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INTEGRAL RECOVERY OF A TIME-DEPENDENT BURGERS COEFFICIENT

We study an inverse coefficient problem for the one-dimensional periodic viscous Burgers equation, in which an unknown time-dependent coefficient multiplies the solution. Additional data are provided by a weighted spatial integral of the state. The periodic weight has first-order Sobolev regularity, and the prescribed observation is bounded away from zero. No sign restriction is imposed on the coefficient. An explicit reconstruction formula reduces the inverse problem to a loaded Burgers equation with a cubic nonlocal term. Bihari's inequality controls the resulting superlinear energy estimate and yields a sufficient local existence threshold depending on the data. Using Fourier Galerkin approximations, a priori estimates, and compactness arguments, we prove existence and uniqueness of a local strong solution and square-integrability in time of the reconstructed coefficient. The prescribed integral observation is recovered under the initial compatibility condition. A numerical experiment with a known exact solution and a sign-changing coefficient illustrates reconstruction accuracy and shows observed second-order spatial convergence for both the state and the coefficient.
Keywords: Burgers equation, inverse coefficient problem, integral overdetermination condition, priori estimates, Bihari inequality.

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Бюргерс теңдеуіндегі уақытқа тәуелді коэффициентті интегралдық деректер бойынша қалпына келтіру

Бұл жұмыста периодикалық шекаралық шарттары бар бірөлшемді тұтқыр Бюргерс теңдеуіндегі уақытқа тәуелді белгісіз коэффициентті анықтаудың кері есебі зерттеледі. Қосымша деректер шешімнің кеңістіктік айнымалысы бойынша салмақталған интегралы арқылы беріледі. Периодикалық салмақ функциясы бірінші ретті Соболев кеңістігіне жатады деп, ал берілген интегралдық бақылаудың абсолют шамасы оң тұрақтыдан кем емес деп ұйғарылады. Коэффициентті қалпына келтірудің айқын формуласы кері есепті шешімге қатысты үшінші дәрежелі локалды емес мүшесі бар жүктелген Бюргерс теңдеуіне келтіруге мүмкіндік береді. Бихари теңсіздігі энергетикалық бағалауда пайда болатын суперсызықтық өсімді бақылауға және шешімнің локалды бар болуы үшін берілгендерге тәуелді жеткілікті уақыт шегін анықтауға мүмкіндік береді. Фурье–Галеркин жуықтаулары, априорлық бағалаулар және компактылыққа негізделген пайымдаулар арқылы локалды күшті шешімнің бар болуы мен жалғыздығы, сондай-ақ қалпына келтірілген коэффициенттің квадратының уақыт бойынша интегралданатыны дәлелденеді. Бастапқы деректердің үйлесімділік шарты орындалған жағдайда берілген интегралдық шарттың орындалатыны көрсетіледі. Белгілі дәл шешімі және таңбасын өзгертетін коэффициенті бар тестілік есеп үшін жүргізілген сандық тәжірибе қалпына келтірудің дәлдігін сипаттап, шешім үшін де, коэффициент үшін де кеңістіктік айнымалысы бойынша екінші ретті жинақталу орынды екендігін көрсетеді.
Түйін сөздер: Бюргерс теңдеуі, коэффициентті анықтаудың кері есебі, интегралдық қосымша шарт, априорлық бағалаулар, Бихари теңсіздігі.

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Восстановление зависящего от времени коэффициента уравнения Бюргерса по интегральным данным

Нами исследуется одна обратная задача определения коэффициента в одномерном вязком уравнении Бюргерса с периодическими граничными условиями, в котором неизвестный зависящий от времени коэффициент умножается на решение. Дополнительные данные задаются взвешенным пространственным интегралом решения. Предполагается, что периодическая весовая функция принадлежит пространству Соболева первого порядка, а заданное интегральное наблюдение отделено от нуля. Явная формула восстановления коэффициента позволяет свести обратную задачу к нагруженному уравнению Бюргерса с кубическим нелокальным членом. Неравенство Бихари позволяет контролировать возникающий в энергетической оценке суперлинейный рост и получить зависящий от данных достаточный временной порог локального существования решения. С использованием приближений Фурье–Галёркина, априорных оценок и аргументов компактности доказываются существование и единственность локального сильного решения, а также квадратичная интегрируемость восстановленного коэффициента по времени. Выполнение заданного интегрального условия устанавливается при соблюдении условия согласованности начальных данных. Численный эксперимент с известным точным решением и меняющим знак коэффициентом иллюстрирует точность восстановления и демонстрирует наблюдаемый второй порядок сходимости по пространству как для решения, так и для коэффициента.
Ключевые слова: уравнение Бюргерса, обратная задача определения коэффициента, интегральное условие переопределения, априорные оценки, неравенство Бихари.

1 Introduction

We consider an inverse coefficient problem for the periodic viscous Burgers equation. The problem is related to inverse parabolic equations with integral overdetermination, recovery of time-dependent lower-order coefficients, and inverse problems for Burgers-type equations.

Time-dependent parameters in semilinear and quasilinear parabolic equations under weighted integral observations were studied by Cannon and Lin [1, 2]. A periodic inverse problem for a time-dependent zeroth-order coefficient with integral overdetermination was considered by Kanca [3]. A quasilinear inverse coefficient problem with periodic boundary and integral overdetermination conditions was also studied by Bağlan and Kanca [4]. Related problems in which a time-dependent lower-order coefficient multiplies the state and is recovered from an integral observation were studied, in different settings, by Azizbayov and Mehraliyev [5], Kozhanov, Abulkayirov, and Ashurova [6], and Altybay and Ruzhansky [7]. The latter two works use a general spatial weight in the observation and are particularly close to the coefficient structure considered below. Reduction of inverse problems to loaded parabolic equations is also used in [8].

In our previous works [9, 10], we studied inverse problems for Burgers equations with integral overdetermination. In [9], the unknown time-dependent coefficient multiplies a prescribed source term, whereas in the present problem it multiplies the state.

Here the unknown coefficient multiplies the solution itself:

$$u_t + uu_x = u_{xx} - a(t)u + f(x,t), \qquad \int_0^\ell N(x)u(x,t)\,dx = E(t).$$

Under the assumption $N \in H^1_{\mathrm{per}}(0,\ell)$, integration against the weight and one integration by parts yield a reconstruction formula involving the two quantities

$$\int_0^\ell N_x(x)u_x(x,t)\,dx, \qquad \int_0^\ell N_x(x)u(x,t)^2\,dx.$$

Substituting this formula into the equation multiplies these nonlocal quantities by the unknown state. In particular, the reduced equation contains the cubic nonlocal term

$$\frac{u(x,t)}{2E(t)}\int_0^\ell N_x(x)u(x,t)^2\,dx.$$

Thus, the relevant distinction from identifying a coefficient multiplying a prescribed source is the dependence of the multiplier of the quadratic integral on the unknown solution.

In the L^2 energy estimate used here, applying Young's inequality to this cubic nonlocal term produces a contribution of order $\|u\|^6_{L^2(0,\ell)}$. The resulting differential inequality is superlinear and cannot be closed directly by Gronwall's inequality. Applying Bihari's inequality to the shifted energy $1 + \|u\|^2_{L^2(0,\ell)}$ yields an explicit data-dependent sufficient threshold T^* for local existence.

The specific contribution of the present work is a local strong-solvability analysis for this weighted periodic Burgers problem with Sobolev-regular data, rather than coefficient identification from integral observations in general. For every $T_0 < T^*$, we prove existence and uniqueness of a strong solution to the reduced problem and recover $a \in L^2(0,T_0)$ from the reconstruction formula. No sign condition is imposed on the unknown coefficient. The proof is based on Galerkin approximations, a priori estimates, compactness, and a uniqueness argument.

Section 2 formulates the problem and states the main results. Section 3 proves existence and uniqueness for the reduced problem and completes the reconstruction of the coefficient. Section 4 presents a numerical experiment illustrating the reconstruction accuracy and the observed second-order convergence under spatial mesh refinement.

2 Problem formulation and main results

Let $Q := (0,\ell)\times(0,T)$. We seek a pair (u,a) satisfying

$$u_t(x,t) + u(x,t)\,u_x(x,t) = u_{xx}(x,t) - a(t)u(x,t) + f(x,t), \qquad (x,t)\in Q, \tag{1a}$$

$$u(0,t) = u(\ell,t), \qquad u_x(0,t) = u_x(\ell,t), \qquad t\in(0,T), \tag{1b}$$

$$u(x,0) = u_0(x), \qquad x\in(0,\ell), \tag{1c}$$

$$\int_0^\ell N(x)\,u(x,t)\,dx = E(t), \qquad t\in[0,T]. \tag{1d}$$

2.1 Functional setting and assumptions

For $k = 1, 2$, let $H^k_{\mathrm{per}}(0,\ell)$ be the closure in $H^k(0,\ell)$ of smooth ℓ-periodic functions. All functions are real-valued. We assume

$$u_0(x) \in H^1_{\mathrm{per}}(0,\ell), \tag{2}$$
$$f(x,t) \in L^2(0,T;L^2(0,\ell)), \tag{3}$$
$$E(t) \in W^{1,2}(0,T), \tag{4}$$
$$|E(t)| \geq E_0 > 0, \qquad t \in [0,T], \tag{5}$$
$$N(x) \in H^1_{\mathrm{per}}(0,\ell). \tag{6}$$

Since $W^{1,2}(0,T) \hookrightarrow C([0,T])$, we identify $E(t)$ with its continuous representative. In particular, the lower bound (5) is understood pointwise on $[0,T]$.

