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ON THE REGULARIZATION OF THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN THE STRIPE

This paper investigates the problem of determining the temperature distribution along the upper boundary of a strip, given prescribed temperature data on its lower boundary. The analysis is carried out within the framework of the Laplace equation, leading to the consideration of a corresponding Cauchy-type problem. The uniqueness of the solution is rigorously established in an appropriate space of generalized functions. Furthermore, quantitative estimates are derived that characterize the relationship between solutions of the associated well-posed and ill-posed problems. On the basis of these estimates, approximation results are obtained, and relevant approximation theorems are proved, providing a theoretical foundation for stable reconstruction of the solution.
Key words: regularization method, Cauchy problem, heat equation, ill-posed problem of PDE, auxiliary equation, stationary solution.

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Жолақтағы Лаплас теңдеуі үшін Коши есебін регуляризациялау туралы

Бұл мақалада жолақтың төменгі шекарасындағы температураның берілген мәндері бойынша оның жоғарғы шекарасындағы температураның таралуын анықтау мәселесі зерттеледі. Талдау Лаплас теңдеуі аясында жүргізіліп, соған сәйкес Коши типті есеп қарастырылады. Жалпыланған функциялардың тиісті кеңістігінде шешімнің бірмәнділігі қатаң түрде дәлелденеді. Сонымен қатар, дұрыс қойылған және дұрыс қойылмаған ілеспе есептердің шешімдері арасындағы байланысты сипаттайтын сандық бағалаулар алынады. Осы бағалаулар негізінде жуықтау нәтижелері тұжырымдалып, тиісті жуықтау теоремалары дәлелденеді. Бұл нәтижелер шешімді орнықты түрде қалпына келтірудің теориялық негізін қалайды.
Кілт сөздер: регуляризация әдісі, Коши есебі, жылу теңдеуі, дербес туындылы теңдеулердің дұрыс қойылмаған есебі, көмекші теңдеу, стационарлық шешім.

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О регуляризации задачи Коши для уравнения Лапласа в полосе

В данной работе исследуется задача определения распределения температуры на верхней границе полосы по заданным значениям температуры на её нижней границе. Исследование проводится в рамках уравнения Лапласа, что приводит к рассмотрению соответствующей задачи типа Коши. В подходящем пространстве обобщённых функций строго доказывается единственность решения. Кроме того, получены количественные оценки, характеризующие взаимосвязь между решениями соответствующих корректно и некорректно поставленных задач. На основе этих оценок получены результаты аппроксимации и доказаны соответствующие теоремы, которые создают теоретическую основу для устойчивого восстановления решения.

Ключевые слова: метод регуляризации, задача Коши, уравнение теплопроводности, некорректно поставленная задача ДУЧП (уравнений в частных производных), вспомогательное уравнение, стационарное решение.

1 Introduction

Consider the process of heating in the following stripe

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < h,\ x \in \mathbb{R}\}.$$

The given temperature is maintained and there is no heat flow at the lower base. We need to find the temperature at the upper base of the stripe.

The process of heating is described by equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad x \in \mathbb{R}, \quad 0 < y < h, t > 0, \tag{1}$$

with boundary conditions

$$u(x, 0, t) = \phi(x, t), \quad \frac{\partial u(x, 0, t)}{\partial y} = 0, \quad -\infty < x < +\infty, \quad t > 0. \tag{2}$$

We need to find the temperature $u(x, y, t)$ on the upper base $y = h$.

The Cauchy problem (1)-(2) has been studied in a large number of mathematical publications, starting with the classical work of Hadamard [2] (see also [3] where an overview of the corresponding literature is made). We know that the Cauchy problem for the elliptic equation is ill-posed problem of PDE (for this reason, see [7], [5]). In most works, the problem of stability and regularization is studied (see [4], [7], [8]). The computational aspect of this problem is discussed in [8]. Quasi-inversion method for Cauchy problem was studied in the cylinder (see [12, 14]). The existence and uniqueness of the Cauchy problem for the elliptic equation was studied (see [9], [10] [11]).

