Бөлім 2 **Раздел 2** **Section 2**

Механика **Механика** **Mechanics**

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DIFFERENTIAL MOTION EQUATIONS OF THE VARIABLE-MASS THREE-BODY PROBLEM WITH REACTIVE FORCES IN A BARYCENTRIC COORDINATE SYSTEM

The present study addresses the three-body problem (point masses) undergoing non-isotropic mass variation while incorporating the influence of reactive forces within different coordinate systems. The primary contribution of this research is the formulation, first obtained, of the governing differential equations of motion for three-body problem with variable mass in the barycentric coordinate system. In the barycentric coordinate system, integrals of linear momentum were obtained, which, following G. N. Duboshin, are referred to as the invariants of the center-of-mass. These invariants constitute the only analytical relationships that remain valid for arbitrary laws of mass variation and provide a direct connection between the coordinates and velocities of the interacting bodies. The proposed mathematical model extends the theoretical framework of celestial mechanics. Consequently, the derived equations and invariants provide a theoretical basis for qualitative investigations, analytical studies, and high-precision numerical simulations of variable-mass three-body systems evolution. Furthermore, the developed approach offers significant potential for investigation of dynamical evolution of many body systems. That is, it makes it possible to derive differential equations describing systems of many bodies with a non-isotropic mass variation and accounting for reactive forces in the barycentric coordinate system. In particular, it can be used to study the long-term dynamic evolution, orbital stability of exoplanet systems subject to arbitrary mass-variation laws and reactive-force effects. Overall, the obtained results provide a comprehensive mathematical foundation for advancing the study of variable-mass dynamical systems and broaden the range of opportunities of modern celestial mechanics.

Keywords: three-body problem, variable mass, reactive force, barycentric coordinate system, the invariants of center of mass.

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Барицентрлік координаталар жүйесінде реактивті күштер бар жағдайда айнымалы массалы үш дене мәселесінің дифференциалдық қозғалыс теңдеулері

Бұл зерттеу жұмысы массалары изотропты емес түрде өзгеретін үш дене (материялық нүктелер) мәселесін әртүрлі координаталар жүйелеріндегі реактивті күштердің әсерін ескере отырып қарастыруға арналған. Зерттеудің негізгі ғылыми нәтижесі – алғаш рет барицентрлік координаталар жүйесінде айнымалы массалары бар үш дене есебінің қозғалысын сипаттайтын негізгі дифференциалдық теңдеулер қорытылып шығарылды. Барицентрлік координаталар жүйесінде қозғалыс мөлшерінің интегралдары алынды. Олар Г. Н. Дубошиннің терминологиясына сәйкес массалар центрінің инварианттары деп аталады. Аталған инварианттар массалардың кез келген өзгеру заңдары кезінде өз күшін сақтайды және өзара әрекеттесетін денелердің координаталары мен жылдамдықтарын байланыстыратын бірегей аналитикалық қатынастар болып табылады. Ұсынылған математикалық модель аспан механикасының теориялық негіздерін кеңейтеді.

Осыған байланысты алынған қозғалыс теңдеулері мен массалар центрінің инварианттары айнымалы массалы үш денелі жүйелердің эволюциясын сапалық, аналитикалық және жоғары дәлдікті сандық әдістермен зерттеу үшін сенімді теориялық негіз қалайды. Сонымен қатар, әзірленген әдіс айнымалы массалы көп денелі жүйелердің динамикалық эволюциясын зерттеуге негіз бола алады. Яғни, барицентрлік координаталар жүйесінде массалардың изотропты емес өзгеруін және реактивті күштердің әсерін ескере отырып, көп денелі жүйелердің қозғалысын сипаттайтын дифференциалдық теңдеулерді алуға мүмкіндік береді. Атап айтқанда, ол массаның кез-келген өзгеру заңдылығы үшін және реактивті күштердің әсерін ескерген жағдайда, экзопланеталық жүйелердің ұзақ мерзімді динамикалық эволюциясын, орбиталық орнықтылығын зерттеу үшін қолданыла алады. Жалпы алғанда, алынған нәтижелер айнымалы массалы динамикалық жүйелерді зерттеуді одан әрі дамытуға арналған кешенді математикалық негіз қалыптастырып, қазіргі заманғы аспан механикасының мүмкіндігін арттырады.
Түйін сөздер: үш дене мәселесі, айнымалы масса, реактивті күш, барицентрлік координаталар жүйесі, массалар центрінің инварианттары.