The compatibility condition is

$$\int_0^\ell N(x)u_0(x)\,dx = E(0). \tag{7}$$

2.2 Reduction to a nonlocal problem

Set

$$\widetilde{f}(t) := \int_0^\ell N(x)f(x,t)\,dx, \qquad t \in (0,T). \tag{8}$$

Then $\widetilde{f}(t) \in L^2(0,T)$.

Multiplying (1a) by $N(x)$ and integrating over one period, we obtain

$$\int_0^\ell N(x)u_{xx}(x,t)\,dx = -\int_0^\ell N_x(x)u_x(x,t)\,dx,$$

$$\int_0^\ell N(x)u(x,t)u_x(x,t)\,dx = -\frac{1}{2}\int_0^\ell N_x(x)u^2(x,t)\,dx.$$

Using the observation condition (1d), this becomes

$$E'(t) = -a(t)E(t) + \widetilde{f}(t) - \int_0^\ell N_x(x)u_x(x,t)\,dx + \frac{1}{2}\int_0^\ell N_x(x)u^2(x,t)\,dx. \tag{9}$$

Because $E(t)$ is bounded away from zero, we may solve this identity for $a(t)$:

$$a(t) = \frac{\widetilde{f}(t) - E'(t)}{E(t)} - \frac{1}{E(t)}\int_0^\ell N_x(x)u_x(x,t)\,dx + \frac{1}{2E(t)}\int_0^\ell N_x(x)u^2(x,t)\,dx. \tag{10}$$

Substituting (10) back into (1a) leads to

$$u_t(x,t)+u(x,t)u_x(x,t)-u_{xx}(x,t)+\frac{u(x,t)(\widetilde{f}(t)-E'(t))}{E(t)} - \frac{u(x,t)}{E(t)}\int_0^\ell N_x(x)u_x(x,t)\,dx+\frac{u(x,t)}{2E(t)}\int_0^\ell N_x(x)u(x,t)^2\,dx=f(x,t). \tag{11}$$

This equation is supplemented with the periodic boundary conditions (1b) and the initial condition (1c).

2.3 Definition of strong solutions

Definition 1 *Let $T_0\in(0,T]$. A function*

$$u\in L^\infty(0,T_0;H^1_{\mathrm{per}}(0,\ell))\cap L^2(0,T_0;H^2_{\mathrm{per}}(0,\ell)),\qquad u_t\in L^2(0,T_0;L^2(0,\ell)),$$

is called a local strong solution of (11), (1b)–(1c) if (11) holds a.e. on $(0,\ell)\times(0,T_0)$ and $u(\cdot,0)=u_0$ in $H^1_{\mathrm{per}}(0,\ell)$.

Definition 2 *Let $T_0\in(0,T]$. A pair (u,a) is called a local strong solution of the inverse problem (1) on $[0,T_0]$ if*

$$u\in L^\infty(0,T_0;H^1_{\mathrm{per}}(0,\ell))\cap L^2(0,T_0;H^2_{\mathrm{per}}(0,\ell)),\qquad u_t\in L^2(0,T_0;L^2(0,\ell)),$$

and

$$a\in L^2(0,T_0).$$

The differential equation (1a) is required to hold a.e. on $(0,\ell)\times(0,T_0)$, the periodic boundary conditions (1b) are satisfied in the trace sense, and $u(\cdot,0)=u_0$ in $H^1_{\mathrm{per}}(0,\ell)$. In addition,

$$\int_0^\ell N(x)u(x,t)\,dx=E(t),\qquad t\in[0,T_0].$$

2.4 Main results

Theorem 1 *Assume (2)–(6), and let T^* be defined by (31). For every $T_0\in(0,T^*)$, the reduced problem (11), (1b), (1c) has a unique local strong solution $u(x,t)$ in the sense of Definition 1, with $u\in C([0,T_0];H^1_{\mathrm{per}}(0,\ell))$. Moreover,*

$$\|u\|^2_{L^\infty(0,T_0;H^1_{\mathrm{per}})}+\|u\|^2_{L^2(0,T_0;H^2_{\mathrm{per}})}+\|u_t\|^2_{L^2(0,T_0;L^2)}\le K(T_0), \tag{12}$$

where $K(T_0)$ depends only on the data and T_0.

Theorem 2 (Inverse problem) *Assume (2)–(7), and let T^* be as in Theorem 1. For every $T_0\in(0,T^*)$, the inverse problem (1) has a unique strong solution pair (u,a) in the sense of Definition 2. The coefficient is given by the reconstruction formula (10). Moreover, $u\in C([0,T_0];H^1_{\mathrm{per}}(0,\ell))$.*

3 Existence and uniqueness for the reduced problem

3.1 Galerkin approximation

We use the standard periodic Fourier basis

$$e_k(x) := \frac{1}{\sqrt{\ell}} \exp\left(\frac{2\pi i k x}{\ell}\right), \qquad k \in \mathbb{Z},$$

and for $n \geq 1$ define $V_n := \operatorname{span}\{e_k : \ |k| \leq n\}$. We look for a Galerkin approximation of the form

$$u_n(x,t) = \sum_{k=-n}^{n} c_{nk}(t) e_k(x) = \frac{1}{\sqrt{\ell}} \sum_{k=-n}^{n} c_{nk}(t) \exp\left(\frac{2\pi i k x}{\ell}\right). \tag{13}$$

Both the basis functions and their derivatives are periodic. We use the complex L^2 inner product $(v,w) := \int_0^\ell v(x)\overline{w(x)}\,dx$.

The Galerkin system is

$$\begin{aligned}(u_{n,t}, \psi_n) + (u_n u_{n,x}, \psi_n) + (u_{n,x}, \psi_{n,x}) + \frac{(u_n, \psi_n)}{E(t)} \Bigg(\widetilde{f}(t) \\ -E'(t) - \int_0^\ell N_x(x) u_{n,x}(x,t)\,dx + \frac{1}{2}\int_0^\ell N_x(x) u_n(x,t)^2\,dx\Bigg) = (f, \psi_n),\end{aligned} \tag{14}$$

for every $\psi_n \in V_n$.

Using $e_{k,x}(x) = \frac{2\pi i k}{\ell} e_k(x)$ and $e_p(x)e_q(x) = \frac{1}{\sqrt{\ell}} e_{p+q}(x)$, and choosing $\psi_n = e_k$, $|k| \leq n$, equation (14) becomes

$$\begin{cases} c'_{nk}(t) + \left(\dfrac{2\pi k}{\ell}\right)^2 c_{nk}(t) + \dfrac{\pi i k}{\ell\sqrt{\ell}} \displaystyle\sum_{\substack{p+q=k \\ |p|,|q|\leq n}} c_{np}(t) c_{nq}(t) \\ \qquad + \dfrac{c_{nk}(t)}{E(t)} \Bigg[\widetilde{f}(t) - E'(t) - \dfrac{1}{\sqrt{\ell}} \displaystyle\sum_{m=-n}^{n} \frac{2\pi i m}{\ell} c_{nm}(t) \int_0^\ell N_x(x) \exp\left(\frac{2\pi i m x}{\ell}\right) dx \\ \qquad\qquad + \dfrac{1}{2\ell} \displaystyle\sum_{p,q=-n}^{n} c_{np}(t) c_{nq}(t) \int_0^\ell N_x(x) \exp\left(\frac{2\pi i (p+q) x}{\ell}\right) dx\Bigg] \\ \qquad = \dfrac{1}{\sqrt{\ell}} \displaystyle\int_0^\ell f(x,t) \exp\left(-\frac{2\pi i k x}{\ell}\right) dx, \qquad |k| \leq n, \\ c_{nk}(0) = \dfrac{1}{\sqrt{\ell}} \displaystyle\int_0^\ell u_0(x) \exp\left(-\frac{2\pi i k x}{\ell}\right) dx. \end{cases} \tag{15}$$

Because $u_n(\cdot,0)$ is the Fourier projection of u_0 onto V_n,

$$\|u_n(\cdot,0)\|_{L^2(0,\ell)} \le \|u_0\|_{L^2(0,\ell)}, \qquad \|u_n(\cdot,0)\|_{H^1(0,\ell)} \le \|u_0\|_{H^1(0,\ell)}, \tag{16}$$

and $u_n(\cdot,0) \to u_0 \quad \text{in } H^1_{\mathrm{per}}(0,\ell)$.

The system (15) has dimension $2n+1$ and is polynomial of degree at most three in the coefficient vector. Its right-hand side is measurable in t and locally Lipschitz with respect to the coefficients. On every ball of radius $R>0$ in coefficient space, the norm of the right-hand side and its Lipschitz constant are bounded by

$$m_{n,R}(t) := C_{n,R}\left(1 + |\widetilde{f}(t) - E'(t)| + \|f(\cdot,t)\|_{L^2(0,\ell)}\right),$$

where $C_{n,R}$ is independent of t. This majorant belongs to $L^1(0,T)$, since $\widetilde{f} - E' \in L^2(0,T)$ and $f \in L^2(0,T;L^2(0,\ell))$. Carathéodory's theorem [14] therefore gives a unique local solution on a maximal interval $[0,\tau_n) \subset [0,T]$. For real-valued data, uniqueness preserves the conjugate symmetry $c_{n,-k}(t) = \overline{c_{nk}(t)}$, so the Galerkin approximation u_n remains real-valued.

