role of a basic correct restriction and is used in the subsequent construction of general regular boundary conditions.

Our main goal is to characterize all regular boundary value problems generated by the considered singular differential expression. The analysis is carried out within the framework of the theory of correct restrictions of maximal operators and regular extensions of minimal operators in a Hilbert space. Furthermore, the boundary criterion for integral operators obtained in [13] is adapted to the singular weighted setting. Using these approaches, we obtain the general form of boundary conditions that generate regular realizations of the considered singular equation.
Keywords: singular second-order differential equation, regular boundary value problem, minimal operator, maximal operator, correct restriction.

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Екінші ретті сингулярлық дифференциалдық теңдеу үшін регулярлық шеттік есептер

Бұл жұмыс интервалдың бір шетінде ерекшеленген екінші ретті сингулярлы дифференциалдық теңдеу үшін регулярлы шеттік есептерді зерттеуге арналған. Мұндай теңдеулер математикалық физиканың әртүрлі есептерінде табиғи түрде туындайды және арнайы талдауды талап етеді.

Жұмыста берілген сингулярлы теңдеуге сәйкес келетін максималды және минималды операторлар енгізіліп, осы контексте туындайтын табиғи салмақтық шекаралық іздер анықталады. Содан кейін сәйкес сингулярлы Коши есебі зерттеліп, оның тиісті салмақтық Соболев кеңістігіндегі бірмәнді шешімділігі орнатылады және кері оператордың айқын бейнесі алынады.

Зерттеудің негізгі мақсаты — қарастырылып отырған сингулярлы дифференциалдық өрнектен туындайтын барлық регулярлы шеттік есептерді сипаттау. Сонымен қатар [13] жұмысында алынған интегралдық операторлар үшін шекаралық критерий сингулярлы салмақтық жағдайға бейімделеді.
Түйін сөздер: екінші ретті сингулярлық дифференциалдық теңдеу, регулярлық шеттік есеп, минимал оператор, максимал оператор, қисынды тарылту.

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Регулярные краевые задачи для сингулярного дифференциального уравнения второго порядка

Данная работа посвящена исследованию регулярных краевых задач для сингулярного дифференциального уравнения второго порядка, обладающего вырождением на одном из концов рассматриваемого интервала. Уравнения подобного типа естественным образом возникают в различных задачах математической физики и требуют специального анализа.

В работе вводятся максимальный и минимальный операторы, ассоциированные с данным сингулярным выражением, и определяются естественные весовые граничные следы. Далее исследуется соответствующая сингулярная задача Коши, устанавливается её единственная разрешимость в соответствующем весовом пространстве Соболева и получено явное представление обратного оператора.

Основная цель работы заключается в описании всех регулярных краевых задач, порождаемых рассматриваемым сингулярным дифференциальным выражением. В частности, граничный критерий для интегральных операторов, полученный в работе [13], адаптирован к сингулярному весовому случаю.
Ключевые слова: сингулярное дифференциальное уравнение второго порядка, регулярная краевая задача, минимальный оператор, максимальный оператор, корректное сужение.

1 Introduction

Extension theory originated in the work of J. von Neumann [1], who first studied the problem of extending a symmetric operator. Later, the theory of extensions of symmetric operators found important applications in boundary value problems for differential equations and in analysis. Many authors have since contributed to this area; we refer, for example, to [2–5].

A substantial advance was made by M. I. Vishik [4], who removed the requirement that the initial operator be symmetric and developed the theory in a more general setting. He applied this approach to the study of general boundary value problems for second-order elliptic differential equations.

A. A. Dezin [6, 7] investigated general questions of the theory of boundary value problems and obtained a description of regular operators. These results played an important role in the development of the theory of extensions and restrictions of differential operators.

The problem of correct extensions and restrictions of operators, not necessarily linear, acting in Banach spaces was initiated by M. Otelbaev and his disciples [8, 9]. Later, B. K. Kokebaev, M. Otelbaev, and A. N. Shynybekov [9, 10] investigated restrictions of normally solvable operators and further developed the theory of contractions and extensions in Banach spaces. Subsequent developments of this approach, together with various applications to differential and nonlinear operators, may be found in [8, 11, 12, 15, 17, 21] and the references cited therein. For recent developments concerning singular Sturm–Liouville operators, their self-adjoint extensions, and spectral properties, see [18–20] and the references therein.

