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INVERSE PROBLEMS FOR A DOUBLY DEGENERATE THIRD-ORDER DIFFERENTIAL EQUATION WITH MULTIPLE CHARACTERISTICS

The paper is devoted to the study of solvability in anisotropic Sobolev spaces of a class of new nonlinear inverse problems with unknown coefficients for third-order differential equations with multiple characteristics. The equations under consideration may contain coefficients that vanish or change sign, which causes degeneracy and may lead to a change in the direction of evolution of the process. Such features significantly increase the analytical complexity of the problems and require a careful functional framework. Within this setting, we investigate both the existence and uniqueness of solutions to the corresponding inverse problems. The obtained results are formulated for regular solutions, understood as functions possessing all Sobolev generalized derivatives that appear in the given differential equations. The analysis is based on techniques of functional analysis and the theory of anisotropic Sobolev spaces, allowing us to handle the mixed-type and degenerate nature of the studied models effectively.
Keywords: inverse problems, third-order differential equations, multiple characteristics, degeneration, unknown coefficient, regular solutions, existence and uniqueness.

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Екі еселенген деградациясы бар және көп сипаттамалы үшінші ретті дифференциалдық теңдеу үшін кері есептер

Бұл мақала көп сипаттамалы үшінші ретті дифференциалдық теңдеулер үшін белгісіз коэффициенттері бар жаңа сызықтық емес кері есептер класын анизотропты Соболев кеңістіктерінде шешілімділігін зерттеуге арналған. Қарастырылып отырған теңдеулерде коэффициенттер нөлге айналуы немесе таңбасын өзгертуі мүмкін, бұл деградацияға әкеліп, процестің эволюция бағытының өзгеруіне себеп болуы ықтимал. Мұндай ерекшеліктер есептердің аналитикалық күрделілігін едәуір арттырып, мұқият таңдалған функционалдық базаны қажет етеді. Осы шеңберде сәйкес кері есептердің шешімдерінің бар болуы мен жалғыздығы зерттеледі. Алынған нәтижелер регуляр шешімдер үшін тұжырымдалады, мұнда регуляр шешім деп берілген дифференциалдық теңдеуге кіретін барлық Соболев бойынша жалпыланған туындылары бар функциялар түсініледі. Зерттеу функционалдық талдау әдістеріне және анизотропты Соболев кеңістіктер теориясына негізделген, бұл қарастырылған модельдердің аралас типті және деградацияланған табиғатын тиімді түрде талдауға мүмкіндік береді.
Түйін сөздер: кері есептер, үшінші ретті дифференциалдық теңдеулер, көп сипаттамалар, деградация, белгісіз коэффициент, регуляр шешімдер, бар болуы және жалғыздығы.

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Обратные задачи для вырожденного дифференциального уравнения третьего порядка с двойным вырождением и кратными характеристиками

Статья посвящена исследованию разрешимости в анизотропных пространствах Соболева нового класса нелинейных обратных задач с неизвестными коэффициентами для дифференциальных уравнений третьего порядка с кратными характеристиками. Рассматриваемые уравнения могут содержать коэффициенты, которые обращаются в нуль или меняют знак, что приводит к вырождению и может вызывать изменение направления эволюции процесса. Такие особенности существенно усложняют аналитическую структуру задач и требуют аккуратного функционального подхода. В рамках данной постановки исследуются существование и единственность решений соответствующих обратных задач. Полученные результаты формулируются для регулярных решений, понимаемых как функции, обладающие всеми обобщёнными производными Соболева, входящими в рассматриваемые дифференциальные уравнения. Анализ основан на методах функционального анализа и теории анизотропных пространств Соболева, что позволяет эффективно работать со смешанным типом и вырожденной природой изучаемых моделей.
Ключевые слова: обратные задачи, дифференциальные уравнения третьего порядка, уравнения с кратными характеристиками, вырождение, неизвестный коэффициент, регулярные решения, существование и единственность.

