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AN INVERSE SOURCE PROBLEM FOR A SECOND-ORDER HYPERBOLIC EQUATION WITH AN INTEGRAL CONDITION

This paper investigates a linear inverse boundary value problem for a second-order hyperbolic equation with an unknown right-hand side depending on time. The problem is posed with initial conditions, a Neumann boundary condition, a nonlocal integral condition, and an additional condition. The aim of the study is to establish conditions for the existence and uniqueness of a classical solution to the inverse problem. Under the imposed zero-mean and compatibility conditions, the original problem with a nonlocal integral condition is reduced to an equivalent auxiliary problem with homogeneous Neumann boundary conditions at both endpoints and an additional relation, for which the uniqueness of the solution is established. The Fourier method is applied to investigate solvability. The solution of the hyperbolic equation is represented as a series in the corresponding system of eigenfunctions, after which the expansion coefficients are determined. Based on the obtained representations and estimates, the necessary properties of the unknown functions are established. It is proved that the obtained functions satisfy the differential equation, the initial conditions, the Neumann boundary condition, the nonlocal integral condition, and the additional condition in the classical sense. As a result, under the formulated requirements on the given functions and the necessary compatibility conditions, the original linear inverse boundary value problem for the considered second-order hyperbolic equation is shown to admit exactly one classical solution.

Keywords: inverse boundary value problem, second-order hyperbolic equation, Fourier method, Volterra integral equation, classical solution.

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Интегралдық шарты бар екінші ретті гиперболалық теңдеу үшін көзді қалпына келтірудің кері есебі

Бұл жұмыста уақытқа тәуелді белгісіз оң жағы бар екінші ретті гиперболалық теңдеу үшін сызықтық кері шеттік есеп зерттеледі. Қарастырылатын есеп бастапқы шарттармен, Нейман шеттік шартымен, локалдық емес интегралдық шартпен және қосымша шартпен беріледі. Зерттеудің мақсаты қойылған кері есептің классикалық шешімінің бар болуы мен жалғыздығының шарттарын анықтау болып табылады. Нөлдік орташа мән және үйлесімділік шарттары орындалғанда, локалдық емес интегралдық шарты бар бастапқы есеп екі шетінде біртекті Нейман шарттары және қосымша қатынасы бар эквивалентті көмекші есепке келтіріліп, оның шешімінің жалғыздығы дәлелденеді. Есептің шешілімділігін зерттеу үшін Фурье әдісі қолданылады. Гиперболалық теңдеудің шешімі сәйкес меншікті функциялар жүйесі бойынша қатар түрінде өрнектеледі, содан кейін жіктеу коэффициенттері анықталады. Алынған өрнектер мен бағалаулар негізінде ізделінетін функциялардың қажетті қасиеттері белгіленеді. Табылған функциялардың дифференциалдық теңдеуді, бастапқы шарттарды, Нейман шеттік шартын, локалдық емес интегралдық шартты және қосымша шартты классикалық мағынада қанағаттандыратыны дәлелденеді. Нәтижесінде берілген функцияларға қойылған талаптар мен қажетті үйлесімділік шарттары орындалғанда, қарастырылатын екінші ретті гиперболалық теңдеу үшін бастапқы сызықтық кері шеттік есептің жалғыз классикалық шешімі бар екені көрсетілді.

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Түйін сөздер: кері шеттік есеп, екінші ретті гиперболалық теңдеу, Фурье әдісі, Вольтерра интегралдық теңдеуі, классикалық шешім.

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Обратная задача восстановления источника для гиперболического уравнения второго порядка с интегральным условием

В работе исследуется линейная обратная краевая задача для гиперболического уравнения второго порядка с неизвестной правой частью, зависящей от времени. Рассматриваемая задача задается начальными условиями, условием Неймана, нелокальным интегральным условием и дополнительным условием. Целью исследования является установление условий существования и единственности классического решения поставленной обратной задачи. При выполнении условий нулевого среднего значения и условий согласования исходная задача с нелокальным интегральным условием сводится к эквивалентной вспомогательной задаче с однородными условиями Неймана на обоих концах и дополнительным соотношением, для которой устанавливается единственность решения. Для исследования разрешимости применяется метод Фурье. Решение гиперболического уравнения представляется в виде ряда по соответствующей системе собственных функций, после чего определяются коэффициенты разложения. На основании полученных представлений и оценок устанавливаются необходимые свойства искомых функций. Доказывается, что найденные функции удовлетворяют дифференциальному уравнению, начальным условиям, условию Неймана, нелокальному интегральному и дополнительному условиям в классическом смысле. В результате при сформулированных требованиях к заданным функциям и выполнении необходимых условий согласования показано, что исходная линейная обратная краевая задача для рассматриваемого гиперболического уравнения второго порядка имеет единственное классическое решение.
Ключевые слова: обратная краевая задача, гиперболическое уравнение второго порядка, метод Фурье, интегральное уравнение Вольтерра, классическое решение.

1 Introduction

Inverse problems for differential equations arise when, in addition to the solution of the equation, one or more unknown quantities, such as coefficients or source terms, are to be determined from additional information. Depending on the way the unknown quantities enter the equation and the additional conditions, such problems may be linear or nonlinear. Inverse problems constitute an important area of modern mathematics and arise in various applications.

