1-бөлім Раздел 1 Section 1

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CORRECT RESTRICTION OF NONLINEAR OPERATORS OF S.L. SOBOLEV

In this work, we first consider the solvability of a linear model boundary value problem for a Sobolev-type differential expression and prove the equivalence of two types of boundary value problems for it. Based on this, we investigate the solvability of nonlinear Sobolev-type differential operators that arise in the dynamics of stratified media. The analysis employs the theory of well-posed operators in Banach spaces, particularly those that can be represented as operator products. Further, two main theorems are formulated and proved: the first theorem establishes the unique solvability of a nonlinear Sobolev-type differential operator in a cylindrical domain; the second therorem generalizes the result of Theorem 1 and considers a Bitsadze–Samarsky-type problem that relates the boundary data to the values of the sought-after function on a smooth surface located inside the cylindrical domain. The study shows that the application of the theory of correct operator restrictions is effective for analyzing complex nonlinear problems and can be extended to more general geometric and physical models.
Keywords: operator, nonlinear operator, correct restriction, Bitsadze–Samarskii-type problem, Sobolev-type differential operator.

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С. Л. Соболевтің сызықты емес операторларының қисынды шектеуі

Бұл жұмыста біз алдымен Соболев типті дифференциалдық өрнек үшін сызықты модельдік шекаралық есептерінің шешілу мүмкіндігін қарастырамыз және оның екі түрлі шекаралық мәндер есебіне эквиваленттілігін дәлелдейміз. Бұған негізделе отырып, біз стратификацияланған орталардың динамикасында пайда болатын Соболев типті сызықты емес дифференциалдық операторлардың шешілу мүмкіндігін зерттейміз. Талдау Банах кеңістіктеріндегі дұрыс қойылған операторлар теориясын қолдану арқылы жүргізіледі, әсіресе операторларды көбейтінді ретінде көрсетуге болатын жағдайларда көрсетіледі. Сонымен қатар, екі негізгі теорема тұжырымдалып, дәлелденеді. Бірінші теорема цилиндрлік обылыста Соболев типті сызықты емес дифференциалдық оператордың жалғыз шешімінің бар екені дәлелденеді. Екінші теорема 1-теореманың нәтижесін жалпылайды және Бицадзе-Самарский типті есепті қарастырады, онда шекаралық шарттар цилиндрлік обылыс ішіндегі тегіс бетке орналасған ізделетін функцияның мәндерімен байланыстырылады. Зерттеу көрсеткендей, дұрыс операторлык шектеулер теориясын қолдану күрделі сызыкты емес маселелерді талдауда тиімді болып, оны геометриялық және физикалық модельдердің кеңірек түрлеріне кеңейтуге болады.
Түйін сөздер: оператор, сызықты емес оператор, дұрыс шектеу, Бицадзе-Самарский типті есептер, Соболев типті дифференциалдық операторлар.

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Корректное ограничение нелинейных операторов С. Л. Соболева

В данной работе сначала рассматривается разрешимость краевой задачи для линейной мо дели, содержащей дифференциальное выражение соболевского типа, и доказывается экви валентность двух типов краевых задач для этой модели. На основе полученных результатов исследуется разрешимость нелинейных дифференциальных операторов соболевского типа, возникающих в динамике стратифицированных сред. Анализ опирается на теорию корректно поставленных операторов в банаховых пространствах, в частности таких, которые допуска ют представление в виде произведения операторов. Далее формулируются и доказываются две основные теоремы: первая теорема устанавливает единственность решения нелинейного дифференциального оператора соболевского типа в цилиндрической области; вторая теорема обобщает результат Теоремы 1 и рассматривает задачу типа Бицадзе-Самарского, в которой граничные данные связываются со значениями искомой функции на гладкой поверхности, расположенной внутри цилиндрической области. Проведённое исследование показывает, что применение теории корректных ограничений операторов является эффективным инструмен том для анализа сложных нелинейных задач и может быть распространено на более общие геометрические и физические модели.
Ключевые слова: оператор, нелинейный оператор , корректное ограничение (оператора), задача типа Бицадзе-Самарского, дифференциальный оператор соболевского типа.