2 Uniqueness of the solution

Note that in general case it is necessary to add to boundary conditions (1)-(2) the initial condition. We are looking for a stationary solution $u = u(x, y)$. In this case equation (1) takes the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad x \in \mathbb{R}, \quad 0 < y < h. \tag{3}$$

with boundary conditions

$$u(x,0) = \phi(x), \quad \frac{\partial u(x,0)}{\partial y} = 0, \quad -\infty < x < \infty. \tag{4}$$

Let $D(\Omega)$ denote the space of functions that are infinitely differentiable in the strip Ω and whose supports do not intersect a neighborhood of the upper boundary $y = h$.

Definition 2.1. Let $\phi \in L_2(\mathbb{R})$. We say that a function $u(x,y) \in L_2(\Omega)$ is a solution to the problem (3)–(4) if, for every test function $v \in D(\Omega)$, the following relation holds:

$$\int\limits_{\Omega} u(x,y) \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) dx\, dy = \int\limits_{-\infty}^{\infty} \phi(x) \frac{\partial v}{\partial y}(x,0)\, dx. \tag{5}$$

(see [5]).

Proposition 2.1. *If the problem* (3)-(4) *has a solution, then this solution is unique.*

Proof. Let $u_1(x,y)$ and $u_2(x,y)$ be two solutions. Set

$$u(x,y) = u_1(x,y) - u_2(x,y).$$

Then for any function $v \in D(\Omega)$ the following equation is valid

$$\int\limits_{\Omega} u(x,y) \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) dx\, dy = 0. \tag{6}$$

As a function $v(x,y)$ we take the function of the following form:

$$v(x,y) = w(y) e^{-isx}, \qquad s \in \mathbb{R}, \tag{7}$$

where the function $w \in C^\infty[0,h]$ vanishes near the point $z = h$.

Then

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \left[w''(y) - s^2 w(y) \right] e^{-isx}.$$

In this case we get from equation (6):

$$\int\limits_{0}^{h} \left[w''(y) - s^2 w(y) \right] dy \int\limits_{-\infty}^{\infty} u(x,y) e^{-isx}\, dx = 0. \tag{8}$$

Consider the expansion of u into a Fourier series:

$$u(x,y) = \int\limits_{-\infty}^{\infty} \widehat{u}(s,y) e^{isx}\, ds, \tag{9}$$

where

$$\widehat{u}(s,y) \;=\; \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} u(x,y)e^{-isx}\,dx. \tag{10}$$

Then we can rewrite equation (8) as

$$\int\limits_{0}^{h}\left[w''(y)-k^2w(y)\right]\widehat{u}(s,y)\,dy \;=\; 0, \quad s\in\mathbb{R}. \tag{11}$$

Fix any $y\in(0,h)$ and choose $\delta>0$ so small that $(y-\delta,y+\delta)\subset[0,h]$.

Let $w_\delta(y)$ be a function from $C^\infty[0,h]$ such that $w'_\delta(y)\le 0$ and $w''_\delta(y)\ge 0$, and outside the δ-neighborhood of point x this function equals

$$w_\delta(y) \;=\; \begin{cases} z-y, & 0 \;\le\; y \;\le\; z-\delta,\\ 0, & z+\delta \;\le\; y \;\le\; h. \end{cases} \tag{12}$$

It is clear that $w''_\delta(y)=0$ for $|y-z|>\delta$ and

$$\int\limits_{0}^{h} w''_\delta(y)\,dy \;=\; w'_\delta(h)-w'_\delta(0) \;=\; 1.$$

Hence, this function $w''_\delta(y)$ is a delta-shaped kernel ([1], Chapter III, Sec. VI).

Further, according to (12),

$$\lim_{\delta\to 0^+} w_\delta(y) \;=\; \begin{cases} z-y, & 0 \;\le\; y \;\le\; z,\\ 0, & z \;\le\; y \;\le\; h. \end{cases} \tag{13}$$

Then

$$\lim_{\delta\to 0^+}\int\limits_{0}^{h}\left[w''_\delta(y)-s^2w_\delta(y)\right]\widehat{u}(s,y)\,dy \;=\; \widehat{u}(s,z)-s^2\int\limits_{0}^{z}(z-y)\widehat{u}(s,y)\,dy.$$

It follows from (11) that the function $\widehat{u}(s,z)$ satisfies equation

$$\widehat{u}(s,z)-s^2\int\limits_{0}^{z}(z-y)\,\widehat{u}(s,y)\,dy \;=\; 0. \tag{14}$$

Since this is a homogeneous Volterra integral equation of the second kind, it has only a trivial solution. Indeed, this directly follows from the fact that the integral operator in this equation is quasinilpotent.