1 Introduction

Third-order differential equations with multiple characteristics are of considerable interest both from the viewpoint of the general theory of nonclassical differential equations and due to their applications. Cases where the coefficients of the equation may vanish or change sign are especially difficult, since this leads to degeneration of the equation and to a change in the direction of evolution.

Boundary value problems for degenerate differential equations with multiple characteristics were studied in the works of A.I. Kozhanov, G.A. Lukina and other authors. In particular, direct problems for third-order equations with multiple characteristics were investigated by A.I. Kozhanov and O.S. Zikirov [1] in the cases where the coefficient of the time derivative may change sign, whereas the coefficient of the highest-order spatial derivative is assumed to be nonnegative.

Unlike the above-mentioned works, coefficient inverse problems are considered in the present paper. The unknown is the function $q(t)$, which enters the equation as a coefficient of a given function $h(x,t)$, while additional information on the solution is prescribed by an overdetermination condition of integral type.

The purpose of this work is to prove existence and uniqueness theorems for regular solutions of the inverse problems under consideration in anisotropic Sobolev spaces.

2 Statement of the Problems

Let $\Omega = (0,1)$, $Q = \Omega \times (0,T)$, $0 < T < +\infty$. Further, let $\varphi(t)$, $\psi(t)$, $c(x,t)$, $f(x,t)$, $h(x,t)$, and $N(x)$ be given functions for $x \in [0,1]$, $t \in [0,T]$.

Inverse Problem I: find the functions $u(x,t)$ and $q(t)$ that satisfy the equation

$$\varphi(t)u_t - \psi(t)u_{xxx} + c(x,t)u = f(x,t) + q(t)h(x,t), \qquad (x,t) \in Q, \tag{1}$$

supplemented with boundary conditions

$$u(0,t) = u_x(0,t) = 0, \qquad t \in (0,T), \tag{2}$$

$$u(1,t) = 0, \qquad t \in (0,T), \tag{3}$$

and the overdetermination condition

$$\int_\Omega N(x)u(x,t)\,dx = 0. \tag{4}$$

Inverse Problem II: Find the functions $u(x,t)$ and $q(t)$ satisfying equation (1) in Q, supplemented with conditions (2), (3),

$$u(x,0) = 0, \qquad x \in \Omega, \tag{5}$$

and the overdetermination condition (4).

Inverse Problem III: Find functions $u(x,t)$ and $q(t)$ satisfying equation (1) in Q, supplemented with conditions (2)–(5) and

$$u(x,T) = 0, \qquad x \in \Omega.$$

Definition 1 *A pair of functions $\{u(x,t), q(t)\}$ is called a regular solution of Inverse Problem I if*

$$u(x,t) \in W_2^{3,1}(Q), \qquad q(t) \in L_2(0,T),$$

equation (1) is satisfied almost everywhere in Q, and conditions (2)–(4) are fulfilled.

Regular solutions of Inverse Problems II and III are defined analogously.

3 Solvability of Inverse Problem I

Throughout what follows, we assume that

$$\beta(t) = \int_\Omega N(x)h(x,t)\,dx \neq 0, \qquad t \in [0,T]. \tag{6}$$

Since the functions $N(x)$ and $h(x,t)$ are prescribed, the function

$$\beta(t) = \int_\Omega N(x)h(x,t)\,dx$$

is known a priori. Since $h(x,t) \in C(\overline{Q})$, it follows that

$$\beta \in C[0,T].$$

Set

$$f_1(x,t) = f(x,t) - \frac{h(x,t)}{\beta(t)} \int_\Omega N(x) f(x,t)\, dx.$$

Furthermore, for a given function $v(x,t)$, define

$$A(t,v) = \int_\Omega \big[\psi(t) N'''(x) + c(x,t) N(x)\big] v(x,t)\, dx.$$

Consider the boundary value problem of finding a function $u(x,t)$ satisfying the equation

$$\varphi(t) u_t - \psi(t) u_{xxx} + c(x,t) u = f_1(x,t) + \frac{h(x,t)}{\beta(t)} A(t,u), \qquad (x,t) \in Q, \tag{7}$$

and conditions (2)-(3).