3.2 A priori estimates

Lemma 1 *For each $T_0 \in (0,T^*)$ there is $K_1(T_0)$, independent of n, such that*

$$\sup_{t\in[0,T_0]} \|u_n(t)\|^2_{L^2(0,\ell)} + \int_0^{T_0} \|u_{n,x}(t)\|^2_{L^2(0,\ell)}\,dt \le K_1(T_0).$$

Proof. Take $\psi_n = u_n$ in (14). By periodicity, $(u_n u_{n,x}, u_n) = \frac{1}{3}\int_0^\ell (u_n^3)_x\,dx = 0$. Consequently,

$$\begin{aligned}\frac{d}{dt}\|u_n\|^2_{L^2(0,\ell)} + 2\|u_{n,x}\|^2_{L^2(0,\ell)} = -\frac{2\|u_n\|^2_{L^2(0,\ell)}}{E(t)}\Big(&\widetilde{f}(t) \\ &- E'(t) - \int_0^\ell N_x(x)u_{n,x}(x,t)\,dx + \frac{1}{2}\int_0^\ell N_x(x)u_n^2(x,t)\,dx\Big) + 2(f,u_n).\end{aligned} \tag{17}$$

For $\varepsilon>0$, Young's inequality gives

$$2|(f,u_n)| \le \frac{1}{\varepsilon}\|f\|^2_{L^2(0,\ell)} + \varepsilon\|u_n\|^2_{L^2(0,\ell)}. \tag{18}$$

For $\varepsilon_1>0$, $|E(t)| \ge E_0$, Cauchy–Schwarz, and Young give

$$\frac{2\|u_n\|^2_{L^2(0,\ell)}}{|E(t)|}\left|\int_0^\ell N_x u_{n,x}\,dx\right| \le \frac{2\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_n\|^2_{L^2(0,\ell)}\|u_{n,x}\|_{L^2(0,\ell)} \tag{19}$$

$$\le \frac{2\|N_x\|^2_{L^2(0,\ell)}}{\varepsilon_1 E_0^2}\|u_n\|^4_{L^2(0,\ell)} + \frac{\varepsilon_1}{2}\|u_{n,x}\|^2_{L^2(0,\ell)}. \tag{20}$$

For $\varepsilon_2>0$, $\|u_n\|^2_{L^4(0,\ell)} \le \|u_n\|_{L^2(0,\ell)}\|u_n\|_{L^\infty(0,\ell)} \le C_\ell\|u_n\|_{L^2(0,\ell)}\|u_n\|_{H^1(0,\ell)}$. Thus

$$\frac{\|u_n\|^2_{L^2(0,\ell)}}{|E(t)|}\left|\int_0^\ell N_x(x)u_n^2(x,t)\,dx\right| \le \frac{\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_n\|^2_{L^2(0,\ell)}\|u_n\|^2_{L^4(0,\ell)}$$

$$\leq \frac{C_\ell \|N_x\|_{L^2(0,\ell)}}{E_0} \|u_n\|^3_{L^2(0,\ell)} \|u_n\|_{H^1(0,\ell)}$$
$$\leq \frac{\varepsilon_2}{2}\|u_n\|^2_{H^1} + \frac{C_\ell^2 \|N_x\|^2_{L^2(0,\ell)}}{2\varepsilon_2 E_0^2}\|u_n\|^6_{L^2(0,\ell)}. \tag{21}$$

Since $\|u_n\|^2_{H^1(0,\ell)} = \|u_n\|^2_{L^2(0,\ell)} + \|u_{n,x}\|^2_{L^2(0,\ell)}$, the two Young estimates contribute $\frac{\varepsilon_1+\varepsilon_2}{2}\|u_{n,x}\|^2_{L^2(0,\ell)}$ to the right-hand side. Choose $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 = 2$. Then $2 - \frac{\varepsilon_1+\varepsilon_2}{2} = 1$, hence

$$\begin{aligned}\frac{d}{dt}\|u_n\|^2_{L^2(0,\ell)} + \|u_{n,x}\|^2_{L^2} \leq \frac{1}{\varepsilon}\|f(t)\|^2_{L^2(0,\ell)} + \left[\varepsilon + \frac{\varepsilon_2}{2} + \frac{2|\widetilde{f}(t) - E'(t)|}{E_0}\right]\|u_n\|^2_{L^2(0,\ell)} \\ + \frac{2\|N_x\|^2_{L^2(0,\ell)}}{\varepsilon_1 E_0^2}\|u_n\|^4_{L^2(0,\ell)} + \frac{C_\ell^2\|N_x\|^2_{L^2(0,\ell)}}{2\varepsilon_2 E_0^2}\|u_n\|^6_{L^2(0,\ell)}.\end{aligned} \tag{22}$$

We fix once and for all the parameters $\varepsilon > 0$ and $\varepsilon_1, \varepsilon_2 > 0$ satisfying $\varepsilon_1 + \varepsilon_2 = 2$, together with a constant C_ℓ such that $\|v\|_{L^\infty(0,\ell)} \leq C_\ell \|v\|_{H^1(0,\ell)}$ for every $v \in H^1(0,\ell)$. These choices are independent of n and t and are retained in the definitions of $c(t)$, $\Phi(t)$, $\Gamma(t)$, and T^* below.

Set

$$c(t) := \varepsilon + \frac{\varepsilon_2}{2} + \frac{2|\widetilde{f}(t) - E'(t)|}{E_0} + \frac{2\|N_x\|^2_{L^2(0,\ell)}}{\varepsilon_1 E_0^2} + \frac{C_\ell^2\|N_x\|^2_{L^2(0,\ell)}}{2\varepsilon_2 E_0^2}. \tag{23}$$

The function $c(t)$ belongs to $L^1(0,T)$, because $\widetilde{f} - E' \in L^2(0,T)$ and the time interval is finite.

We now use the elementary estimate

$$\|u_n\|^2_{L^2(0,\ell)} + \|u_n\|^4_{L^2(0,\ell)} + \|u_n\|^6_{L^2(0,\ell)} \leq \left(1 + \|u_n\|^2_{L^2(0,\ell)}\right)^3. \tag{24}$$

Indeed,

$$\left(1 + \|u_n\|^2_{L^2(0,\ell)}\right)^3 - \left(\|u_n\|^2_{L^2(0,\ell)} + \|u_n\|^4_{L^2(0,\ell)} + \|u_n\|^6_{L^2(0,\ell)}\right)$$
$$= 1 + 2\|u_n\|^2_{L^2(0,\ell)} + 2\|u_n\|^4_{L^2(0,\ell)} \geq 0.$$

Thus (22) gives $\frac{d}{dt}\|u_n\|^2_{L^2(0,\ell)} \leq \frac{1}{\varepsilon}\|f(t)\|^2_{L^2(0,\ell)} + c(t)\left(1 + \|u_n\|^2_{L^2(0,\ell)}\right)^3$. Introduce the shifted energy

$$y_n(t) := 1 + \|u_n(t)\|^2_{L^2(0,\ell)}. \tag{25}$$

Then

$$y_n'(t) \leq \frac{1}{\varepsilon}\|f(t)\|^2_{L^2(0,\ell)} + c(t) y_n(t)^3. \tag{26}$$

Set

$$\Phi(t) := 1 + \|u_0\|^2_{L^2(0,\ell)} + \frac{1}{\varepsilon}\int_0^t \|f(s)\|^2_{L^2(0,\ell)}\, ds, \qquad \Gamma(t) := \int_0^t c(s)\, ds. \tag{27}$$

Then

$$y_n(t) \leq \Phi(t) + \int_0^t c(s) y_n^3(s)\, ds. \tag{28}$$

Applying Bihari's inequality [11], we obtain

$$y_n(t) \le \frac{\Phi(t)}{\sqrt{1-2\Phi^2(t)\Gamma(t)}}, \tag{29}$$

if

$$2\Phi^2(t)\Gamma(t) < 1. \tag{30}$$

This suggests defining

$$T^* := \sup\big\{t \in (0,T] : 2\Phi^2(t)\Gamma(t) < 1\big\}. \tag{31}$$

Since $\Gamma(0) = 0$ and $2\Phi^2(t)\Gamma(t)$ is continuous and nondecreasing, $T^* > 0$. For $T_0 < T^*$, the inequality $2\Phi^2(T_0)\Gamma(T_0) < 1$ holds. Hence, (29) gives

$$\sup_{0\le t\le T_0} \|u_n(t)\|^2_{L^2(0,\ell)} \le C(T_0).$$

Integrating (22) gives $\int_0^{T_0} \|u_{n,x}(t)\|^2_{L^2(0,\ell)}\,dt \le C(T_0)$. The preceding bound prevents blow-up of the finite-dimensional solution before T_0; hence it extends to $[0,T_0]$, and

$$\sup_{t\in[0,T_0]} \|u_n(t)\|^2_{L^2(0,\ell)} + \int_0^{T_0} \|u_{n,x}(t)\|^2_{L^2(0,\ell)}\,dt \le K_1(T_0), \tag{32}$$

with $K_1(T_0)$ independent of n. □

Lemma 2 *For each $T_0 \in (0,T^*)$ there is $K_2(T_0)$, independent of n, such that*

$$\sup_{0\le t\le T_0} \|u_n(t)\|^2_{H^1(0,\ell)} + \int_0^{T_0} \|u_n(t)\|^2_{H^2(0,\ell)}\,dt \le K_2(T_0).$$