Hence, $\widehat{u}(s,y)=0$ for almost all $y\in[0,h]$. Consequently, according to Parseval equality, $u=0$ as element of Hilbert space $L_2(\Omega)$.

The proof of Proposition 2.1 is complete. □

3 Conditional solution

We are looking for the solution in the form

$$u(x,y) \;=\; \int\limits_{-\infty}^{\infty} \widehat{u}(s,y)e^{isx}\; ds.$$

To find the function $\widehat{u}(s,y)$ we use the condition (4):

$$\widehat{u}''(s,y) \;-\; s^2\widehat{u}(s,y) \;=\; 0, \quad \widehat{u}(s,0) = \widehat{\phi}(x), \quad \widehat{u}'(s,0) = 0.$$

Then

$$\widehat{u}(s,y) \;=\; \widehat{\phi}\cosh|s|y.$$

Hence,

$$u(x,y) \;=\; \int\limits_{-\infty}^{\infty} \widehat{\phi}\cosh|s|y\; e^{isx}\; ds. \tag{15}$$

It is clear that this solution exists only when the function $\widehat{\phi}$ tend to zero exponentially.

Nevertheless, we can take advantage of the experimental fact that the solution exists for $y = h$ and belongs to the class $L_2(\mathbb{R})$. We will search for a solution under this additional condition.

Set

$$\psi(x) \;=\; u(x,h). \tag{16}$$

Consider the expansion of ψ:

$$\psi(x) \;=\; \int\limits_{-\infty}^{\infty} \widehat{\psi}(s)\; e^{isx}\; ds. \tag{17}$$

Thus, according to Parseval's equality, the additional condition we have introduced is as follows

$$\|\psi\|^2_{L_2(\mathbb{R})} \;=\; 2\pi \int\limits_{-\infty}^{\infty} |\widehat{\psi}(s)|^2\; ds. \tag{18}$$

Since

$$\psi(x) \;=\; u(x,h) \;=\; \int\limits_{-\infty}^{\infty} \widehat{\phi}(s)\cosh|s|h\; e^{isx} ds \tag{19}$$

we get

$$\widehat{\psi}(s) \ = \ \widehat{\phi}(s)\cosh|s|h. \tag{20}$$

Hence, according to (15),

$$u(x,y) \ = \ \int_{-\infty}^{\infty} \widehat{\psi}(s)\frac{\cosh|s|y}{\cosh|s|h}e^{isx}\ ds. \tag{21}$$

4 Auxiliary equation

Consider the auxiliary equation

$$\frac{\partial^2 u}{\partial x^2} \ + \ \frac{\partial^2 u}{\partial y^2} \ + \ \alpha\frac{\partial^4 u}{\partial x^4} = \ 0, \quad -\infty < x < \infty, \quad 0 < y < h, \tag{22}$$

with the same boundary values (4).

We can look for the solution in the form

$$u(x,y) \ = \ \int_{-\infty}^{\infty} \widehat{u}(s,y)e^{isx}\ ds.$$

The function is $\widehat{u}(s,y) = \widehat{\phi}(s)\cos\mu_s y$ and required solution is

$$u_\alpha(x,y) \ = \ \int_{-\infty}^{\infty} \widehat{\phi}(s)\cos\mu_s y\ e^{isx}\ ds, \tag{23}$$

where

$$\mu_s \ = \ \mu_s(\alpha) \ = \ \sqrt{\alpha s^4 - s^2}. \tag{24}$$

Proposition 4.1. *The following inequality*

$$|\cos\mu_s y| \ \le \ \cosh|s|y, \quad 0 \le y \le h, \quad s \in \mathbb{R}, \tag{25}$$

is valid.

Proof. In case where $|s| > \sqrt{\dfrac{1}{\alpha}}$ the values of μ_k are real. Hence, the following estimate

$$|\cos\mu_s y| \ \le \ 1 \ \le \cosh|s|y \tag{26}$$

is valid.