Equation (7) is obtained from equation (1) and the overdetermination condition (4) by eliminating the unknown coefficient $q(t)$. Therefore, the study of Inverse Problem I can be reduced to the investigation of boundary value problem (7), (2), (3).

In this problem, equation (7) is a loaded integro-differential equation. Solvability of problem (7), (2), (3) in $W_2^{3,1}(Q)$ will be established by means of the regularization method and the continuation with respect to a parameter method.

Let ε be a positive number and let $\lambda \in [0,1]$. Consider the family of boundary value problems: find a function $u(x,t)$ satisfying in Q the equation

$$-\varepsilon u_{xxxxxx} - \varepsilon u_{tt} + \varphi(t) u_t - \psi(t) u_{xxx} + c(x,t) u = f_1(x,t) + \lambda \frac{h(x,t)}{\beta(t)} A(t,u) \tag{8}$$

and conditions (2), (3), together with

$$u_t(x,0) = u_t(x,T) = 0, \qquad x \in \Omega, \tag{9}$$

$$u_{xxx}(0,t) = u_{xxx}(1,t) = u_{xxxx}(1,t) = 0, \qquad t \in (0,T). \tag{10}$$

Introduce the notation

$$c_1 = \min_{\overline{Q}} \left[c(x,t) - \frac{1}{2} \varphi'(t) \right],$$

$$R_1 = \max_{\overline{Q}} \left\{ \frac{h^2(x,t)}{\beta^2(t)} \int_\Omega [c(y,t) N(y) + \psi(t) N'''(y)]^2 dy \right\}.$$

Lemma 1 *Assume that condition (6) holds and*

$$\varphi(t) \in C^1([0,T]), \qquad \varphi(0) \le 0, \qquad \varphi(T) \ge 0; \tag{11}$$

$$\psi(t) \in C([0,T]), \qquad \psi(t) > 0 \quad \text{for } t \in (0,T]; \tag{12}$$

$$c(x,t) \in C(\overline{Q}), \qquad h(x,t) \in C(\overline{Q}), \qquad N(x) \in W_2^3(\Omega); \tag{13}$$

$$c_1 > R_1. \tag{14}$$

Then every solution $u(x,t)$ of boundary value problem (8), (2), (3), (9), (10) belonging to the space $W_2^{6,2}(Q)$ satisfies the estimate

$$\varepsilon \int_Q (u_{xxx}^2 + u_t^2)\,dxdt + \int_Q u^2\,dxdt \le M_1 \int_Q f^2\,dxdt, \tag{15}$$

where the constant M_1 depends only on the functions $\varphi(t)$, $\psi(t)$, $c(x,t)$, $h(x,t)$, and $N(x)$.

Proof 1 *The proof is based on the analysis of the identity*

$$\int_Q \{-\varepsilon u_{xxxxxx} - \varepsilon u_{tt} + \varphi(t)u_t - \psi(t)u_{xxx} + c(x,t)u\}u\,dxdt$$
$$= \int_Q \left\{ f_1(x,t) + \lambda \frac{h(x,t)}{\beta(t)} A(t,u) \right\} u\,dxdt,$$

together with the boundary conditions, Hölder's inequality, and assumptions (11)–(14).