The integral condition

$$\int_0^1 u(x,t)\,dx = 0$$

specifies that the spatial mean of the solution vanishes at each time t. In the context of vibration processes, where $u(x,t)$ represents the displacement of a medium, this condition expresses a balance between positive and negative displacements over the spatial interval. The additional condition

$$u(x_0,t) = h(t)$$

specifies the displacement at a fixed spatial point x_0. This observation provides additional information for determining the unknown time-dependent source factor $a(t)$ together with the solution $u(x,t)$.

The general theory of inverse problems has been developed in the fundamental works of A.N. Tikhonov [1], M.M. Lavrentiev [2, 3], V.K. Ivanov [4], A.M. Denisov [5], V. Isakov [6], and their followers.

Inverse problems for second-order hyperbolic equations with integral and nonlocal conditions have been investigated in several studies. Mehraliyev and Isgendarova [7] studied an inverse problem involving an unknown time-dependent coefficient in a second-order hyperbolic equation with an integral condition. The original problem was reduced to an equivalent problem, and the Fourier method was used to obtain a system of integral equations whose solvability was established.

Mehraliyev and Isgenderova [8] investigated an inverse boundary value problem for a second-order hyperbolic equation with an integral condition of the first kind. Their problem involves the recovery of an unknown coefficient multiplying the solution and is formulated with nonclassical boundary conditions. The analysis is based on a biorthogonal system and the Fourier method, while the existence and uniqueness of a classical solution are established using the contraction mapping principle. Related inverse problems for second-order hyperbolic equations were also considered by Mehraliyev [9, 10].

Mehraliyev and Azizbayov [11] investigated a time-nonlocal inverse coefficient problem for a second-order hyperbolic equation with an integral overdetermination condition. Their problem involves nonlocal initial conditions connecting the values of the solution and its time derivative at $t = 0$ and $t = T$. The unknown time-dependent coefficient multiplies the solution itself. By reducing the original problem to an auxiliary system and applying the Fourier method and the contraction mapping principle, the authors established the existence and uniqueness of a classical solution under restrictions on the time interval.

E. Azizbayov [12, 13] also investigated inverse coefficient identification problems for hyperbolic equations subject to nonlocal integral conditions.

The identification of unknown source terms in hyperbolic equations has also been investigated in classical studies. Cannon and Dunninger [14] considered the determination of an unknown forcing function in a hyperbolic equation from additional data. Prilepko, Orlovsky, and Vasin [15] presented a systematic treatment of inverse problems for partial differential equations, including questions of existence, uniqueness, and stability.

Kozhanov and Safiullova [16] investigated linear inverse problems for parabolic and hyperbolic equations involving the simultaneous determination of the solution and an unknown right-hand side under additional overdetermination conditions, including final and integral conditions. They established the existence and uniqueness of regular solutions. Kozhanov and Pulkina [17] studied initial-boundary value problems for multidimensional second-order hyperbolic equations with nonlocal boundary conditions of integral type and established the existence and uniqueness of regular solutions.

Mehraliyev [18] investigated an inverse boundary value problem for a second-order elliptic equation with an unknown time-dependent coefficient under an integral condition and an additional observation. The existence and uniqueness of a classical solution were established. Although this problem concerns an elliptic equation, it is related to the present study through the use of an integral condition and additional information for the simultaneous determination

of the solution and an unknown time-dependent function.

More recently, S.A. Ould Beinane, N. Alotaibi, G. Alsahli, and A. Ilyas [19] considered an inverse problem for the wave equation

$$u_{tt}(x,t) - u_{xx}(x,t) = a(t)f(x,t)$$

with the nonlocal boundary conditions

$$u(0,t) = 0, \qquad u(1,t) + u(\alpha,t) = 0, \qquad 0 < \alpha < 1,$$

where the unknown function $a(t)$ is determined together with $u(x,t)$ under the additional integral condition

$$\int_0^1 u(x,t)\,dx = E(t).$$

For this inverse problem, conditions for the existence and uniqueness of a solution were established.

The present study follows the general Fourier-based approach developed by Mehraliyev and his co-authors. In particular, the regularity assumptions imposed in Theorem 3 and the estimates for the Fourier coefficients are similar in structure to assumptions H2)–H4) and estimates (3.10)–(3.11) in [11]. These techniques are adapted to the boundary conditions and the source term considered in the present problem.

The main contribution of the present study is the establishment of existence and uniqueness of a classical solution for the considered linear inverse source problem.

A distinctive feature of the present problem is that the unknown time-dependent source factor $a(t)$ multiplies a prescribed spatially and temporally dependent function $g(x,t)$, rather than the unknown solution $u(x,t)$, as in [7].

This distinction changes the structure of the inverse problem: the present formulation is linear with respect to the unknown pair (u,a) and leads to a single linear Volterra integral equation of the second kind for the recovery of $a(t)$.

The main contribution consists in establishing the existence and uniqueness of a classical solution for this source structure under the considered combination of a homogeneous Neumann boundary condition, a zero-mean spatial integral condition, and an additional pointwise observation, without imposing a smallness restriction on the finite time interval.

