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DIRECT AND INVERSE SOURCE PROBLEMS FOR GENERALIZED FRACTIONAL DIFFERENTIAL EQUATIONS

In this paper, we investigate the question of solution existence for higher order fractional differential equations that involve both Riemann-Liouville and Caputo type fractional derivatives. To prove our main results, we use the Laplace transform method, which provides a powerful tool for dealing with fractional operators and finding explicit formulas for solutions. Moreover, we investigate some inverse source problems for the class of higher order fractional differential equations under consideration. We study the problem of determining unknown sources in the equations under some additional conditions imposed on the solutions, called the over-determination condition $y(T) = h$. The results obtained in this study contribute to the development of the theory of fractional calculus and its applications in various fields of mathematical physics and engineering sciences, in which fractional differential equations arise in a natural way to describe memory and hereditary properties of various phenomena.
Keywords: direct problem, inverse source problem, Riemann-Liouville fractional derivative, Caputo fractional derivative, Laplace transform.

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Жалпыланған бөлшек ретті дифференциалдық теңдеулер үшін тура және кері есептер

Бұл мақалада біз Риман–Лиувилл және Капуто типті бөлшек туындылары бар жоғары ретті бөлшек дифференциалдық теңдеулер үшін шешімнің бар болу мәселесін зерттейміз. Негізгі нәтижелерді дәлелдеу барысында бөлшек операторлармен жұмыс істеуге және шешімдердің айқын формулаларын табуға мүмкіндік беретін қуатты әдіс ретінде Лаплас түрлендіруі қолданылады. Сонымен қатар, қарастырылып отырған жоғары ретті бөлшек дифференциалдық теңдеулер класы үшін кейбір кері есептер зерттеледі. Атап айтқанда, шешімге қойылатын қосымша шарттар, яғни $y(T) = h$, негізінде теңдеудің оң жағын анықтау мәселесі қарастырылады. Зерттеу нәтижелері бөлшек ретті туындылар теориясының дамуына, сондай-ақ математикалық физика мен инженерлік ғылымдардың әртүрлі салаларындағы қолданбалы есептерін шешуге үлес қосады, өйткені бөлшек ретті дифференциалдық теңдеулер көптеген құбылыстардың жады және тұқымқуалаушылық қасиеттерін сипаттауда табиғи түрде пайда болады.
Түйін сөздер: тура есеп, теңдеудің оң жағын қалпына келтіруге арналған кері есеп, Риман–Лиувилл бөлшек ретті туындысы, Капуто бөлшек ретті туындысы, Лаплас түрлендіруі.

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Прямые и обратные задачи источника для обобщённых дробных дифференциальных уравнений

В данной работе исследуется вопрос существования решений дробных дифференциальных уравнений высокого порядка, содержащих дробные производные типов Римана–Лиувилля и Капуто. Для доказательства основных результатов используется метод преобразования Лапласа, который является эффективным инструментом для работы с дробными операторами и позволяет получать явные формулы решений. Кроме того, рассматриваются некоторые обратные задачи определения источника для класса исследуемых дробных дифференциальных уравнений высокого порядка. Изучается задача восстановления неизвестных источников в уравнениях при наличии дополнительных условий, наложенных на решения, так называемого условия переопределения $y(T) = h$. Полученные результаты вносят вклад в развитие теории дробного исчисления и её приложений в различных областях математической физики и инженерных наук, где дробные дифференциальные уравнения естественным образом возникают при описании эффектов памяти и наследственных свойств различных явлений.
Ключевые слова: прямая задача, обратная задача определения источника, дробная производная Римана–Лиувилля, дробная производная Капуто, преобразование Лапласа.

1 Introduction

In their research [10], S.-D. Lin and C.-H. Lu (2013) consider the equation

$$y''(x) + a\,{}^C D_{0+}^{\alpha} y(x) + b\,y(x) = 0,$$

under the initial conditions

$$y(0) = c_0, \qquad y'(0) = c_1,$$

and prove the solvability of this equation together with some of its particular cases. Here, the operator ${}^C D_{0+}^{\alpha}$ denotes the Caputo fractional derivative.

Later, in 2019, this problem was generalized by G. Bozkurt, D. Albayrak, and N. Dernekin [4], who presented solutions to a wider range of fractional differential equations of the form

$${}^C D_{0+}^{\alpha} y(x) + a\,{}^C D_{0+}^{\beta} y(x) + b\,y(x) = f(x).$$

and

$$D_{0+}^{\alpha} y(x) + a\,D_{0+}^{\beta} y(x) + b\,y(x) = f(x),$$

as well as their corresponding particular instances. In this setting, the operator D_{0+}^{α} represents the Riemann–Liouville fractional derivative.

The study in [4] includes specific cases like the fractional damping vibration equation for a single degree of freedom and the general Bagley–Torvik equation. The authors demonstrated that these equations can be solved efficiently using tools from fractional calculus. Other solution methods for such equations can also be found in (see [1, 2, 5]).