The classification of boundary value problems generated by a fixed differential expression is one of the central questions in the theory of differential operators. In the operator-theoretic approach, such problems are described in terms of restrictions of maximal operators and extensions of minimal operators. More precisely, if L_0 is the minimal operator generated by a differential expression l, and $\widehat{L}$ is the corresponding maximal operator, then an invertible operator $L \subset \widehat{L}$ is called a correct restriction of $\widehat{L}$, whereas an invertible operator $L \supset L_0$ is called a correct extension of L_0. If both inclusions $L_0 \subset L \subset \widehat{L}$ hold, then L is called a regular extension. Therefore, the problem of describing all regular boundary value problems for a given differential expression can be reformulated as the problem of describing all regular extensions of the corresponding minimal operator.

An effective tool in this direction is the kernel representation of inverse operators. If a boundary value problem is correct, then its inverse is often represented by an integral operator

$$u = L^{-1}f = \int_{\Omega} K(x,\xi)f(\xi)\,d\xi,$$

and the structure of the boundary conditions may be recovered from the properties of the kernel $K(x,\xi)$. In this way, the theory connects correct boundary value problems with Green-type kernels and with distributional identities involving the underlying differential expression.

A systematic criterion of this type was established by Kal'menov and Otelbaev [13]. The core of their method consists of three criterion-type results. The first one gives a characterization of a correct restriction in terms of the kernel identity

$$L(x,D)K(x,\xi) = \delta(x-\xi).$$

The second one provides the corresponding adjoint criterion through the condition

$$L^{+}(\xi,D)K(x,\xi) = \delta(x-\xi).$$

Finally, combining these two assertions, they obtain a criterion for regular extensions in terms of the simultaneous validity of both delta-identities. Thus, the paper [13] provides a method for passing from integral kernels to correct and regular boundary realizations(see also [14]).

The purpose of the present paper is to adapt this criterion scheme to the singular weighted ordinary differential equation

$$(x^{\beta}u')' + \lambda u = f(x), \qquad 0 < x < 1,$$

where $0 < \beta < 1, \qquad \lambda \in \mathbb{R}$.

Our main goal is to describe all regular boundary value problems generated by this singular differential expression. Thus, we are interested not only in the solvability of a particular problem, but in the complete operator-theoretic description of all regular realizations of the equation.

The paper is organized as follows. In Section 2, we provide the necessary background on the restrictions and extensions of operators in Hilbert spaces and establish an analogue of the Kal'menov–Otelbaev criterion for our specific operator. Section 3 is devoted to the characterization of all regular (well-posed) boundary value problems for the singular equation.

2 Preliminaries and auxiliary statements

2.1 Restrictions and extensions of operators in Hilbert spaces

In this subsection we recall basic notions from the theory of restrictions and extensions of operators in Hilbert spaces. These tools will be used in Section 3 to characterize the general boundary conditions that yield regular (well-posed) boundary value problems for the one-dimensional singular ordinary differential equation arising from the spectral reduction.

Let H be a Hilbert space and let L be a linear operator in H with domain $D(L)$ and range $R(L)$. We adopt the terminology of [15, p. 9050]. Given operators A and B in H, we say that A is a *restriction* of B (equivalently, B is an *extension* of A) if $D(A) \subset D(B)$ and $Au = Bu$ for all $u \in D(A)$; in this case we write $A \subset B$. A closed operator L_0 is called *minimal* if $R(L_0) \neq H$ and $L_0^{-1} : R(L_0) \to H$ exists and is bounded, while a closed operator $\widehat{L}$ is called *maximal* if $R(\widehat{L}) = H$ and $\ker(\widehat{L}) \neq \{0\}$. A closed operator L is said to be *correct* if it is bijective and its inverse $L^{-1} : H \to H$ is bounded. A correct operator L is a *correct extension* of a minimal operator L_0 (respectively, a *correct restriction* of a maximal operator $\widehat{L}$) if $L_0 \subset L$ (respectively, $L \subset \widehat{L}$). Finally, L is called a *boundary correct extension* of L_0 with respect to $\widehat{L}$ if $L_0 \subset L \subset \widehat{L}$.