Under the imposed zero-mean and compatibility conditions, the original problem is reduced to an equivalent auxiliary problem with homogeneous Neumann boundary conditions at both endpoints. This reduction allows the application of the cosine Fourier method and leads to a single linear Volterra integral equation of the second kind for the unknown source factor. In contrast, the analysis in [7] is based on a biorthogonal system and a coupled system of integral equations, whose solvability is established by the contraction mapping principle under the additional restriction

$$(A(T) + 2)^2 B(T) < 1.$$

The present approach establishes the existence and uniqueness of a classical solution for any finite $T > 0$, without imposing a smallness condition on the time interval or restricting the solution to a prescribed ball. Thus, the contribution of the present study lies in the classical solvability result for the considered linear inverse source problem rather than in the introduction of a new Fourier-based analytical method.

2 Problem Formulation and an Equivalent Reformulation

We consider the following inverse boundary value problem

$$u_{tt}(x,t) - u_{xx}(x,t) = a(t)g(x,t) + f(x,t) \tag{1}$$

for $D_T = \{(x,t) : 0 < x < 1,\ 0 \le t \le T\}$. At $t = 0$, we prescribe

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad 0 \le x \le 1. \tag{2}$$

The boundary condition at $x = 0$ is

$$u_x(0,t) = 0, \quad 0 \le t \le T. \tag{3}$$

We also require

$$\int_0^1 u(x,t)\,dx = 0, \quad 0 \le t \le T. \tag{4}$$

At a fixed point $x_0 \in [0,1]$, the solution satisfies

$$u(x_0,t) = h(t), \quad 0 \le t \le T. \tag{5}$$

Here $f(x,t)$, $g(x,t)$, $\varphi(x)$, $\psi(x)$, and $h(t)$ are prescribed. We seek $u(x,t)$ and $a(t)$.

Definition 1 *A pair of functions $\{u(x,t), a(t)\}$, where $u(x,t) \in C^2(\overline{D}_T)$ and $a(t) \in C[0,T]$, is termed a classical solution of the inverse boundary value problem (1)–(5) if (1) holds in D_T, the initial conditions (2) hold on $[0,1]$, and the relations (3)–(5) hold throughout $[0,T]$.*

Theorem 1 *Let $f(x,t) \in C(\overline{D}_T)$, $\int_0^1 f(x,t)\,dx = 0$, $0 \le t \le T$, $g(x,t) \in C(\overline{D}_T)$, $\int_0^1 g(x,t)\,dx = 0$, $0 \le t \le T$, $\varphi(x) \in C[0,1]$, $\psi(x) \in C[0,1]$, $h(t) \in C^2[0,T]$ assume that the following compatibility relations hold:*

$$\int_0^1 \varphi(x)\,dx = 0, \qquad \int_0^1 \psi(x)\,dx = 0, \tag{6}$$

$$\varphi(x_0) = h(0), \qquad \psi(x_0) = h'(0). \tag{7}$$

Then solving problem (1)–(5) in the classical sense is equivalent to finding $u(x,t) \in C^2(\overline{D}_T)$ and $a(t) \in C[0,T]$ satisfying (1)–(3) together with

$$u_x(1,t) = 0, \quad 0 \le t \le T, \tag{8}$$

$$h''(t) - u_{xx}(x_0,t) = a(t)g(x_0,t) + f(x_0,t), \quad 0 \le t \le T. \tag{9}$$

Proof. Assume that the pair $\{u(x,t),a(t)\}$ solves (1)–(5) in the classical sense. Integration of (1) over the spatial interval $[0,1]$ yields:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)\,dx - u_x(1,t) + u_x(0,t) = a(t)\int_0^1 g(x,t)\,dx + \int_0^1 f(x,t)\,dx, \quad 0 \le t \le T. \tag{10}$$

In view of the conditions

$$\int_0^1 f(x,t)\,dx = 0, \qquad \int_0^1 g(x,t)\,dx = 0, \qquad 0 \le t \le T,$$

and taking into account (3) and (4), it follows from (10) that (8) holds.

Evaluating equation (1) at $x = x_0$ gives

$$\frac{d^2}{dt^2}u(x_0,t) - u_{xx}(x_0,t) = a(t)g(x_0,t) + f(x_0,t), \qquad 0 \le t \le T. \tag{11}$$

Since $h \in C^2[0,T]$, two successive differentiations of (5) with respect to t yield

$$u_{tt}(x_0,t) = h''(t), \qquad 0 \le t \le T. \tag{12}$$

Substitution of (12) into (11) establishes (9).

Next, let $\{u(x,t),a(t)\}$ satisfy problem (1)–(3), (8), (9). Then from (10), taking into account

$$\int_0^1 f(x,t)\,dx = \int_0^1 g(x,t)\,dx = 0, \qquad 0 \le t \le T,$$

and (3), (8), we obtain:

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)\,dx = 0, \qquad 0 \le t \le T. \tag{13}$$

In view of (2), taking into account (6), it is easy to see that

$$\int_0^1 u(x,0)\,dx = \int_0^1 \varphi(x)\,dx = 0, \qquad \int_0^1 u_t(x,0)\,dx = \int_0^1 \psi(x)\,dx = 0. \tag{14}$$

The trivial solution is the unique solution of (13), (14). Consequently,

$$\int_0^1 u(x,t)\,dx = 0, \qquad 0 \le t \le T,$$

so the integral requirement (4) holds.

