IRSTI 27.31.44 DOI: https://doi.org/10.26577/JMMCS206110426

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SOLVABILITY AND CONSTRUCTIVE SOLUTION OF A THREE-POINT BOUNDARY VALUE PROBLEM FOR A HYPERBOLIC EQUATION WITH PIECEWISE CONSTANT ARGUMENT AND A DISTRIBUTED PARAMETER

The present study addresses a three-point boundary value problem for a linear hyperbolic system involving a piecewise constant argument and an unknown distributed parameter. The problem is supplemented by initial and boundary conditions and a three-point condition involving partial derivatives of the unknown function and the distributed parameter. With the aid of suitably chosen auxiliary functions, the initial problem is transformed into an equivalent family of three-point boundary value problems for functional differential systems involving functional parameters. Subsequently, Dzhumabaev's parametrization method is employed to reduce the resulting problems to systems of linear algebraic equations for the introduced parameters. A readily verifiable sufficient condition ensuring the unique solvability of the auxiliary problem is established through the nonsingularity of the associated block matrix, and a corresponding a priori bound for the solution is derived. A constructive algorithm is proposed that involves the sequential solution of Cauchy problems and systems of linear algebraic equations. The successive approximation technique is employed to derive sufficient conditions guaranteeing the existence and uniqueness of the solution to the original problem. These findings establish a theoretical framework for the numerical analysis of boundary value problems for hyperbolic equations involving piecewise constant arguments and unknown distributed parameters.

Keywords: hyperbolic equation, piecewise constant argument, distributed parameter, multipoint boundary condition, functional parametrization method.

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Бөлікті-тұрақты аргументі және үлестірілген параметрі бар гиперболалық теңдеу үшін үш нүктелі шеттік есептің шешілімділігі және конструктивті шешімі

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Бұл жұмыста бөлік-тұрақты аргументі және белгісіз үлестірілген параметрі бар сызықтық гиперболалық жүйе үшін үш нүктелі шеттік есеп зерттеледі. Есеп бастапқы және шеттік шарттармен, сондай-ақ белгісіз функцияның дербес туындылары мен үлестірілген параметрді қамтитын үш нүктелі шартпен толықтырылған. Тиісті түрде таңдалған көмекші функциялардың көмегімен бастапқы есеп функционалдық параметрлері бар функционалдық-дифференциалдық жүйелер үшін үш нүктелі шеттік есептердің эквивалентті үйіріне түрлендіріледі. Одан әрі алынған есептерді енгізілген параметрлерге қатысты сызықтық алгебралық теңдеулер жүйелеріне келтіру үшін Жұмабаевтың параметрлеу әдісі қолданылады. Сәйкес блоктық матрицаның айрықша еместігі терминдерінде көмекші есептің бірмәнді шешілімділігін қамтамасыз ететін оңай тексерілетін жеткілікті шарт тұжырымдалып, оның шешімі үшін тиісті априорлық баға алынады. Коши есептері мен сызықтық алгебралық теңдеулер жүйелерін тізбектей шешуге негізделген конструктивті алгоритм ұсынылады. Тізбектей жуықтау әдісінің көмегімен бастапқы есеп шешімінің бар болуы мен жалғыздығына кепілдік беретін жеткілікті шарттар алынды. Алынған нәтижелер бөлік-тұрақты аргументтері және белгісіз үлестірілген параметрлері бар гиперболалық теңдеулер үшін шеттік есептерді сандық зерттеудің теориялық негізін құрайды.
Түйін сөздер: гиперболалық теңдеу, бөлікті-тұрақты аргумент, үлестірілген параметр, көпнүктелі шеттік шарт, функционалдық параметрлеу әдісі.

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Разрешимость и конструктивное решение трехточечной краевой задачи для гиперболического уравнения с кусочно-постоянным аргументом и распределенным параметром

В настоящей работе исследуется трехточечная краевая задача для линейной гиперболической системы с кусочно-постоянным аргументом и неизвестным распределенным параметром. Задача дополняется начальным и краевым условиями, а также трехточечным условием, содержащим частные производные неизвестной функции и распределенный параметр. С помощью соответствующим образом выбранных вспомогательных функций исходная задача преобразуется в эквивалентное семейство трехточечных краевых задач для функционально-дифференциальных систем с функциональными параметрами. Затем для сведения полученных задач к системам линейных алгебраических уравнений относительно введенных параметров применяется метод параметризации Джумабаева. В терминах невырожденности соответствующей блочной матрицы устанавливается легко проверяемое достаточное условие однозначной разрешимости вспомогательной задачи и выводится соответствующая априорная оценка решения. Предлагается конструктивный алгоритм, основанный на последовательном решении задач Коши и систем линейных алгебраических уравнений. С помощью метода последовательных приближений получены достаточные условия, гарантирующие существование и единственность решения исходной задачи. Полученные результаты составляют теоретическую основу для численного исследования краевых задач для гиперболических уравнений с кусочно-постоянными аргументами и неизвестными распределенными параметрами.
Ключевые слова: гиперболическое уравнение, кусочно-постоянный аргумент, распределенный параметр, многоточечное краевое условие, метод функциональной параметризации.

1 Introduction

Functional differential equations constitute an important class of mathematical models for processes whose evolution depends not only on their current state but also on states

attained at preceding or prescribed moments. Such equations arise naturally in control theory, mathematical biology, population dynamics, engineering, and other fields involving aftereffects and delayed responses.The theoretical foundations of functional differential equations and delay differential equations have been thoroughly investigated; we refer, for instance, to [1–3] and the literature cited therein.

Differential equations involving piecewise constant arguments constitute a significant subclass of such equations. These equations combine characteristic features of differential and difference equations and provide natural models for systems whose states are observed, controlled, or updated only at discrete moments. The systematic study of such equations was initiated in [4, 5], while further developments and applications are presented in [6]. Since the argument is discontinuous, the time interval must be partitioned into appropriate subintervals, and the derivative of a solution is generally only piecewise continuous. Consequently, the analysis of existence, uniqueness, stability, and numerical approximation requires methods that differ from those used for differential equations with continuous arguments.

Boundary value problems for hyperbolic equations arise in mathematical models of wave propagation, transport processes, and other phenomena depending on two independent variables. Multipoint and nonlocal conditions establish relationships between the values of an unknown function, or those of its derivatives, at several distinct points of the domain. However, the presence of such conditions creates additional difficulties in establishing the existence, uniqueness, and continuous dependence of solutions.

Numerous analytical and computational approaches have been proposed for the investigation of nonlocal boundary value problems for hyperbolic equations. In [7], numerical-analytical techniques were introduced for constructing periodic solutions to partial differential equations. Criteria for the well-posedness and unique solvability of nonlocal boundary value problems for hyperbolic systems were obtained in [8–10]. Related problems involving integral boundary conditions and impulsive effects were subsequently investigated in [11, 12]. Further developments in this area, including periodic problems with finite time delay, generalized integral conditions, and periodic boundary conditions, are presented in [13–17].

Multipoint boundary value problems for systems of differential equations with generalized piecewise constant arguments were studied in [18], where the parameterization method was employed to derive solvability conditions and constructive solution algorithms. Nevertheless, to the best of our knowledge, three-point boundary value problems involving simultaneously a generalized piecewise constant argument, derivative conditions, and an unknown distributed parameter have not yet been addressed. The interaction of these features requires new solvability conditions and a constructive procedure for determining both the solution of the hyperbolic system and the unknown distributed parameter.