In the early 1980s, M. Otelbaev and his collaborators established abstract results which make it possible to describe all correct extensions of a given minimal operator in terms of the inverse of one fixed known correct extension. An analogous description was obtained for all correct restrictions of a given maximal operator. For the reader's convenience, we briefly recall the principal assertions of these results.

Theorem 1 (see [8, 9]) *Let $\widehat{L}$ be a maximal linear operator in a Hilbert space H, and let L be a fixed correct restriction of $\widehat{L}$. Let K be an arbitrary bounded linear operator in H, satisfying $R(K) \subset \ker \widehat{L}$. Then the operator L_K^{-1}, defined by*

$$L_K^{-1} f = L^{-1} f + K f, \tag{1}$$

gives the inverses of all correct restrictions L_K of the maximal operator $\widehat{L}$, that is, $L_K \subset \widehat{L}$.

Similarly, let L_0 be a minimal operator in the Hilbert space H, and let L be a fixed correct extension of L_0. Suppose that K satisfies the conditions $R(L_0) \subset \ker K$, and

$$\ker\bigl(L^{-1} + K\bigr) = \{0\}. \tag{2}$$

Then the operator L_K^{-1}, again defined by (1), describes the inverses of all correct extensions L_K of the minimal operator L_0.

Finally, let L be a fixed boundary correct extension of the minimal operator L_0, that is,

$$L_0 \subset L \subset \widehat{L}.$$

Assume that K be an arbitrary bounded linear operator in H, satisfies $R(L_0) \subset \ker K$, and $R(K) \subset \ker \widehat{L}$. Then the operator L_K^{-1} defined by (1), describes the inverses of all boundary correct extensions L_K of the minimal operator L_0.

The existence of at least one boundary correct extension L was proved by M. I. Vishik [4].

2.2 Criterion for regular extensions

Consider the differential expression

$$\ell\phi := (t^\beta \phi')' + \lambda\phi, \qquad 0 < t < 1,$$

where $0 < \beta < 1$ and $\lambda \in \mathbb{R}$. In this case, the formal adjoint expression coincides with ℓ, that is, $\ell^+ = \ell$.

Denote by $\widehat{L}$ the maximal operator in $L_2(0,1)$ generated by the expression ℓ, namely, $\widehat{L}\phi := \ell\phi$, with domain

$$D(\widehat{L}) := \Big\{\phi \in L_2(0,1) : \phi \in AC[0,1],\ t^\beta \phi' \in AC[0,1],\ \ell\phi \in L_2(0,1)\Big\}.$$

Note that $\phi \in AC[0,1]$ implies $\phi' \in L_2(0,1)$. These conditions ensure that ϕ is regular enough to apply ℓ in a classical (not just distributional) sense. The domain $D(\widehat{L})$ is maximal in the sense that it is the largest subspace on which ℓ maps back into $L_2(0,1)$; see [18] for a detailed treatment.

Since $0 < \beta < 1$, the traces $\phi(0)$, $\lim_{t\to 0+} t^\beta \phi'(t)$, $\phi(1)$, $\phi'(1)$ are well defined for every $\phi \in D(\widehat{L})$. Define the minimal operator L_0 by $L_0\phi := \ell\phi$,

$$D(L_0) := \left\{\phi \in D(\widehat{L}) : \phi(0) = 0,\ \lim_{t\to 0+} t^\beta \phi'(t) = 0,\ \phi(1) = 0,\ \phi'(1) = 0\right\}.$$

We also introduce the singular Cauchy problem

$$L_C\phi := \ell\phi$$

with domain

$$D(L_C) := \left\{\phi \in D(\widehat{L}) : \phi(0) = 0,\ \lim_{t\to 0+} t^\beta \phi'(t) = 0\right\}.$$