Subtracting (9) from (11) yields

$$\frac{d^2}{dt^2}\left(u(x_0,t) - h(t)\right) = 0, \qquad 0 \le t \le T. \tag{15}$$

The initial conditions (2) and the compatibility relations (7) give:

$$u(x_0,0) - h(0) = \varphi(x_0) - h(0) = 0, \qquad u_t(x_0,0) - h'(0) = \psi(x_0) - h'(0) = 0. \tag{16}$$

Relations (15) and (16) imply (5), completing the proof of the theorem.

The following theorem holds.

Theorem 2 *If $a(t) \in C[0,T]$ is known and problem (1)–(3), (8) has a solution, then it is unique in the class $C^2(\overline{D}_T)$.*

Proof 1 *Suppose that, for the same given function $a(t)$, there exist two solutions $u_1(x,t)$, $u_2(x,t)$, and consider the difference*

$$v(x,t) = u_1(x,t) - u_2(x,t).$$

The function $v(x,t)$ obviously satisfies the homogeneous equation

$$v_{tt}(x,t) - v_{xx}(x,t) = 0, \qquad (x,t) \in D_T \tag{17}$$

and the conditions:

$$v(x,0) = 0, \qquad v_t(x,0) = 0, \qquad 0 \le x \le 1, \tag{18}$$

$$v_x(0,t) = 0, \qquad v_x(1,t) = 0, \qquad 0 \le t \le T. \tag{19}$$

We prove that the function $v(x,t)$ is identically equal to zero.

We multiply equation (17) throughout by $2v_t(x,t)$ and then integrate each side over the interval $0 \le x \le 1$, obtaining:

$$2\int_0^1 \left(v_{tt}(x,t) - v_{xx}(x,t)\right) v_t(x,t)\, dx = 0, \qquad 0 \le t \le T. \tag{20}$$

By using the boundary conditions (19), we have:

$$2\int_0^1 v_{tt}(x,t) v_t(x,t)\, dx = \frac{d}{dt}\int_0^1 v_t^2(x,t)\, dx, \qquad 0 \le t \le T,$$

$$2\int_0^1 v_{xx}(x,t) v_t(x,t)\, dx = -\frac{d}{dt}\int_0^1 v_x^2(x,t)\, dx, \qquad 0 \le t \le T.$$

Then, from (20) we have:

$$\frac{d}{dt}\int_0^1 \left(v_t^2(x,t) + v_x^2(x,t)\right)\, dx = 0, \qquad 0 \le t \le T.$$

Consequently,

$$\int_0^1 \left(v_t^2(x,t) + v_x^2(x,t)\right)\, dx = \int_0^1 \left(v_t^2(x,0) + v_x^2(x,0)\right)\, dx.$$

Since $v(x,0) = 0$, we have $v_x(x,0) = 0$. Taking also into account $v_t(x,0) = 0$, it follows that

$$\int_0^1 \left(v_t^2(x,t) + v_x^2(x,t)\right)\, dx = 0, \qquad 0 \le t \le T.$$

Hence $v_t(x,t) \equiv 0$, $v_x(x,t) \equiv 0$. Therefore, the identity $v(x,t) = \text{const} = C_0$ follows. Using condition (18), we have $v(x,0) = C_0 = 0$. Thus, it is proved that $v(x,t) = 0$. Therefore, if there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of problem (1)–(3), (8), then $u_1(x,t) = u_2(x,t)$. Hence, if a solution of problem (1)–(3), (8) exists, then it is unique. The theorem is proved.

3 On the Solvability of an Inverse Boundary Value Problem

To investigate the solvability of problem (1)–(3), (8), (9), we consider the corresponding spectral problem

$$-y''(x) = \lambda^2 y(x), \qquad y'(0) = y'(1) = 0.$$

Its eigenvalues and corresponding eigenfunctions are

$$\lambda_k = k\pi, \qquad y_k(x) = \cos(k\pi x), \qquad k = 0, 1, 2, \dots.$$

For problem (1)–(3), (8), (9), we seek $u(x,t)$, the first component of the solution pair $\{u(x,t), a(t)\}$, in the following form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x, \qquad \lambda_k = k\pi, \tag{21}$$

where

$$u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x \, dx, \qquad k = 0, 1, 2, \dots,$$

$$m_0 = 1, \qquad m_k = 2, \quad k = 1, 2, \dots.$$

A formal application of the Fourier method to (1) and (2) gives the following relations:

$$u_k''(t) + \lambda_k^2 u_k(t) = F_k(t; a), \qquad k = 0, 1, 2, \dots; \ 0 \le t \le T, \tag{22}$$

$$u_k(0) = \varphi_k, \qquad u_k'(0) = \psi_k, \qquad k = 0, 1, 2, \dots, \tag{23}$$

where

$$F_k(t; a) = f_k(t) + a(t) g_k(t), \quad f_k(t) = m_k \int_0^1 f(x,t) \cos \lambda_k x \, dx,$$

$$g_k(t) = m_k \int_0^1 g(x,t) \cos \lambda_k x \, dx, \quad \varphi_k = m_k \int_0^1 \varphi(x) \cos \lambda_k x \, dx,$$

$$\psi_k = m_k \int_0^1 \psi(x) \cos \lambda_k x \, dx, \quad k = 0, 1, 2, \dots.$$