Lemma 2 *Assume that conditions (6) and (11)–(14) hold and that*

$$\frac{\partial^k c(x,t)}{\partial x^k} \in C(\overline{Q}), \qquad \frac{\partial^k h(x,t)}{\partial x^k} \in C(\overline{Q}), \qquad k = 0,1,2,3; \tag{16}$$

$$h(0,t) = h(1,t) = h_x(0,t) = 0, \qquad t \in [0,T]. \tag{17}$$

In addition, suppose that

$$\frac{\partial^k f(x,t)}{\partial x^k} \in L_2(Q), \qquad k = 0,1,2,3,$$

and

$$f(0,t) = f(1,t) = f_x(0,t) = 0, \qquad t \in [0,T].$$

Then every solution $u(x,t)$ of problem (8), (2), (3), (9), (10) belonging to $W_2^{6,2}(Q)$ satisfies the estimate

$$\varepsilon \int_Q (u_{xxxxxx}^2 + u_{xxxt}^2)\,dxdt + \int_Q u_{xxx}^2\,dxdt \le M_2 \int_Q (f^2 + f_{xxx}^2)\,dxdt, \tag{18}$$

where the constant M_2 depends only on the functions $\varphi(t)$, $\psi(t)$, $c(x,t)$, $h(x,t)$, and $N(x)$.

Proof 2 *Consider the identity*

$$-\int_Q \{-\varepsilon u_{xxxxxx} - \varepsilon u_{tt} + \varphi(t)u_t - \psi(t)u_{xxx} + c(x,t)u\} u_{xxxxxx}\, dxdt$$
$$= -\int_Q \left\{ f_1 + \lambda \frac{h(x,t)}{\beta(t)} A(t,u) \right\} u_{xxxxxx}\, dxdt. \tag{19}$$

After integrating by parts three times with respect to x in all terms of the above identity except the first term on the left-hand side, and integrating by parts with respect to t in the terms involving u_{tt} and u_t, we use assumptions (14), (16), and (17), Hölder's inequality, and estimate (15) to derive (18).

Define

$$c_2 = \min_{\overline{Q}} \left[c(x,t) + \frac{1}{2}\varphi'(t) \right].$$

Lemma 3 *Suppose that conditions (6), (13), (14), (16), and (17) hold and*

$$\psi(t) \in C^1([0,T]), \qquad c_t(x,t) \in C(\overline{Q}), \qquad h_t(x,t) \in C(\overline{Q}); \tag{20}$$

$$c_2 > R_1. \tag{21}$$

In addition, if

$$\frac{\partial^k f(x,t)}{\partial x^k} \in L_2(Q), \qquad k = 0,1,2,3,$$

$f_t(x,t) \in L_2(Q)$, *and*

$$f(0,t) = f(1,t) = f_x(0,t) = 0, \qquad t \in [0,T],$$

then every solution $u(x,t)$ of problem (8), (2), (3), (9), (10) belonging to $W_2^{6,2}(Q)$ satisfies

$$\varepsilon \int_Q u_{tt}^2\, dxdt + \int_Q u_t^2\, dxdt \le M_3 \int_Q (f^2 + f_t^2 + f_{xxx}^2)\, dxdt, \tag{22}$$

where the constant M_3 depends only on the functions $\varphi(t)$, $\psi(t)$, $c(x,t)$, $h(x,t)$, and $N(x)$.

Proof 3 *The proof follows from the following identity*

$$-\int_Q \{-\varepsilon u_{xxxxxx} - \varepsilon u_{tt} + \varphi(t)u_t - \psi(t)u_{xxx} + c(x,t)u\} u_{tt}\, dxdt$$

$$= -\int_Q \left\{ f_1(x,t) + \lambda \frac{h(x,t)}{\beta(t)} A(t,u) \right\} u_{tt}\, dxdt.$$

Integrating by parts with respect to both variables x and t, using the assumptions of the lemma, applying Hölder's inequality and taking into account estimate (15), one obtains estimate (22).

The estimates established in Lemmas 1–3 are sufficient for proving solvability of boundary value problem (7), (2), (3), and subsequently of Inverse Problem I.

Theorem 1 *Suppose that conditions (6), (11)–(14), (16), (17), (20), and (21) are satisfied. Then, if* $f(x,t) \in W_2^{3,1}(Q)$ *and*

$$f(0,t) = f(1,t) = f_x(0,t) = 0, \qquad t \in [0,T],$$

boundary value problem (7), (2), (3) admits a unique solution $u(x,t) \in W_2^{3,1}(Q)$.