It is worth noting that the results of [4] were further generalized in [3], where the solvability of fractional differential equations involving Hilfer fractional derivatives, which interpolate between the Riemann–Liouville and Caputo derivatives, was investigated. In particular, the equation

$$D_{0+}^{(\alpha,\beta)\mu} y(x) + a\,D_{0+}^{(\gamma,\delta)\varepsilon} y(x) + b\,y(x) = f(x)$$

subject to the conditions

$$\lim_{x\to 0}\left(\frac{d}{dx}\right)^{n-k-1} I_{0+}^{(1-\mu)(n-\beta)}y(x) = c_k, \qquad k = 0, 1, \ldots, n-1,$$

and

$$\lim_{x\to 0}\left(\frac{d}{dx}\right)^{m-r-1} I_{0+}^{(1-\varepsilon)(m-\delta)}y(x) = b_r, \qquad r = 0, 1, \ldots, m-1,$$

was considered.

In the present paper, we investigate the solvability of the three-term fractional differential equations

$$D_{0+}^{\alpha}y(x) + a\, D_{0+}^{\beta}y(x) + b\, D_{0+}^{\mu}y(x) = f(x)$$

and

$${}^{C}D_{0+}^{\alpha}y(x) + a\,{}^{C}D_{0+}^{\beta}y(x) + b\,{}^{C}D_{0+}^{\mu}y(x) = f(x),$$

thereby extending the two-term models considered in [4] to a more general three-term setting.

Furthermore, we investigate the corresponding inverse source problems

$$D_{0+}^{\alpha}y(x) + a\, D_{0+}^{\beta}y(x) + b\, D_{0+}^{\mu}y(x) = f$$

and

$${}^{C}D_{0+}^{\alpha}y(x) + a\,{}^{C}D_{0+}^{\beta}y(x) + b\,{}^{C}D_{0+}^{\mu}y(x) = f,$$

where both the solution y and the source term f are unknown.

We also refer the reader to [4, 7, 11] for related results on various classes of evolutionary equations involving Riemann–Liouville and Caputo fractional derivatives.

2 Preliminaries

In this section, some of the fundamental concepts and definitions considered essential for the development of the results presented in this paper are introduced (see [6,9,11]). Relevant theorems are also stated, and their proofs are given. The belief of the authors is that the ideas presented in this section are adequate for a comprehensive understanding of the subsequent material.

Definition 1 *[9, p. 18] For a function $g(x)$ defined on $x \geq 0$, its Laplace transform, denoted $\mathcal{L}\{g(x)\}(s)$, is given in terms of the integral*

$$\mathcal{L}\{g(x)\}(s) = G(s) = \int_0^{\infty} g(x)e^{-sx}\, dx,$$

where s is a complex variable. The Laplace transform is said to exist provided that this integral converges.

Definition 2 *For functions $f(x)$ and $g(x)$, their convolution is expressed through the integral*

$$(f * g)(x) = \int_0^x f(\tau)g(x-\tau)d\tau.$$

Remark 1 *Using the Laplace transform to the convolution, we obtain*

$$\mathcal{L}(f * g)(t) = F(s)G(s).$$

Definition 3 *[9, p. 69] The Riemann–Liouville operator of fractional integration of order α, where α is complex number with a positive real part $\Re(\alpha) > 0$, of a function g is given by*

$$\mathrm{I}^{\alpha}_{a+}g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} g(\tau)d\tau$$

and

$$\mathrm{I}^{\alpha}_{b-}g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} g(\tau)d\tau,$$

where $\mathrm{I}^{\alpha}_{a+}g$ denotes the left-sided Riemann–Liouville fractional integral and $\mathrm{I}^{\alpha}_{b-}g$ denotes the right-sided Riemann–Liouville fractional integral.

Definition 4 *[9, p. 70] The Riemann–Liouville operator corresponding to a fractional derivative of order α ($0 \le n-1 < \Re(\alpha) \le n$) of a function g is expressed as*

$$D^{\alpha}_{a+}g(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} g(\tau)d\tau = \left(\frac{d}{dx}\right)^n \mathrm{I}^{n-\alpha}_{a+}g(x)$$

and

$$D^{\alpha}_{b-}g(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha-1} g(\tau)d\tau = (-1)^n \left(\frac{d}{dx}\right)^n \mathrm{I}^{n-\alpha}_{b-}g(x),$$

where $D^{\alpha}_{a+}g$ denotes the left-sided Riemann–Liouville fractional derivative while $D^{\alpha}_{b-}g$ represents the right-sided one.

Definition 5 *[9, p. 92] For a function g, the Caputo fractional derivative of order α ($0 \le n-1 < \Re(\alpha) \le n$) is given by*

$${}^{C}D^{\alpha}_{a+}g(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n g(\tau)d\tau$$

and

$${}^{C}D^{\alpha}_{b-}g(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (\tau-x)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n g(\tau)d\tau,$$

where ${}^{C}D^{\alpha}_{a+}g$ denotes the left-sided Caputo fractional derivative while ${}^{C}D^{\alpha}_{b-}g$ represents the right-sided one.

Lemma 1 *The Laplace transform of the left-hand sided Riemann-Liouville fractional derivative is determined by*

$$\mathcal{L}\{D_{0+}^{\alpha}g(x)\} = s^{\alpha}G(s) - \sum_{k=0}^{n-1} s^{k} \lim_{x\to 0} D_{0+}^{\alpha-k-1} g(x),$$

where $\alpha \in \mathbb{C}$, $0 \leq n-1 < \Re(\alpha) \leq n$.