This study focuses on a three-point boundary value problem for a linear hyperbolic system involving a generalized piecewise constant argument and an unknown distributed parameter. The system contains the functional term

$$\frac{\partial \mathbf{u}(\gamma(t), x)}{\partial x},$$

where $\gamma(t)$ is a piecewise constant function. The three-point condition relates the values of $\partial \mathbf{u}(t, x)/\partial x$ at $t_0 = 0$, t_1, and $t_2 = T$ to the unknown distributed parameter $\boldsymbol{\theta}(x)$. By defining

the auxiliary functions

$$\mathbf{v}(t,x)=\frac{\partial \mathbf{u}(t,x)}{\partial x},\qquad \mathbf{w}(t,x)=\frac{\partial \mathbf{u}(t,x)}{\partial t},$$

the initial problem is transformed into an equivalent family of three-point boundary value problems for functional differential systems containing a distributed parameter. The resulting family is investigated using Dzhumabaev's parameterization method [19, 20]. In accordance with the structure of the piecewise constant argument, the time interval is partitioned into subintervals, and additional functional parameters are introduced at the partition points and at the points determining the functional argument. As a result, the auxiliary boundary value problems are converted into systems of linear algebraic equations for the newly introduced parameters.

The nonsingularity of the associated block matrix provides a directly verifiable sufficient condition guaranteeing the existence of a unique solution to the auxiliary problem. An a priori estimate for its solution is also obtained. Based on this result, a sequence of successive approximations is constructed for the original hyperbolic problem, and sufficient conditions for its convergence are established. As a consequence, sufficient conditions guaranteeing the existence and uniqueness of the solution pair $\big(\mathbf{u}(t,x),\boldsymbol{\theta}(x)\big)$ are established. Moreover, the constructive character of the proposed method makes it suitable for numerical implementation.

The principal results of the study can be summarized as follows. First, the considered three-point boundary value problem for a hyperbolic system is shown to be equivalent to a family of boundary value problems for functional differential systems containing an unknown distributed parameter. The parametrization method developed by Dzhumabaev is then extended to this family, allowing the introduced functional parameters to be determined from a system of linear algebraic equations. Based on the nonsingularity of the associated block matrix, a directly verifiable criterion ensuring the unique solvability of the auxiliary problem is established. An a priori bound for the solution of the parameterized problem is also derived. Finally, a constructive iterative procedure is proposed, and conditions guaranteeing the existence and uniqueness of the solution to the original problem are obtained.

The paper is structured as follows. We begin by stating the three-point boundary value problem and introducing the definition of its classical solution. The original problem is subsequently reformulated as an equivalent problem involving auxiliary functions. Finally, the parametrization method is applied to the resulting family of functional differential equations, and sufficient criteria ensuring unique solvability are derived. Subsequently, a constructive solution algorithm is presented. Finally, the convergence of the successive approximations is proved, which yields the existence and uniqueness of a classical solution to the original hyperbolic problem.

2 Problem Statement

Our analysis focuses on a three-point boundary value problem for a linear hyperbolic system involving a piecewise constant argument and an unknown distributed parameter. Let

$$\overline{\Omega}=[0,T]\times[0,\omega]$$

be the domain of the problem.

The problem consists in finding a vector-valued function $\mathbf{u}(t,x) = \operatorname{col}(u_1(t,x),\dots,u_n(t,x))$ and an unknown distributed parameter $\boldsymbol{\theta}(x) = \operatorname{col}(\theta_1(x),\dots,\theta_n(x))$ satisfying the delay hyperbolic equation

$$\frac{\partial^2 \mathbf{u}}{\partial t \partial x} = A(t,x)\frac{\partial \mathbf{u}}{\partial x} + B(t,x)\frac{\partial \mathbf{u}}{\partial t} + C(t,x)\mathbf{u} + D(t,x)\frac{\partial \mathbf{u}(\gamma(t),x)}{\partial x} + K(t,x)\boldsymbol{\theta}(x) + f(t,x), \quad (1)$$

$$(t,x) \in \Omega = [0,T) \times [0,\omega],$$

the parameter-dependent boundary condition

$$M_1(x)\frac{\partial \mathbf{u}(0,x)}{\partial x} + M_2(x)\frac{\partial \mathbf{u}(t_1,x)}{\partial x} + M_3(x)\frac{\partial \mathbf{u}(T,x)}{\partial x} + M_4(x)\boldsymbol{\theta}(x) = \varphi_1(x), \quad (2)$$

$$0 = t_0 < t_1 < t_2 = T, \quad x \in [0,\omega],$$

and the initial and boundary conditions

$$\mathbf{u}(0,x) = \varphi_2(x), \qquad x \in [0,\omega], \quad (3)$$

$$\mathbf{u}(t,0) = \psi(t), \qquad t \in [0,T]. \quad (4)$$

The following assumptions are imposed:

- the matrix-valued functions $A, B, C, D, K\colon \Omega \to \mathbb{R}^{n\times n}$ and the vector-valued function $f\colon \Omega \to \mathbb{R}^n$ are continuous;
- the matrix-valued functions $M_i\colon [0,\omega] \to \mathbb{R}^{n\times n}$, $i = \overline{1,4}$, and the vector-valued function $\varphi_1\colon [0,\omega] \to \mathbb{R}^n$ are continuous;
- the vector-valued functions $\varphi_2\colon [0,\omega] \to \mathbb{R}^n$ and $\psi\colon [0,T] \to \mathbb{R}^n$ are continuously differentiable on their respective domains;
- the piecewise constant argument $\gamma\colon [0,T] \to [0,T]$ is defined by

  $$\gamma(t) = \xi_k, \qquad t \in [t_{k-1}, t_k), \qquad k = 1,2,$$

  where ξ_k are fixed constants satisfying

  $$t_{k-1} < \xi_k < t_k, \qquad k = 1,2;$$

- the initial and boundary data satisfy the compatibility condition at the corner point $(0,0)$:

  $$\varphi_2(0) = \psi(0).$$

We shall use the following function spaces. The space of all continuous mappings from Ω into $\mathbb{R}^n$ is denoted by $\mathbb{C}(\Omega,\mathbb{R}^n)$. For $\mathbf{u} \in \mathbb{C}(\Omega,\mathbb{R}^n)$, its norm is defined by

$$\|\mathbf{u}\|_0 := \max_{(t,x)\in\Omega} \|\mathbf{u}(t,x)\|.$$

By $\mathbb{C}^1([0,\omega],\mathbb{R}^n)$ we mean the space of continuously differentiable vector-valued functions on $[0,\omega]$. This space is endowed with the norm

$$\|\varphi\|_{0,1} := \max\left\{\max_{x\in[0,\omega]}\|\varphi(x)\|, \max_{x\in[0,\omega]}\|\varphi'(x)\|\right\}.$$

Similarly, $\mathbb{C}^1([0,T],\mathbb{R}^n)$ stands for the space of continuously differentiable functions $\boldsymbol{\psi}\colon [0,T]\to\mathbb{R}^n$, with the norm

$$\|\psi\|_{1,0} := \max\left\{\max_{t\in[0,T]}\|\psi(t)\|, \max_{t\in[0,T]}\|\psi'(t)\|\right\}.$$

A pair $(\mathbf{u}^*,\boldsymbol{\theta}^*)$ is said to be a solution of problem (1)–(4) if $\mathbf{u}^*\in\mathbb{C}(\overline{\Omega},\mathbb{R}^n)$ and $\boldsymbol{\theta}^*\in\mathbb{C}([0,\omega],\mathbb{R}^n)$. It is also required that the partial derivative $\partial\mathbf{u}^*/\partial x$ belong to $\mathbb{C}(\Omega,\mathbb{R}^n)$. The partial derivative $\partial\mathbf{u}^*/\partial t$ and the mixed derivative $\partial^2\mathbf{u}^*/(\partial t\,\partial x)$ must exist at every $(t,x)\in\Omega$, except possibly at the points (t_{r-1},x), $r=1,2$, where the corresponding one-sided mixed derivatives are assumed to exist for every $x\in[0,\omega]$. Finally, upon setting $\boldsymbol{\theta}(x)=\boldsymbol{\theta}^*(x)$, the function $\mathbf{u}^*(t,x)$ must satisfy equation (1) at all points where the mixed derivative is defined, as well as the initial and boundary conditions (2)–(4).