Proof 4 *For* $\lambda = 0$ *and fixed* ε*, boundary value problem (8), (2), (3), (9), (10) is solvable in* $W_2^{3,1}(Q)$ *for every function* $f \in L_2(Q)$*. This fact can be established by the classical Galerkin method with a specially chosen basis [4], [5] and a priori estimates (15), (18), (22), whose right-hand side has the form const* $\|f\|^2_{L_2(Q)}$*. Such estimates obtained by analyzing the identities used in the proofs of Lemmas 1–3; in this case no integration by parts in the terms containing* $f(x,t)$ *is required.*

Further, similar estimates for fixed ε *hold for all* $\lambda \in [0,1]$*. From these estimates, the solvability of problem (8), (2), (3), (9), (10) for* $\lambda = 0$*, and the continuation with respect to a parameter theorem [6, Ch. III, §14], it follows that this problem is solvable in* $W_2^{6,2}(Q)$ *for* $f(x,t) \in L_2(Q)$ *for all* $\lambda \in [0,1]$*, in particular for* $\lambda = 1$*.*

For solutions $u_\varepsilon(x,t)$ *of problem (8), (2), (3), (9), (10) corresponding to* $\lambda = 1$*, the a priori estimates (15), (18), (22) are uniform with respect to* ε*. From these estimates, the reflexivity of Hilbert spaces and compactness of the embedding* $W_2^1(Q) \subset L_2(Q)$ *(see [5], [7], [8]), there exist sequences* $\{\varepsilon_m\}_{m=1}^{\infty}$ *of positive numbers and* $\{u_m(x,t)\}_{m=1}^{\infty}$ *of solutions of problem (8), (2), (3), (9), (10) for* $\lambda = 1$ *and* $\varepsilon = \varepsilon_m$ *such that, as* $m \to \infty$*,*

$$\varepsilon_m \to 0,$$

$$\varepsilon_m u_{mxxxxxx}(x,t) \to 0 \quad \text{weakly in } L_2(Q),$$

$$\varepsilon_m u_{mtt}(x,t) \to 0 \quad \text{weakly in } L_2(Q),$$

$$u_m(x,t) \to u(x,t) \quad \text{weakly in } W_2^{3,1}(Q),$$

$$u_m(x,t) \to u(x,t) \quad \text{almost everywhere in } \overline{Q}.$$

These convergences imply that the limit function $u(x,t)$ *is a solution of problem (7), (2), (3) belonging to* $W_2^{3,1}(Q)$*.*

Uniqueness of solutions of problem (7), (2), (3) in $W_2^{3,1}(Q)$ *follows, for example, from estimate (15), which remains valid for* $\varepsilon = 0$*.*

The solvability theorem established above for the loaded degenerate integro-differential equation with multiple characteristics is of independent interest. At the same time, after a slight modification it can be used to prove solvability of Inverse Problem I.

In what follows, we assume that

$$c(x,t) = c_0(t) + c_1(x,t), \qquad (x,t) \in \overline{Q}. \tag{23}$$

Set

$$\widetilde{R}_1 = \max_{\overline{Q}} \left\{ \frac{h^2(x,t)}{\beta^2(t)} \int_\Omega [c_1(y,t)N(y) - \psi(t)N'''(y)]^2 dy \right\}.$$