Lemma 2 *[10, Remark 1.1, p. 3] The Laplace transform of the left-hand sided Caputo fractional derivative is determined by*

$$\mathcal{L}\{{}^{C}D_{0+}^{\alpha}g(x)\} = s^{\alpha}G(s) - \sum_{k=1}^{n} s^{\alpha-k} \lim_{x\to 0}\left(\frac{d}{dx}\right)^{k-1} g(x),$$

where $\alpha \in \mathbb{C}$, $0 \leq n-1 < \Re(\alpha) \leq n$.

Definition 6 *[12] The Mittag-Leffler function* $E_{\alpha,\beta}(z)$ *is formulated as the series*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+\beta)}, \quad \Re(\alpha) > 0,$$

where $\beta, \alpha \in \mathbb{C}$.

Definition 7 *[12] The generalized Mittag-Leffler function* $E_{\alpha,\beta}^{\gamma}(z)$ *is formulated as the series*

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n} z^{n}}{\Gamma(\alpha n+\beta) n!}, \quad \Re(\alpha) > 0,$$

where β, α, γ *are complex numbers, and* $(\gamma)_n$ *is given by* $(\gamma)_n = \Gamma(\gamma+n)/\Gamma(\gamma)$ *and* $(\gamma)_0 = 1$.

Remark 2 *[9, p. 45] The function* $E_{\alpha,\beta}^{\gamma}(z)$ *is an entire and its order is determined by* $[\Re(\alpha)]^{-1}$.

Lemma 3 *[12] For* $\Re(s) > 0$, $\Re(\beta) > 0$, $\lambda \in \mathbb{C}$, *and* $|\lambda s^{-\alpha}| < 1$, *the Laplace transform of the function* $x^{\beta-1}E_{\alpha,\beta}^{\gamma}(\lambda x^{\alpha})$ *is defined by*

$$\mathcal{L}\{x^{\beta-1}E_{\alpha,\beta}^{\gamma}(\lambda x^{\alpha})\} = \frac{1}{s^{\beta}(1-\lambda s^{-\alpha})^{\gamma}}. \tag{1}$$

Definition 8 *We use the notation* $\mathcal{C}^{\infty}$ *to represent the space of functions* $g \in C^{\infty}(\mathbb{R}_+)$ *such that* $\mathcal{L}g$ *is an entire function.*

3 Main part

In this section, we will formulate the principal problem being addressed in this paper and prove its solvability. In order to solve the problem, the Laplace transform along with its inverse will be used. The results stated in the previous section will be heavily employed.

Lemma 4 *We suppose that* $a, b \in \mathbb{R}$, $\alpha, \beta, \mu \in \mathbb{C}$, $\Re(\alpha) > \Re(\beta) > \Re(\mu) > 0$, *and*

$$\left| \frac{b}{s^{\alpha-\mu} + a s^{\beta-\mu}} \right| < 1.$$

Then, we obtain

$$\frac{1}{s^\alpha + a s^\beta + b s^\mu} = \sum_{k=0}^{\infty} \frac{(-b)^k}{s^{\alpha k - \mu k + \alpha} \left(1 + a s^{\beta-\alpha}\right)^{k+1}}. \tag{2}$$

Proof. By transforming the left-hand side of the equation (2), we show that it coincides with the right-hand side. To this end, we start from the left-hand side and proceed as follows.

$$\begin{aligned} \frac{1}{s^\alpha + a s^\beta + b s^\mu} &= \frac{s^{-\mu}}{s^{\alpha-\mu} + a s^{\beta-\mu} + b} \\ &= \frac{s^{-\mu}}{\left(s^{\alpha-\mu} + a s^{\beta-\mu}\right)\left(1 + \frac{b}{s^{\alpha-\mu} + a s^{\beta-\mu}}\right)} \\ &= \frac{s^{-\mu}}{s^{\alpha-\mu} + a s^{\beta-\mu}} \left(1 + \frac{b}{s^{\alpha-\mu} + a s^{\beta-\mu}}\right)^{-1}. \end{aligned}$$

By using the series expansion

$$\left(1 + \frac{b}{s^{\alpha-\mu} + a s^{\beta-\mu}}\right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{-b}{s^{\alpha-\mu} + a s^{\beta-\mu}}\right)^k,$$

we obtain

$$\frac{1}{s^\alpha + a s^\beta + b s^\mu} = \frac{s^{-\mu}}{s^{\alpha-\mu} + a s^{\beta-\mu}} \sum_{k=0}^{\infty} \left(\frac{-b}{s^{\alpha-\mu} + a s^{\beta-\mu}}\right)^k = \sum_{k=0}^{\infty} \frac{(-b)^k s^{-\mu}}{\left(s^{\alpha-\mu} + a s^{\beta-\mu}\right)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{(-b)^k s^{-\mu}}{s^{(\alpha-\mu)(k+1)} \left(1 + a s^{\beta-\alpha}\right)^{k+1}} = \sum_{k=0}^{\infty} \frac{(-b)^k}{s^{\alpha k - \mu k + \alpha} \left(1 + a s^{\beta-\alpha}\right)^{k+1}}.$$

The proof is complete.