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Дифференциальные уравнения движения задачи трех тел с переменными массами с учетом реактивных сил в барицентрической системе координат

Настоящее исследование посвящено задаче трёх тел (точек), изменяющихся неизотропным образом, с учётом влияния реактивных сил в различных системах координат. Основным научным результатом работы является впервые полученная формулировка дифференциальных уравнений движения задачи трёх тел с переменными массами в барицентрической системе координат. В барицентрической системе координат получены интегралы количества движения, которых следуя Г. Н. Дубошина названы инвариантами центра масс. Эти инварианты представляют собой единственные аналитические соотношения, сохраняющие справедливость при произвольных законах изменения масс и непосредственно связывающие координаты и скорости взаимодействующих тел. Предложенная математическая модель расширяет теоретические основы небесной механики. Полученные дифференциальные уравнения движения и инварианты центра масс создают теоретическую основу для проведения качественного анализа, аналитических исследований и высокоточного численного моделирования эволюции систем трех тел с переменными массами. Кроме того, разработанный подход обладает значительным потенциалом для исследования динамической эволюции многотельных систем. В частности, он позволяет вывести дифференциальные уравнения движения, описывающие системы, состоящие из большого числа взаимодействующих тел с неизотропным изменением масс, с учётом действия реактивных сил в барицентрической системе координат. Предложенная математическая модель также может быть использована для исследования долговременной динамической эволюции и орбитальной устойчивости экзопланетных систем при произвольных законах изменения масс и воздействии реактивных сил. В целом полученные результаты формируют целостную математическую основу для дальнейшего развития теории динамических систем с переменной массой и существенно расширяют возможности современной небесной механики при исследовании сложных гравитационных систем.
Ключевые слова: задача трех тел, переменная масса, реактивная сила, барицентрическая система координат, инварианты центра масс.

1 Introduction

The importance of the classical Newtonian three-point-body problem in the celestial mechanics of sciences cannot be overestimated. The problem has ten classical integrals, however, this problem still does not have a practically useful general solution [1, 2]. The five exact particular solutions established by Euler (1767) and Lagrange (1772) have many applications in astronomy, space engineering and technology [3–5]. The relatively recently established particular eight-shaped solution has its own specifics [2, 6, 7].

Observational astronomical data indicate that the masses of celestial bodies vary throughout their evolutionary processes. Consequently, the investigation of different aspects of the variable-mass three-body problem has become an important and relevant area of research [8–11]. The differential equations of motion describing the variable-mass three-point-body problem are non-autonomous system and mathematically highly complex. They neither possess a Laplace plane nor admit a first integral in the general case [11]. The three-body problem with variable masses, assuming isotropic mass variation at different rates for each body, was examined in [12]. The problem involving bodies whose masses vary anisotropically and at different rates was investigated in Jacobi coordinates in [13].

This study examines the three-body problem for bodies with non-isotropically varying masses while accounting for the effects of reactive forces in different coordinate systems.

The primary objective of this study is to formulate the differential equations governing the motion of the variable-mass three-body problem while accounting for the effects of reactive forces within a barycentric coordinate framework and to determine the invariants associated with the center of mass in this coordinate system.

In the article, based on the Meshchersky equation [14], differential equations describing the problem under consideration are formulated in absolute and relative coordinate systems, as well as in Jacobi coordinates. Further, on the basis of the written equations, using the analytical geometry of point masses [15], for the first time, the differential equations describing the motion of the system have been formulated in the barycentric coordinate system. Another important contribution of this study is the first derivation of the center-of-mass invariants in the barycentric coordinate system. These invariants provide the sole analytical relations between the bodies' coordinates and velocities for arbitrary laws of mass variation.