Clearly, $L_0 \subset L_C \subset \widehat{L}$. The solution to the singular Cauchy problem admits the representation

$$\begin{aligned}\phi(t) =& L_C^{-1} f = \frac{\pi}{2-\beta}\, t^{\frac{1-\beta}{2}} \left[-J_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta} t^{\frac{2-\beta}{2}}\right) \int_0^t f(\tau)\, \tau^{\frac{1-\beta}{2}} Y_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta} \tau^{\frac{2-\beta}{2}}\right) d\tau \right.\\ &\left. + Y_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta} t^{\frac{2-\beta}{2}}\right) \int_0^t f(\tau)\, \tau^{\frac{1-\beta}{2}} J_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta} \tau^{\frac{2-\beta}{2}}\right) d\tau \right],\end{aligned} \tag{3}$$

Lemma 1 *Let $0 < \beta < 1$. Then, for every $f \in L_2(0,1)$, the problem*

$$(t^\beta \phi')' + \lambda\phi = f, \qquad 0 < t < 1, \tag{4}$$

subject to the weighted initial conditions

$$\phi(0) = 0, \qquad \lim_{t\to 0+} t^\beta \phi'(t) = 0, \tag{5}$$

admits a unique solution $\phi \in D(L_C)$.

Consequently, L_C is a correct restriction of the maximal operator $\widehat{L}$ and, at the same time, a correct extension of the minimal operator L_0. In particular, the inverse operator L_C^{-1} exists on all of $L_2(0,1)$ and is a Volterra integral operator.

The next result gives a kernel criterion for correct restrictions of $\widehat{L}$, and regular extensions of the minimal operator L_0, see [13].

Lemma 2 *Let $K(t,\tau)$ be a measurable kernel such that*

$$\sup_{\tau\in(0,1)}\int_0^1 |K(t,\tau)|^2\,dt<\infty.$$

Consider the integral operator

$$(Tf)(t):=\int_0^1 K(t,\tau)f(\tau)\,d\tau.$$

Then T is generated by a correct restriction of the maximal operator $\widehat{L}$ if and only if

$$\big((t^\beta\partial_t)\partial_t+\lambda\big)K(t,\tau)=\delta(t-\tau)$$

in the sense of distributions with respect to the variable t, where $\tau\in(0,1)$ is a parameter.

Theorem 2 *Assume that the kernel $K(t,\tau)$ satisfies*

$$\sup_{t\in(0,1)}\int_0^1 |K(t,\tau)|^2\,d\tau+\sup_{\tau\in(0,1)}\int_0^1 |K(t,\tau)|^2\,dt<\infty.$$

Let

$$(Tf)(t):=\int_0^1 K(t,\tau)f(\tau)\,d\tau.$$

Then T is generated by a regular extension of the minimal operator L_0 if and only if

$$\big((t^\beta\partial_t)\partial_t+\lambda\big)K(t,\tau)=\delta(t-\tau),\qquad \big((\tau^\beta\partial_\tau)\partial_\tau+\lambda\big)K(t,\tau)=\delta(t-\tau)$$

in the sense of distributions with respect to the variables t and τ, respectively.

3 Characterization of regular boundary conditions

Fix $0<\beta<1$, $\lambda>0$, and let $f\in L^2(0,1)$. Consider the singular equation

$$\widehat{l}\phi:=\frac{d}{dt}\big(t^\beta\phi'(t)\big)+\lambda\phi(t)=f(t),\quad t\in(0,1). \tag{6}$$

For later use, we introduce the fundamental solutions associated with this singular expression:

$$\Phi_J(t):=t^{\frac{1-\beta}{2}}J_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta}t^{\frac{2-\beta}{2}}\right),\quad \Phi_Y(t):=t^{\frac{1-\beta}{2}}Y_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta}t^{\frac{2-\beta}{2}}\right), \tag{7}$$

where $\nu := \frac{1-\beta}{2-\beta}$. Then a general solution of the equation $\widehat{l}\phi = f$ can be written as

$$\phi(t) = -\frac{\pi}{2-\beta} \int_0^t f(s)\,(\Phi_Y(s)\,\Phi_J(t) - \Phi_Y(t)\,\Phi_J(s))\,ds + C_1\,\Phi_J(t) + C_2\,\Phi_Y(t), \tag{8}$$

where $C_1, C_2 \in \mathbb{R}$.