Proof. Set $\psi_n = -u_{n,xx}$ in (14). Since $-(u_n, u_{n,xx}) = \|u_{n,x}\|^2_{L^2(0,\ell)}$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{n,x}\|^2_{L^2(0,\ell)} + \|u_{n,xx}\|^2_{L^2(0,\ell)} &= (u_n u_{n,x}, u_{n,xx}) \\ &\quad + (f, -u_{n,xx}) - \frac{\widetilde{f}(t) - E'(t)}{E(t)}\|u_{n,x}\|^2_{L^2(0,\ell)} \\ &\quad + \frac{\|u_{n,x}\|^2_{L^2(0,\ell)}}{E(t)}\int_0^\ell N_x(x)u_{n,x}(x,t)\,dx \\ &\quad - \frac{\|u_{n,x}\|^2_{L^2(0,\ell)}}{2E(t)}\int_0^\ell N_x(x)u_n^2(x,t)\,dx.\end{aligned} \tag{33}$$

For $\varepsilon_3, \varepsilon_4 > 0$, Hölder, Cauchy–Schwarz, Young's inequality, $|E(t)| \ge E_0$, and $\|u_n\|^2_{L^4(0,\ell)} \le C_\ell \|u_n\|_{L^2(0,\ell)}\|u_n\|_{H^1(0,\ell)}$ give

$$\begin{aligned}|(u_n u_{n,x}, u_{n,xx})| &\le \|u_n\|_{L^\infty(0,\ell)}\|u_{n,x}\|_{L^2(0,\ell)}\|u_{n,xx}\|_{L^2(0,\ell)} \le \\ &\frac{\varepsilon_3}{2}\|u_{n,xx}\|^2_{L^2(0,\ell)} + \frac{1}{2\varepsilon_3}\|u_n\|^2_{L^\infty(0,\ell)}\|u_{n,x}\|^2_{L^2(0,\ell)}.\end{aligned}$$

$$|(f,-u_{n,xx})| \le \frac{\varepsilon_4}{2}\|u_{n,xx}\|^2_{L^2(0,\ell)} + \frac{1}{2\varepsilon_4}\|f\|^2_{L^2(0,\ell)}.$$

$$\left|\frac{\widetilde{f}-E'}{E}\right| \|u_{n,x}\|^2_{L^2(0,\ell)} \le \frac{|\widetilde{f}-E'|}{E_0}\|u_{n,x}\|^2_{L^2(0,\ell)}.$$

$$\left|\int_0^\ell N_x(x)u_{n,x}(x,t)\,dx\right| \le \|N_x\|_{L^2(0,\ell)}\|u_{n,x}\|_{L^2(0,\ell)}.$$

$$\left|\int_0^\ell N_x(x)u_n^2(x,t)\,dx\right| \le \|N_x\|_{L^2(0,\ell)}\|u_n\|^2_{L^4(0,\ell)}.$$

$$\left|\frac{\|u_{n,x}\|^2_{L^2(0,\ell)}}{E}\int_0^\ell N_x(x)u_{n,x}(x,t)\,dx\right| \le \frac{\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_{n,x}\|^3_{L^2(0,\ell)}.$$

$$\begin{aligned}\left|\frac{\|u_{n,x}\|^2_{L^2(0,\ell)}}{2E}\int_0^\ell N_x(x)u_n^2(x,t)\,dx\right| &\le \frac{\|N_x\|_{L^2(0,\ell)}}{2E_0}\|u_{n,x}\|^2_{L^2(0,\ell)}\|u_n\|^2_{L^4(0,\ell)}\\ &\le \frac{C_\ell\|N_x\|_{L^2(0,\ell)}}{2E_0}\|u_n\|_{L^2(0,\ell)}\|u_n\|_{H^1(0,\ell)}\|u_{n,x}\|^2_{L^2(0,\ell)}.\end{aligned}$$

Insert these bounds into (33), choose $\varepsilon_3+\varepsilon_4=1$, and multiply by 2:

$$\frac{d}{dt}\|u_{n,x}\|^2_{L^2(0,\ell)} + \|u_{n,xx}\|^2_{L^2(0,\ell)} \le B_n(t)\|u_{n,x}\|^2_{L^2(0,\ell)} + \frac{1}{\varepsilon_4}\|f(t)\|^2_{L^2(0,\ell)}, \tag{34}$$

with

$$\begin{aligned}B_n(t) := \frac{1}{\varepsilon_3}\|u_n(t)\|^2_{L^\infty(0,\ell)} &+ \frac{2|\widetilde{f}(t)-E'(t)|}{E_0} + \frac{2\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_{n,x}(t)\|_{L^2(0,\ell)}\\ &+ \frac{C_\ell\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_n(t)\|_{L^2(0,\ell)}\|u_n(t)\|_{H^1(0,\ell)}.\end{aligned} \tag{35}$$

By (32), $H^1(0,\ell)\hookrightarrow L^\infty(0,\ell)$, and Cauchy–Schwarz in time,

$$\int_0^{T_0}\|u_n(t)\|^2_{H^1(0,\ell)}\,dt \le T_0\sup_{0\le t\le T_0}\|u_n(t)\|^2_{L^2(0,\ell)} + \int_0^{T_0}\|u_{n,x}(t)\|^2_{L^2(0,\ell)}\,dt \le C(T_0),$$

$$\int_0^{T_0}\|u_n(t)\|^2_{L^\infty(0,\ell)}\,dt \le C\int_0^{T_0}\left(\|u_n(t)\|^2_{L^2(0,\ell)} + \|u_{n,x}(t)\|^2_{L^2(0,\ell)}\right)dt \le C(T_0),$$

$$\int_0^{T_0}\|u_{n,x}(t)\|_{L^2(0,\ell)}\,dt \le T_0^{1/2}\left(\int_0^{T_0}\|u_{n,x}(t)\|^2_{L^2(0,\ell)}\,dt\right)^{1/2} \le C(T_0),$$

$$\int_0^{T_0}\|u_n(t)\|_{L^2(0,\ell)}\|u_n(t)\|_{H^1(0,\ell)}\,dt \tag{36}$$

$$\le \sup_{0\le t\le T_0}\|u_n(t)\|_{L^2(0,\ell)}\,T_0^{1/2}\left(\int_0^{T_0}\|u_n(t)\|^2_{H^1(0,\ell)}\,dt\right)^{1/2} \le C(T_0). \tag{37}$$

Since $\widetilde{f}-E' \in L^2(0,T) \subset L^1(0,T_0)$, $\int_0^{T_0} B_n(t)\,dt \le C(T_0)$ uniformly in n. Also, $\|u_{n,x}(0)\|_{L^2(0,\ell)} \le \|u_{0,x}\|_{L^2(0,\ell)}$. Gronwall applied to (34) gives $\sup_{0\le t\le T_0}\|u_{n,x}(t)\|^2_{L^2(0,\ell)} \le C(T_0)$. Together with (32), $\sup_{0\le t\le T_0}\|u_n(t)\|^2_{H^1(0,\ell)} \le C(T_0)$. Integrating (34) gives $\int_0^{T_0}\|u_{n,xx}(t)\|^2_{L^2(0,\ell)}\,dt \le C(T_0)$. Thus

$$\sup_{t\in[0,T_0]}\|u_n(t)\|^2_{H^1(0,\ell)} + \int_0^{T_0}\|u_n(t)\|^2_{H^2(0,\ell)}\,dt \le K_2(T_0). \tag{38}$$

□

Lemma 3 *For every $T_0 \in (0,T^*)$, the sequence $\{u_{n,t}\}$ is bounded in $L^2(0,T_0;L^2(0,\ell))$ uniformly in n.*

Proof. For a.e. $t\in(0,T_0)$, $u_{n,t}(t)\in V_n$, so

$$\|u_{n,t}(t)\|_{L^2} = \sup_{\substack{\psi_n\in V_n\\ \|\psi_n\|_{L^2}=1}} |(u_{n,t}(t),\psi_n)|.$$

By (14), periodicity, $(u_{n,x},\psi_{n,x}) = -(u_{n,xx},\psi_n)$, and $|E(t)|\ge E_0$, we obtain

$$\begin{aligned}\|u_{n,t}(t)\|_{L^2(0,\ell)} &\le \|u_{n,xx}(t)\|_{L^2(0,\ell)}\\ &\quad+\|u_n(t)u_{n,x}(t)\|_{L^2(0,\ell)}+\|f(t)\|_{L^2(0,\ell)} + \frac{|\widetilde{f}(t)-E'(t)|}{E_0}\|u_n(t)\|_{L^2(0,\ell)}\\ &\quad+\frac{\|u_n(t)\|_{L^2(0,\ell)}}{E_0}\left|\int_0^\ell N_x(x)u_{n,x}(x,t)\,dx\right| + \frac{\|u_n(t)\|_{L^2(0,\ell)}}{2E_0}\left|\int_0^\ell N_x(x)u_n^2(x,t)\,dx\right|.\end{aligned}$$