Theorem 1 *Let* $a, b \in \mathbb{R}$, $\alpha, \beta, \mu \in \mathbb{C}$, $n, m, \ell \in \mathbb{N}$, $n \geq \Re(\alpha) > n-1 \geq m \geq \Re(\beta) > m-1 \geq \ell \geq \Re(\mu) > \ell - 1 \geq 0$, *and*

$$\left| \frac{b}{s^{\alpha-\mu} + a s^{\beta-\mu}} \right| < 1.$$

Then the following fractional differential equation

$$D_{0+}^{\alpha}y(t) + aD_{0+}^{\beta}y(t) + bD_{0+}^{\mu}y(t) = f(t), \qquad t > 0, \tag{3}$$

under the given initial conditions

$$\lim_{t\to 0} D_{0+}^{\alpha-r-1}y(t) = c_r \quad \textit{for} \quad r = 0, 1, 2, \ldots, n-1, \tag{4}$$

$$\lim_{t\to 0} D_{0+}^{\beta-j-1}y(t) = b_j \quad \textit{for} \quad j = 0, 1, 2, \ldots, m-1, \tag{5}$$

and

$$\lim_{t\to 0} D_{0+}^{\mu-i-1}y(t) = a_i \quad \textit{for} \quad i = 0, 1, 2, \ldots, \ell-1, \tag{6}$$

has a unique solution, defined by

$$\begin{aligned}
y(t) &= f(t) * \sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k+\alpha}^{k+1}(-at^{\alpha-\beta}) \\
&\quad + \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k-r+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k-r+\alpha}^{k+1}(-at^{\alpha-\beta}) \\
&\quad + a\sum_{j=0}^{m-1} b_j \sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k-j+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k-j+\alpha}^{k+1}(-at^{\alpha-\beta}) \\
&\quad + b\sum_{i=0}^{\ell-1} a_i \sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k-i+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k-i+\alpha}^{k+1}(-at^{\alpha-\beta}),
\end{aligned}$$

where $f \in C^{\infty}(\mathbb{R}_+)$.

Proof. To prove this, we apply the Laplace transform method.Taking the Laplace transform of equation (3) and using

$$\mathcal{L}\{D_{0+}^{\alpha}y(t)\} = s^{\alpha}Y(s) - \sum_{r=0}^{n-1} s^r \lim_{t\to 0} D_{0+}^{\alpha-r-1}y(t) = s^{\alpha}Y(s) - \sum_{r=0}^{n-1} s^r c_r,$$

$$\mathcal{L}\{D_{0+}^{\beta}y(t)\} = s^{\beta}Y(s) - \sum_{j=0}^{m-1} s^j \lim_{t\to 0} D_{0+}^{\beta-j-1}y(t) = s^{\beta}Y(s) - \sum_{j=0}^{m-1} s^j b_j,$$

$$\mathcal{L}\{D_{0+}^{\mu}y(t)\} = s^{\mu}Y(s) - \sum_{i=0}^{\ell-1} s^i \lim_{t\to 0} D_{0+}^{\mu-i-1}y(t) = s^{\mu}Y(s) - \sum_{i=0}^{\ell-1} s^i a_i,$$

we obtain

$$s^{\alpha}Y(s) - \sum_{r=0}^{n-1} s^r c_r + as^{\beta}Y(s) - a\sum_{j=0}^{m-1} s^j b_j + bs^{\mu}Y(s) - b\sum_{i=0}^{\ell-1} s^i a_i = F(s).$$

Solving the above equation with respect to the unknown function $Y(s)$, we obtain the following expression

$$Y(s)=\frac{F(s)+\sum_{r=0}^{n-1}s^{r}c_{r}+a\sum_{j=0}^{m-1}s^{j}b_{j}+b\sum_{i=0}^{\ell-1}s^{i}a_{i}}{s^{\alpha}+as^{\beta}+bs^{\mu}}$$

or

$$Y(s)=\frac{1}{s^{\alpha}+as^{\beta}+bs^{\mu}}\left(F(s)+\sum_{r=0}^{n-1}s^{r}c_{r}+a\sum_{j=0}^{m-1}s^{j}b_{j}+b\sum_{i=0}^{\ell-1}s^{i}a_{i}\right).$$

Now, using Lemma 4, we have

$$Y(s)=\sum_{k=0}^{\infty}\frac{(-b)^{k}}{s^{\alpha k-\mu k+\alpha}\left(1+as^{\beta-\alpha}\right)^{k+1}}\left(F(s)+\sum_{r=0}^{n-1}s^{r}c_{r}+a\sum_{j=0}^{m-1}s^{j}b_{j}+b\sum_{i=0}^{\ell-1}s^{i}a_{i}\right)$$

$$=F(s)\cdot\sum_{k=0}^{\infty}\frac{(-b)^{k}}{s^{\alpha k-\mu k+\alpha}\left(1+as^{\beta-\alpha}\right)^{k+1}}+\sum_{r=0}^{n-1}c_{r}\sum_{k=0}^{\infty}\frac{(-b)^{k}}{s^{\alpha k-\mu k-r+\alpha}\left(1+as^{\beta-\alpha}\right)^{k+1}}$$

$$+a\sum_{j=0}^{m-1}b_{j}\sum_{k=0}^{\infty}\frac{(-b)^{k}}{s^{\alpha k-\mu k-j+\alpha}\left(1+as^{\beta-\alpha}\right)^{k+1}}+b\sum_{i=0}^{\ell-1}a_{i}\sum_{k=0}^{\infty}\frac{(-b)^{k}}{s^{\alpha k-\mu k-i+\alpha}\left(1+as^{\beta-\alpha}\right)^{k+1}}.$$