2 Derivation of the Differential Equations of Motion in an Absolute and Relative Coordinate Systems

2.1 Derivation of the Differential Equations of Motion in an Absolute Coordinate System

Let us consider a system of three point masses P_0, P_1 and P_2 , (hereinafter referred to as bodies), with variable masses in an absolute coordinate system. Assume that the masses of the bodies vary with time and, in the general case, vary non-isotropically according to arbitrary different laws

$$m_i = m_i(t), \quad \frac{\dot{m}_i}{m_i} \neq \frac{\dot{m}_j}{m_j}, \quad i \neq j, \quad i, j = 0, 1, 2. \tag{1}$$

From the Meshchersky equation [14] for bodies with variable masses, we obtain

$$m_i \ddot{\vec{R}}_i^* = grad_{\vec{R}_i^*} U^* + \vec{F}_{i,reac}^*, \quad (i = 0, 1, 2) \tag{2}$$

where $\vec{R}_i^*$ is the radius vector of the body P_i. The reactive force is written as

$$\vec{F}_{i,reac}^* = \dot{m}_i \vec{V}_i, \quad \vec{V}_i = \vec{u}_i - \dot{\vec{R}}_i^* \tag{3}$$

$\vec{u}_i$ is the absolute velocity, $\vec{V}_i$ is the relative velocity of the separating (joining) particles. The force function of the Newtonian interaction has the form

$$U^* = f\left(\frac{m_0 m_1}{R_{01}^*} + \frac{m_0 m_2}{R_{02}^*} + \frac{m_1 m_2}{R_{12}^*}\right), \tag{4}$$

where f is the gravitational constant, R_{01}^*, R_{02}^*, R_{12}^* are the distances between the bodies.

2.2 Differential equations of motion in a relative coordinate system

Let's displace from the absolute system to the relative coordinate system whose origin is located at the center of the body P_0, using the well-known transformation

$$\vec{R}_i = \vec{R}_i^* - \vec{R}_0^* = \vec{R}_{0i}^* \quad i = 1, 2 \tag{5}$$

From the system of equation (2) we obtain

$$\ddot{\vec{R}}_1 = \frac{1}{\mu_{01}} grad_{\vec{R}_1} U + \vec{F}_{01}, \tag{6}$$

$$\ddot{\vec{R}}_2 = \frac{1}{\mu_{02}} grad_{\vec{R}_2} U + \vec{F}_{02}, \tag{7}$$

where are denoted

$$\mu_{01} = \frac{1}{m_0} + \frac{1}{m_1} = \frac{m_1 + m_0}{m_0 m_1}, \quad \mu_{02} = \frac{1}{m_0} + \frac{1}{m_2} = \frac{m_2 + m_0}{m_0 m_2}, \tag{8}$$

$$\vec{F}_{01} = \left(\frac{\dot{m}_1}{m_1}\vec{V}_1 - \frac{\dot{m}_0}{m_0}\vec{V}_0\right), \quad \vec{F}_{02} = \left(\frac{\dot{m}_2}{m_2}\vec{V}_2 - \frac{\dot{m}_0}{m_0}\vec{V}_0\right), \tag{9}$$

$$U = f\left(\frac{m_0 m_1}{R_{01}} + \frac{m_0 m_2}{R_{02}} + \frac{m_1 m_2}{R_{12}}\right). \tag{10}$$

3 Differential equations of motion in Jacobi coordinate system

Using the analytical geometry of point masses [15], we obtain formulas for the transformation [13] from the relative coordinate system to Jacobi coordinate system

$$\vec{r}_{1jac} = \vec{R}_1, \quad \vec{r}_{2jac} = \vec{R}_2 - \nu_1 \vec{r}_{1jac}, \tag{11}$$

where $\vec{r}_{1jac}$, $\vec{r}_{2jac}$ is the radius-vectors of the bodies P_1 and P_2, respectively, in Jacobi coordinate system. As a result, the differential equations describing the motion of the problem under consideration in Jacobi coordinates are obtained

$$\mu_1 \ddot{\vec{r}}_{1jac} = grad_{\vec{r}_{1jac}} U + \mu_1 \vec{F}_{1jac}, \tag{12}$$

$$\mu_2 \ddot{\vec{r}}_{2jac} = grad_{\vec{r}_{2jac}} U - \mu_2 \left(2\dot{\nu}_1 \dot{\vec{r}}_{1jac} + \ddot{\nu}_1 \vec{r}_{1jac} \right) + \mu_2 \vec{F}_{2jac}, \tag{13}$$

where are denoted

$$\mu_1 = m_1 \frac{m_0}{m_0 + m_1} = \frac{1}{\mu_{01}}, \quad \mu_2 = m_2 \frac{m_0 + m_1}{m_0 + m_1 + m_2}, \quad \nu_1 = \frac{m_1}{m_0 + m_1}, \tag{14}$$

$$\vec{F}_{1jac} = \left(\frac{\dot{m}_1}{m_1} \vec{V}_1 - \frac{\dot{m}_0}{m_0} \vec{V}_0 \right), \tag{15}$$

$$\vec{F}_{2jac} = \left(\frac{\dot{m}_2}{m_2} \vec{V}_2 - \frac{\dot{m}_0}{m_0} \vec{V}_0 \right) - \nu_1 \left(\frac{\dot{m}_1}{m_1} \vec{V}_1 - \frac{\dot{m}_0}{m_0} \vec{V}_0 \right). \tag{16}$$