1 Introduction

The first investigations of pseudoparabolic equations are associated with the name of S. L. Sobolev [1]. In 1954, he published a paper that essentially contained such an equation in the context of solving a new problem in mathematical physics. This classical work by Sobolev was devoted to the study of small oscillations of a rotating ideal fluid. Initially, there was no specific term for this class of equations. The term "pseudoparabolic equations"appeared later, in the late 1960s. It was introduced by the mathematician T. T. Ting in [2], where he compared conventional parabolic equations with a new type of equations that include an additional time derivative in the diffusion term. In the early 1970s, a number of works laid the foundation for the theory of pseudoparabolic equations. In particular, in 1970, R. Showalter and T. Ting systematically studied this class and effectively established the term “pseudoparabolic equations” in [3]. Sobolev-type equations are evolutionary equations that involve a time derivative inside a spatial differential operator. In other words, the derivative with respect to time t appears under the differential operator, such as the Laplacian Δ. Such models describe processes with memory or inertia effects in diffusion, which arise in various areas of physics [4], [5]. A large number of works [6–19] are devoted to studying the existence and uniqueness of generalized solutions to pseudoparabolic equations. The approaches used in these studies include energy and variational methods, a priori estimates, the Galerkin method, monotone operator theory, and functional-analytic techniques. The use of cryogenic liquids in geophysics, oceanology, atmospheric physics, and technology has increased interest in the study of wave liquids, under independent conditions, and the dynamics of stratified liquids in [4, 5, 18]. This interest is driven not only by practical applications but also by the profound theoretical challenges these problems present. The problems of continuum mechanics and hydromechanics, along with the development of new mathematical directions, will become an incentive to test the effectiveness of new theories. This paper presents applications of the general theory of operator restrictions, particularly for problems related to the dynamics of stratified fluids. It is known that the hydrodynamic current function satisfies the following type of differential equation in [18]. The solvability of the initial-boundary value problem for the aforementioned differential equation has been analyzed, and the formulation of additional boundary value problems has been proposed in [18]. The critical significance of this work is further elaborated in [19].

2 Materials and methods

The Materials and Methods should be described with sufficient detail to allow others to replicate and build on the published results. Please note that the publication of your manuscript implies that you must make all materials, data, computer code, and protocols associated with the publication available to readers. Please disclose at the submission stage any restrictions on the availability of materials or information. New methods and protocols should be described in detail, while well-established methods can be briefly described and appropriately cited. Research manuscripts reporting large datasets that are deposited in a publicly available database should specify where the data have been deposited and provide the relevant accession numbers. If the accession numbers have not yet been obtained at the time of submission, please state that they will be provided during review. They must be provided prior to publication. Interventionary studies involving animals or humans, and other studies that require ethical approval, must list the authority that provided approval and the corresponding ethical approval code. In this section, where applicable, authors are required to disclose details of how generative artificial intelligence (GenAI) has been used in this paper (e.g., to generate text, data, or graphics, or to assist in study design, data collection, analysis, or interpretation). The use of GenAI for superficial text editing (e.g., grammar, spelling, punctuation, and formatting) does not need to be declared.

3 Results

The following lemma is first proved.

Lemma 1 *The initial-boundary value problem*

$$\begin{cases} \dfrac{\partial^2}{\partial z^2}\Delta u = f(x,y,z), \\ \Delta u\big|_{z=0} = 0, \\ \dfrac{\partial}{\partial z}\Delta u\Big|_{z=0} = 0, \\ u\big|_{\Gamma} = 0 \end{cases} \tag{1}$$

is equivalent to the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial^2}{\partial z^2}\Delta u = f(x,y,z), \\ u\big|_{z=0} = 0, \\ \dfrac{\partial u}{\partial z}\Big|_{z=0} = 0, \\ u\big|_{\Gamma} = 0. \end{cases} \tag{2}$$