Cauchy–Schwarz and $H^1(0,\ell)\hookrightarrow L^\infty(0,\ell)$ give

$$\left|\int_0^\ell N_x(x)u_{n,x}(x,t)\,dx\right| \le \|N_x\|_{L^2(0,\ell)}\|u_{n,x}\|_{L^2(0,\ell)},$$

$$\left|\int_0^\ell N_x(x)u_n^2(x,t)\,dx\right| \le \|N_x\|_{L^2(0,\ell)}\|u_n\|^2_{L^4(0,\ell)} \le C_\ell\|N_x\|_{L^2(0,\ell)}\|u_n\|_{L^2(0,\ell)}\|u_n\|_{H^1(0,\ell)},$$

$$\begin{aligned}\|u_{n,t}(t)\|_{L^2(0,\ell)} &\le \|u_{n,xx}(t)\|_{L^2(0,\ell)} + \|u_n(t)u_{n,x}(t)\|_{L^2(0,\ell)} + \|f(t)\|_{L^2(0,\ell)}\\ &\quad+\frac{|\widetilde{f}(t)-E'(t)|}{E_0}\|u_n(t)\|_{L^2(0,\ell)} + \frac{\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_n(t)\|_{L^2(0,\ell)}\|u_{n,x}(t)\|_{L^2(0,\ell)}\\ &\quad+\frac{C_\ell\|N_x\|_{L^2(0,\ell)}}{2E_0}\|u_n(t)\|^2_{L^2(0,\ell)}\|u_n(t)\|_{H^1(0,\ell)}.\end{aligned} \tag{39}$$

Squaring and using $(a_1+\cdots+a_6)^2 \le 6\sum_{j=1}^{6} a_j^2$ gives

$$\begin{aligned}\|u_{n,t}(t)\|^2_{L^2(0,\ell)} &\le 6\|u_{n,xx}(t)\|^2_{L^2(0,\ell)} + 6\|u_n(t)u_{n,x}(t)\|^2_{L^2(0,\ell)} + 6\|f(t)\|^2_{L^2(0,\ell)}\\ &\quad + \frac{6|\widetilde{f}(t)-E'(t)|^2}{E_0^2}\|u_n(t)\|^2_{L^2(0,\ell)} + \frac{6\|N_x\|^2_{L^2(0,\ell)}}{E_0^2}\|u_n(t)\|^2_{L^2(0,\ell)}\|u_{n,x}(t)\|^2_{L^2(0,\ell)}\\ &\quad + \frac{3C_\ell^2\|N_x\|^2_{L^2(0,\ell)}}{2E_0^2}\|u_n(t)\|^4_{L^2(0,\ell)}\|u_n(t)\|^2_{H^1(0,\ell)}.\end{aligned}$$

Integration over $(0,T_0)$ gives

$$\begin{aligned}\int_0^{T_0}\|u_{n,t}\|^2_{L^2(0,\ell)}\,dt &\le 6\int_0^{T_0}\|u_{n,xx}\|^2_{L^2}\,dt + 6\int_0^{T_0}\|u_nu_{n,x}\|^2_{L^2(0,\ell)}\,dt\\ &\quad + 6\int_0^{T_0}\|f\|^2_{L^2(0,\ell)}\,dt + \frac{6}{E_0^2}\int_0^{T_0}|\widetilde{f}-E'|^2\|u_n\|^2_{L^2(0,\ell)}\,dt\\ &\quad + \frac{6\|N_x\|^2_{L^2(0,\ell)}}{E_0^2}\int_0^{T_0}\|u_n\|^2_{L^2(0,\ell)}\|u_{n,x}\|^2_{L^2(0,\ell)}\,dt\\ &\quad + \frac{3C_\ell^2\|N_x\|^2_{L^2(0,\ell)}}{2E_0^2}\int_0^{T_0}\|u_n\|^4_{L^2(0,\ell)}\|u_n\|^2_{H^1(0,\ell)}\,dt.\end{aligned}$$

By (32), (38), Hölder, and the $L^2(0,T_0;L^\infty(0,\ell))$ estimate from Lemma 2,

$$\begin{aligned}&\int_0^{T_0}\|u_{n,xx}(t)\|^2_{L^2(0,\ell)}\,dt \le C(T_0), \qquad \int_0^{T_0}\|f(t)\|^2_{L^2(0,\ell)}\,dt \le \|f\|^2_{L^2(0,T;L^2(0,\ell))},\\ &\int_0^{T_0}\|u_nu_{n,x}\|^2_{L^2(0,\ell)}\,dt \le \sup_{0\le t\le T_0}\|u_{n,x}(t)\|^2_{L^2(0,\ell)}\int_0^{T_0}\|u_n(t)\|^2_{L^\infty(0,\ell)}\,dt \le C(T_0),\\ &\int_0^{T_0}|\widetilde{f}-E'|^2\|u_n\|^2_{L^2(0,\ell)}\,dt \le \sup_{0\le t\le T_0}\|u_n(t)\|^2_{L^2(0,\ell)}\int_0^{T_0}|\widetilde{f}-E'|^2\,dt \le C(T_0),\\ &\int_0^{T_0}\|u_n\|^2_{L^2(0,\ell)}\|u_{n,x}\|^2_{L^2(0,\ell)}\,dt \le \sup_{0\le t\le T_0}\|u_n(t)\|^2_{L^2(0,\ell)}\int_0^{T_0}\|u_{n,x}(t)\|^2_{L^2(0,\ell)}\,dt \le C(T_0),\\ &\int_0^{T_0}\|u_n\|^4_{L^2(0,\ell)}\|u_n\|^2_{H^1(0,\ell)}\,dt \le \sup_{0\le t\le T_0}\|u_n(t)\|^4_{L^2(0,\ell)}\int_0^{T_0}\|u_n(t)\|^2_{H^1(0,\ell)}\,dt \le C(T_0). \qquad (40)\end{aligned}$$

Therefore

$$\int_0^{T_0}\|u_{n,t}(t)\|^2_{L^2(0,\ell)}\,dt \le C(T_0), \qquad \|u_{n,t}\|_{L^2(0,T_0;L^2(0,\ell))} \le C(T_0). \qquad (41)$$

The lemma follows. □

3.3 Existence

Fix $T_0 \in (0, T^*)$. The estimates of Lemmas 2 and 3 imply

$$\|u_n\|_{L^\infty(0,T_0;H^1(0,\ell))} + \|u_n\|_{L^2(0,T_0;H^2(0,\ell))} + \|u_{n,t}\|_{L^2(0,T_0;L^2(0,\ell))} \le K(T_0), \tag{42}$$

where $K(T_0)$ is independent of n. After passing to a subsequence, if necessary,

$$u_n \rightharpoonup u \qquad \text{in } L^2(0,T_0;H^2_{\mathrm{per}}(0,\ell)),$$
$$u_n \overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty(0,T_0;H^1_{\mathrm{per}}(0,\ell)), \tag{43}$$
$$u_{n,t} \rightharpoonup u_t \qquad \text{in } L^2(0,T_0;L^2(0,\ell)). \tag{44}$$

Since $H^2_{\mathrm{per}}(0,\ell) \hookrightarrow\hookrightarrow H^1_{\mathrm{per}}(0,\ell) \hookrightarrow L^2(0,\ell)$, the Aubin–Lions–Simon compactness theorem [13], together with $H^1(0,\ell) \hookrightarrow L^\infty(0,\ell)$, yields

$$u_n \to u \qquad \text{in } L^2(0,T_0;H^1(0,\ell)), \tag{45}$$
$$u_n \to u \qquad \text{in } L^2(0,T_0;L^2(0,\ell)),$$
$$u_{n,x} \to u_x \qquad \text{in } L^2(0,T_0;L^2(0,\ell)), \tag{46}$$
$$u_n \to u \qquad \text{in } L^2(0,T_0;L^\infty(0,\ell)).$$

In particular,

$$\|u_n^2 - u^2\|_{L^2(0,T_0;L^2(0,\ell))} \le \|u_n - u\|_{L^2(0,T_0;L^\infty(0,\ell))}\|u_n + u\|_{L^\infty(0,T_0;L^2(0,\ell))} \longrightarrow 0. \tag{47}$$

Let $\psi \in C^\infty([0,T_0];H^1_{\mathrm{per}}(0,\ell))$ be arbitrary, and let

$$\psi_n(\cdot,t) = \sum_{k=-n}^{n} (\psi(\cdot,t), e_k)e_k \in V_n.$$

Then

$$\sup_{0\le t\le T_0} \|\psi_n(\cdot,t) - \psi(\cdot,t)\|_{H^1(0,\ell)} \longrightarrow 0, \qquad \|\psi_n\|_{L^\infty(0,T_0;H^1(0,\ell))} \le C. \tag{48}$$

To pass to the limit in the Galerkin identity, it is enough to verify

$$\begin{aligned}
\lim_{n\to\infty} \Bigg| \int_0^{T_0} \Bigg[& (u_{n,t},\psi_n) - (u_t,\psi) + (u_n u_{n,x},\psi_n) - (uu_x,\psi) \\
& + (u_{n,x},\psi_{n,x}) - (u_x,\psi_x) + \frac{\widetilde{f} - E'}{E}\big((u_n,\psi_n) - (u,\psi)\big) \\
& - \frac{(u_n,\psi_n)}{E}\int_0^\ell N_x(x)u_{n,x}(x,t)\,dx + \frac{(u,\psi)}{E}\int_0^\ell N_x(x)u_x(x,t)\,dx \\
& + \frac{(u_n,\psi_n)}{2E}\int_0^\ell N_x(x)u_n^2(x,t)\,dx - \frac{(u,\psi)}{2E}\int_0^\ell N_x(x)u^2(x,t)\,dx \\
& - (f,\psi_n) + (f,\psi)\Bigg] dt \Bigg| = 0.
\end{aligned} \tag{49}$$

Consider first the time-derivative term.

$$\left| \int_0^{T_0} \big[(u_{n,t},\psi_n) - (u_t,\psi)\big]\,dt \right|$$

$$\le \|u_{n,t}\|_{L^2(0,T_0;L^2(0,\ell))}\|\psi_n-\psi\|_{L^2(0,T_0;L^2(0,\ell))} + \left|\int_0^{T_0}(u_{n,t}-u_t,\psi)\,dt\right| \longrightarrow 0, \tag{50}$$

by Lemma 3, (48), and (44).