Theorem 2 *Suppose that conditions (6), (11)–(14), (16), (17), (20), (21), and (23) hold and, in addition,*

$$\min(c_1, c_2) > \widetilde{R}_1; \tag{24}$$

$$N(0) = N(1) = N'(0) = 0; \tag{25}$$

$$c_0(t) - \frac{1}{2}\varphi'(t) > 0, \qquad t \in [0, T]. \tag{26}$$

Then, if $f(x,t) \in W_2^{3,1}(Q)$ and $f(0,t) = f(1,t) = f_x(0,t) = 0$ for $t \in [0,T]$, Inverse Problem I admits a unique solution $\{u(x,t), q(t)\}$ such that

$$u(x,t) \in W_2^{3,1}(Q), \qquad q(t) \in L_2([0,T]).$$

Proof 5 *For a given function $v(x,t)$, define*

$$A_1(t,v) = \int_\Omega [\psi(t)N'''(x) + c_1(x,t)N(x)]v(x,t)\,dx.$$

Consider the boundary value problem of finding a function $u(x,t)$ satisfying in Q the equation

$$\varphi(t)u_t - \psi(t)u_{xxx} + c(x,t)u = f_1(x,t) + \frac{h(x,t)}{\beta(t)}A_1(t,u)$$

and conditions (2), (3). Repeating the proof of Theorem 1, we obtain that this problem has a solution $u(x,t) \in W_2^{3,1}(Q)$. Define

$$q(t) = \frac{1}{\beta(t)}\left[A_1(t,u) - \int_\Omega N(x)f(x,t)\,dx\right].$$

Then

$$q(t) = \frac{1}{\beta(t)}\left\{\int_\Omega [c_1(x,t)N(x) + \psi(t)N'''(x)]u(x,t)\,dx - \int_\Omega N(x)f(x,t)\,dx\right\}.$$

It is clear that $q(t) \in L_2([0,T])$ and that the functions $u(x,t)$ and $q(t)$ satisfy equation (1) in Q.

Multiplying equation (1) by $N(x)$ and integrating over Ω, after straightforward transformations, we obtain

$$\varphi(t)\left(\int_\Omega N(x)u(x,t)\,dx\right)_t + c_0(t)\int_\Omega N(x)u(x,t)\,dx = 0.$$

From this equality and condition (26) it follows that

$$\int_\Omega N(x)u(x,t)\,dx = 0, \qquad t \in (0,T).$$

Thus condition (4) is satisfied, and the pair $\{u(x,t), q(t)\}$ is the required solution of Inverse Problem I.

Uniqueness of solutions of Inverse Problem I in $W_2^{3,1}(Q) \times L_2([0,T])$ is obvious.

4 Solvability of Inverse Problems II and III

The proofs of solvability for Inverse Problems II and III, as well as the corresponding auxiliary problems, are carried out similarly to the proofs of Theorems 1 and 2. The differences concern only the assumptions imposed on the functions $\varphi(t)$, $h(x,t)$, and $f(x,t)$. Therefore, omitting the detailed derivations, we present only the final results.

Theorem 3 *Suppose that conditions (12)–(14), (16), (17), (20), and (21) hold and, in addition,*

$$\varphi(0) > 0, \qquad \varphi(T) \geq 0; \tag{27}$$

$$h(x,0) = 0, \qquad x \in \overline{\Omega}. \tag{28}$$

Then, if $f(x,t) \in W_2^{3,1}(Q)$, $f(0,t) = f(1,t) = f_x(0,t) = 0$ *for* $t \in [0,T]$, *and* $f(x,0) = 0$ *for* $x \in \Omega$, *the boundary value problem*

$$\varphi(t)u_t - \psi(t)u_{xxx} + c(x,t)u = f_1(x,t) + \frac{h(x,t)}{\beta(t)}A(t,u), \tag{29}$$

$$u(0,t) = u(1,t) = u_x(1,t) = 0, \qquad t \in (0,T), \tag{30}$$

$$u(x,0) = 0, \qquad x \in \Omega, \tag{31}$$

admits a unique solution $u(x,t) \in W_2^{3,1}(Q)$.