4 Differential equations of motion in a barycentric coordinate system

4.1 Transformation formulas of barycentric coordinates by Jacobi coordinates

Formulas for the transformation from Jacobi coordinates to barycentric coordinate system are obtained using the point masses' geometry [11], [15]

$$\vec{r}_{0b} = -\frac{m_1}{m_0 + m_1} \vec{r}_{1jac} - \frac{m_2}{m_0 + m_1 + m_2} \vec{r}_{2jac}, \tag{17}$$

$$\vec{r}_{1b} = \frac{m_0}{m_0 + m_1} \vec{r}_{1jac} - \frac{m_2}{m_0 + m_1 + m_2} \vec{r}_{2jac}, \tag{18}$$

$$\vec{r}_{2b} = \frac{m_0 + m_1}{m_0 + m_1 + m_2} \vec{r}_{2jac} \tag{19}$$

where $\vec{r}_{0b}$, $\vec{r}_{1b}$, $\vec{r}_{2b}$ are the barycentric radius-vectors of the three bodies. We denote

$$\nu_0 = \frac{m_0}{m_0 + m_1}. \tag{20}$$

Using the notation (14) and (20), we rewrite the transformation formulas (17)-(19) in the following compact form

$$\vec{r}_{0b} = -\nu_1 \vec{r}_{1jac} - \mu_2 \frac{\nu_0}{m_0} \vec{r}_{2jac}, \tag{21}$$

$$\vec{r}_{1b} = \nu_0 \vec{r}_{1jac} - \mu_2 \frac{\nu_1}{m_1} \vec{r}_{2jac}, \tag{22}$$

$$\vec{r}_{2b} = \frac{\mu_2}{m_2} \vec{r}_{2jac}. \tag{23}$$

The inverse conversion formulas have the form

$$\vec{r}_{1jac} = \frac{m_1}{\mu_1} \vec{r}_{1b} + \frac{\nu_1 m_2}{\mu_1} \vec{r}_{2b}, \quad \vec{r}_{2jac} = \frac{m_2}{\mu_2} \vec{r}_{2b}. \tag{24}$$

4.2 The invariant of the center of mass

The transformation formulas (21)-(23) are written as

$$m_0 \vec{r}_{0b} = -\mu_1 \vec{r}_{1jac} - \nu_0 \mu_2 \vec{r}_{2jac}, \tag{25}$$

$$m_1 \vec{r}_{1b} = \mu_1 \vec{r}_{1jac} - \nu_1 \mu_2 \vec{r}_{2jac}, \tag{26}$$

$$m_2 \vec{r}_{2b} = \mu_2 \vec{r}_{2jac}. \tag{27}$$

From formulas (25)-(27) follows an important analytical relation - the invariants of the center of mass of the problem under investigation within the barycentric coordinate system

$$m_0 \vec{r}_{0b} + m_1 \vec{r}_{1b} + m_2 \vec{r}_{2b} = 0. \tag{28}$$

4.3 Derivation of the Equations of Motion in the Barycentric Coordinate System

Differentiating expressions (25)-(27) twice, we obtain

$$\frac{d^2}{dt^2}\left(m_0 \vec{r}_{0b}\right) = -\left(\ddot{\mu}_1 \vec{r}_{1jac} + 2\dot{\mu}_1 \dot{\vec{r}}_{1jac} + \mu_1 \ddot{\vec{r}}_{1jac}\right) - \left(\frac{d^2}{dt^2}\left(\nu_0 \mu_2\right) \vec{r}_{2jac} + 2\frac{d}{dt}\left(\nu_0 \mu_2\right) \dot{\vec{r}}_{2jac} + \nu_0 \mu_2 \ddot{\vec{r}}_{2jac}\right), \tag{29}$$