3 Reduction to an Equivalent Problem and a Solution Algorithm

To obtain an equivalent formulation of problem (1)–(4), we introduce the auxiliary functions

$$\mathbf{v}(t,x)=\frac{\partial\mathbf{u}(t,x)}{\partial x},\qquad \mathbf{w}(t,x)=\frac{\partial\mathbf{u}(t,x)}{\partial t}. \tag{5}$$

Then problem (1)–(4) can be equivalently represented as the following problem:

$$\begin{aligned}\frac{\partial\mathbf{v}}{\partial t} &= A(t,x)\mathbf{v}+D(t,x)\mathbf{v}(\gamma(t),x)+K(t,x)\boldsymbol{\theta}(x)+B(t,x)\mathbf{w}(t,x)\\ &\quad + C(t,x)\mathbf{u}(t,x)+f(t,x),\qquad (t,x)\in\Omega,\end{aligned} \tag{6}$$

$$\sum_{j=1}^{3}M_j(x)\mathbf{v}(t_{j-1},x)+M_4(x)\boldsymbol{\theta}(x)=\varphi_1(x),\qquad x\in[0,\omega], \tag{7}$$

$$\mathbf{v}(t_0,x)=\varphi_2'(x),\quad x\in[0,\omega], \tag{8}$$

$$\mathbf{u}(t,x)=\psi(t)+\int_0^x\mathbf{v}(t,\eta)\,d\eta,\qquad \mathbf{w}(t,x)=\psi'(t)+\int_0^x\frac{\partial\mathbf{v}(t,\eta)}{\partial t}\,d\eta. \tag{9}$$

where the relations in (9) are derived from condition (4) of the original problem (1)–(4) and relations (5).

A quadruple $(\mathbf{v}^*,\mathbf{u}^*,\mathbf{w}^*,\boldsymbol{\theta}^*)$, where $\mathbf{v}^*,\mathbf{u}^*,\mathbf{w}^*\in\mathbb{C}(\Omega,\mathbb{R}^n)$ and $\boldsymbol{\theta}^*\in\mathbb{C}([0,\omega],\mathbb{R}^n)$, is said to be a solution of problem (6)–(9) provided that $\mathbf{v}^*(t,x)$ is continuously differentiable with respect to t, that is,

$$\frac{\partial\mathbf{v}^*}{\partial t}\in\mathbb{C}(\Omega,\mathbb{R}^n),$$

and the functions $\mathbf{u}^*(t,x)$ and $\mathbf{w}^*(t,x)$ are connected with $\mathbf{v}^*(t,x)$ through the integral relations (9).

Accordingly, problems (1)–(4) can be reformulated as an equivalent problem comprising a family of systems differential equations with generalized piecewise constant argument with a distributed parameter together with the corresponding integral relations.

By treating $\mathbf{u}(t,x)$ and $\mathbf{w}(t,x)$ as prescribed functions, problem (6)–(9) is interpreted as a family of three-point boundary value problems for ordinary differential systems with a piecewise constant argument and a distributed parameter. Meanwhile, the integral identities (9) express $\mathbf{u}(t,x)$ and $\mathbf{w}(t,x)$ through $\mathbf{v}(t,x)$ and its partial derivative $\dfrac{\partial \mathbf{v}(t,x)}{\partial t}$. These observations lead to an iterative construction of the solution to problem (6)–(9), which is described in the algorithm below.

Algorithm $\mathcal{A}$.

Step 0. (*a*) At the initial step, the right-hand side of system (6) is evaluated with $\mathbf{u}(t,x) = \boldsymbol{\psi}(t)$, $\mathbf{w}(t,x) = \dot{\boldsymbol{\psi}}(t)$, which yields the pair $\big(\mathbf{v}^{(0)}(t,x), \boldsymbol{\theta}^{(0)}(x)\big)$, where $\mathbf{v}^{(0)}(t,x) \in \mathbb{C}\big(\Omega, \mathbb{R}^n\big)$, $\boldsymbol{\theta}^{(0)}(x) \in \mathbb{C}\big([0,\omega], \mathbb{R}^n\big)$ from the family of problems (6)–(8);

(*b*) Using the integral relations (9) with $\mathbf{v}(t,x) = \mathbf{v}^{(0)}(t,x)$, $\frac{\partial \mathbf{v}(t,x)}{\partial t} = \frac{\partial \mathbf{v}^{(0)}(t,x)}{\partial t}$, we determine $\mathbf{u}^{(0)}(t,x), \mathbf{w}^{(0)}(t,x) \in \mathbb{C}\big(\Omega, \mathbb{R}^n\big)$.

Step k. Assume that the approximations $\mathbf{u}^{(k-1)}(t,x)$ and $\mathbf{w}^{(k-1)}(t,x)$ have already been constructed.

(*a*) Substituting $\mathbf{u}(t,x) = \mathbf{u}^{(k-1)}(t,x)$, $\mathbf{w}(t,x) = \mathbf{w}^{(k-1)}(t,x)$ into the right-hand side of equation (6) and solving problem (6)–(8), we obtain the next approximation $\big(\mathbf{v}^{(k)}(t,x), \boldsymbol{\theta}^{(k)}(x)\big)$, where $\mathbf{v}^{(k)}(t,x) \in \mathbb{C}(\Omega, \mathbb{R}^n)$ and $\boldsymbol{\theta}^{(k)}(x) \in \mathbb{C}([0,\omega], \mathbb{R}^n)$.

(*b*) The approximations $\mathbf{u}^{(k)}(t,x)$ and $\mathbf{w}^{(k)}(t,x)$ are then recovered from the integral relations (9) by setting $\mathbf{v}(t,x) = \mathbf{v}^{(k)}(t,x)$, $\frac{\partial \mathbf{v}(t,x)}{\partial t} = \frac{\partial \mathbf{v}^{(k)}(t,x)}{\partial t}$. Consequently, $\mathbf{u}^{(k)}(t,x)$, $\mathbf{w}^{(k)}(t,x) \in \mathbb{C}(\Omega, \mathbb{R}^n)$.

The implementation of Algorithm $\mathcal{A}$ relies on the unique solvability of the family of three-point boundary value problems (6)–(8) for ordinary differential systems involving a piecewise constant argument and a distributed parameter, when $\mathbf{u}(t,x)$ and $\mathbf{w}(t,x)$ are regarded as prescribed functions. This issue will be addressed in the next section. The convergence conditions for Algorithm $\mathcal{A}$ will be presented in Section 5.

4 Analysis of a Family of Three-Point Boundary Value Problems for Differential Equations with a Piecewise Constant Argument and a Distributed Parameter

This section is devoted to the study of a family of three-point boundary value problems for ordinary differential equations involving a piecewise constant argument and a distributed

parameter:

$$\frac{\partial \mathbf{v}}{\partial t} = A(t,x)\mathbf{v} + D(t,x)\mathbf{v}(\gamma(t),x) + K(t,x)\boldsymbol{\theta}(x) + g(t,x), \qquad (t,x) \in \Omega, \tag{10}$$

$$\sum_{j=1}^{3} M_j(x)\mathbf{v}(t_{j-1},x) + M_4(x)\boldsymbol{\theta}(x) = \varphi_1(x), \qquad x \in [0,\omega], \tag{11}$$

$$\mathbf{v}(0,x) = \varphi_2'(x), \qquad x \in [0,\omega], \tag{12}$$

where the n-vector function $g(t,x) \in \mathbb{C}(\Omega,\mathbb{R}^n)$.

A pair $\big(\mathbf{v}^*(t,x),\boldsymbol{\theta}^*(x)\big)$ is said to solve the family of problems (10)–(12) if $\mathbf{v}^*(t,x) \in \mathbb{C}\big(\Omega,\mathbb{R}^n\big)$, $\frac{\partial \mathbf{v}^*(t,x)}{\partial t} \in \mathbb{C}\big(\Omega,\mathbb{R}^n\big)$, and $\boldsymbol{\theta}^*(x) \in \mathbb{C}\big([0,\omega],\mathbb{R}^n\big)$. Moreover, the function $\mathbf{v}^*(t,x)$ satisfies equation (10) and conditions (11) and (12) by $\boldsymbol{\theta}(x) = \boldsymbol{\theta}^*(x)$ for every $x \in [0,\omega]$.