Here $f(x,y,z) \in C(Q)$ *and*

$$Q = \left\{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2 \le 1, \ 0 \le z \le 1\right\}, \qquad \Gamma = \partial G,$$

$$G = \left\{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 \le 1\right\}.$$

Furthermore, using this lemma, we analyze the unique solvability of problem (2), as well as Bitsaadze–Samarskii-type problems for a nonlinear Sobolev-type differential operator

$$Au = \frac{\partial^2}{\partial z^2}\left(u^{2n}\Delta u + \rho u\right)$$

in the cylindrical domain Q, where the solutions are written explicitly. To prove the lemma and the corresponding theorems concerning the nonlinear operator A, we first describe uniquely solvable boundary value problems for the following model equation of Sobolev type in the cylindrical domain Q:

$$\frac{\partial^2}{\partial z^2}\Delta u = f(x,y,z).$$

Here we essentially rely on abstract theorems that make it possible to describe all possible proper restrictions of operators represented in the form of a product [20]. Let us consider the following partial differential operator

$$Au = \frac{\partial^2}{\partial z^2}\Delta u$$

in the cylindrical domain

$$Q = \left\{(x,y,z) \in \mathbb{R}^3 : \ x^2+y^2 \le 1, \ 0 \le z \le 1\right\}.$$

Define L to be the closure of the operator, initially defined on a suitable space, with respect to the norm of the space $C(Q)$. We introduce a norm on the manifold $D(L)$ and denote the resulting Banach space by M:

$$\|u\|_M = \|u\|_{C(Q)} + \|Lu\|_{C(Q)}, \qquad u \in D(L).$$

The operator M is defined as the closure of $M_0 u = \Delta u$ in the norm of the space M, where M_0 is initially defined on $C^\infty(G)$, with

$$G = \left\{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 \le 1\right\}.$$

Consequently, the operator A can be represented as $A = L \cdot M$ ([20]). Let $\widetilde{L}$ denote the restriction of the operator L to the domain

$$D(\widetilde{L}) = \left\{ u \in D(L) : \ u\big|_{z=0} = 0 \right\}.$$

Then, the unique solution of the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial z^2} = f(x,y,z), \qquad f(x,y,z) \in C(Q), \\ u\big|_{z=0} = 0, \\ \dfrac{\partial u}{\partial z}\Big|_{z=0} = 0 \end{cases}$$

can be represented in the following form:

$$u(x,y,z) = \widetilde{L}^{-1} f = \int_0^z (z-t)\, f(x,y,t)\, dt.$$

Consider the operator $\widetilde{M}$ as the restriction of M to the domain

$$D(\widetilde{M}) = \left\{ u \in D(M) : \ u\big|_{\Gamma} = 0 \right\}.$$

Then, for any $f(x,y,z) \in C(Q)$, the problem

$$\Delta u = f(x,y,z), \qquad f(x,y,z) \in C(Q), \qquad u\big|_{\Gamma} = 0,$$

is uniquely solvable, and its unique solution is given by

$$u(x,y,z) = \widetilde{M}^{-1} f = \iint\limits_G J(x,y,\xi,\eta)\, f(\xi,\eta,z)\, d\xi\, d\eta,$$

where $J(x,y,\xi,\eta)$ is the Green's function defined on a circular domain [23].