$$\frac{d^2}{dt^2}\left(m_1 \vec{r}_{1b}\right) = \ddot{\mu}_1 \vec{r}_{1jac} + 2\dot{\mu}_1 \dot{\vec{r}}_{1jac} + \mu_1 \ddot{\vec{r}}_{1jac} - \left(\frac{d^2}{dt^2}\left(\nu_1 \mu_2\right) \vec{r}_{2jac} + 2\frac{d}{dt}\left(\nu_1 \mu_2\right) \dot{\vec{r}}_{2jac} + \nu_1 \mu_2 \ddot{\vec{r}}_{2jac}\right), \tag{30}$$

$$\frac{d^2}{dt^2}\left(m_2 \vec{r}_{2b}\right) = \ddot{\mu}_2 \vec{r}_{2jac} + 2\dot{\mu}_2 \dot{\vec{r}}_{2jac} + \mu_2 \ddot{\vec{r}}_{2jac}. \tag{31}$$

Using formulas (24)and (12)-(13), substituting the corresponding expressions in the right part of equation (29), we obtain the differential equation for body in the barycentric coordinate system in the form

$$\frac{d^2}{dt^2}\left(m_0\vec{r}_{0b}\right) = grad_{\vec{r}_{0b}}U + A_{01}\dot{\vec{r}}_{1b} + B_{01}\vec{r}_{1b} + A_{02}\dot{\vec{r}}_{2b} + B_{02}\vec{r}_{2b} + \vec{F}_{0b}, \tag{32}$$

where are denoted

$$U = f\left(\frac{m_0m_1}{r_{01}} + \frac{m_0m_2}{r_{02}} + \frac{m_1m_2}{r_{12}}\right), \tag{33}$$

$$A_{01} = -2\dot{\mu}_1\frac{m_1}{\mu_1} + 2\dot{\nu}_1\nu_0\mu_2\frac{m_1}{\mu_1}, \tag{34}$$

$$B_{01} = -\ddot{\mu}_1\frac{m_1}{\mu_1} - 2\dot{\mu}_1\frac{d}{dt}\left(\frac{m_1}{\mu_1}\right) + 2\dot{\nu}_1\nu_0\mu_2\frac{d}{dt}\left(\frac{m_1}{\mu_1}\right) + \nu_0\ddot{\nu}_1\mu_2\frac{m_1}{\mu_1}, \tag{35}$$

$$A_{02} = -2\dot{\mu}_1\frac{\nu_1m_2}{\mu_1} - 2\frac{d}{dt}\left(\nu_0\mu_2\right)\frac{m_2}{\mu_2} + 2\dot{\nu}_1\nu_0\mu_2\frac{\nu_1m_2}{\mu_1}, \tag{36}$$

$$\begin{gathered}B_{02} = -\ddot{\mu}_1\frac{\nu_1m_2}{\mu_1} - 2\dot{\mu}_1\frac{d}{dt}\left(\frac{\nu_1m_2}{\mu_1}\right) - \frac{d^2}{dt^2}\left(\nu_0\mu_2\right)\frac{m_2}{\mu_2} - \\ -2\frac{d}{dt}\left(\nu_0\mu_2\right)\frac{d}{dt}\left(\frac{m_2}{\mu_2}\right) + 2\dot{\nu}_1\nu_0\mu_2\frac{d}{dt}\left(\frac{\nu_1m_2}{\mu_1}\right) + \nu_0\ddot{\nu}_1\mu_2\frac{\nu_1m_2}{\mu_1},\end{gathered} \tag{37}$$

$$\vec{F}_{0b} = -\mu_1\vec{F}_{1jac} - \nu_0\mu_2\vec{F}_{2jac}. \tag{38}$$

In exactly the same way, using the relations (30),(24), (12)-(13) we obtain the differential equation for body in the barycentric coordinate system

$$\frac{d^2}{dt^2}\left(m_1\vec{r}_{1b}\right) = grad_{\vec{r}_{1b}}U + A_{11}\dot{\vec{r}}_{1b} + B_{11}\vec{r}_{1b} + A_{12}\dot{\vec{r}}_{2b} + B_{12}\vec{r}_{2b} + \vec{F}_{1b}, \tag{39}$$