Then applying inverse Laplace transform to the $Y(s)$, and using Remark 1, and Lemma 3, we get the solution of problem (3)-(6) defined by

$$\begin{aligned}y(t)&=f(t)*\sum_{k=0}^{\infty}(-b)^{k}t^{\alpha k-\mu k+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k+\alpha}^{k+1}(-at^{\alpha-\beta})\\&+\sum_{r=0}^{n-1}c_{r}\sum_{k=0}^{\infty}(-b)^{k}t^{\alpha k-\mu k-r+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k-r+\alpha}^{k+1}(-at^{\alpha-\beta})\\&+a\sum_{j=0}^{m-1}b_{j}\sum_{k=0}^{\infty}(-b)^{k}t^{\alpha k-\mu k-j+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k-j+\alpha}^{k+1}(-at^{\alpha-\beta})\\&+b\sum_{i=0}^{\ell-1}a_{i}\sum_{k=0}^{\infty}(-b)^{k}t^{\alpha k-\mu k-i+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k-i+\alpha}^{k+1}(-at^{\alpha-\beta}).\end{aligned}$$

The proof is complete.

Corollary 1 *Let* $a,b\in\mathbb{R}$, $\alpha,\beta,\mu\in\mathbb{C}$, $n,m,\ell\in\mathbb{N}$, $n\geq\Re(\alpha)>n-1\geq m\geq\Re(\beta)>m-1\geq\ell\geq\Re(\mu)>\ell-1\geq 0$,

$$\left|\frac{b}{s^{\alpha-\mu}+as^{\beta-\mu}}\right|<1,$$

and assume that

$$C_{11}=\sum_{k=0}^{\infty}(-b)^{k}T^{\alpha k-\mu k+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k+\alpha}^{k+1}(-aT^{\alpha-\beta})\neq 0.$$

Then the following fractional inverse source problem

$$D^{\alpha}_{0+}y(t) + aD^{\beta}_{0+}y(t) + bD^{\mu}_{0+}y(t) = f, \qquad 0 < t < T, \tag{7}$$

with the initial conditions

$$\lim_{t\to 0} D^{\alpha-r-1}_{0+}y(t) = c_r \quad \textit{for} \quad r = 0, 1, 2, \dots, n-1, \tag{8}$$

$$\lim_{t\to 0} D^{\beta-j-1}_{0+}y(t) = b_j \quad \textit{for} \quad j = 0, 1, 2, \dots, m-1, \tag{9}$$

$$\lim_{t\to 0} D^{\mu-i-1}_{0+}y(t) = a_i \quad \textit{for} \quad i = 0, 1, 2, \dots, \ell-1, \tag{10}$$

and the over-determination condition

$$y(T) = h_1, \tag{11}$$

has a unique solution pair $(y(t), f)$*, defined by*

$$f = \frac{h_1 - C_{12} - a\cdot C_{13} - b\cdot C_{14}}{C_{11}}$$

and

$$\begin{aligned} y(t) = {} & \left(\frac{h_1 - C_{12} - a\cdot C_{13} - b\cdot C_{14}}{C_{11}}\right)\cdot \sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k+\alpha-1}E^{k+1}_{\alpha-\beta,\alpha k-\mu k+\alpha}(-at^{\alpha-\beta}) \\ & + \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k-r+\alpha-1}E^{k+1}_{\alpha-\beta,\alpha k-\mu k-r+\alpha}(-at^{\alpha-\beta}) \\ & + a\sum_{j=0}^{m-1} b_j \sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k-j+\alpha-1}E^{k+1}_{\alpha-\beta,\alpha k-\mu k-j+\alpha}(-at^{\alpha-\beta}) \\ & + b\sum_{i=0}^{\ell-1} a_i \sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k-i+\alpha-1}E^{k+1}_{\alpha-\beta,\alpha k-\mu k-i+\alpha}(-at^{\alpha-\beta}), \end{aligned}$$

where

$$C_{12} = \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty}(-b)^k T^{\alpha k-\mu k-r+\alpha-1}E^{k+1}_{\alpha-\beta,\alpha k-\mu k-r+\alpha}(-aT^{\alpha-\beta}),$$

$$C_{13} = \sum_{j=0}^{m-1} b_j \sum_{k=0}^{\infty}(-b)^k T^{\alpha k-\mu k-j+\alpha-1}E^{k+1}_{\alpha-\beta,\alpha k-\mu k-j+\alpha}(-aT^{\alpha-\beta}),$$

and

$$C_{14} = \sum_{i=0}^{\ell-1} a_i \sum_{k=0}^{\infty}(-b)^k T^{\alpha k-\mu k-i+\alpha-1}E^{k+1}_{\alpha-\beta,\alpha k-\mu k-i+\alpha}(-aT^{\alpha-\beta}).$$