To describe all regular boundary value problems, we consider all correct restrictions of the maximal operator. By Otelbaev's theorem (see Section 2.1), every correct restriction has an inverse of the form

$$L_K^{-1} = L^{-1} + K, \qquad R(K) \subset \ker(\widehat{l}),$$

where L is a fixed correct restriction of $\widehat{l}$.

In the present situation,

$$\ker(\widehat{l}) = \operatorname{span}\{\Phi_J, \Phi_Y\}.$$

We now seek boundary conditions that define a restriction $L_K \subset \widehat{l}$ such that, for every $f \in L^2(0,1)$, the equation $L_K\phi = f$ admits a unique solution $\phi \in D(L_K)$ and the solution operator $L_K^{-1} : L^2(0,1) \to L^2(0,1)$ is bounded. In this case the coefficients C_1 and C_2 in (8) depend linearly and continuously on f, i.e.

$$C_1 = C_1(f), \qquad C_2 = C_2(f), \tag{9}$$

where $C_1(\cdot)$ and $C_2(\cdot)$ are bounded linear functionals on $L^2(0,1)$. By the Riesz representation theorem, there exist $\sigma_1, \sigma_2 \in L^2(0,1)$ such that

$$C_1(f) = \int_0^1 f(s)\,\overline{\sigma_1(s)}\,ds, \qquad C_2(f) = \int_0^1 f(s)\,\overline{\sigma_2(s)}\,ds. \tag{10}$$

Since $\ker(\widehat{l}) = \operatorname{span}\{\Phi_J, \Phi_Y\}$ for the corresponding maximal operator $\widehat{l}$, the correction term in the inverse can be taken to be of finite rank, and we may represent C_1 and C_2 in the form

$$\begin{aligned} C_1(f) &= \int_0^1 \big(q_1\Phi_J(s) + q_2\Phi_Y(s)\big)\,f(s)\,ds, \\ C_2(f) &= \int_0^1 \big(q_3\Phi_J(s) + q_4\Phi_Y(s)\big)\,f(s)\,ds, \end{aligned} \tag{11}$$

with constants $q_1, q_2, q_3, q_4 \in \mathbb{R}$.

Substituting (11) into (8) yields the representation

$$\begin{aligned} \phi(t) = &-\frac{\pi}{2-\beta}\,\Phi_J(t) \int_0^t f(s)\,\Phi_Y(s)\,ds + \frac{\pi}{2-\beta}\,\Phi_Y(t) \int_0^t f(s)\,\Phi_J(s)\,ds \\ &+ \Phi_J(t) \int_0^1 \big(q_1\Phi_J(s) + q_2\Phi_Y(s)\big)\,f(s)\,ds \\ &+ \Phi_Y(t) \int_0^1 \big(q_3\Phi_J(s) + q_4\Phi_Y(s)\big)\,f(s)\,ds. \end{aligned} \tag{12}$$

Or equivalently,

$$\phi(t) = I_1(t) + I_2(t) + I_3(t) + I_4(t),$$

where

$$\begin{aligned}
I_1(t) &:= -\frac{\pi}{2-\beta}\,\Phi_J(t)\int_0^t f(s)\,\Phi_Y(s)\,ds,\\
I_2(t) &:= \frac{\pi}{2-\beta}\,\Phi_Y(t)\int_0^t f(s)\,\Phi_J(s)\,ds,\\
I_3(t) &:= \Phi_J(t)\int_0^1 \big(q_1\Phi_J(s) + q_2\Phi_Y(s)\big)f(s)\,ds,\\
I_4(t) &:= \Phi_Y(t)\int_0^1 \big(q_3\Phi_J(s) + q_4\Phi_Y(s)\big)f(s)\,ds.
\end{aligned}$$

Now we substitute $f = (t^\beta\phi')' + \lambda\phi$ into the integral terms in (12) and apply integration by parts.