$$A_{11} = 2\dot{\mu}_1\frac{m_1}{\mu_1} + 2\dot{\nu}_1\nu_1\mu_2\frac{m_1}{\mu_1}, \tag{40}$$

$$B_{11} = \ddot{\mu}_1\frac{m_1}{\mu_1} + 2\dot{\mu}_1\frac{d}{dt}\left(\frac{m_1}{\mu_1}\right) + 2\dot{\nu}_1\nu_1\mu_2\frac{d}{dt}\left(\frac{m_1}{\mu_1}\right) + \nu_1\ddot{\nu}_1\mu_2\frac{m_1}{\mu_1}, \tag{41}$$

$$A_{12} = 2\dot{\mu}_1\frac{\nu_1m_2}{\mu_1} - 2\frac{d}{dt}\left(\nu_1\mu_2\right)\frac{m_2}{\mu_2} + 2\dot{\nu}_1\nu_1\mu_2\frac{\nu_1m_2}{\mu_1}, \tag{42}$$

$$B_{12} = \ddot{\mu}_1 \frac{\nu_1 m_2}{\mu_1} + 2\dot{\mu}_1 \frac{d}{dt}\left(\frac{\nu_1 m_2}{\mu_1}\right) - \frac{d^2}{dt^2}(\nu_1\mu_2)\frac{m_2}{\mu_2} - \\ -2\frac{d}{dt}(\nu_1\mu_2)\frac{d}{dt}\left(\frac{m_2}{\mu_2}\right) + 2\dot{\nu}_1\nu_1\mu_2\frac{d}{dt}\left(\frac{\nu_1 m_2}{\mu_1}\right) + \nu_1\ddot{\nu}_1\mu_2\frac{\nu_1 m_2}{\mu_1}, \tag{43}$$

$$\vec{F}_{1b} = \mu_1 \vec{F}_{1jac} - \nu_1\mu_2\vec{F}_{2jac}. \tag{44}$$

The differential equation for body P_2 in the barycentric coordinate system follows from the relations (31), (24), (13)

$$\frac{d^2}{dt^2}(m_2\vec{r}_{2b}) = grad_{\vec{r}_{2b}}U + A_{21}\dot{\vec{r}}_{1b} + B_{21}\vec{r}_{1b} + A_{22}\dot{\vec{r}}_{2b} + B_{22}\vec{r}_{2b} + \vec{F}_{2b}, \tag{45}$$

$$A_{21} = -2\dot{\nu}_1\mu_2\frac{m_1}{\mu_1}, \tag{46}$$

$$B_{21} = -2\dot{\nu}_1\mu_2\frac{d}{dt}\left(\frac{m_1}{\mu_1}\right) - \ddot{\nu}_1\mu_2\frac{m_1}{\mu_1}, \tag{47}$$

$$A_{22} = 2\dot{\mu}_2\frac{m_2}{\mu_2} - 2\dot{\nu}_1\mu_2\frac{\nu_1 m_2}{\mu_1}, \tag{48}$$

$$B_{22} = \ddot{\mu}_2\frac{m_2}{\mu_2} + 2\dot{\mu}_2\frac{d}{dt}\left(\frac{m_2}{\mu_2}\right) - 2\dot{\nu}_1\mu_2\frac{d}{dt}\left(\frac{\nu_1 m_2}{\mu_1}\right) - \ddot{\nu}_1\mu_2\frac{\nu_1 m_2}{\mu_1}, \tag{49}$$

$$\vec{F}_{2b} = \mu_2\vec{F}_{2jac}. \tag{50}$$

It should be noted that the forms of equations (32), (39), and (45) obtained are convenient for further analysis. Let's rewrite the equations (32), (39), (45) in the form of

$$\frac{d^2}{dt^2}(m_0\vec{r}_{0b}) = \vec{Q}_0, \qquad \frac{d^2}{dt^2}(m_1\vec{r}_{1b}) = \vec{Q}_1, \qquad \frac{d^2}{dt^2}(m_2\vec{r}_{2b}) = \vec{Q}_2, \tag{51}$$

$$\vec{Q}_0 = grad_{\vec{r}_{0b}}U + A_{01}\dot{\vec{r}}_{1b} + B_{01}\vec{r}_{1b} + A_{02}\dot{\vec{r}}_{2b} + B_{02}\vec{r}_{2b} + \vec{F}_{0b}, \tag{52}$$

$$\vec{Q}_1 = grad_{\vec{r}_{1b}}U + A_{11}\dot{\vec{r}}_{1b} + B_{11}\vec{r}_{1b} + A_{12}\dot{\vec{r}}_{2b} + B_{12}\vec{r}_{2b} + \vec{F}_{1b}, \tag{53}$$

$$\vec{Q}_2 = grad_{\vec{r}_{2b}} U + A_{21}\dot{\vec{r}}_{1b} + B_{21}\vec{r}_{1b} + A_{22}\dot{\vec{r}}_{2b} + B_{22}\vec{r}_{2b} + \vec{F}_{2b}. \tag{54}$$