Let $|s| \le \sqrt{\dfrac{1}{\alpha}}$. Then $\mu_s^2 \le 0$ and

$$|\mu_s^2| \;=\; s^2 - \alpha s^4 \;\le\; s^2.$$

Hence,

$$|\mu_s| \;\le\; |s|.$$

Then

$$|\cos \mu_s y| \;=\; \cosh |\mu_s| y \le\; \cosh |s| y.$$

The proof of Proposition 4.1 is complete. □

Proposition 4.2. *Assume that* $\psi \in L_2(\mathbb{R})$. *Then the solutions* (15) *and* (23) *of the problems* (3) *and* (22), *respectively, with boundary conditions* (4) *satisfy the condition*

$$\lim_{\alpha \to 0^+} \|u_\alpha(x,y) - u(x,y)\|_{L_2(\mathbb{R})} \;=\; 0 \tag{27}$$

uniformly with respect to $y \in [0,h]$.

Proof 1 *We have*

$$u(x,y) - u_\alpha(x,y) \;=\; \int_{-\infty}^{\infty} \widehat{\phi}(s)[\cosh |s| y - \cos \mu_s y]\; e^{isx}.$$

Then, according to (20),

$$u(x,y) - u_\alpha(x,y) \;=\; \int_{-\infty}^{\infty} \frac{\widehat{\psi}(s)}{\cosh |s| h}[\cosh |s| y - \cos \mu_s y]\; e^{isx}\; ds. \tag{28}$$

According to Parseval equation,

$$\|u(x,y) - u_\alpha(x,y)\|^2_{L_2(\mathbb{R})} \;=\; 2\pi \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(s)|^2}{\cosh^2 |s| h} |\cosh |s| y - \cos \mu_s y|^2 ds. \tag{29}$$

Set

$$w_s(y,\alpha) \;=\; \frac{|\cosh |s| y - \cos \mu_s y|^2}{\cosh^2 |s| h}, \tag{30}$$

(recall that $\mu_s = \mu_s(\alpha)$*).*

Then

$$\|u(x,y) - u_\alpha(x,y)\|^2_{L_2(\mathbb{R})} \;=\; 2\pi \int_{-\infty}^{\infty} |\widehat{\psi}(s)|^2\, w_s(y,\alpha)\; ds, \tag{31}$$

where $0 \le y \le h$, *and* $\alpha > 0$.

According to Proposition 4.1,

$$|\cosh|s|y - \cos\mu_s y|^2 \;\le\; |2\cosh|s|y|^2 \;\le\; 4\cosh^2|s|h.$$

Hence,

$$w_s(y,\alpha) \;\le\; 4, \quad 0 \le y \le h, \quad \alpha > 0. \tag{32}$$

In addition,

$$\lim_{\alpha\to 0^+} \mu_s(\alpha) \;=\; |s|i.$$

and therefore

$$\lim_{\alpha\to 0^+} \cos\mu_s y \;=\; \cosh|s|y.$$

Hence, for any $y \in [0,h]$

$$\lim_{\alpha\to 0^+} w_s(y,\alpha) \;=\; 0. \tag{33}$$

Since $\psi \in L_2(\mathbb{R})$ *the following integral converges:*

$$\int_{-\infty}^{\infty} |\widehat{\psi}(s)|^2 \; ds = \frac{1}{2\pi}\,\|\psi(s)\|^2_{L_2(\mathbb{R})}.$$

Hence, it follows from (32) *and* (33) *and from theorem of Weierstrass, that the right hand side of* (31) *tends to 0 subject to* $\alpha \to 0$ *uniformly with respect to* $y \in [0,h]$.
The proof of Proposition 4.2 is complete.

□

Corollary. *Assume that* $\psi \in L_2(\mathbb{R})$. *Then*

$$\lim_{\alpha\to 0^+} \|u_\alpha(x,h) - u(x,h)\|_{L_2(\mathbb{R})} \;=\; 0. \tag{34}$$

□

Note that the Proposition 4.2 only asserts the convergence of the auxiliary solution to exact one. In order to estimate the approximation error, it is necessary to require additional smoothness from the solution.

Denote by $L_2^\beta(\mathbb{R})$ the Sobolev space with the norm

$$\|f\|^2_\beta \;=\; 2\pi \int_{-\infty}^{\infty} |\widehat{f}(s)|^2 \big(1 + |s|^2\big)^\beta \; ds, \tag{35}$$

where $\widehat{f}(s)$ is the Fourier transform of the function $f(x)$.