By the properties of operators expressed as products, the operator

$$\widetilde{A}^{-1} = \widetilde{M}^{-1} \widetilde{L}^{-1}$$

is invertible, and $\widetilde{A}$ defines a well-posed restriction of A. Consequently, the initial-boundary value problem (1) has a unique solution, which can be written as

$$u(x,y,z) = \widetilde{A}^{-1} f = \iint\limits_G \int_0^z (z-t)\, J(x,y,\xi,\eta)\, f(\xi,\eta,t)\, dt\, d\xi\, d\eta. \tag{3}$$

3.1 Proof of Lemma 1

Proof 1 *Given that the unique solution of problem* (1) *is represented by expression* (3)*, it is sufficient to demonstrate that this function simultaneously constitutes the unique solution of problem* (2)*.*

Consider the equation

$$\frac{\partial^2}{\partial z^2}\Delta u = f(x,y,z), \qquad f(x,y,z) \in C(Q).$$

This equation possesses a solution given by (3)*, and a direct verification confirms that this function satisfies all the conditions of problem* (2)*. To establish the uniqueness of the solution, assume that problem* (2) *admits two solutions* $u_1(x,y,z)$ *and* $u_2(x,y,z)$*. Then their difference* $u = u_1 - u_2$ *satisfies the following homogeneous problem:*

$$\begin{cases} \dfrac{\partial^2}{\partial z^2}\Delta u = 0, \\ u\big|_{\Gamma} = 0, \\ u\big|_{z=0} = 0, \\ \dfrac{\partial u}{\partial z}\Big|_{z=0} = 0. \end{cases} \tag{4}$$

We show that problem (4) *has only the trivial solution.*

In fact, as shown in [20], the solution to the homogeneous equation corresponding to problem (4) *can be represented in the form*

$$u(x,y,z) = \iint\limits_G J(x,y,\xi,\eta)\big[\varphi_1(\xi,\eta)z + \varphi_2(\xi,\eta)\big]\,d\xi\,d\eta + \psi(x,y,z),$$

where $\varphi_i(x,y) \in C^2(G)$*,* $i = 1,2$*, are arbitrary functions such that* $\varphi_1(\xi,\eta)z + \varphi_2(\xi,\eta) \in KerL$*, and* $\psi(x,y,z)$ *is a harmonic function with respect to* x *and* y *in* Q*, i.e.,* $\psi(x,y,z) \in KerM$*.*

The functions $\varphi_1(\xi,\eta)$*,* $\varphi_2(\xi,\eta)$*, and* $\psi(x,y,z)$ *must be chosen to satisfy the boundary conditions of problem* (2)*, in particular,*

$$u\big|_{\Gamma} = \psi(x,y,z)\big|_{\Gamma} = 0.$$

*Then, according to [23]*pp. 377, the condition* $\psi(x,y,z)\big|_{\Gamma} = 0$ *implies*

$$\psi(x,y,z) = 0.$$

Therefore, we obtain

$$u(x,y,z) = \iint\limits_G J(x,y,\xi,\eta)\big[\varphi_1(\xi,\eta)z + \varphi_2(\xi,\eta)\big]\,d\xi\,d\eta.$$

Setting $z = 0$*, we get*

$$u(x,y,0) = \iint\limits_G J(x,y,\xi,\eta)\,\varphi_2(\xi,\eta)\,d\xi\,d\eta.$$

Now, consider the function $v(x, y)$, which is a solution of the problem

$$\begin{cases} \Delta v = \varphi_2(x, y), \\ v\big|_\Gamma = 0. \end{cases} \tag{5}$$

It follows that

$$v(x, y) = \iint\limits_G J(x, y, \xi, \eta)\, \varphi_2(\xi, \eta)\, d\xi\, d\eta. \tag{6}$$

On the other hand, it is known that problem (5) *has a unique solution. Therefore, from equality* (6) *we deduce that*

$$\varphi_2(x, y) = 0.$$

Similarly, we obtain

$$\varphi_1(x, y) = 0.$$

Hence, problem (2) *is uniquely solvable, and its solution is expressed in the form* (3)*. The lemma is proved.*

Let l denote a smooth surface specified by the equation $z = \gamma(x, y)$ which lies within the domain Q. We assume that every line parallel to the Oz-axis intersects this surface at exactly one point, where $(x, y) \in G$ and $z \in [0, 1]$.