Proof. According to Theorem 1, the solution of problem (7)-(10) admits the representation

$$\begin{aligned} y(t) &= f \cdot \sum_{k=0}^{\infty} (-b)^k t^{\alpha k-\mu k+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k+\alpha}(-at^{\alpha-\beta}) \\ &+ \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k-\mu k-r+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k-r+\alpha}(-at^{\alpha-\beta}) \\ &+ a \sum_{j=0}^{m-1} b_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k-\mu k-j+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k-j+\alpha}(-at^{\alpha-\beta}) \\ &+ b \sum_{i=0}^{\ell-1} a_i \sum_{k=0}^{\infty} (-b)^k t^{\alpha k-\mu k-i+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k-i+\alpha}(-at^{\alpha-\beta}). \end{aligned}$$

Using the additional condition $y(T) = h_1$, we obtain

$$\begin{aligned} y(T) &= f \cdot \sum_{k=0}^{\infty} (-b)^k T^{\alpha k-\mu k+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k+\alpha}(-aT^{\alpha-\beta}) \\ &+ \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k T^{\alpha k-\mu k-r+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k-r+\alpha}(-aT^{\alpha-\beta}) \\ &+ a \sum_{j=0}^{m-1} b_j \sum_{k=0}^{\infty} (-b)^k T^{\alpha k-\mu k-j+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k-j+\alpha}(-aT^{\alpha-\beta}) \\ &+ b \sum_{i=0}^{\ell-1} a_i \sum_{k=0}^{\infty} (-b)^k T^{\alpha k-\mu k-i+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k-i+\alpha}(-aT^{\alpha-\beta}) \\ &=: f \cdot C_{11} + C_{12} + a \cdot C_{13} + b \cdot C_{14} \end{aligned}$$

from which the unknown function f can be determined as

$$f = \frac{h_1 - C_{12} - a \cdot C_{13} - b \cdot C_{14}}{C_{11}}.$$

After substituting f into $y(t)$, we get

$$\begin{aligned} y(t) &= \left(\frac{h_1 - C_{12} - a \cdot C_{13} - b \cdot C_{14}}{C_{11}} \right) \cdot \sum_{k=0}^{\infty} (-b)^k t^{\alpha k-\mu k+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k+\alpha}(-at^{\alpha-\beta}) \\ &+ \sum_{r=0}^{n-1} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k-\mu k-r+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k-r+\alpha}(-at^{\alpha-\beta}) \\ &+ a \sum_{j=0}^{m-1} b_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k-\mu k-j+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k-j+\alpha}(-at^{\alpha-\beta}) \\ &+ b \sum_{i=0}^{\ell-1} a_i \sum_{k=0}^{\infty} (-b)^k t^{\alpha k-\mu k-i+\alpha-1} E^{k+1}_{\alpha-\beta,\alpha k-\mu k-i+\alpha}(-at^{\alpha-\beta}). \end{aligned}$$

This concludes the proof.

Theorem 2 *Let $a, b \in \mathbb{R}$, $\alpha, \beta, \mu \in \mathbb{C}$, $n, m, \ell \in \mathbb{N}$, $n \geq \Re(\alpha) > n-1 \geq m \geq \Re(\beta) > m-1 \geq \ell \geq \Re(\mu) > \ell - 1 \geq 0$, and*

$$\left|\frac{b}{s^{\alpha-\mu} + as^{\beta-\mu}}\right| < 1.$$

Then the following fractional differential equation

$$ {}^C D^{\alpha}_{0+} y(t) + a\,{}^C D^{\beta}_{0+} y(t) + b\,{}^C D^{\mu}_{0+} y(t) = f(t), \qquad t > 0, \tag{12}$$

under the given initial conditions

$$\lim_{t\to 0}\left(\frac{d}{dt}\right)^{r-1} y(t) = c_r \quad for \quad r = 1, 2, \ldots, n, \tag{13}$$

$$\lim_{t\to 0}\left(\frac{d}{dt}\right)^{j-1} y(t) = b_j \quad for \quad j = 1, 2, \ldots, m, \tag{14}$$

and

$$\lim_{t\to 0}\left(\frac{d}{dt}\right)^{i-1} y(t) = a_i \quad for \quad i = 1, 2, \ldots, \ell, \tag{15}$$

has a unique solution, expressed by

$$\begin{aligned} y(t) &= f(t) * \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + \alpha - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + \alpha}(-at^{\alpha-\beta}) \\ &+ \sum_{r=1}^{n} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + r - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + r}(-at^{\alpha-\beta}) \\ &+ a \sum_{j=1}^{m} b_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + j - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + j}(-at^{\alpha-\beta}) \\ &+ b \sum_{i=1}^{\ell} a_i \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + i - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + i}(-at^{\alpha-\beta}), \end{aligned}$$

where $f \in C^{\infty}(\mathbb{R}_+)$.