When $x \in [0,\omega]$ is held fixed, relations (10)–(12) define a linear three-point boundary value problem for an ordinary differential system with a piecewise constant argument, in which $\boldsymbol{\theta}(x)$ plays the role of an unknown parameter. Considering this problem for every $x \in [0,\omega]$ gives rise to a family of linear three-point boundary value problems with the unknown distributed parameter $\boldsymbol{\theta}(x)$.

We now apply the parametrization method [19] to the family of problems (10)–(12). Let Δ denote the partition of the domain Ω given by

$$\Omega = \Omega_1 \cup\ \Omega_2, \quad \text{where } \Omega_r = [t_{r-1},t_r) \times [0,\omega] \text{ for } r = 1,2.$$

For each $x \in [0,\omega]$, we introduce the functional parameters

$$\lambda_r(x) := \mathbf{v}(t_{r-1},x), \quad r = 1,2, \quad \lambda_3(x) := \lim_{t\to T-0} \mathbf{v}(t,x), \quad \lambda_4(x) := \boldsymbol{\theta}(x),$$

$$\mu_1(x) := \mathbf{v}(\xi_1,x), \quad \mu_2(x) := \mathbf{v}(\xi_2,x).$$

and on each subdomain Ω_r, for $r = 1,2$, we write

$$\mathbf{v}(t,x) = \mathbf{y}_r(t,x) + \lambda_r(x).$$

With this substitution, problem (10)–(12) can be transformed into the following equivalent problem with parameters:

$$\frac{\partial \mathbf{y}_r}{\partial t} = A(t,x)\big(\mathbf{y}_r + \lambda_r(x)\big) + D(t,x)\mu_r(x) + K(t,x)\lambda_4(x) + g(t,x), \quad (t,x) \in \Omega_r, \tag{13}$$

$$\mathbf{y}_r(t_{r-1},x) = 0, \qquad r = 1,2, \tag{14}$$

$$\sum_{j=1}^{4} M_j(x)\lambda_j(x) = \varphi_1(x), \qquad x \in [0,\omega], \tag{15}$$

$$\lambda_1(x) = \varphi_2'(x), \qquad x \in [0,\omega], \tag{16}$$

$$\lambda_s(x) + \lim_{t\to t_s-0} \mathbf{y}_s(t,x) = \lambda_{s+1}(x), \qquad s = 1,2, \qquad x \in [0,\omega], \tag{17}$$

$$\lambda_i(x) + \mathbf{y}_i(\xi_i,x) = \mu_i(x), \quad i = 1,2, \qquad x \in [0,\omega]. \tag{18}$$

We denote by $\mathbb{C}(\Omega, \Delta, \mathbb{R}^{2n})$ the space consisting of vector-valued functions $\mathbf{y}([t], x) = \big(\mathbf{y}_1(t,x), \mathbf{y}_2(t,x)\big)$, such that $\mathbf{y}_r \in \mathbb{C}(\Omega_r, \mathbb{R}^n)$, $r = 1, 2$, and have finite left-hand limits at the points t_r, $r = 1, 2$, for every $x \in [0, \omega]$.

A pair $(\mathbf{y}^*, \mathbf{\Lambda}^*)$ is said to solve problem (13)–(18) if

$$\mathbf{y}^*([t], x) = \big(\mathbf{y}_1^*(t,x), \mathbf{y}_2^*(t,x)\big) \in \mathbb{C}(\Omega, \Delta, \mathbb{R}^{2n})$$

and

$$\Lambda^*(x) = \big(\lambda_1^*(x), \dots, \lambda_4^*(x), \mu_1^*(x), \mu_2^*(x)\big) \in \mathbb{C}([0, \omega], \mathbb{R}^{6n}),$$

and the pair (y^*, Λ^*) satisfies Eq. (13) and the conditions (14)–(18) for all $x \in [0, \omega]$.

Once the parameters $\lambda_1(x), \dots, \lambda_4(x), \mu_1^*(x), \mu_2^*(x)$ are fixed, the solutions of the initial value problems (13) and (14) admit the following representation:

$$\begin{aligned}\mathbf{y}_r(t,x) = \Phi_r(t,x) \int\limits_{t_{r-1}}^{t} \Phi_r^{-1}(s,x)\Big[A(s,x)\lambda_r(x) + D(s,x)\mu_r(x) \\ + K(s,x)\lambda_4(x) + g(s,x)\Big]ds, \quad (t,x) \in \Omega_r, \quad r = 1,2,\end{aligned} \tag{19}$$

where $\Phi_r(t,x)$, $r = 1, 2$, denotes the fundamental matrix of the homogeneous system

$$\frac{\partial \mathbf{y}_r}{\partial t} = A(t,x)\mathbf{y}_r \qquad (t,x) \in \Omega_r, \quad r = 1,2.$$

Introducing the notations

$$a_r(P,t,x) = \Phi_r(t,x) \int\limits_{t_{r-1}}^{t} \Phi_r^{-1}(s,x)P(s,x)ds, \quad (t,x) \in \Omega_r, \quad r = 1,2, \tag{20}$$

where $P(t,x)$ is either a $(n \times n)$-matrix or a n- vector-function.

Using (20), we can rewrite the integral representation (19) of $\mathbf{y}_r(t,x)$ in the more compact form

$$\mathbf{y}_r(t,x) = a_r(A,t,x)\lambda_r + a_r(D,t,x)\mu_r(x) + a_r(K,t,x)\lambda_4 + a_r(g,t,x), \tag{21}$$

$(t,x) \in \Omega_r, \quad r = 1,2.$

From (21) we determine $\lim\limits_{t \to t_r - 0} \mathbf{y}_r(t,x)$, $r = 1, 2$. Using (15)–(18), we construct a linear algebraic system for determining the unknown parameters:

$$Q_*(\Delta, x)\Lambda(x) = -\mathcal{F}_*(\Delta, x), \quad \Lambda(x) \in \mathbb{R}^{6n}, \tag{22}$$

where

$$Q_*(\Delta,x) = \begin{pmatrix} M_1(x) & M_2(x) & M_3(x) & M_4(x) & O & O \\ I & O & O & O & O & O \\ I{+}a_1(A,t_1,x) & -I & O & a_1(K,t_1,x) & a_1(D,t_1,x) & O \\ O & I{+}a_2(A,t_2,x) & -I & a_2(K,t_2,x) & O & a_2(D,t_2,x) \\ I{+}a_1(A,\xi_1,x) & O & O & a_1(K,\xi_1,x) & -I{+}a_1(D,\xi_1,x) & O \\ O & I{+}a_2(A,\xi_2,x) & O & a_2(K,\xi_2,x) & O & -I{+}a_2(D,\xi_2,x) \end{pmatrix},$$

$$\mathcal{F}_*(\Delta,x) = \big(-\varphi_1(x), -\varphi_2'(x), a_1(g,t_1,x), a_2(g,t_2,x), a_1(g,\xi_1,x), a_2(g,\xi_2,x)\big).$$

Thus, we obtain the system of equations (21) and (22) for determining the pair $(\mathbf{y}([t],x),\Lambda(x))$.

Next, we derive sufficient conditions guaranteeing the unique solvability of the family of three-point boundary value problems (10)–(12) for ordinary differential equations involving a piecewise constant argument and a distributed parameter.