Lemma 2 *The inverse operator to A_1, denoted by*

$$A_1^{-1} = \widetilde{M}^{-1} L_1^{-1},$$

is defined through the following boundary value problem:

$$\begin{cases} \dfrac{\partial^2}{\partial z^2} \Delta u = f(x, y, z), \qquad f \in C(Q), \\ u\big|_\Gamma = 0, \\ u\big|_{z=0} + u\big|_{z=\gamma(x,y)} = 0, \\ \dfrac{\partial u}{\partial z}\Big|_{z=0} = 0. \end{cases} \tag{7}$$

This problem admits a unique solution, which can be expressed in the form

$$\begin{aligned} u(x, y, z) = &\iint\limits_G \int_0^z (z - t)\, J(x, y, \xi, \eta) f(\xi, \eta, t)\, dt\, d\xi\, d\eta \\ &- \frac{1}{2} \iint\limits_G \int_0^{\gamma(x,y)} (\gamma(x, y) - t)\, J(x, y, \xi, \eta) f(\xi, \eta, t)\, dt\, d\xi\, d\eta. \end{aligned} \tag{8}$$

Proof 2 *The operator L_1 is defined as the restriction of L to the domain*

$$D(L_1) = \{u \in D(L) : u\big|_{z=0} + u\big|_{z=\gamma(x,y)} = 0,\ \partial u/\partial z\big|_{z=0} = 0\}.$$

The solution to the problem

$$\frac{\partial^2 u}{\partial z^2} = f(x,y,z), \qquad f \in C(Q),$$

with the boundary conditions $u|_{z=0} + u|_{z=\gamma(x,y)} = 0$, $\partial u/\partial z|_{z=0} = 0$, can be represented as

$$u(x,y,z) = L_1^{-1} f = \int_0^z (z-t) f(x,y,t)\, dt + w_1(x,y)z + w_2(x,y).$$

By the boundary condition $\partial u/\partial z|_{z=0} = 0$, it follows that $w_1(x,y) = 0$. Now we introduce a function $w_2(x,y)$ defined to satisfy

$$u(x,y,0) = w_2(x,y), \qquad u(x,y,\gamma(x,y)) = \int_0^{\gamma(x,y)} f(x,y,t)\, dt + w_2(x,y).$$

From this, we obtain

$$w_2(x,y) = -\frac{1}{2} \int_0^{\gamma(x,y)} (\gamma(x,y) - t) f(x,y,t)\, dt.$$

Accordingly, the solution can be written as

$$u(x,y,z) = L_1^{-1} f = \int_0^z (z-t) f(x,y,t)\, dt - \frac{1}{2} \int_0^{\gamma(x,y)} (\gamma(x,y) - t) f(x,y,t)\, dt.$$

Then, by applying the theory of well-posed restrictions of operators expressed as products [20], we obtain a rigorous proof of Lemma 2.

Using the lemmas proven above, the following theorems can be easily proved.

Theorem 1 *The boundary value problem*

$$\begin{cases} Au = f(x,y,z), \\ u\big|_{\Gamma} = 0, \\ u\big|_{z=0} = 0, \\ \dfrac{\partial u}{\partial z}\Big|_{z=0} = 0 \end{cases} \tag{9}$$

is well-posed for every $f(x,y,z) \in C(Q)$, and the unique solution of problem (9) *is given by*

$$A^{-1} f = \left((2n+1) \iint\limits_G \int_0^z (z-t)\, J(x,y,\xi,\eta)\, f(\xi,\eta,t)\, dt\, d\xi\, d\eta \right)^{\frac{1}{2n+1}}. \tag{10}$$

Proof 3 *Consider the bijective operator* $N : C(Q) \to C(Q)$, *defined by* $Nu = u^{2n+1}$, *where* n *is a natural number. According to the abstract theorem proven in [22], the operator product*

$$\widetilde{A}u = \widetilde{L}\widetilde{M}\widetilde{N}$$

is well-defined and possesses a continuous inverse $\widetilde{A}^{-1}$ *on the entire space* $C(Q)$. *Thus,* $\widetilde{A}$ *constitutes a well-posed restriction of the operator* A *corresponding to the boundary value problem* (9)*. According to Lemma 1, the unique solution of this problem is given by formula* (10)*. The theorem is proved.*