Proof. To prove this, we apply the Laplace transform method.Taking the Laplace transform of equation (12) and using

$$\mathcal{L}\{{}^C D^{\alpha}_{0+} y(t)\} = s^{\alpha} Y(s) - \sum_{r=1}^{n} s^{\alpha-r} \lim_{t\to 0}\left(\frac{d}{dt}\right)^{r-1} y(t) = s^{\alpha} Y(s) - \sum_{r=1}^{n} s^{\alpha-r} c_r,$$

$$\mathcal{L}\{{}^C D^{\beta}_{0+} y(t)\} = s^{\beta} Y(s) - \sum_{j=1}^{m} s^{\beta-j} \lim_{t\to 0}\left(\frac{d}{dt}\right)^{j-1} y(t) = s^{\beta} Y(s) - \sum_{j=1}^{m} s^{\beta-j} b_j,$$

and

$$\mathcal{L}\{{}^{C}D_{0+}^{\mu}y(t)\} = s^{\mu}Y(s) - \sum_{i=1}^{\ell} s^{\mu-i}\lim_{t\to 0}\left(\frac{d}{dt}\right)^{i-1} y(t) = s^{\mu}Y(s) - \sum_{i=1}^{\ell} s^{\mu-i}a_i,$$

we obtain

$$s^{\alpha}Y(s) - \sum_{r=1}^{n} s^{\alpha-r}c_r + as^{\beta}Y(s) - a\sum_{j=1}^{m} s^{\beta-j}b_j + bs^{\mu}Y(s) - b\sum_{i=1}^{\ell} s^{\mu-i}a_i = F(s).$$

Simple arithmetic operations imply

$$Y(s) = \frac{F(s) + \sum_{r=1}^{n} s^{\alpha-r}c_r + a\sum_{j=1}^{m} s^{\beta-j}b_j + b\sum_{i=1}^{\ell} s^{\mu-i}a_i}{s^{\alpha} + as^{\beta} + bs^{\mu}}$$

or

$$Y(s) = \frac{1}{s^{\alpha} + as^{\beta} + bs^{\mu}}\left(F(s) + \sum_{r=1}^{n} s^{\alpha-r}c_r + a\sum_{j=1}^{m} s^{\beta-j}b_j + b\sum_{i=1}^{\ell} s^{\mu-i}a_i\right).$$

Then using the Lemma 4 to the above expression, we get

$$Y(s) = \sum_{k=0}^{\infty}\frac{(-b)^k}{s^{\alpha k-\mu k+\alpha}\left(1+as^{\beta-\alpha}\right)^{k+1}}\left(F(s) + \sum_{r=1}^{n} s^{\alpha-r}c_r + a\sum_{j=1}^{m} s^{\beta-j}b_j + b\sum_{i=1}^{\ell} s^{\mu-i}a_i\right)$$

or

$$Y(s) = F(s)\cdot\sum_{k=0}^{\infty}\frac{(-b)^k}{s^{\alpha k-\mu k+\alpha}\left(1+as^{\beta-\alpha}\right)^{k+1}} = \sum_{r=1}^{n} c_r\sum_{k=0}^{\infty}\frac{(-b)^k}{s^{\alpha k-\mu k+r}\left(1+as^{\beta-\alpha}\right)^{k+1}}$$

$$+a\sum_{j=1}^{m} b_j\sum_{k=0}^{\infty}\frac{(-b)^k}{s^{\alpha k-\mu k+j}\left(1+as^{\beta-\alpha}\right)^{k+1}} + b\sum_{i=1}^{\ell} a_i\sum_{k=0}^{\infty}\frac{(-b)^k}{s^{\alpha k-\mu k+\alpha}\left(1+as^{\beta-\alpha}\right)^{k+1}}.$$

Then applying the inverse Laplace transform, we get the solution of problem (12)-(15), given by

$$\begin{aligned}
y(t) &= f(t) * \sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k+\alpha-1}E_{\alpha-\beta,\alpha k-\mu k+\alpha}^{k+1}(-at^{\alpha-\beta}) \\
&+ \sum_{r=1}^{n} c_r\sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k+r-1}E_{\alpha-\beta,\alpha k-\mu k+r}^{k+1}(-at^{\alpha-\beta}) \\
&+ a\sum_{j=1}^{m} b_j\sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k+j-1}E_{\alpha-\beta,\alpha k-\mu k+j}^{k+1}(-at^{\alpha-\beta}) \\
&+ b\sum_{i=1}^{\ell} a_i\sum_{k=0}^{\infty}(-b)^k t^{\alpha k-\mu k+i-1}E_{\alpha-\beta,\alpha k-\mu k+i}^{k+1}(-at^{\alpha-\beta}).
\end{aligned}$$

The proof is completed.

Corollary 2 *Let* $a, b \in \mathbb{R}$, $\alpha, \beta, \mu \in \mathbb{C}$, $n, m, \ell \in \mathbb{N}$, $n \geq \Re(\alpha) > n-1 \geq m \geq \Re(\beta) > m-1 \geq \ell \geq \Re(\mu) > \ell - 1 \geq 0$,

$$\left| \frac{b}{s^{\alpha-\mu} + a s^{\beta-\mu}} \right| < 1,$$

and we assume that

$$C_{21} = \sum_{k=0}^{\infty} (-b)^k T^{\alpha k - \mu k + \alpha - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + \alpha}(-aT^{\alpha-\beta}) \neq 0.$$