Let us introduce the notations

$$\alpha(x)=\max_{(t,x)\in\Omega}\|A(t,x)\|,\qquad d(x)=\max_{(t,x)\in\Omega}\|D(t,x)\|,\qquad k(x)=\max_{(t,x)\in\Omega}\|K(t,x)\|,$$

$$h_\Delta=\max_{r=1,2}(t_r-t_{r-1}),\qquad L_\Delta(x)=h_\Delta e^{\alpha(x)h_\Delta}.$$

Theorem 1 *Let, for some partition* $\Delta=\{0=t_0<t_1<t_2=T\}$, *the matrix* $Q_*(\Delta,x)\colon\mathbb{R}^{6n}\to\mathbb{R}^{6n}$ *be invertible for every* $x\in[0,\omega]$ *and satisfy*

$$\|Q_*^{-1}(\Delta,x)\|\leqslant\gamma_*(\Delta,x).$$

Then, for any $g\in C(\Omega,\mathbb{R}^n)$, $\varphi_1\in C([0,\omega],\mathbb{R}^n)$, $\varphi_2\in C^1([0,\omega],\mathbb{R}^n)$, *problem* (13)–(18) *has a unique solution* $(\mathbf{y}^*,\Lambda^*)$. *Moreover, this solution satisfies the estimates*

$$\|\mathbf{y}^*([\cdot],x)\|_2\leqslant\mathcal{K}_1(x)\max\left\{\|g\|_0,\,\|\varphi_1\|_{0,1},\,\|\varphi_2\|_{0,1}\right\},\qquad x\in[0,\omega],\tag{23}$$

$$\|\Lambda^*(x)\|\leqslant\mathcal{K}_2(x)\max\left\{\|g\|_0,\,\|\varphi_1\|_{0,1},\,\|\varphi_2\|_{0,1}\right\},\qquad x\in[0,\omega],\tag{24}$$

where

$$\mathcal{K}_1(x)=L_\Delta(x)\big[1+(\alpha(x)+d(x)+k(x))\gamma_*(\Delta,x)\big]\max\{1,L_\Delta(x)\},$$

$$\mathcal{K}_2(x)=\gamma_*(\Delta,x)\max\{1,L_\Delta(x)\}.$$

The functions $\mathcal{K}_1(x)$ *and* $\mathcal{K}_2(x)$ *are independent of* $g(t,x)$, $\varphi_1(x)$, *and* $\varphi_2(x)$.

Proof 1 *Let* $g\in C(\Omega,\mathbb{R}^n)$, $\varphi_1\in C([0,\omega],\mathbb{R}^n)$, $\varphi_2\in C^1([0,\omega],\mathbb{R}^n)$ *be given. For a fixed* $x\in[0,\omega]$, *introduce the vector of functional parameters*

$$\Lambda(x)=\operatorname{col}\big(\lambda_1(x),\lambda_2(x),\lambda_3(x),\lambda_4(x),\mu_1(x),\mu_2(x)\big)\in\mathbb{R}^{6n}.$$

For prescribed values of the functional parameters, equations (13) *and the initial conditions* (14) *form Cauchy problems on the subintervals* $[t_{r-1},t_r)$, $r=1,2$. *Their solutions are given by representation* (21). *Substituting this representation into conditions* (15)–(18), *we obtain the linear algebraic system* (22) *for the unknown vector* $\Lambda(x)$.

Under the stated assumption, the matrix $Q_*(\Delta,x)$ *is invertible for every* $x\in[0,\omega]$. *Therefore, system* (22) *has the unique solution*

$$\Lambda^*(x)=-Q_*^{-1}(\Delta,x)\mathcal{F}_*(\Delta,x).$$

Substituting $\Lambda^*(x)$ *into representation* (21), *we uniquely determine the functions* $\mathbf{y}_r^*(t,x)$, $r=1,2$. *By construction, the pair* $\big(\mathbf{y}^*([\cdot],x),\Lambda^*(x)\big)$ *satisfies equations* (13), *the initial conditions* (14), *and conditions* (15)–(18). *Hence, problem* (13)–(18) *has a solution.*

We now derive estimates for this solution. Let $\Phi_r(t,x)$ *be the fundamental matrix of the homogeneous system*

$$\frac{\partial \mathbf{y}}{\partial t} = A(t,x)\mathbf{y}.$$

For $t_{r-1} \leqslant s \leqslant t \leqslant t_r$*, the standard estimate for the transition matrix yields*

$$\|\Phi_r(t,x)\Phi_r^{-1}(s,x)\| \leqslant e^{\alpha(x)(t-s)} \leqslant e^{\alpha(x)h_\Delta}.$$

Consequently, for every continuous matrix- or vector-valued function $P(t,x)$*, we have*

$$\|a_r(P,t,x)\| \leqslant e^{\alpha(x)h_\Delta} \int_{t_{r-1}}^{t} \|P(s,x)\|\,ds \leqslant h_\Delta e^{\alpha(x)h_\Delta}\|P\|_0 = L_\Delta(x)\|P\|_0. \tag{25}$$

Using the explicit form of the right-hand side $\mathcal{F}_*(\Delta,x)$ *of system* (22) *and estimate* (25)*, we obtain*

$$\|\mathcal{F}_*(\Delta,x)\| \leqslant \max\{1, L_\Delta(x)\} \max\left\{\|g\|_0,\ \|\varphi_1\|_{0,1},\ \|\varphi_2\|_{0,1}\right\}. \tag{26}$$

Since $\Lambda^*(x) = -Q_*^{-1}(\Delta,x)\mathcal{F}_*(\Delta,x)$*, it follows that*

$$\|\Lambda^*(x)\| \leqslant \gamma_*(\Delta,x)\|\mathcal{F}_*(\Delta,x)\|. \tag{27}$$

Combining estimates (26) *and* (27)*, we obtain estimate* (24)*.*

Next, representation (21) *and estimate* (25) *imply that*

$$\begin{aligned}\|\mathbf{y}_r^*(t,x)\| \leqslant{}& L_\Delta(x)\alpha(x)\|\lambda_r^*(x)\| + L_\Delta(x)d(x)\|\mu_r^*(x)\| \\ &+ L_\Delta(x)k(x)\|\lambda_4^*(x)\| + L_\Delta(x)\|g\|_0, \qquad r = 1,2.\end{aligned}$$

Taking the maximum over $r = 1,2$ *and* $t \in [t_{r-1}, t_r)$*, and observing that every component of* $\Lambda^*(x)$ *is bounded by* $\|\Lambda^*(x)\|$*, we obtain estimate* (23) *from estimate* (24) *and the definition of* $\mathcal{K}_1(x)$*.*

It remains to prove uniqueness. Suppose that problem (13)–(18) *has two solutions*

$$\left(\mathbf{y}^{(1)}, \Lambda^{(1)}\right) \quad \textit{and} \quad \left(\mathbf{y}^{(2)}, \Lambda^{(2)}\right).$$

Set $\widehat{\mathbf{y}}([t],x) = \mathbf{y}^{(1)}([t],x) - \mathbf{y}^{(2)}([t],x)$, $\widehat{\Lambda}(x) = \Lambda^{(1)}(x) - \Lambda^{(2)}(x)$*. Then the pair* $\left(\widehat{\mathbf{y}}, \widehat{\Lambda}\right)$ *satisfies the corresponding homogeneous problem. In particular, the functional parameters satisfy*

$$Q_*(\Delta,x)\widehat{\Lambda}(x) = 0.$$

Since $Q_*(\Delta,x)$ *is invertible, we have* $\widehat{\Lambda}(x) = 0$*. Applying representation* (21) *to the homogeneous problem, we then obtain* $\widehat{\mathbf{y}}_r(t,x) = 0$, $r = 1,2$*. Therefore,*

$$\mathbf{y}^{(1)}([t],x) = \mathbf{y}^{(2)}([t],x), \quad \Lambda^{(1)}(x) = \Lambda^{(2)}(x).$$

Consequently, problem (13)–(18) *admits at most one solution.*

Combining the equivalence between problems (10)–(12) and (13)–(18) with Theorem 1, we obtain the following statement.