By solving (22)–(23), we arrive at

$$u_0(t) = \varphi_0 + t\psi_0 + \int_0^t (t - \tau) F_0(\tau; a) \, d\tau, \tag{24}$$

$$u_k(t) = \varphi_k \cos \lambda_k t + \frac{1}{\lambda_k} \psi_k \sin \lambda_k t + \frac{1}{\lambda_k} \int_0^t F_k(\tau; a) \sin \lambda_k (t - \tau) \, d\tau, \quad k = 1, 2, \dots; \ 0 \le t \le T.$$

(25)

Taking into account expressions (24) and (25), from (21) we obtain

$$u(x,t) = \varphi_0 + t\psi_0 + \int_0^t (t-\tau)F_0(\tau;a)\,d\tau + \sum_{k=1}^{\infty}\left\{\varphi_k \cos\lambda_k t + \frac{1}{\lambda_k}\psi_k \sin\lambda_k t + \frac{1}{\lambda_k}\int_0^t F_k(\tau;a)\sin\lambda_k(t-\tau)\,d\tau\right\}\cos\lambda_k x. \tag{26}$$

In view of (21), from (9) it follows that

$$a(t) = [g(x_0,t)]^{-1}\left\{h''(t) - f(x_0,t) + \sum_{k=1}^{\infty}\lambda_k^2 u_k(t)\cos\lambda_k x_0\right\}. \tag{27}$$

An equation determining the second solution component $a(t)$ in problem (1)–(3), (8), (9) follows upon replacing the expression in (27) by its representation (25):

$$a(t) = [g(x_0,t)]^{-1}\left\{h''(t) - f(x_0,t) + \sum_{k=1}^{\infty}\lambda_k^2\Big[\varphi_k\cos\lambda_k t + \frac{1}{\lambda_k}\psi_k\sin\lambda_k t + \frac{1}{\lambda_k}\int_0^t F_k(\tau;a)\sin\lambda_k(t-\tau)\,d\tau\Big]\cos\lambda_k x_0\right\}. \tag{28}$$

Therefore, the construction of a solution of problem (1)–(3), (8), (9) is reduced to solving equation (28) and verifying the required regularity of the function defined by (26).

The following theorem can be proved. Here and below, derivatives in L_2 are understood in the weak sense.

It should be noted that the auxiliary problem (1)–(3), (8), (9) is investigated under more general assumptions than those required for the equivalence established in Theorem 1. In particular, the zero-mean conditions on the prescribed data are not imposed in Theorem 3. Therefore, the zero Fourier mode $u_0(t)$ is not necessarily identically zero and must be retained in the Fourier representation and the corresponding estimates. These zero-mean conditions are imposed subsequently in Theorem 4 to establish the solvability of the original inverse problem (1)–(5).

Theorem 3 *For problem* (1)–(3), (8), (9), *impose the following assumptions on the prescribed data:*

1. $\varphi(x) \in C^2[0,1]$, $\varphi'''(x) \in L_2(0,1)$, $\varphi'(0) = 0$, $\varphi'(1) = 0$.

2. $\psi(x) \in C^1[0,1]$, $\psi''(x) \in L_2(0,1)$, $\psi'(0) = 0$, $\psi'(1) = 0$.

3. $f(x,t), f_x(x,t) \in C(\overline{D}_T)$, $f_{xx}(x,t) \in L_2(D_T)$, $f_x(0,t) = 0$, $f_x(1,t) = 0$, $0 \le t \le T$.

4. $g(x,t), g_x(x,t) \in C(\overline{D}_T)$, $g_{xx}(x,t) \in C([0,T];L_2(0,1))$, $g_x(0,t) = 0$, $g_x(1,t) = 0$, $0 \le t \le T$.

5. $h(t) \in C^2[0,T]$, $g(x_0,t) \ne 0$, $0 \le t \le T$.

Then problem (1)–(3), (8), (9) *has a unique solution, with* $u(x,t) \in C^2(\overline{D}_T), a(t) \in C[0,T]$.

Proof 2 *Equation* (28) *can be written in the form:*

$$a(t) = b(t) + \int_0^t G(t,\tau)a(\tau)\,d\tau, \tag{29}$$

where

$$b(t) = [g(x_0,t)]^{-1}\Bigg\{h''(t) - f(x_0,t) + \sum_{k=1}^{\infty}\lambda_k^2\Bigg[\varphi_k\cos\lambda_k t + \frac{1}{\lambda_k}\psi_k\sin\lambda_k t + \frac{1}{\lambda_k}\int_0^t f_k(\tau)\sin\lambda_k(t-\tau)\,d\tau\Bigg]\cos\lambda_k x_0\Bigg\},$$

$$G(t,\tau) = [g(x_0,t)]^{-1}\sum_{k=1}^{\infty}\lambda_k g_k(\tau)\sin\lambda_k(t-\tau)\cos\lambda_k x_0.$$

Now, taking into account the conditions of Theorem 3, it can be shown that

$$\begin{aligned}\|b(t)\|_{C[0,T]} \le\ & \left\|[g(x_0,t)]^{-1}\right\|_{C[0,T]}\Bigg\{\|h''(t) - f(x_0,t)\|_{C[0,T]} \\ & + \sqrt{2}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\Big[\|\varphi'''(x)\|_{L_2(0,1)} + \|\psi''(x)\|_{L_2(0,1)} \\ & + \sqrt{T}\|f_{xx}(x,t)\|_{L_2(D_T)}\Big]\Bigg\}.\end{aligned} \tag{30}$$

$$|G(t,\tau)| \le \sqrt{2}\left\|[g(x_0,t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\|g_{xx}(x,t)\|_{C([0,T];L_2(0,1))}. \tag{31}$$

We now justify the continuity of the functions $b(t)$ and $G(t,\tau)$ appearing in equation (29).