For the convection term, periodicity gives $(u_nu_{n,x},\psi_n) = -\frac{1}{2}(u_n^2,\psi_{n,x})$ and $(uu_x,\psi) = -\frac{1}{2}(u^2,\psi_x)$. Thus

$$\left|\int_0^{T_0}\big[(u_nu_{n,x},\psi_n)-(uu_x,\psi)\big]\,dt\right| \le \frac{1}{2}\|u_n^2-u^2\|_{L^2(0,T_0;L^2(0,\ell))}\|\psi_{n,x}\|_{L^2(0,T_0;L^2(0,\ell))}$$
$$+\frac{1}{2}\|u^2\|_{L^2(0,T_0;L^2(0,\ell))}\|\psi_{n,x}-\psi_x\|_{L^2(0,T_0;L^2(0,\ell))} \longrightarrow 0. \tag{51}$$

The diffusion term is handled in the same way:

$$\left|\int_0^{T_0}\big[(u_{n,x},\psi_{n,x})-(u_x,\psi_x)\big]\,dt\right| \le \|u_{n,x}-u_x\|_{L^2(0,T_0;L^2(0,\ell))}\|\psi_{n,x}\|_{L^2(0,T_0;L^2(0,\ell))}$$
$$+\|u_x\|_{L^2(0,T_0;L^2(0,\ell))}\|\psi_{n,x}-\psi_x\|_{L^2(0,T_0;L^2(0,\ell))} \longrightarrow 0. \tag{52}$$

For the scalar products appearing in the loaded terms, we have

$$\|(u_n,\psi_n)-(u,\psi)\|_{L^2(0,T_0)} \le \|\psi_n\|_{L^\infty(0,T_0;L^2(0,\ell))}\|u_n-u\|_{L^2(0,T_0;L^2(0,\ell))}$$
$$+\|u\|_{L^\infty(0,T_0;L^2(0,\ell))}\|\psi_n-\psi\|_{L^2(0,T_0;L^2(0,\ell))} \longrightarrow 0. \tag{53}$$

Since $|E(t)| \ge E_0$ and $\widetilde{f}-E' \in L^2(0,T_0)$,

$$\left\|\frac{\widetilde{f}-E'}{E}\right\|_{L^2(0,T_0)} \le \frac{1}{E_0}\|\widetilde{f}-E'\|_{L^2(0,T_0)} < \infty.$$

Thus

$$\left|\int_0^{T_0}\frac{\widetilde{f}-E'}{E}\big((u_n,\psi_n)-(u,\psi)\big)\,dt\right| \le \left\|\frac{\widetilde{f}-E'}{E}\right\|_{L^2(0,T_0)}\|(u_n,\psi_n)-(u,\psi)\|_{L^2(0,T_0)} \longrightarrow 0. \tag{54}$$

For the derivative nonlocal term, add and subtract $\frac{(u,\psi)}{E(t)}\int_0^\ell N_x(x)u_{n,x}(x,t)\,dx$. Then

$$\left|-\int_0^{T_0}\frac{(u_n,\psi_n)}{E}\int_0^\ell N_x(x)u_{n,x}(x,t)\,dx\,dt + \int_0^{T_0}\frac{(u,\psi)}{E}\int_0^\ell N_x(x)u_x(x,t)\,dx\,dt\right|$$
$$\le \frac{1}{E_0}\|(u_n,\psi_n)-(u,\psi)\|_{L^2(0,T_0)}\left\|\int_0^\ell N_x(x)u_{n,x}(x,t)\,dx\right\|_{L^2(0,T_0)}$$
$$+\frac{1}{E_0}\|(u,\psi)\|_{L^2(0,T_0)}\left\|\int_0^\ell N_x(x)(u_{n,x}(x,t)-u_x(x,t))\,dx\right\|_{L^2(0,T_0)}$$
$$\le \frac{\|N_x\|_{L^2(0,\ell)}}{E_0}\|(u_n,\psi_n)-(u,\psi)\|_{L^2(0,T_0)}\|u_{n,x}\|_{L^2(0,T_0;L^2(0,\ell))}$$

$$+\frac{\|N_x\|_{L^2(0,\ell)}}{E_0}\|(u,\psi)\|_{L^2(0,T_0)}\|u_{n,x}-u_x\|_{L^2(0,T_0;L^2(0,\ell))}\longrightarrow 0. \tag{55}$$

Here the first derivative norm is bounded by Lemma 1, while the last tends to zero by (46).

For the nonlocal term involving u^2,

$$\begin{aligned}
&\left|\int_0^{T_0}\frac{(u_n,\psi_n)}{2E}\int_0^{\ell}N_x(x)u_n^2(x,t)\,dx\,dt-\int_0^{T_0}\frac{(u,\psi)}{2E}\int_0^{\ell}N_x(x)u^2(x,t)\,dx\,dt\right|\\
&\quad\le\frac{1}{2E_0}\|(u_n,\psi_n)-(u,\psi)\|_{L^2(0,T_0)}\left\|\int_0^{\ell}N_x(x)u_n^2(x,t)\,dx\right\|_{L^2(0,T_0)}\\
&\qquad+\frac{1}{2E_0}\|(u,\psi)\|_{L^2(0,T_0)}\left\|\int_0^{\ell}N_x(x)(u_n^2(x,t)-u^2(x,t))\,dx\right\|_{L^2(0,T_0)}\\
&\quad\le\frac{\|N_x\|_{L^2(0,\ell)}}{2E_0}\|(u_n,\psi_n)-(u,\psi)\|_{L^2(0,T_0)}\|u_n^2\|_{L^2(0,T_0;L^2(0,\ell))}\\
&\qquad+\frac{\|N_x\|_{L^2(0,\ell)}}{2E_0}\|(u,\psi)\|_{L^2(0,T_0)}\|u_n^2-u^2\|_{L^2(0,T_0;L^2(0,\ell))}\longrightarrow 0.
\end{aligned} \tag{56}$$

The first norm involving u_n^2 is bounded, and the last tends to zero by (47).

Finally,

$$\left|\int_0^{T_0}\big[-(f,\psi_n)+(f,\psi)\big]dt\right|\le\|f\|_{L^2(0,T_0;L^2)}\|\psi_n-\psi\|_{L^2(0,T_0;L^2)}\longrightarrow 0. \tag{57}$$

Combining the preceding convergences proves (49). Passing to the limit in the Galerkin identity therefore gives

$$\begin{aligned}
&(u_t,\psi)+(uu_x,\psi)+(u_x,\psi_x)+\frac{\widetilde{f}-E'}{E}(u,\psi)\\
&\qquad-\frac{(u,\psi)}{E}\int_0^{\ell}N_x(x)u_x(x,t)\,dx+\frac{(u,\psi)}{2E}\int_0^{\ell}N_x(x)u^2(x,t)\,dx=(f,\psi)
\end{aligned}$$

for every admissible ψ. Hence u satisfies (11) for a.e. $(x,t)\in(0,\ell)\times(0,T_0)$.

It remains to identify the initial value. From (42), $H^1_{\mathrm{per}}(0,\ell)\Subset L^2(0,\ell)$, and Simon's compactness theorem give

$$\begin{aligned}
u_n&\to u && \text{in } C([0,T_0];L^2(0,\ell)),\\
u_n(\cdot,0)&\to u_0 && \text{in } H^1_{\mathrm{per}}(0,\ell),
\end{aligned}$$

so $u(\cdot,0)=u_0$ in $L^2(0,\ell)$. Moreover, by Lions–Magenes [12]

$$\begin{aligned}
&u\in L^\infty(0,T_0;H^1_{\mathrm{per}})\cap L^2(0,T_0;H^2_{\mathrm{per}}), && u_t\in L^2(0,T_0;L^2),\\
&u\in C([0,T_0];H^1_{\mathrm{per}}(0,\ell)).
\end{aligned} \tag{58}$$

Consequently $u(\cdot,0)=u_0$ in $H^1_{\mathrm{per}}(0,\ell)$, and u is a strong solution on $[0,T_0]$. Since $T_0<T^*$ was arbitrary, the solution exists on every compact subinterval of $[0,T^*)$.