Theorem 2 *Suppose that all the hypotheses of Theorem 1 are fulfilled. Then problem* (10)–(12) *admits exactly one solution pair* $\big(\mathbf{v}^*(t,x), \boldsymbol{\theta}^*(x)\big)$, *for which the following estimates are valid:*

$$\|\mathbf{v}^*(t,x)\| \leqslant (\mathcal{K}_1(x) + \mathcal{K}_2(x)) \max\left\{\|g\|_0,\ \|\varphi_1\|_{0,1},\ \|\varphi_2\|_{0,1}\right\}, \qquad x \in [0,\omega], \tag{28}$$

$$\|\boldsymbol{\theta}^*(x)\| \leqslant \mathcal{K}_2(x) \max\left\{\|g\|_0,\ \|\varphi_1\|_{0,1},\ \|\varphi_2\|_{0,1}\right\}, \qquad x \in [0,\omega]. \tag{29}$$

Next, we propose a constructive algorithm for the numerical solution of problem (10)–(12), based on solving Cauchy problems:

$$\frac{\partial \mathbf{y}}{\partial t} = A(t,x)\mathbf{y} + P(t,x), \quad \mathbf{y}(t_{r-1},x) = 0 \quad (t,x) \in \Omega_r, \quad r = 1,2, \tag{30}$$

Let $a_r(P,t,x)$ be the unique solution to problem (30) on each r-th interval, whose explicit form was given in (20).

Algorithm $\mathcal{B}$.

1. Consider the partition: $0 = t_0 < t_1 < t_2 = T$, and divide each subdomain Ω_r, $r = 1,2$, into N_r parts with a step size of $h_r = (t_r - t_{r-1})/N_r$.

2. Solve the following problem (30) with $P(t,x) \in \{A(t,x), D(t,x), K(t,x), g(t,x)\}$ find the functions $a_r^{h_r}(A,t,x)$, $a_r^{h_r}(D,t,x)$, $a_r^{h_r}(K,t,x)$, $a_r^{h_r}(g,t,x)$, $(t,x) \in \Omega_r$, $r = 1,2$.

3. We formulate the system of linear algebraic equations for the parameters:

$$Q_*^{h_r}(\Delta,x)\Lambda(x) = -\mathcal{F}_*^{h_r}(\Delta,x), \quad \Lambda(x) \in \mathbb{R}^{6n}. \tag{31}$$

Solving (31), we find $\Lambda^{h_r}(x) = \big(\lambda_1^{h_r}(x), \lambda_2^{h_r}(x), \lambda_3^{h_r}(x), \lambda_4^{h_r}(x), \mu_1^{h_r}(x), \mu_2^{h_r}(x)\big)$.

4. Solving the following Cauchy problems

$$\frac{\partial \mathbf{v}}{\partial t} = A(t,x)\mathbf{v} + D(t,x)\mu_r^{h_r}(x) + K(t,x)\lambda_4^{h_r}(x) + g(t,x) \quad (t,x) \in \Omega_r,$$

$$\mathbf{v}(t_{r-1},x) = \lambda_r^{h_r}, \quad x \in [0,\omega], \quad r = 1,2.$$

and find the values of the $\mathbf{v}^{h_r}(t,x)$ and $\boldsymbol{\theta}^{h_r}(x)$ at the remaining subintervals' points.

As a result, the proposed algorithm gives us the numerical solution to the problem (10)–(12).

5 Convergence conditions for Algorithm $\mathcal{A}$ and Main Results

The next theorem establishes sufficient conditions under which Algorithm $\mathcal{A}$ converges and problem (6)–(9) admits a unique solution.

Theorem 3 *Assume that all the hypotheses of Theorem 2 hold. Then problem* (6)–(9) *admits a unique solution* $\big(\mathbf{v}^*(t,x), \mathbf{u}^*(t,x), \mathbf{w}^*(t,x), \boldsymbol{\theta}^*(x)\big)$. *Furthermore, the solution satisfies estimates* (28) *and* (29).

Proof. Under the hypotheses of the theorem, Theorem 2 ensures that the family of boundary value problems (10)–(12) is uniquely solvable. We apply Algorithm $\mathcal{A}$ to construct the solution of problem (6)–(9). At the zeroth iteration, we first solve the following auxiliary problem:

$$\frac{\partial \mathbf{v}}{\partial t} = A(t,x)\mathbf{v} + D(t,x)\mathbf{v}(\gamma(t),x) + K(t,x)\boldsymbol{\theta}(x) + B(t,x)\psi'(t) \tag{32}$$
$$+ C(t,x)\psi(t) + f(t,x), \qquad (t,x) \in \Omega,$$

$$\sum_{j=1}^{3} M_j(x)\mathbf{v}(t_{j-1},x) + M_4(x)\boldsymbol{\theta}(x) = \varphi_1(x), \qquad x \in [0,\omega], \tag{33}$$

$$\mathbf{v}(t_0,x) = \varphi_2'(x), \qquad x \in [0,\omega]. \tag{34}$$

By assumption, problem (32)–(34) has a unique solution $\big(\mathbf{v}^{(0)}(t,x), \boldsymbol{\theta}^{(0)}(x)\big)$, and estimates (28) and (29) hold for this solution; that is,

$$\sup_{t\in[0,T]} \|\mathbf{v}^{(0)}(t,x)\| \leqslant \mathcal{K}_3(x) \max\Big\{ \max_{t\in[0,T]} \|B(t,x)\dot{\psi}(t)+C(t,x)\psi(t)+f(t,x)\|, \|\varphi_1\|_{0,1}, \|\varphi_2\|_{0,1}\Big\},$$

$$\|\boldsymbol{\theta}^{(0)}(x)\| \leqslant \mathcal{K}_2(x) \max\Big\{ \max_{t\in[0,T]} \|B(t,x)\dot{\psi}(t) + C(t,x)\psi(t) + f(t,x)\|, \|\varphi_1\|_{0,1}, \|\varphi_2\|_{0,1}\Big\},$$

where $\mathcal{K}_3(x) = \mathcal{K}_1(x) + \mathcal{K}_2(x)$.

Moreover, the following estimate holds:

$$\sup_{t\in[0,T]} \left\| \frac{\partial \mathbf{v}^{(0)}(t,x)}{\partial t} \right\| \leqslant \left[\max_{t\in[0,T]} \|A(t,x)\| \mathcal{K}_3(x) + \max_{t\in[0,T]} \big(\|D(t,x)\| + \|K(t,x)\|\big)\mathcal{K}_2(x) + 1 \right]$$
$$\times \max\Big\{ \max_{t\in[0,T]} \|B(t,x)\dot{\psi}(t) + C(t,x)\psi(t) + f(t,x)\|, \|\varphi_1\|_{0,1}, \|\varphi_2\|_{0,1}\Big\},$$

Let the functions $\mathbf{u}^{(k-1)}(t,x)$ and $\mathbf{w}^{(k-1)}(t,x)$ be already determined. At the k-th step, $k = 1, 2, \ldots$, we obtain $(\mathbf{v}^{(k)}(t,x), \boldsymbol{\theta}^{(k)}(x))$ by solving (6)–(9) with

$$\mathbf{u}(t,x) = \mathbf{u}^{(k-1)}(t,x), \qquad \mathbf{w}(t,x) = \mathbf{w}^{(k-1)}(t,x).$$

After $\mathbf{v}^{(k)}(t,x)$ has been computed, the next approximations to $\mathbf{u}(t,x)$ and $\mathbf{w}(t,x)$ are recovered from (9) as follows:

$$\mathbf{u}^{(k)}(t,x) = \psi(t) + \int_0^x \mathbf{v}^{(k)}(t,\eta)\, d\eta, \qquad \mathbf{w}^{(k)}(t,x) = \dot{\psi}(t) + \int_0^x \frac{\partial \mathbf{v}^{(k)}(t,\eta)}{\partial t}\, d\eta. \tag{35}$$