Proposition 4.3. *Assume that $\psi \in L_2^\beta(\mathbb{R})$, where $0 < \beta \leq 3$. Then the solutions* (15) *and* (23) *of the problems* (3) *and* (22), *respectively, with boundary conditions* (4) *for every* $\alpha, 0 < \alpha \leq 2^{-6/\beta} h^2$ *satisfy the estimate*

$$\|u_\alpha(x,y) - u(x,y)\|_{L_2(\mathbb{R})} \ \leq \ \sqrt{2\pi}\ \alpha^{\beta/3} h^{\beta/3} \|\psi\|_\beta \tag{36}$$

uniformly with respect to $y \in [0,h]$.

Proof. We use the equation (31). Divide the integral (31) into two parts:

$$\int_{-\infty}^{\infty} |\widehat{\psi}(s)|^2\, w_s(y,\alpha)\ ds = \int_{s^2 \leq 1/\alpha} + \int_{s^2 > 1/\alpha} = I_1 + I_2. \tag{37}$$

1. To estimate I_1 note that in case $s < \sqrt{\frac{1}{\alpha}}$ the multiplier $w_s(y,\alpha)$ defined by (30) has the form

$$w_s(y,\alpha) \ = \ \frac{|\cosh|s|y - \cosh(|\mu_s|y)|^2}{\cosh^2|s|h} \tag{38}$$

In this case

$$|s| - |\mu_s| = \frac{s^2 - |\mu_s|^2}{|s| + |\mu_s|} \ = \ \frac{\alpha s^4}{k + |\mu_s|} \ \leq \ \frac{\alpha s^4}{|s|} \ = \ \alpha|s|^3. \tag{39}$$

Further, using the Lagrange's mean value theorem for $0 \leq a < b$ we get

$$\cosh b - \cosh a \ = \ (b-a)\sinh\xi \ \leq \ (b-a)\sinh b \ \leq (b-a)\cosh b.$$

Then it follows from (39)

$$\cosh|s|y - \cosh(|\mu_s|y) \ \leq \ (|s| - |\mu_s|)y\cosh|s|y \ \leq \ \alpha|s|^3 z\cosh|s|y.$$

Hence,

$$w_s(y,\alpha) \ \leq \ h^2\alpha^2 s^6. \tag{40}$$

Further, since $|s| \geq |\mu_k|$ then $\cosh|s|y \geq \cosh|\mu_s|y$ and

$$0 \ \leq \ \cosh|s|y - \cosh|\mu_s|y \ \leq \ \cosh|s|y.$$

Then

$$w_s(y,\alpha) \ \leq \ 1. \tag{41}$$

It follows from (40) and (41) that for every λ from interval $0 \leq \lambda \leq 1$ the estimate

$$w_s(y,\alpha) \ \leq \ (h^2\alpha^2 s^6)^\lambda,$$

is valid.

Choosing $\lambda = \beta/3$, we get

$$I_1 \;\le\; \int\limits_{s^2\le 1/\alpha} |\widehat{\psi}(s)|^2\, w_s(y,\alpha)\; ds \;\le\; \alpha^{2\beta/3} h^{2\beta/3} \int\limits_{s^2\le 1/\alpha} |\widehat{\psi}(s)|^2\, s^{2\beta}\; ds. \tag{42}$$

2. To estimate I_2 we use the inequality (32). Then we have

$$I_2 \;=\; \int\limits_{s^2>1/\alpha} |\widehat{\psi}(s)|^2\, w_s(y,\alpha)\; ds \;\le\; 4 \int\limits_{s^2>1/\alpha} |\widehat{\psi}(s)|^2\; ds \;\le\; 4\alpha^{\beta} \int\limits_{s^2>1/\alpha} |\widehat{\psi}(s)|^2 s^{2\beta}\; ds. \tag{43}$$

Set

$$C(\alpha) \;=\; \max\{h^{\beta/3},\; 2\alpha^{\beta/6}\}. \tag{44}$$

Then we get from (31), (37), (42), and (43):

$$\|u_\alpha(x,y) - u(x,y)\|_{L_2(\mathbb{R})} \;\le\; \sqrt{2\pi}\; C(\alpha)\alpha^{\beta/3}\|\psi\|_\beta \tag{45}$$