Let's show that

$$\vec{Q}_0 + \vec{Q}_1 + \vec{Q}_2 = 0. \tag{55}$$

It is obvious that

$$grad_{\vec{r}_{0b}} U + grad_{\vec{r}_{1b}} U + grad_{\vec{r}_{2b}} U = 0. \tag{56}$$

Indeed, considering that $\nu_0 + \nu_1 = 1$, direct calculations give

$$\vec{F}_{0b} + \vec{F}_{1b} + \vec{F}_{2b} = 0. \tag{57}$$

In exactly the same way, we get

$$A_{01} + A_{11} + A_{21} = 0, \quad B_{01} + B_{11} + B_{21} = 0, \tag{58}$$

$$A_{02} + A_{12} + A_{22} = 0, \quad B_{02} + B_{12} + B_{22} = 0. \tag{59}$$

Then it follows from equations (51)

$$\frac{d^2}{dt^2}\left(m_0\vec{r}_{0b} + m_1\vec{r}_{1b} + m_2\vec{r}_{2b}\right) = 0, \tag{60}$$

according to (28), this is natural for the barycentric coordinate system. Therefore, we can write

$$m_0\dot{\vec{r}}_{0b} + m_1\dot{\vec{r}}_{1b} + m_2\dot{\vec{r}}_{2b} + \dot{m}_0\vec{r}_{0b} + \dot{m}_1\vec{r}_{1b} + \dot{m}_2\vec{r}_{2b} = 0. \tag{61}$$

Equations (28) and (61) are the only analytical relations connecting the coordinates and velocities of three bodies in the barycentric coordinate system under any laws of mass change.

It should be noted that, owing to the center-of-mass invariant, it is actually possible to consider only two of the three vector equations (32), (39), and (45).

When the masses of the bodies vary isotropically and the reactive forces vanish, the center-of-mass invariants are conserved [11]. Naturally, in the case of constant masses, the equations of motion and the center-of-mass invariants obtained above reduce to the well-known equations [1], [2], [16].

5 Conclusion and discussions

In this paper, the differential equations of motion for the variable-mass three-body problem in the presence of reactive forces are first formulated in the absolute and relative coordinate systems. Subsequently, on the basis of the equations expressed in the relative coordinate system, the corresponding equations of motion are derived in Jacobi coordinates.

Furthermore, using the equations of motion obtained in Jacobi coordinates, the differential equations governing the motion of the variable-mass three-body problem with reactive forces are formulated in the barycentric coordinate system. To the best of our knowledge, this represents the first derivation of these equations within such a framework. For the first time, integrals of linear momentum have been obtained in the barycentric coordinate system, following G. N. Duboshin [16], these integrals are referred to as the center-of-mass invariants. These invariants constitute the only analytical relations linking the coordinates and velocities of the bodies under arbitrary laws of mass variation in the variable-mass three-body problem with reactive forces.

It should be emphasized that the absolute coordinate system is adopted as the primary inertial reference frame. In contrast, the relative, Jacobi, and barycentric coordinate systems are non-inertial. The barycentric coordinate system provides a convenient framework for investigating the dynamical evolution of an isolated self-consistent system composed of three mutually gravitating bodies with variable masses.

The differential equations of motion derived in the barycentric coordinate system, together with the established center-of-mass invariants, may be employed in a wide range of qualitative, analytical, and numerical investigations of the variable-mass three-body problem in the presence of reactive forces.

The new results obtained can be applied to the study of exoplanetary systems [17] with variable masses [18] - [19].

Authors contribution

M.M.: methodology, supervision. M.M. and A.K.: writing - original draft preparation. M.M., A.P. and A.K.: editing after review. All authors have read and agreed to the published version of the manuscript.

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