3.4 Uniqueness

Fix $T_0 < T^*$, and let $u_1(x,t)$ and $u_2(x,t)$ be two strong solutions of (11), (1b)–(1c). Set $w(x,t) := u_1(x,t) - u_2(x,t)$. Since $u_1(x,t)u_{1,x}(x,t) - u_2(x,t)u_{2,x}(x,t) = w(x,t)\,u_{1,x}(x,t) + u_2(x,t)w_x(x,t)$ and $u_1^2(x,t) - u_2^2(x,t) = (u_1(x,t) + u_2(x,t))w(x,t)$, the difference w satisfies

$$\begin{aligned} w_t(x,t) + w(x,t)\,u_{1,x}(x,t) + u_2(x,t)w_x(x,t) &= w_{xx}(x,t) \\ -\frac{w(x,t)}{E(t)}\left(\widetilde{f} - E' - \int_0^\ell N_x(x)u_{1,x}(x,t)\,dx + \frac{1}{2}\int_0^\ell N_x(x)u_1^2(x,t)\,dx\right) & \\ +\frac{u_2(x,t)}{E(t)}\int_0^\ell N_x(x)w_x(x,t)\,dx - \frac{u_2(x,t)}{2E(t)}\int_0^\ell N_x(x)(u_1(x,t)+u_2(x,t))w(x,t)\,dx. & \end{aligned} \tag{59}$$

Testing (59) with $w(x,t)$ and using periodicity gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|_{L^2(0,\ell)}^2 + \|w_x\|_{L^2(0,\ell)}^2 &= -\int_0^\ell (w(x,t)u_{1,x}(x,t) + u_2(x,t)w_x(x,t))w(x,t)\,dx \\ &\quad - \frac{\|w\|_{L^2(0,\ell)}^2}{E(t)}\left(\widetilde{f} - E' - \int_0^\ell N_x(x)u_{1,x}(x,t)\,dx + \frac{1}{2}\int_0^\ell N_x(x)u_1^2(x,t)\,dx\right) \\ &\quad + \frac{(u_2,w)}{E(t)}\int_0^\ell N_x(x)w_x(x,t)\,dx - \frac{(u_2,w)}{2E(t)}\int_0^\ell N_x(x)(u_1(x,t)+u_2(x,t))w(x,t)\,dx. \end{aligned} \tag{60}$$

For $\varepsilon_5, \varepsilon_6 > 0$, periodic integration by parts, Cauchy–Schwarz, Hölder, Young's inequality, and $|E(t)| \geq E_0$ give

$$\begin{aligned} &\left|\int_0^\ell \big(w(x,t)u_{1,x}(x,t) + u_2(x,t)w_x(x,t)\big)w(x,t)\,dx\right| \\ &\qquad = \left|\int_0^\ell \big(u_2(x,t) - 2u_1(x,t)\big)w(x,t)w_x(x,t)\,dx\right| \\ &\qquad \leq \frac{\varepsilon_5}{2}\|w_x\|_{L^2(0,\ell)}^2 + \frac{\big(2\|u_1\|_{L^\infty(0,\ell)} + \|u_2\|_{L^\infty(0,\ell)}\big)^2}{2\varepsilon_5}\|w\|_{L^2(0,\ell)}^2. \end{aligned} \tag{61}$$

$$\begin{aligned}
&\frac{\|w\|^2_{L^2(0,\ell)}}{|E(t)|}\left|\widetilde{f}-E'-\int_0^\ell N_x(x)u_{1,x}(x,t)\,dx+\frac{1}{2}\int_0^\ell N_x(x)u_1^2(x,t)\,dx\right|\\
&\quad\le\frac{\|w\|^2_{L^2(0,\ell)}}{E_0}\left[|\widetilde{f}-E'|+\|N_x\|_{L^2(0,\ell)}\|u_{1,x}\|_{L^2(0,\ell)}+\frac{1}{2}\|N_x\|_{L^2(0,\ell)}\|u_1^2\|_{L^2(0,\ell)}\right],\\
&\frac{|(u_2,w)|}{|E(t)|}\left|\int_0^\ell N_x(x)w_x(x,t)\,dx\right|\le\frac{\varepsilon_6}{2}\|w_x\|^2_{L^2(0,\ell)}+\frac{\|N_x\|^2_{L^2(0,\ell)}}{2\varepsilon_6E_0^2}\|u_2\|^2_{L^2(0,\ell)}\|w\|^2_{L^2(0,\ell)},\\
&\frac{|(u_2,w)|}{2|E(t)|}\left|\int_0^\ell N_x(x)(u_1(x,t)+u_2(x,t))w(x,t)\,dx\right|\le\frac{\|N_x\|_{L^2}}{2E_0}\|u_2\|_{L^2(0,\ell)}\|u_1+u_2\|_{L^\infty(0,\ell)}\|w\|^2_{L^2(0,\ell)}.
\end{aligned}\tag{62}$$

Here $\int_0^\ell w^2(x,t)u_{1,x}(x,t)\,dx=-2\int_0^\ell u_1(x,t)w(x,t)w_x(x,t)\,dx$.

Substituting the above estimates into (60) yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(0,\ell)}+\|w_x\|^2_{L^2(0,\ell)}\le\frac{\varepsilon_5+\varepsilon_6}{2}\|w_x\|^2_{L^2(0,\ell)}+\Bigg[&\frac{(2\|u_1\|_{L^\infty(0,\ell)}+\|u_2\|_{L^\infty(0,\ell)})^2}{2\varepsilon_5}\\
&+\frac{|\widetilde{f}-E'|}{E_0}+\frac{\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_{1,x}\|_{L^2(0,\ell)}+\frac{\|N_x\|_{L^2(0,\ell)}}{2E_0}\|u_1^2\|_{L^2(0,\ell)}\\
&+\frac{\|N_x\|^2_{L^2(0,\ell)}}{2\varepsilon_6E_0^2}\|u_2\|^2_{L^2(0,\ell)}+\frac{\|N_x\|_{L^2(0,\ell)}}{2E_0}\|u_2\|_{L^2(0,\ell)}\|u_1+u_2\|_{L^\infty(0,\ell)}\Bigg]\|w\|^2_{L^2(0,\ell)}.
\end{aligned}\tag{63}$$

Choose $\varepsilon_5+\varepsilon_6=1$. Then $1-\frac{\varepsilon_5+\varepsilon_6}{2}=\frac{1}{2}$, and, after absorption and multiplication by 2,

$$\frac{d}{dt}\|w(t)\|^2_{L^2(0,\ell)}+\|w_x(t)\|^2_{L^2(0,\ell)}\le\Lambda(t)\|w(t)\|^2_{L^2(0,\ell)},\tag{64}$$

where

$$\begin{aligned}
\Lambda(t):=&\frac{(2\|u_1(t)\|_{L^\infty(0,\ell)}+\|u_2(t)\|_{L^\infty(0,\ell)})^2}{\varepsilon_5}+\frac{2|\widetilde{f}(t)-E'(t)|}{E_0}\\
&+\frac{2\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_{1,x}(t)\|_{L^2(0,\ell)}+\frac{\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_1(t)^2\|_{L^2(0,\ell)}\\
&+\frac{\|N_x\|^2_{L^2(0,\ell)}}{\varepsilon_6E_0^2}\|u_2(t)\|^2_{L^2(0,\ell)}+\frac{\|N_x\|_{L^2(0,\ell)}}{E_0}\|u_2(t)\|_{L^2(0,\ell)}\|u_1(t)+u_2(t)\|_{L^\infty(0,\ell)}.
\end{aligned}\tag{65}$$

Since $u_i\in L^\infty(0,T_0;H^1_{\mathrm{per}}(0,\ell))$ and $H^1(0,\ell)\hookrightarrow L^\infty(0,\ell)$, we have

$$\|u_i(t)\|_{L^\infty(0,\ell)}+\|u_i(t)\|_{L^2(0,\ell)}+\|u_{i,x}(t)\|_{L^2(0,\ell)}\le C,$$

$$\|u_1(t)^2\|_{L^2(0,\ell)}\le\|u_1(t)\|_{L^\infty(0,\ell)}\|u_1(t)\|_{L^2(0,\ell)}\le C.$$

Consequently, $\Lambda(t)\le C\big(1+|\widetilde{f}(t)-E'(t)|\big)$. Since $\widetilde{f}-E'\in L^2(0,T_0)\subset L^1(0,T_0)$, it follows that $\Lambda\in L^1(0,T_0)$.