The proof of Proposition 4.3 is complete. □

5 Approximate boundary value problem

Assume that the boundary value is measured with some error $\delta > 0$. It means that instead ϕ we have the function ϕ_δ such that

$$\|\phi(x) - \phi_\delta(x)\|_{L_2(\mathbb{R})} \;\le\; \delta. \tag{46}$$

Let $u_{\delta,\alpha}(x,y)$ be a solution to the equation (22) with boundary condition

$$u_{\delta,\alpha}(x,0) \;=\; \phi_\delta(x). \tag{47}$$

Proposition 5.1. *For any $\alpha > 0$ and $\delta > 0$ the following estimate*

$$\|u_{\delta,\alpha}(x,h) - u_\alpha(x,h)\|_{L_2(\mathbb{R})} \;\le\; 2\sqrt{\pi}\,\delta\cosh\frac{h}{\sqrt{\alpha}} \tag{48}$$

is valid.

Proof 2 *Let*

$$\phi_\delta(x) \;=\; \int\limits_{-\infty}^{\infty} \widehat{\phi}_\delta(s)\, e^{isx}\; ds.$$

Then, according to (23),

$$u_{\delta,\alpha}(x,h) \;-\; u_\alpha(x,h) \;=\; \int\limits_{-\infty}^{\infty} (\widehat{\phi}_\delta(s) - \widehat{\phi}(s)) \cos \mu_s h \, e^{isx} ds.$$

Further we use Parseval's equality:

$$\|u_{\delta,\alpha}(\cdot,h) - u_\alpha(\cdot,h)\|^2 \;=\; 2\pi \int\limits_{-\infty}^{\infty} |\widehat{\phi}_\delta(s) - \widehat{\phi}(s)|^2 \cos^2 \mu_s h \; ds.$$

Divide the integral into two parts:

$$\int\limits_{-\infty}^{\infty} |\widehat{\phi}_\delta(s) - \widehat{\phi}(s)|^2 \cos^2 \mu_s h \; ds \;=\; \int\limits_{s^2 \le 1/\alpha} \;+\; \int\limits_{s^2 > 1/\alpha} \;=\; I_1 \;+\; I_2.$$

According to Proposition 4.1,

$$I_1 \le \int\limits_{s^2 \le 1/\alpha} |\widehat{\phi}_\delta - \widehat{\phi}(s)|^2 \cosh^2 |s| h \; ds \;\le \cosh^2 \frac{h}{\sqrt{\alpha}} \int\limits_{-\infty}^{\infty} |\widehat{\phi}_\delta(s) - \widehat{\phi}(s)|^2 \; ds \;=$$

$$=\; \cosh^2 \frac{h}{\sqrt{\alpha}} \; \|\phi_\delta - \phi\|^2 \;\le\; \delta^2 \cosh^2 \frac{h}{\sqrt{\alpha}}. \tag{49}$$

Further, according to (46) *and first inequality* (26)*,*

$$I_2 \;\le\; \int\limits_{s^2 > 1/\alpha} |\widehat{\phi}_\delta(s) - \widehat{\phi}(s)|^2 \; ds \;\le\; \|\phi(x) - \phi_\delta(x)\|^2_{L_2(\mathbb{R})} \;\le\; \delta^2. \tag{50}$$

The required estimate follows from (49) *and* (50)*:*

$$\|u_{\delta,\alpha}(x,h) - u_\alpha(x,h)\|^2 \;\le\; 2\pi\delta^2 \,\mathrm{ch}^2 \frac{h}{\sqrt{\alpha}} \; + 2\pi\delta^2 \;\le\; 4\pi\delta^2 \; \mathrm{ch}^2 \frac{h}{\sqrt{\alpha}}.$$

The proof of Proposition 5.1 is complete. □

Corollary. *Under the conditions of the Proposition 5.1, the following estimate*

$$\|u_{\delta,\alpha}(x,h) - u_\alpha(x,h)\| \;\le\; 2\sqrt{\pi}\, \delta \, e^{\frac{h}{\sqrt{\alpha}}} \tag{51}$$

is valid.

Proposition 5.2. *Assume that* $\psi \in L_2(\mathbb{R})$*. Set*

$$\alpha \;=\; \alpha(\delta) \;=\; \frac{\lambda^2}{\ln^2 \delta}, \tag{52}$$

where λ is an arbitrary constant such that $\lambda > h$. Then for any δ from the interval $0 < \delta < 1$ the estimate

$$\|u_{\delta,\alpha}(x,h) - u_\alpha(x,h)\| \ \leq \ 2\sqrt{\pi}\,\delta^{\left(1-\frac{h}{\lambda}\right)} \tag{53}$$

is valid.