By the assumptions of Theorem 3, $g_{xx} \in C([0,T];L^2(0,1))$. Using the Cauchy–Schwarz inequality and Parseval's inequality, we obtain

$$\sum_{k=1}^{\infty}\lambda_k|g_k(\tau)| \le \left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\lambda_k^4|g_k(\tau)|^2\right)^{\frac{1}{2}} \le \sqrt{2}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\right)^{\frac{1}{2}}\|g_{xx}(\cdot,\tau)\|_{L^2(0,1)}.$$

Consequently, the series defining $G(t,\tau)$ converges absolutely and uniformly on

$$\Delta_T = \{(t,\tau) : 0 \le \tau \le t \le T\}.$$

Since each term of this series is continuous and $g(x_0,t) \neq 0$ on $[0,T]$, we conclude that

$$G \in C(\Delta_T).$$

We now establish the continuity of $b(t)$. *By the Cauchy–Schwarz inequality and Parseval's inequality, we have*

$$\sum_{k=1}^{\infty}\lambda_k^2|\varphi_k| \le \left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\lambda_k^6|\varphi_k|^2\right)^{\frac{1}{2}} \le \sqrt{2}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\right)^{\frac{1}{2}}\|\varphi'''\|_{L^2(0,1)},$$

and

$$\sum_{k=1}^{\infty}\lambda_k|\psi_k| \le \left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\lambda_k^4|\psi_k|^2\right)^{\frac{1}{2}} \le \sqrt{2}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\right)^{\frac{1}{2}}\|\psi''\|_{L^2(0,1)}.$$

Furthermore, applying the Cauchy–Schwarz inequality with respect to time, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\lambda_k\int_0^t|f_k(\tau)|\,d\tau &\le \sqrt{T}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\right)^{\frac{1}{2}}\left(\int_0^T\sum_{k=1}^{\infty}\lambda_k^4|f_k(\tau)|^2\,d\tau\right)^{\frac{1}{2}}\\ &\le \sqrt{2T}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\right)^{\frac{1}{2}}\|f_{xx}\|_{L^2(D_T)}.\end{aligned}$$

Consequently, all the series appearing in the definition of $b(t)$ *converge absolutely and uniformly on* $[0,T]$. *Since* $h''(t)$, $f(x_0,t)$, *and* $[g(x_0,t)]^{-1}$ *are continuous on* $[0,T]$, *we conclude that*

$$b \in C[0,T].$$

Since $b \in C[0,T]$ *and* $G \in C(\Delta_T)$, *the linear Volterra integral equation* (29) *has a unique solution* $a \in C[0,T]$ *for every finite* $T > 0$.

We now show that the function $u(x,t)$ *defined by* (26) *belongs to* $C^2(\overline{D}_T)$. *Indeed, from* (24) *and* (25), $u_k(t) \in C^2[0,T]$, $k = 0,1,2,\dots$, *and*

$$\|u_0(t)\|_{C[0,T]} \le |\varphi_0|+T|\psi_0|+T\sqrt{T}\left(\int_0^T|f_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}}+T\sqrt{T}\|a(t)\|_{C[0,T]}\left(\int_0^T|g_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}},$$

$$\begin{aligned}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} &\le 2\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_k|)^2\right)^{\frac{1}{2}}+2\left(\sum_{k=1}^{\infty}(\lambda_k^2|\psi_k|)^2\right)^{\frac{1}{2}}\\ &\quad+2\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|f_k(\tau)|)^2\,d\tau\right)^{\frac{1}{2}}\\ &\quad+2\sqrt{T}\|a(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^2|g_k(\tau)|)^2\,d\tau\right)^{\frac{1}{2}}\end{aligned}$$

or

$$\begin{aligned}\|u_0(t)\|_{C[0,T]} &\le \|\varphi(x)\|_{L_2(0,1)}+T\|\psi(x)\|_{L_2(0,1)}+T\sqrt{T}\|f(x,t)\|_{L_2(D_T)}\\ &\quad+T\sqrt{T}\|a(t)\|_{C[0,T]}\|g(x,t)\|_{L_2(D_T)},\end{aligned}$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le 2\|\varphi'''(x)\|_{L_2(0,1)} + 2\|\psi''(x)\|_{L_2(0,1)} + 2\sqrt{T}\|f_{xx}(x,t)\|_{L_2(D_T)} + 2\sqrt{T}\|a(t)\|_{C[0,T]}\|g_{xx}(x,t)\|_{L_2(D_T)}.$$

From the representation (21) *we have*

$$u_x(x,t) = -\sum_{k=1}^{\infty}\lambda_k u_k(t)\sin\lambda_k x, \qquad u_{xx}(x,t) = -\sum_{k=1}^{\infty}\lambda_k^2 u_k(t)\cos\lambda_k x.$$

By the Cauchy–Schwarz inequality,

$$\sum_{k=1}^{\infty}|u_k(t)| \le \left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^6}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},$$

$$\sum_{k=1}^{\infty}\lambda_k|u_k(t)| \le \left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^4}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},$$

and

$$\sum_{k=1}^{\infty}\lambda_k^2|u_k(t)| \le \left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}.$$

The estimates above establish that $u(x,t)$ *and its derivatives of orders one and two with respect to* x *are continuous on* $\overline{D}_T$.