Theorem 4 *Suppose that conditions (6), (12)–(14), (16), (17), (20), (21), and (23)–(28) hold. Then, if* $f(x,t) \in W_2^{3,1}(Q)$, $f(0,t) = f(1,t) = f_x(0,t) = 0$ *for* $t \in [0,T]$, *and* $f(x,0) = 0$ *for* $x \in \overline{\Omega}$, *Inverse Problem II admits a unique solution* $\{u(x,t), q(t)\}$ *such that*

$$u(x,t) \in W_2^{3,1}(Q), \qquad q(t) \in L_2([0,T]).$$

Theorem 5 *Suppose that conditions (12)–(14), (16), (17), (20), and (21) hold and, in addition,*

$$\varphi(0) > 0, \qquad \varphi(T) < 0; \tag{32}$$

$$h(x,0) = h(x,T) = 0, \qquad x \in \overline{\Omega}. \tag{33}$$

Then, if $f(x,t) \in W_2^{3,1}(Q)$, $f(0,t) = f(1,t) = f_x(0,t) = 0$ *for* $t \in [0,T]$, *and* $f(x,0) = f(x,T) = 0$ *for* $x \in \overline{\Omega}$, *the boundary value problem for equation (29) with conditions (30), (31), and the additional condition*

$$u(x,T) = 0, \qquad x \in \Omega,$$

admits a unique solution $u(x,t) \in W_2^{3,1}(Q)$.

Theorem 6 *Suppose that conditions (6), (12)–(14), (16), (17), (20), (21), (23)–(26), (32), and (33) hold. Moreover, if* $f(x,t) \in W_2^{3,1}(Q)$*,* $f(0,t) = f(1,t) = f_x(0,t) = 0$ *for* $t \in [0,T]$*, and* $f(x,0) = f(x,T) = 0$ *for* $x \in \overline{\Omega}$*, then Inverse Problem III admits a unique solution* $\{u(x,t), q(t)\}$ *such that*

$$u(x,t) \in W_2^{3,1}(Q), \qquad q(t) \in L_2([0,T]).$$

The proofs of Theorems 3 and 5, corresponding to the auxiliary problems, are based on the regularization of equation (29) by equation (8) with condition (10), and additionally with the condition

$$u(x,0) = u_t(x,T) = 0, \qquad x \in \Omega,$$

in the proof of Theorem 3, and with the condition

$$u(x,0) = u(x,T) = 0, \qquad x \in \Omega,$$

in the proof of Theorem 5. The proofs use a priori estimates and the continuation with respect to a parameter theorem. The estimates themselves are derived by analyzing the identities used in the proofs of Lemmas 1–3.

Theorems 4 and 6 are proved in the same way as Theorem 2.

Thus, solvability of the inverse problems is established.

5 Comments and Further Remarks

New results on solvability of linear inverse problems for differential equations with an arbitrarily varying direction of evolution have been obtained. Indeed, the function $\varphi(t)$ may change sign an arbitrary number of times on the interval $[0,T]$ and may vanish on a set of positive measure. These results admit further development and may be strengthened and generalized. Some possible extensions are listed below.

1. Similar methods make it possible to obtain solvability results for inverse problems of finding functions $u(x,t)$ and $q(t)$ connected in the rectangle Q by the equation

$$\varphi(t)u_t + (-1)^p \psi(t) \frac{\partial^{2p+1} u}{\partial x^{2p+1}} + c(x,t)u = f(x,t) + q(t)h(x,t), \tag{34}$$

where $p \geq 1$ is an integer, and the functions $\varphi(t)$ and $\psi(t)$ have the same properties as in Inverse Problems I–III. Moreover, both equation (1) and equation (34) may include all lower-order derivatives with respect to x with variable coefficients. The solvability conditions become rather cumbersome, but the essence of the results remains unchanged.

2. Boundary conditions in inverse problems for equation (1) or equation (34) may be modified; for example, one may study problems with boundary conditions similar to those considered in [1].

3. In Inverse Problems I–III, the function $\psi(t)$ may vanish only at $t = 0$. Obviously, a more general situation may be admitted, namely when $\psi(t)$ vanishes at any finite number of points of the interval $[0,T]$.

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