To investigate convergence, introduce the increments $\Delta\mathbf{v}^{(k)}(t,x) = \mathbf{v}^{(k)}(t,x) - \mathbf{v}^{(k-1)}(t,x)$, $\Delta\mathbf{u}^{(k)}(t,x) = \mathbf{u}^{(k)}(t,x) - \mathbf{u}^{(k-1)}(t,x)$, $\Delta\mathbf{w}^{(k)}(t,x) = \mathbf{w}^{(k)}(t,x) - \mathbf{w}^{(k-1)}(t,x)$, $\Delta\boldsymbol{\theta}^{(k)}(x) = \boldsymbol{\theta}^{(k)}(x) - \boldsymbol{\theta}^{(k-1)}(x)$. Using the unique solvability of the problem (10)–(12), we obtain the following estimates:

$$\max\left\{ \sup_{t\in[0,T]} \|\Delta\mathbf{v}^{(k+1)}(t,x)\|, \sup_{t\in[0,T]} \left\| \frac{\partial \mathbf{v}^{(k+1)}(t,x)}{\partial t} \right\| \right\}$$
$$\leqslant \mathcal{K}_4(x) \max\left\{ \sup_{t\in[0,T]} \|\Delta\mathbf{w}^{(k)}(t,x)\|, \sup_{t\in[0,T]} \|\Delta\mathbf{u}^{(k)}(t,x)\| \right\}, \tag{36}$$

where

$$\mathcal{K}_4(x) = \left(\max_{t\in[0,T]} \|B(t,x)\| + \max_{t\in[0,T]} \|C(t,x)\|\right) \max\Big\{\mathcal{K}_3(x), \alpha(x)\mathcal{K}_3(x) + \max_{t\in[0,T]} \big(\|D(t,x)\| + \|K(t,x)\|\big)\mathcal{K}_2(x) + 1\Big\}.$$

In view of (35), we also have

$$\begin{aligned} &\max\left\{\sup_{t\in[0,T]} \|\Delta\mathbf{w}^{(k)}(t,x)\|, \sup_{t\in[0,T]} \|\Delta\mathbf{u}^{(k)}(t,x)\|\right\} \\ &\leqslant \int_0^x \max\left\{\sup_{t\in[0,T]} \|\Delta\mathbf{v}^{(k)}(t,s)\|, \sup_{t\in[0,T]} \left\|\frac{\partial\mathbf{v}^{(k)}(t,x)}{\partial t}\right\|\right\} dx. \end{aligned} \tag{37}$$

Combining (36) and (37), we arrive at the fundamental recursive estimate

$$\begin{aligned} &\max\left\{\sup_{t\in[0,T]} \|\Delta\mathbf{v}^{(k+1)}(t,x)\|, \sup_{t\in[0,T]} \left\|\frac{\partial\mathbf{v}^{(k+1)}(t,x)}{\partial t}\right\|\right\} \\ &\leqslant \mathcal{K}_4(x) \int_0^x \max\left\{\sup_{t\in[0,T]} \|\Delta\mathbf{v}^{(k)}(t,s)\|, \sup_{t\in[0,T]} \left\|\frac{\partial\mathbf{v}^{(k)}(t,s)}{\partial t}\right\|\right\} ds. \end{aligned} \tag{38}$$

Estimate (38) ensures that both $\{\mathbf{v}^{(k)}\}$ and $\{\partial\mathbf{v}^{(k)}/\partial t\}$ converge in $\mathbb{C}(\Omega,\mathbb{R}^n)$ as $k\to\infty$. Relation (37) then yields convergence of the sequences $\{\mathbf{u}^{(k)}\}$ and $\{\mathbf{w}^{(k)}\}$ in the same space. Consequently, the limit functions $\mathbf{v}^*(t,x)$, $\frac{\partial\mathbf{v}^*(t,x)}{\partial t}$, $\mathbf{u}^*(t,x)$, $\mathbf{w}^*(t,x)$ belong to $\mathbb{C}(\Omega,\mathbb{R}^n)$. Together with $\boldsymbol{\theta}^*(x)$, these functions satisfy the initial–boundary value problem (6)–(9).

It remains to verify uniqueness. Assume that $\big\{\widetilde{\mathbf{v}}(t,x), \widetilde{\mathbf{u}}(t,x), \widetilde{\mathbf{w}}(t,x), \widetilde{\boldsymbol{\theta}}(x)\big\}$ solves the homogeneous problem associated with (6)–(9); that is, $f(t,x)=0$, $\varphi_1(x)=0$, $\varphi_2(x)=0$, $\psi(t)=0$. The assumed uniqueness for the family (6)–(9) implies $\widetilde{\mathbf{v}}(t,x)=0$, $\widetilde{\boldsymbol{\theta}}(x)=0$. Substitution into (9) consequently gives $\widetilde{\mathbf{u}}(t,x)=0$, $\widetilde{\mathbf{w}}(t,x)=0$, $(t,x)\in\Omega$.

Hence, the homogeneous problem corresponding to (6)–(9) possesses no nonzero solutions. Theorem 3 is proved.

The preceding existence argument and this uniqueness result lead to the following statement.

Theorem 4 *Let, for some partition* $\Delta = \{0 = t_0 < t_1 < t_2 = T\}$, *the matrix* $Q_*(\Delta,x)\colon \mathbb{R}^{6n}\to\mathbb{R}^{6n}$ *be invertible for every* $x\in[0,\omega]$ *and satisfy*

$$\|Q_*^{-1}(\Delta,x)\| \leqslant \gamma_*(\Delta,x).$$

Then, for any $g\in C(\Omega,\mathbb{R}^n)$, $\varphi_1\in C([0,\omega],\mathbb{R}^n)$, $\varphi_2\in C^1([0,\omega],\mathbb{R}^n)$, *the boundary value problem* (1)–(4) *admits exactly one solution pair* $\big(\mathbf{u}^*(t,x), \boldsymbol{\theta}^*(x)\big)$.

The preceding arguments also establish Theorem 4. Existence follows from the convergence of the successive approximations and the equivalence of the original and reduced

problems. Uniqueness is established by applying the corresponding integral estimate to the difference of two solutions.

The constructive nature of the proof provides a basis for the numerical implementation of the proposed method. The next section considers a test problem with a known exact solution and distributed parameter. The numerical experiment illustrates the verification of the sufficient conditions and the convergence of the iterative procedure.

6 Numerical experiment

We illustrate Algorithms $\mathcal{A}$ and $\mathcal{B}$ on a test problem with a known exact solution. Let $\Omega = [0,1] \times [0,0.2]$, $t_0 = 0, \quad t_1 = 0.5, \quad t_2 = 1$, and let the piecewise constant argument be given by

$$\gamma(t) = \begin{cases} 0.25, & 0 \le t < 0.5, \\ 0.75, & 0.5 \le t < 1. \end{cases}$$

Consider problem (1)–(4), whose data are given as follows:

$$A(t,x) = \begin{pmatrix} x & t \\ 0 & xt \end{pmatrix}, \quad B(t,x) = \begin{pmatrix} t^2 & 0 \\ -1 & xt \end{pmatrix}, \quad C(t,x) = \begin{pmatrix} t & 0 \\ 1 & 2xt \end{pmatrix}, \quad D(t,x) = \begin{pmatrix} 1 & t \\ 0 & 3x \end{pmatrix},$$

$$K(t,x) = \begin{pmatrix} 1 & 2 \\ t & x \end{pmatrix}, \quad M_1(x) = \begin{pmatrix} 1 & x \\ 2 & 0 \end{pmatrix}, \quad M_2(x) = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}, \quad M_3(x) = \begin{pmatrix} 1-x & 2 \\ 0 & -1 \end{pmatrix},$$

$$M_4(x) = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}, \quad \varphi_1(x) = \begin{pmatrix} x(6.5x - 3x^2 + 5) \\ 2x^3 + 0.25x^2 + x - 1 \end{pmatrix}, \quad \varphi_2(x) = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \quad \psi(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

$$f(t,x) = \begin{cases} f_1(t,x), & 0 \le t < 0.5, \\ f_2(t,x), & 0.5 \le t < 1, \end{cases}$$

where, for $r = 1, 2$,

$$f_r(t,x) = \begin{pmatrix} c_r x^2 - 2t^2x^3 - 3tx^3 - 4tx - d_r t - t^3 - x - 2 \\ 2t - t^3x - 2t^2x^2 - 2t^2x - 3tx^3 - 2tx^2 - 2tx - 2t^3x^2 + x^3 - 6x^2 - e_r x \end{pmatrix},$$

with $(c_1, d_1, e_1) = (2.25, 0.0625, 1.1875), \qquad (c_2, d_2, e_2) = (0.75, 0.5625, 2.6875).$