Theorem 2 *Let the operator* A_1 *be defined by the boundary value problem*

$$\begin{cases} Au = f(x, y, z), \\ u\big|_{\Gamma} = 0, \\ u\big|_{z=0} + u\big|_{z=\gamma(x,y)} = 0, \\ \dfrac{\partial u}{\partial z}\Big|_{z=0} = 0 \end{cases} \tag{11}$$

where l *is a smooth surface given by the equation* $z = \gamma(x, y)$, *located within the domain* Q, *such that every line parallel to the* Oz*-axis intersects this surface at most at one point, with* $(x, y) \in G$ *and* $z \in [0, 1]$.

Then, there exists a bounded inverse operator A_1^{-1} *defined on the entire space* $C(Q)$, *and the unique solution of problem* (11) *is given by*

$$A_1^{-1}f = \Big((2n + 1)Hf\Big)^{\frac{1}{2n+1}}, \tag{12}$$

where

$$\begin{aligned} Hf = &\iint\limits_G \int_0^z (z - t)\, J(x, y, \xi, \eta)\, f(\xi, \eta, t)\, dt\, d\xi\, d\eta \\ &- \frac{1}{2}\iint\limits_G \int_0^{\gamma(x,y)} (\gamma(x, y) - t)\, J(x, y, \xi, \eta)\, f(\xi, \eta, t)\, dt\, d\xi\, d\eta. \end{aligned} \tag{13}$$

Proof 4 *Following the previous construction, consider the bijective mapping* $N : C(Q) \to C(Q)$, *defined by* $Nu = u^{2n+1}$, *where* n *is a natural number. Then, the operator*

$$A_1u = L_1\widetilde{M}N$$

is well-defined and possesses a continuous inverse A_1^{-1} *on the entire space* $C(Q)$. *Thus,* A_1 *constitutes a well-posed restriction of* A *corresponding to the boundary value problem* (11), *whose unique solution, according to Lemma 2, is given by formula* (12).

Here, instead of the condition

$$u\big|_{z=0} + u\big|_{z=\gamma(x,y)} = 0,$$

the following condition should be imposed:

$$u^{2n+1}\big|_{z=0} + u^{2n+1}\big|_{z=\gamma(x,y)} = 0.$$

This condition can be readily reduced to the previous form by a simple factorization method. Hence, the theorem is proved [20].

Conclusion

The approach presented in this work, which is grounded in the theory of well-posed reductions, substantially broadens the range of potential applications. In particular, it provides an efficient approach for the development of explicitly solvable models in physics and engineering. The significance of explicitly solvable models in physics is widely acknowledged, and interest in them has been steadily increasing in recent years. Notably, operator theory, and in particular the theory of well-posed reductions, offers substantial potential for advancing such modeling [20, 22, 23]. Among the mathematical problems related to this approach, it is important to note its successful use in the well-posed formulation of boundary value problems, problems of the Bitsadze–Samarskii type, various multipoint problems for nonlinear equations of mathematical physics, and also for loaded differential equations [20, 23]. In the present work, we prove the solvability of nonlinear boundary value problems involving Sobolev-type differential operators. The approach is based on the abstract theory of well-posed operator restrictions in Banach spaces, particularly for operators represented as products. The main results are obtained by combining this theory with the properties of Green's functions in planar domains and the structural characteristics of Sobolev-type operators. An auxiliary lemma is also proved, which confirms the equivalence of the problem formulations and ensures uniqueness [20, 23]. Overall, this study demonstrates the applicability of the general theory of well-posed operator restrictions to complex nonlinear problems in mathematical physics. Thus, the proposed methods not only provide a solid foundation for addressing such problems but also open avenues for future research involving more general geometries, coupled systems, and nonlocal boundary interactions.

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Conflicts of Interest

The authors declare no conflict of interest.

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