Then the following fractional inverse source problem

$${}^C D^{\alpha}_{0+} y(t) + a\, {}^C D^{\beta}_{0+} y(t) + b\, {}^C D^{\mu}_{0+} y(t) = f, \qquad 0 < t < T, \tag{16}$$

under the given initial conditions

$$\lim_{t\to 0} \left(\frac{d}{dt}\right)^{r-1} y(t) = c_r \quad \text{for} \quad r = 1, 2, \ldots, n, \tag{17}$$

$$\lim_{t\to 0} \left(\frac{d}{dt}\right)^{j-1} y(t) = b_j \quad \text{for} \quad j = 1, 2, \ldots, m, \tag{18}$$

$$\lim_{t\to 0} \left(\frac{d}{dt}\right)^{i-1} y(t) = a_i \quad \text{for} \quad i = 1, 2, \ldots, \ell, \tag{19}$$

and the over-determination condition

$$y(T) = h_2, \tag{20}$$

has one and only one solution pair $(y(t), f)$, *defined by*

$$f = \frac{h_2 - C_{22} - a \cdot C_{23} - b \cdot C_{24}}{C_{21}}$$

and

$$\begin{aligned} y(t) = &\left(\frac{h_2 - C_{22} - a \cdot C_{23} - b \cdot C_{24}}{C_{21}}\right) \cdot \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + \alpha - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + \alpha}(-at^{\alpha-\beta}) \\ &+ \sum_{r=1}^{n} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + r - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + r}(-at^{\alpha-\beta}) \\ &+ a \sum_{j=1}^{m} b_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + j - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + j}(-at^{\alpha-\beta}) \\ &+ b \sum_{i=1}^{\ell} a_i \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + i - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + i}(-at^{\alpha-\beta}), \end{aligned}$$

where

$$C_{22} = \sum_{r=1}^{n} c_r \sum_{k=0}^{\infty} (-b)^k T^{\alpha k - \mu k + r - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + r}(-aT^{\alpha-\beta}),$$

$$C_{23} = \sum_{j=1}^{m} b_j \sum_{k=0}^{\infty} (-b)^k T^{\alpha k - \mu k + j - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + j}(-aT^{\alpha-\beta}),$$

and

$$C_{24} = \sum_{i=1}^{\ell} a_i \sum_{k=0}^{\infty} (-b)^k T^{\alpha k - \mu k + i - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + i}(-aT^{\alpha-\beta}).$$

Proof. According to Theorem 2, the solution of problem (16)-(19) has the representation

$$\begin{aligned} y(t) = f \cdot \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + \alpha - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + \alpha}(-at^{\alpha-\beta}) \\ + \sum_{r=1}^{n} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + r - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + r}(-at^{\alpha-\beta}) \\ + a \sum_{j=1}^{m} b_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + j - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + j}(-at^{\alpha-\beta}) \\ + b \sum_{i=1}^{\ell} a_i \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + i - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + i}(-at^{\alpha-\beta}). \end{aligned}$$

Using the additional condition $y(T) = h_2$, we obtain

$$\begin{aligned} y(T) = f \cdot \sum_{k=0}^{\infty} (-b)^k T^{\alpha k - \mu k + \alpha - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + \alpha}(-aT^{\alpha-\beta}) \\ + \sum_{r=1}^{n} c_r \sum_{k=0}^{\infty} (-b)^k T^{\alpha k - \mu k + r - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + r}(-aT^{\alpha-\beta}) \\ + a \sum_{j=1}^{m} b_j \sum_{k=0}^{\infty} (-b)^k T^{\alpha k - \mu k + j - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + j}(-aT^{\alpha-\beta}) \\ + b \sum_{i=1}^{\ell} a_i \sum_{k=0}^{\infty} (-b)^k T^{\alpha k - \mu k + i - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + i}(-aT^{\alpha-\beta}) \\ =: f \cdot C_{21} + C_{22} + a \cdot C_{23} + b \cdot C_{24}. \end{aligned}$$

from which the unknown function f can be determined as

$$f = \frac{h_2 - C_{22} - a \cdot C_{23} - b \cdot C_{24}}{C_{21}}.$$

After substituting f into $y(t)$, we obtain

$$y(t) = \left(\frac{h_2 - C_{22} - a \cdot C_{23} - b \cdot C_{24}}{C_{21}}\right) \cdot \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + \alpha - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + \alpha}(-a t^{\alpha - \beta})$$
$$+ \sum_{r=1}^{n} c_r \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + r - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + r}(-a t^{\alpha - \beta})$$
$$+ a \sum_{j=1}^{m} b_j \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + j - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + j}(-a t^{\alpha - \beta})$$
$$+ b \sum_{i=1}^{\ell} a_i \sum_{k=0}^{\infty} (-b)^k t^{\alpha k - \mu k + i - 1} E^{k+1}_{\alpha-\beta, \alpha k - \mu k + i}(-a t^{\alpha - \beta}).$$

This completes the proof.

Conclusion

In this paper, we investigated the solvability of three-term fractional differential equations involving both Riemann–Liouville and Caputo fractional derivatives. By applying the Laplace transform method, explicit representations of the solutions were obtained in terms of generalized Mittag–Leffler functions.

Furthermore, inverse source problems associated with these equations were studied. Under suitable assumptions, explicit formulas for the unknown source term and the corresponding solution were derived. The obtained results extend several known results for two-term fractional differential equations and contribute to the theory of multi-term fractional differential equations and related inverse problems.

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