We now establish the continuity of the mixed derivative $u_{tx}(x,t)$. *Differentiating* (25) *with respect to* t, *we obtain*

$$u_k'(t) = -\lambda_k\varphi_k\sin(\lambda_k t) + \psi_k\cos(\lambda_k t) + \int_0^t F_k(\tau;a)\cos(\lambda_k(t-\tau))\,d\tau.$$

For every $k \ge 1$, *formula* (25) *yields*

$$\lambda_k^2\|u_k'\|_{C[0,T]} \le \lambda_k^3|\varphi_k| + \lambda_k^2|\psi_k| + \lambda_k^2\int_0^T|F_k(\tau;a)|\,d\tau.$$

By the Minkowski and Cauchy–Schwarz inequalities,

$$\left(\sum_{k=1}^{\infty}\lambda_k^4\|u_k'\|_{C[0,T]}^2\right)^{\frac{1}{2}} \le \left(\sum_{k=1}^{\infty}\lambda_k^6|\varphi_k|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\lambda_k^4|\psi_k|^2\right)^{\frac{1}{2}} + \sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\lambda_k^4|F_k(\tau;a)|^2\,d\tau\right)^{\frac{1}{2}}.$$

Since $F_k(t;a) = f_k(t) + a(t)g_k(t)$, *the Parseval inequality and the assumptions of Theorem 3 yield*

$$\left(\int_0^T\sum_{k=1}^{\infty}\lambda_k^4|F_k(t;a)|^2\,dt\right)^{\frac{1}{2}} \le \sqrt{2}\,\|f_{xx}\|_{L^2(D_T)} + \sqrt{2T}\,\|a\|_{C[0,T]}\|g_{xx}\|_{C([0,T];L^2(0,1))}.$$

Consequently, there exists a constant $C > 0$ such that

$$\left(\sum_{k=1}^{\infty} \lambda_k^4 \|u_k'\|_{C[0,T]}^2\right)^{\frac{1}{2}} \le C.$$

For any integers $M \ge N \ge 1$, we have

$$\left|\sum_{k=N}^{M} \lambda_k u_k'(t)\sin(\lambda_k x)\right| \le \left(\sum_{k=N}^{M} \frac{1}{\lambda_k^2}\right)^{\frac{1}{2}} \left(\sum_{k=N}^{M} \lambda_k^4 |u_k'(t)|^2\right)^{\frac{1}{2}} \le C\left(\sum_{k=N}^{\infty} \frac{1}{\lambda_k^2}\right)^{\frac{1}{2}} \longrightarrow 0 \quad (N \to \infty).$$

Thus, the series

$$-\sum_{k=1}^{\infty} \lambda_k u_k'(t)\sin(\lambda_k x)$$

converges absolutely and uniformly on $\overline{D}_T$.

Since the series for $u_t(x,t)$ also converges uniformly, termwise differentiation with respect to x is justified. Therefore,

$$u_{tx}(x,t) = -\sum_{k=1}^{\infty} \lambda_k u_k'(t)\sin(\lambda_k x) \in C(\overline{D}_T).$$

We now establish the existence and continuity of the second derivative with respect to time. Let

$$F(x,t) = a(t)g(x,t) + f(x,t).$$

By the assumptions of Theorem 3,

$$F \in C(\overline{D}_T).$$

Since the continuity of u_{xx} has already been established, the function

$$v(x,t) = u_{xx}(x,t) + F(x,t)$$

belongs to $C(\overline{D}_T)$. For each fixed $t \in [0,T]$, the cosine Fourier coefficients of $v(\cdot,t)$ satisfy

$$v_k(t) = -\lambda_k^2 u_k(t) + F_k(t;a) = u_k''(t), \qquad k \ge 1,$$

and

$$v_0(t) = F_0(t;a) = u_0''(t).$$

For each fixed $t \in [0,T]$, the cosine Fourier coefficients of both sides of the following identity coincide by virtue of (22)–(25). Since the cosine system is complete in $L^2(0,1)$, we obtain

$$u(\cdot,t) = \varphi + t\psi + \int_0^t (t-\tau)v(\cdot,\tau)\,d\tau$$

in $L^2(0,1)$. *Since both sides are continuous with respect to* x, *this identity holds pointwise on* $\overline{D}_T$. *Differentiating twice with respect to* t, *we obtain*

$$u_{tt}(x,t) = v(x,t) = u_{xx}(x,t) + a(t)g(x,t) + f(x,t).$$

Therefore,

$$u_{tt} \in C(\overline{D}_T).$$

Together with the previously established continuity of u, u_x, u_{xx}, u_t, *and* u_{tx}, *this proves that*

$$u \in C^2(\overline{D}_T).$$

It is easy to verify that equation (1) *and conditions* (2)–(3), (8), (9) *are satisfied in the usual sense. Thus, the pair of functions* $\{u(x,t), a(t)\}$ *is a solution of problem* (1)–(3), (8), (9). *To establish uniqueness, let* (u_1, a_1) *and* (u_2, a_2) *be two classical solutions of problem* (1)–(3), (8), (9) *corresponding to the same prescribed data.*