Proof 3 *Note that*

$$\delta^{-1} \ = \ e^{\frac{\lambda}{\sqrt{\alpha}}}, \quad \text{and} \quad \delta^{-\frac{h}{\lambda}} \ = \ e^{\frac{h}{\sqrt{\alpha}}}.$$

Therefore,

$$\delta\, e^{\frac{h}{\sqrt{\alpha}}} \ = \ \delta \cdot \delta^{-\frac{h}{\lambda}} \ = \ \delta^{\left(1-\frac{h}{\lambda}\right)}.$$

Then the required estimate (53) follows from (51).

The proof of Proposition 5.2 is complete.

6 Main results

In the case when the sought function is known only that it belongs to space $L_2(\mathbb{R})$, the following convergence result is valid, without estimating the convergence rate.

Theorem 1 Assume that the function $\psi(x) = u(x,h)$ exists and belongs to $L_2(\mathbb{R})$. Set

$$\alpha \ = \ \alpha(\delta) \ = \ \frac{\lambda^2}{\ln^2\delta}, \quad 0 < \delta < 1, \tag{54}$$

where $\lambda > h$.

Then

$$\lim_{\delta \to 0^+} \|u_{\delta,\alpha}(x,h) - \psi(x)\| \ = \ 0. \tag{55}$$

Proof 4 *We can write*

$$\|u_{\delta,\alpha}(x,h) - \psi(x)\| \ \leq$$

$$\leq \ \|u_{\delta,\alpha}(x,h) - u_\alpha(x,h)\| \ + \ \|u_\alpha(x,h) - \psi(x)\|. \tag{56}$$

According to Proposition 5.2,

$$\|u_{\delta,\alpha}(x,h) - \psi(x)\| \ \leq$$

$$\leq \ 2\sqrt{\pi}\,\delta^{\left(1-\frac{h}{\lambda}\right)} \ + \ \|u_\alpha(x,h) - \psi(x)\|. \tag{57}$$

Therefore, (55) follows from (34) and from condition $\lambda > h$.

The proof of Theorem 6.1 is complete.

Theorem 2 Assume that the function $\psi \in L_2^\beta(\mathbb{R})$, where $0 < \beta \leq 3$, satisfies condition

$$\|\psi\|_\beta \;\leq\; 1. \tag{58}$$

Let $\alpha = \alpha(\delta)$ is defined by equation (54). Then for any δ from the interval $0 < \delta < 1$ the estimate

$$\|u_{\delta,\alpha}(x,h) - \psi(x)\| \;\leq\; C(h,\beta)\left(\ln\frac{1}{\delta}\right)^{-2\beta/3}, \quad 0<\delta<1, \tag{59}$$

is valid.

Proof 5 *It follows from* (56), (51), *and* (36), *that*

$$\|u_{\delta,\alpha}(x,h) - \psi(x)\| \;\leq\; 2\sqrt{\pi}\,\delta^{\left(1-\frac{h}{\lambda}\right)} \;+\; \sqrt{2\pi}\;\alpha^{\beta/3}h^{\beta/3}\|\psi\|_\beta.$$

Taking into account equation (54) *and condition* (58), *we can write*

$$\|u_{\delta,\alpha}(x,h) - \psi(x)\| \;\leq\; 2\sqrt{\pi}\,\delta^{\left(1-\frac{h}{\lambda}\right)} \;+\; \sqrt{2\pi}\;h^{\beta/3}\left(\frac{\lambda}{\ln\frac{1}{\delta}}\right)^{2\beta/3}. \tag{60}$$

It is clear that

$$\delta^{\left(1-\frac{h}{\lambda}\right)} \;\leq\; C\left(\frac{1}{\ln\frac{1}{\delta}}\right)^{2\beta/3}, \quad 0<\delta<1.$$

Then required estimate (59) *follows from* (60).
The proof of Theorem 6.2 is complete.