In particular, for $I_1(t)$ we obtain

$$\int_0^t f(s)\,\Phi_Y(s)\,ds = \Big(s^\beta\big(\phi'(s)\Phi_Y(s) - \phi(s)\Phi_Y'(s)\big)\Big)\Big|_{s=0}^{s=t}. \tag{13}$$

It remains to evaluate the limit as $s \to 0^+$. Using the standard asymptotics of Bessel functions at the origin, one finds that

$$\lim_{s\to 0^+} s^\beta\Phi_Y(s)\phi'(s) = A\lim_{s\to 0^+} s^\beta\phi'(s), \qquad -\lim_{s\to 0^+} s^\beta\Phi_Y'(s)\phi(s) = CB\,\phi(0), \tag{14}$$

where

$$A := \frac{(2-\beta)^\nu}{(\sqrt{\lambda})^\nu\sin(\nu\pi)\Gamma(1-\nu)}, \qquad B := \frac{(\sqrt{\lambda})^\nu(2-\beta)^{1-\nu}}{\Gamma(\nu)}, \qquad C := \cot\big((1-\nu)\pi\big).$$

Moreover, using the differentiation formula for Φ_Y, we have

$$-\,t^\beta\Phi_Y'(t) = \sqrt{\lambda}\,t^{1/2}\,Y_{\nu-1}\Big(b\,t^{\frac{2-\beta}{2}}\Big), \qquad b := \frac{2\sqrt{\lambda}}{2-\beta}. \tag{15}$$

Combining (13)–(15) yields

$$\int_0^t f(s)\,\Phi_Y(s)\,ds = t^\beta\Phi_Y(t)\phi'(t) + \sqrt{\lambda}\,t^{1/2}Y_{\nu-1}\Big(b\,t^{\frac{2-\beta}{2}}\Big)\,\phi(t) - A\lim_{s\to 0^+} s^\beta\phi'(s) - CB\,\phi(0). \tag{16}$$

Substituting (16) into the definition of $I_1(t)$, we obtain

$$I_1(t) = -\frac{\pi}{2-\beta}\,\Phi_J(t)\Big[t^\beta\Phi_Y(t)\phi'(t) + \sqrt{\lambda}\,t^{1/2}Y_{\nu-1}\Big(b\,t^{\frac{2-\beta}{2}}\Big)\,\phi(t) - A\lim_{s\to 0^+} s^\beta\phi'(s) - CB\,\phi(0)\Big]. \tag{17}$$

Using the standard Bessel asymptotics as $s \to 0^+$, the boundary contributions at $s = 0$ depend only on the traces

$$T_1 := \lim_{t\to 0^+} t^\beta \phi'(t), \qquad T_3 := \phi(0),$$

whereas evaluation at the right endpoint produces terms involving

$$T_2 := \phi'(1), \qquad T_4 := \phi(1).$$

Proceeding in the same way for I_2, I_3, and I_4, the representation (12) yields an identity of the form

$$\mathcal{A}(T_1,T_2,T_3,T_4)\, J_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta}\, t^{\frac{2-\beta}{2}}\right) + \mathcal{B}(T_1,T_2,T_3,T_4)\, Y_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta}\, t^{\frac{2-\beta}{2}}\right) = 0, \quad t \in (0,1],$$

where $\mathcal{A}$ and $\mathcal{B}$ are linear forms in (T_1,T_2,T_3,T_4) independent of t. Since J_ν and Y_ν are linearly independent, it follows that $\mathcal{A} = \mathcal{B} = 0$, which yields two linear relations among the boundary traces. Equivalently, these relations can be written in matrix form as

$$A(\lambda)\, T = 0, \qquad T := \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix}, \tag{18}$$

where $A(\lambda) \in \mathbb{R}^{2\times 4}$ and its entries $a_{ij}(\lambda)$ are given explicitly below