For each $i = 1, 2$, *fix the function* $a_i(t)$ *and define* $\widetilde{u}_i(x,t)$ *by the Fourier representation* (26), *with* $a(t) = a_i(t)$. *By the regularity estimates established above, the series defining* $\widetilde{u}_i$ *and its required derivatives converge uniformly on* $\overline{D}_T$. *Consequently,* $\widetilde{u}_i \in C^2(\overline{D}_T)$. *Moreover,* $\widetilde{u}_i$ *satisfies equation* (1), *the initial conditions* (2), *and the homogeneous Neumann boundary conditions* (3) *and* (8), *with the prescribed function* $a_i(t)$. *Since* u_i *is also a classical solution of the same direct problem, Theorem 2 implies that*

$$u_i(x,t) = \widetilde{u}_i(x,t), \qquad (x,t) \in \overline{D}_T, \quad i = 1, 2.$$

Therefore, representation (26) *holds for each classical solution* u_i. *In particular, the uniformly convergent series for the second spatial derivative represents* $u_{i,xx}(x,t)$. *Substituting this representation into the additional condition* (9), *we conclude that both* $a_1(t)$ *and* $a_2(t)$ *satisfy the same linear Volterra integral equation* (29). *Since equation* (29) *has a unique solution in* $C[0,T]$, *it follows that*

$$a_1(t) = a_2(t), \qquad 0 \le t \le T.$$

By Theorem 2, the corresponding solutions of the hyperbolic equation coincide:

$$u_1(x,t) = u_2(x,t), \qquad (x,t) \in \overline{D}_T.$$

Thus, the solution pair (u, a) *is unique. The theorem is proved. By means of Theorems 1 and 3, we obtain the unique solvability of problem* (1)–(5).

Theorem 4 *Suppose that the hypotheses of Theorem 3 hold and that the data satisfy the compatibility relations* (6) *and* (7). *Assume, in addition, that both source functions have zero spatial mean for every* $t \in [0,T]$:

$$\int_0^1 f(x,t)\,dx = \int_0^1 g(x,t)\,dx = 0.$$

Under these assumptions, there exists exactly one pair $(u, a) \in C^2(\overline{D}_T) \times C[0,T]$ *that solves problem* (1)–(5) *in the classical sense.*

4 Examples

In this section, we present an illustrative example. The inverse problem under consideration is governed by the equation

$$u_{tt}(x,t) - u_{xx}(x,t) = a(t)e^t \cos 3\pi x + t^6 \cos 3\pi x,$$

in the domain $D_T = \{(x,t) : 0 < x < 1,\ 0 \le t \le T\}$, with the initial conditions

$$u(x,0) = -7\cos 3\pi x, \qquad u_t(x,0) = -2\cos 3\pi x, \qquad 0 \le x \le 1.$$

At the left endpoint, the spatial derivative is required to vanish:

$$u_x(0,t) = 0, \qquad 0 \le t \le T.$$

The solution must also have zero spatial mean at every time:

$$\int_0^1 u(x,t)\,dx = 0, \qquad 0 \le t \le T.$$

At the right endpoint, its value is prescribed by

$$u(1,t) = t^3 + 2t^2 + 2t + 7, \qquad 0 \le t \le T.$$

The functions to be determined are $u(x,t)$ and $a(t)$. In this example, $x_0 = 1$. The prescribed data satisfy all the assumptions of Theorem 4, including the zero-mean conditions, the compatibility relations (6) and (7), and the nonvanishing condition $g(1,t) = -e^t \neq 0$ for $t \in [0,T]$. The two components of the solution can be expressed explicitly as follows:

$$u(x,t) = -(t^3 + 2t^2 + 2t + 7)\cos 3\pi x,$$

and

$$a(t) = -e^{-t}\left[9\pi^2(t^3 + 2t^2 + 2t + 7) + t^6 + 6t + 4\right].$$

Direct substitution shows that the obtained functions satisfy the differential equation, the initial conditions, the Neumann boundary condition, the spatial integral condition, and the additional pointwise observation. Thus, the example illustrates the existence and uniqueness result established in Theorem 4.

5 Conclusion

In this paper, a linear inverse boundary value problem for a second-order hyperbolic equation with an unknown time-dependent source factor has been investigated. Under the imposed zero-mean and compatibility conditions, the original problem with a spatial integral condition of the first kind has been shown to be equivalent to an auxiliary problem with homogeneous Neumann boundary conditions at both endpoints. By applying the Fourier method, the determination of the unknown source factor has been reduced to a linear Volterra integral equation of the second kind. The existence and uniqueness of a classical solution have been established for any finite time interval. An illustrative example has been provided to demonstrate the obtained results. Further research may focus on the existence and uniqueness of classical solutions to nonlinear inverse problems for second-order hyperbolic equations.

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