7 Example

Examples 1 *Let $h = 1$ and let the boundary data be given by*

$$\phi(x) = e^{-x^2}, \quad x \in \mathbb{R}. \tag{61}$$

Then $\phi \in L_2(\mathbb{R})$ since

$$\|\phi\|^2_{L_2(\mathbb{R})} = \int_{-\infty}^{\infty} e^{-2x^2}\,dx = \sqrt{\frac{\pi}{2}} < \infty. \tag{62}$$

The Fourier transform of ϕ is

$$\widehat{\phi}(s) = \frac{1}{2\sqrt{\pi}}\,e^{-s^2/4}, \quad s \in \mathbb{R}. \tag{63}$$

The exact solution of the Cauchy problem (3)–(4) *is given by formula* (15)*:*

$$u(x,y) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi}}\,e^{-s^2/4}\cosh(|s|y)\,e^{isx}\,ds. \tag{64}$$

The value at the upper boundary $y = h = 1$ is

$$\psi(x) = u(x,1) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi}}\, e^{-s^2/4} \cosh(|s|)\, e^{isx}\, ds, \tag{65}$$

so that, according to (20),

$$\widehat{\psi}(s) = \widehat{\phi}(s) \cosh|s| = \frac{1}{2\sqrt{\pi}}\, e^{-s^2/4} \cosh|s|. \tag{66}$$

We verify that $\psi \in L_2(\mathbb{R})$. Indeed, since $\cosh|s| \le e^{|s|}$, *we have*

$$\|\psi\|^2_{L_2(\mathbb{R})} = 2\pi \int_{-\infty}^{\infty} |\widehat{\psi}(s)|^2\, ds = \frac{1}{2} \int_{-\infty}^{\infty} e^{-s^2/2} \cosh^2|s|\, ds \le \frac{1}{2} \int_{-\infty}^{\infty} e^{2|s| - s^2/2}\, ds < \infty, \tag{67}$$

since $2|s| - s^2/2 \to -\infty$ *as* $|s| \to \infty$.

Moreover, $\psi \in L_2^{\beta}(\mathbb{R})$ for all $0 < \beta \le 3$. Indeed,

$$\|\psi\|^2_{\beta} = 2\pi \int_{-\infty}^{\infty} |\widehat{\psi}(s)|^2 (1+s^2)^{\beta}\, ds = \frac{1}{2} \int_{-\infty}^{\infty} e^{-s^2/2} \cosh^2|s|\, (1+s^2)^{\beta}\, ds < \infty, \tag{68}$$

since the Gaussian factor $e^{-s^2/2}$ dominates any polynomial growth as $|s| \to \infty$.

By rescaling, without loss of generality we may assume $\|\psi\|_{\beta} \le 1$.

The regularized solution (23) *corresponding to the auxiliary equation* (22) *takes the form*

$$u_{\alpha}(x,y) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi}}\, e^{-s^2/4} \cos(\mu_s y)\, e^{isx}\, ds, \tag{69}$$

where $\mu_s = \mu_s(\alpha) = \sqrt{\alpha s^4 - s^2}$ as defined in (24).

Now suppose the boundary data is measured with error $\delta > 0$, so that instead of $\phi(x)$ we observe $\phi_{\delta}(x)$ satisfying

$$\|\phi_{\delta} - \phi\|_{L_2(\mathbb{R})} \le \delta. \tag{70}$$

Set $\lambda = 2 > h = 1$ and choose the regularization parameter as in (54)*:*

$$\alpha(\delta) = \frac{4}{\ln^2 \delta}, \quad 0 < \delta < 1. \tag{71}$$

Then the following results hold:

(i) *Since $\psi \in L_2(\mathbb{R})$, by Theorem 6.1,*

$$\lim_{\delta \to 0^+} \|u_{\delta,\alpha}(\cdot,1) - \psi(\cdot)\|_{L_2(\mathbb{R})} = 0. \tag{72}$$

(ii) *Since $\psi \in L_2^{\beta}(\mathbb{R})$ with $\|\psi\|_{\beta} \le 1$, by Theorem 6.2, for any $0 < \beta \le 3$,*

$$\|u_{\delta,\alpha}(\cdot,1) - \psi(\cdot)\|_{L_2(\mathbb{R})} \le C(1,\beta) \left(\ln \frac{1}{\delta}\right)^{-2\beta/3}, \quad 0 < \delta < 1, \tag{73}$$

which provides an explicit logarithmic rate of convergence of the regularized solution to the exact boundary value ψ.

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