$$\begin{aligned}
a_{11} &= \left(-\frac{\pi}{2-\beta} + q_4\right) \frac{(2-\beta)^\nu}{\sin(\nu\pi)\Gamma(1-\nu)}, \\
a_{12} &= q_1(\sqrt{\lambda})^\nu J_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta}\right) + q_4(\sqrt{\lambda})^\nu Y_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta}\right), \\
a_{13} &= \left[q_1 + q_4\cot((1-\nu)\pi) - \frac{\pi\cot((1-\nu)\pi)}{2-\beta}\right] \frac{(\sqrt{\lambda})^{2\nu}(2-\beta)^{1-\nu}}{\Gamma(\nu)}, \\
a_{14} &= -\left[q_1(\sqrt{\lambda})^{1-\nu} J_{\nu-1}\left(\frac{2\sqrt{\lambda}}{2-\beta}\right) + q_4(\sqrt{\lambda})^{1-\nu} Y_{\nu-1}\left(\frac{2\sqrt{\lambda}}{2-\beta}\right)\right], \\
\\
a_{21} &= q_2 \frac{(2-\beta)^\nu}{\sin(\nu\pi)\Gamma(1-\nu)}, \\
a_{22} &= q_3(\sqrt{\lambda})^\nu J_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta}\right) + q_2(\sqrt{\lambda})^\nu Y_\nu\left(\frac{2\sqrt{\lambda}}{2-\beta}\right), \\
a_{23} &= \left[\frac{\pi}{2-\beta} + q_3 + q_2\cot((1-\nu)\pi)\right] \frac{(\sqrt{\lambda})^{2\nu}(2-\beta)^{1-\nu}}{\Gamma(\nu)}, \\
a_{24} &= -\left[q_3(\sqrt{\lambda})^{1-\nu} J_{\nu-1}\left(\frac{2\sqrt{\lambda}}{2-\beta}\right) + q_2(\sqrt{\lambda})^{1-\nu} Y_{\nu-1}\left(\frac{2\sqrt{\lambda}}{2-\beta}\right)\right].
\end{aligned}$$

We say that the boundary condition (18) is *regular* (equivalently, it defines a correct restriction of the maximal operator associated with l) if the two boundary relations are independent. In terms of the boundary matrix

$$A(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) & a_{13}(\lambda) & a_{14}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & a_{23}(\lambda) & a_{24}(\lambda) \end{pmatrix},$$

this amounts to the nondegeneracy condition

$$\operatorname{rank} A(\lambda) = 2. \tag{19}$$

Equivalently, there exist indices $1 \le i < j \le 4$ such that the corresponding 2×2 minor is nonzero, i.e.

$$\det \begin{pmatrix} a_{1i}(\lambda) & a_{1j}(\lambda) \\ a_{2i}(\lambda) & a_{2j}(\lambda) \end{pmatrix} \neq 0. \tag{20}$$

Equation (18) provides two linear constraints on (T_1, T_2, T_3, T_4) and hence specifies a domain $D(L) \subset D(\widehat{L})$ for a restriction of the maximal operator corresponding to l. By Theorem 1, such a restriction is regular (i.e. has a bounded inverse on $L^2(0,1)$) exactly when the boundary conditions are nondegenerate. We formulate this characterization in the next theorem.

Theorem 3 *Let $0 < \beta < 1$ and $\lambda > 0$, and let*

$$L_A : \qquad D(L_A) = \{\phi \in D(\widehat{L}) : A(\lambda)T(\phi) = 0\},$$

where

$$T_1 = \lim_{t \to 0^+} t^\beta \phi'(t), \qquad T_2 = \phi'(1), \qquad T_3 = \phi(0), \qquad T_4 = \phi(1).$$

Then the following are equivalent:

1. *L_A is a correct restriction of $\widehat{L}$ (equivalently, a regular extension of L_0);*
2. *the homogeneous problem*

   $$(t^\beta \phi')' + \lambda\phi = 0, \qquad A(\lambda)T(\phi) = 0$$

   has only the trivial solution;
3. $\operatorname{rank} A(\lambda) = 2$.

In this case, for every $f \in L^2(0,1)$ there exists a unique solution $\phi \in D(L_A)$ of

$$(t^\beta \phi')' + \lambda\phi = f,$$

and this solution depends continuously on f.

Corollary 1 *The boundary condition* (18) *obtained above is regular. Indeed, the corresponding operator is a correct restriction of the maximal operator by Theorem 1.*

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Conflict of interest

This work does not have any conflicts of interest.

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