Finally, $w(\cdot,0)=0$. Dropping the nonnegative term $\|w_x\|^2_{L^2}$ in (64) and applying

Gronwall's inequality gives

$$\|w(t)\|^2_{L^2(0,\ell)} \le \|w(0)\|^2_{L^2(0,\ell)} \exp\left(\int_0^t \Lambda(s)\,ds\right) = 0.$$

Hence $u_1 = u_2$ on $[0, T_0]$. Since $T_0 < T^*$ was arbitrary, uniqueness holds on every compact subinterval of $[0, T^*)$. □

It remains to recover the coefficient and verify the prescribed integral observation. Fix $T_0 < T^*$ and let u be given by Theorem 1. Define a by (10). Then

$$|a(t)| \le \frac{1}{E_0}\left(|\widetilde{f} - E'| + \|N_x\|_{L^2(0,\ell)}\|u_x\|_{L^2(0,\ell)} + \frac{1}{2}\|N_x\|_{L^2(0,\ell)}\|u\|^2_{L^4(0,\ell)}\right).$$

Since $\|u\|^2_{L^4(0,\ell)} \le C\|u\|_{L^2(0,\ell)}\|u\|_{H^1(0,\ell)}$ and $u \in L^\infty(0, T_0; H^1(0,\ell))$, we obtain $a \in L^2(0, T_0)$. Substitution of (10) into (11) gives (1a).

Set $F(t) := \int_0^\ell N(x)u(x,t)\,dx$. Then $F \in H^1(0, T_0)$ and

$$F' = -aF + \widetilde{f} - \int_0^\ell N_x(x)u_x(x,t)\,dx + \frac{1}{2}\int_0^\ell N_x(x)u^2(x,t)\,dx.$$

Comparing this identity with (10), $G := F - E$ satisfies $G' = -aG$. By (7), $G(0) = 0$, and since $a \in L^1(0, T_0)$, Gronwall's inequality yields $G \equiv 0$. Hence $\int_0^\ell N(x)u(x,t)\,dx = E(t)$ for all $t \in [0, T_0]$.

Finally, let $(\widetilde{u}, \widetilde{a})$ be any other strong solution pair in the sense of Definition 2. Then $\widetilde{u}$ satisfies the same reduced problem. By Theorem 1, $\widetilde{u} = u$, and (10) gives $\widetilde{a} = a$ a.e. This proves Theorem 2. □

4 Numerical experiments

We illustrate the reconstruction on a smooth test with a known exact solution and examine its accuracy under spatial mesh refinement. Set $\ell = 2\pi$ and $T = 1$, and choose

$$N(x) = 1 + \tfrac{1}{2}\sin x, \qquad u_*(x,t) = e^{-t}v(x), \qquad a_*(t) = \cos(\pi t), \qquad v(x) = 2 + \sin x + \cos x. \tag{66}$$

The corresponding input data are

$$\begin{aligned} &u_0(x) = v(x), && E(t) = \tfrac{9\pi}{2}e^{-t}, \quad E'(t) = -E(t), \\ &f(x,t) = -2e^{-t} + e^{-2t}v(x)(\cos x - \sin x) + e^{-t}\cos(\pi t)v(x). \end{aligned} \tag{67}$$

Direct substitution verifies the equation, periodic boundary conditions and integral observation. In particular, $E(t) \ge 9\pi/(2e) > 0$. The coefficient changes sign at $t = 1/2$, and both nonlocal quantities are nonzero:

$$\int_0^{2\pi} N_x u_{*,x}\,dx = \tfrac{\pi}{2}e^{-t}, \qquad \int_0^{2\pi} N_x u_*^2\,dx = 2\pi e^{-2t}.$$

For this numerical illustration, we use a conservative finite-difference discretization. Let $x_j = jh$, $h = 2\pi/M$, $j = 0, \dots, M-1$, with indices understood modulo M, and define

$$(D_0U)_j = \frac{U_{j+1} - U_{j-1}}{2h}, \qquad (D_2U)_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}, \qquad Q_h(Z) = h\sum_{j=0}^{M-1} Z_j.$$

Writing $N_j = N(x_j)$, we solve

$$\begin{aligned} R_j(U,t) &= (D_2U)_j - \Big(D_0\tfrac{U^2}{2}\Big)_j + f(x_j,t), \\ a_h(t) &= \frac{Q_h\big(NR(U,t)\big) - E'(t)}{E(t)}, \\ U_j'(t) &= R_j(U,t) - a_h(t)U_j(t), \qquad U_j(0) = u_0(x_j). \end{aligned} \tag{68}$$

All vector products and squares are componentwise. The coefficient is reconstructed from the current numerical state and the prescribed data; a_* is used only to construct the test and evaluate its error. This discrete weighted balance is consistent with the continuous reconstruction formula. Moreover, $G_h(t) := Q_h(NU(t)) - E(t)$ satisfies $G_h' = -a_hG_h$. Since $G_h(0) = 0$ for the selected data and grids, the observation is preserved at the semidiscrete level.

Time integration is performed using the BDF solver in SciPy's `solve_ivp`, with relative and absolute tolerances 10^{-10} and 10^{-12}. At $t_m = m/200$, $m = 0, \dots, 200$, we evaluate

$$\varepsilon_u(M) = \max_m \left(h\sum_{j=0}^{M-1} |U_j(t_m) - u_*(x_j,t_m)|^2 \right)^{1/2}, \qquad \varepsilon_a(M) = \max_m |a_h(t_m) - a_*(t_m)|. \tag{69}$$

The observed orders are $p_u = \log_2[\varepsilon_u(M/2)/\varepsilon_u(M)]$ and analogously for p_a.

Table 1: Spatial mesh refinement for the exact-data test. Errors are evaluated at 201 equally spaced times in $[0, 1]$.

M	ε_u	p_u	ε_a	p_a
32	9.3445×10^{-3}	—	1.0686×10^{-3}	—
64	2.3319×10^{-3}	2.003	2.6759×10^{-4}	1.998
128	5.8271×10^{-4}	2.001	6.6924×10^{-5}	1.999
256	1.4566×10^{-4}	2.000	1.6733×10^{-5}	2.000

Table 1 shows observed second-order spatial convergence for both the state and the recovered coefficient. On the finest grid, reducing both time tolerances by a factor of ten changes the state by less than 1.5×10^{-9} in the sampled maximum discrete L^2 norm and the coefficient by less than 4.0×10^{-11}. These changes are much smaller than the reported errors. Figure 1 compares the exact and numerical solutions for $M = 64$.

The exact solution pair (u_*, a_*) exists on $[0, 1]$ by construction, independently of the sufficient local existence interval obtained from the a priori estimate. The experiment verifies reconstruction accuracy for exact data and provides a mesh-refinement check, without asserting stability with respect to noisy observations or global solvability for general data.

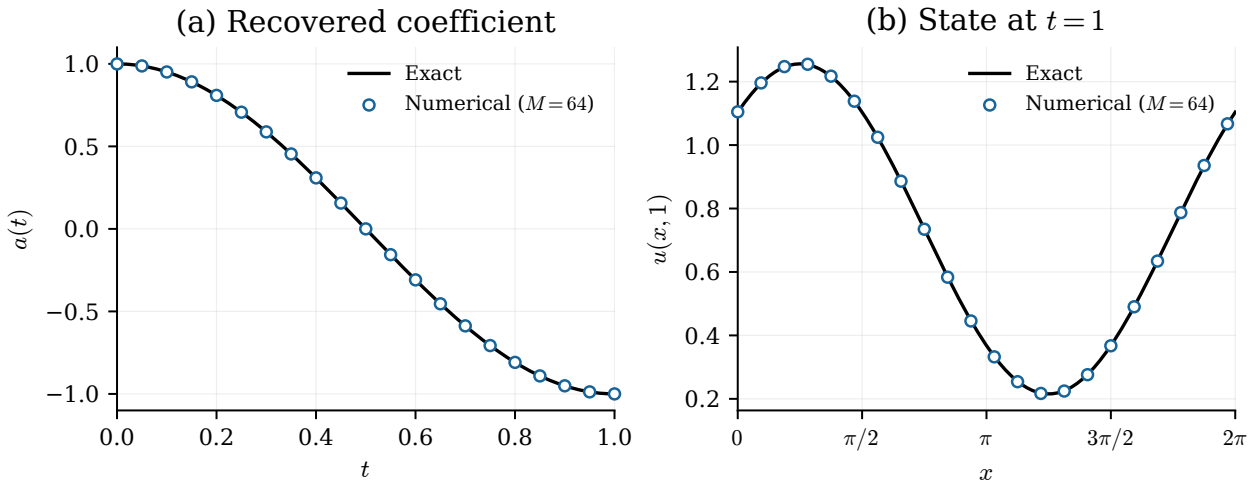


Fig. 1: Exact solutions (solid lines) and numerical values (markers) for $M = 64$: (a) the recovered coefficient; (b) the state at $t = 1$. Markers are shown at selected output times and spatial nodes.

Conclusion

The weighted integral observation yields an explicit reconstruction formula for $a(t)$ and reduces the inverse problem to a nonlinear loaded Burgers equation. The resulting nonlocal term involving u^2 produces a contribution of order $\|u\|_{L^2}^6$ in the basic energy estimate, which is controlled locally in time by Bihari's inequality. For every $T_0 < T^*$, the reduced problem has a unique strong solution, the reconstructed coefficient belongs to $L^2(0, T_0)$, and the prescribed identity $\int_0^\ell N(x)u(x,t)\,dx = E(t)$ is recovered.

The threshold T^* is produced by the present a priori estimate and is not asserted to be the maximal existence time. The present estimates establish local existence and uniqueness, but do not settle global existence on the full prescribed time interval.

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Authors contribution

K.M.: investigation, formal analysis, writing – original draft preparation. **Y.M.**: methodology, validation, writing – original draft preparation, writing – review and editing. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest

The authors declare that they have no conflict of interest.

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