The exact solution and the exact distributed parameter are

$$\mathbf{u}^*(t,x) = \begin{pmatrix} x^3 t \\ xt^2 + x^2 + t \end{pmatrix}, \qquad \boldsymbol{\theta}^*(x) = \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Consequently,

$$\mathbf{v}^*(t,x) = \mathbf{u}_x^*(t,x) = \begin{pmatrix} 3x^2 t \\ t^2 + 2x \end{pmatrix}, \qquad \mathbf{w}^*(t,x) = \mathbf{u}_t^*(t,x) = \begin{pmatrix} x^3 \\ 2xt + 1 \end{pmatrix}.$$

The Cauchy problems in Algorithm $\mathcal{B}$ were solved by an adaptive fifth-order Runge–Kutta method on the grid $t_i = 0.025i$, $i = 0, 1, \ldots, 40$. The calculations were performed for $h_x = 0.1$, 0.05, and 0.025, using the composite trapezoidal rule for the integrals with respect to x.

Starting with

$$\mathbf{u}^{(0)}(t,x) = \boldsymbol{\psi}(t), \qquad \mathbf{w}^{(0)}(t,x) = \dot{\boldsymbol{\psi}}(t),$$

we define, for $k = 0, 1, \ldots, 7$,

$$g^{(k)}(t,x) = B(t,x)\mathbf{w}^{(k-1)}(t,x) + C(t,x)\mathbf{u}^{(k-1)}(t,x) + f(t,x).$$

At each point x_j, Algorithm $\mathcal{B}$ is applied to

$$\frac{d\mathbf{v}^{(k)}}{dt} = A(t,x_j)\mathbf{v}^{(k)} + D(t,x_j)\boldsymbol{\mu}_r^{(k)}(x_j) + K(t,x_j)\boldsymbol{\theta}^{(k)}(x_j) + g^{(k)}(t,x_j),$$

for $t \in [t_{r-1}, t_r)$, $r = 1, 2$, and the approximations are reconstructed by

$$\mathbf{u}^{(k)}(t,x) = \boldsymbol{\psi}(t) + \int_0^x \mathbf{v}^{(k)}(t,\eta)\,d\eta, \quad \mathbf{w}^{(k)}(t,x) = \dot{\boldsymbol{\psi}}(t) + \int_0^x \frac{\partial \mathbf{v}^{(k)}(t,\eta)}{\partial t}\,d\eta.$$

Define

$$E_z^{(k)} = \max_{i,j} \|\mathbf{z}^{(k)}(t_i,x_j) - \mathbf{z}^*(t_i,x_j)\|_\infty, \qquad z \in \{u, v, w\},$$

$$E_\theta^{(k)} = \max_{j} \|\boldsymbol{\theta}^{(k)}(x_j) - \boldsymbol{\theta}^*(x_j)\|_\infty, \qquad \delta_u^{(k)} = \max_{i,j} \|\mathbf{u}^{(k)}(t_i,x_j) - \mathbf{u}^{(k-1)}(t_i,x_j)\|_\infty.$$

Table 1: Convergence of the successive approximations for $h_x = 0.025$

k	$E_v^{(k)}$	$E_u^{(k)}$	$E_\theta^{(k)}$	$\delta_u^{(k)}$
0	$2.09442 \cdot 10^{-1}$	$1.39557 \cdot 10^{-2}$	$6.55641 \cdot 10^{-2}$	$2.36472 \cdot 10^{-1}$
1	$4.66447 \cdot 10^{-2}$	$2.26061 \cdot 10^{-3}$	$1.26407 \cdot 10^{-2}$	$1.16950 \cdot 10^{-2}$
2	$7.73137 \cdot 10^{-3}$	$3.61049 \cdot 10^{-4}$	$2.07016 \cdot 10^{-3}$	$1.89956 \cdot 10^{-3}$
3	$1.19494 \cdot 10^{-3}$	$1.11533 \cdot 10^{-4}$	$3.19150 \cdot 10^{-4}$	$2.49516 \cdot 10^{-4}$
4	$3.36755 \cdot 10^{-4}$	$8.40833 \cdot 10^{-5}$	$9.01306 \cdot 10^{-5}$	$2.74496 \cdot 10^{-5}$
5	$2.42377 \cdot 10^{-4}$	$8.14447 \cdot 10^{-5}$	$6.49692 \cdot 10^{-5}$	$2.63858 \cdot 10^{-6}$
6	$2.33310 \cdot 10^{-4}$	$8.12164 \cdot 10^{-5}$	$6.25528 \cdot 10^{-5}$	$2.28263 \cdot 10^{-7}$
7	$2.32520 \cdot 10^{-4}$	$8.11983 \cdot 10^{-5}$	$6.23422 \cdot 10^{-5}$	$1.80867 \cdot 10^{-8}$

The spatial convergence order is calculated by $p_h = \log_2 \frac{E(h_x)}{E(h_x/2)}$.

The numerical results confirm the convergence of the successive approximations constructed by Algorithm $\mathcal{A}$. By the seventh iteration, the difference between two consecutive approximations becomes negligible. The mesh-refinement results show that the errors in $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\boldsymbol{\theta}$ decrease approximately by a factor of four when the spatial step is halved. Therefore, the observed order of convergence with respect to the spatial variable is approximately two. This agrees with the theoretical accuracy of the composite trapezoidal rule used in the reconstruction formulas. Thus, the numerical experiment confirms both the efficiency of the proposed algorithms and the expected spatial accuracy of the computational method.

Table 2: Mesh refinement after seven iterations

h_x	E_v	p_v	E_u	p_u	E_w	p_w	E_θ	p_θ
0.100	$4.17773 \cdot 10^{-3}$	–	$1.34974 \cdot 10^{-3}$	–	$1.54313 \cdot 10^{-3}$	–	$1.13439 \cdot 10^{-3}$	–
0.050	$1.00663 \cdot 10^{-3}$	2.053	$3.30577 \cdot 10^{-4}$	2.030	$3.75437 \cdot 10^{-4}$	2.039	$2.72785 \cdot 10^{-4}$	2.056
0.025	$2.32520 \cdot 10^{-4}$	2.114	$8.11983 \cdot 10^{-5}$	2.026	$9.20691 \cdot 10^{-5}$	2.028	$6.23422 \cdot 10^{-5}$	2.129

7 Conclusion

This study has addressed a linear hyperbolic system subject to a three-point condition, where the equation contains a generalized piecewise constant argument and a parameter function depending on the spatial variable. The introduction of new dependent variables made it possible to reformulate the initial problem as a parameterized collection of functional differential problems. Dzhumabaev's parametrization technique was then used to incorporate the boundary relations into a finite-dimensional algebraic system. The nonsingularity of the resulting block matrix provides a directly testable criterion for well-posedness and also leads to an a priori bound for the solution. This framework yields an iterative computational procedure in which initial value problems and linear algebraic systems are solved alternately. The proved convergence of this procedure guarantees that both the unknown function and the distributed parameter are determined uniquely. Thus, the developed approach can serve as a foundation for constructing numerical methods for this class of hyperbolic problems.

8 Acknowledgements

This research is partially supported by the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP26194447).

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E.B.: investigation, formal analysis, methodology, validation. **N.I.**: investigation, formal analysis, methodology, validation, writing — review and editing. **A.T.**: investigation, writing — original draft. **A.K.**: investigation, writing — original draft.

All authors have read and agreed to the published version of the manuscript.

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Received: October 12, 2026
Accepted: November 7, 2026